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Lipschitz stability for an inverse problem for the 2D-Sellers model on a manifold

Abstract. In this paper, we are interested in some inverse problem that consists in recovering the so-called insolation function in the 2-D Sellers model on a Riemannian manifold that materializes the Earth's surface. For this nonlinear problem, we obtain a Lipschitz stability result in the spirit of the result by Imanuvilov-Yamamoto in the case of the determination of the source term in the linear heat equation. The paper complements an analogous study by Tort-Vancostenoble in the case of the 1-D Sellers model.

Keywords. PDEs on manifolds, nonlinear parabolic equations, climate models, inverse problems, Carleman estimates.

Mathematics Subject Classification (2010): 58J35, 35K55.

1 - Introduction

1.1 - The Sellers model

In this paper, we are interested in some inverse problem that consists in recovering the so-called insolation function q in the nonlinear Sellers climate model. The case of the 1-D Sellers model has been considered in [39]. Here we focus on the 2-D Sellers model on the Earth's surface:

$$(1) \quad \begin{cases} u_t - \Delta_{\mathcal{M}} u = \overbrace{r(t)q(x)\beta(u)}^{\mathcal{R}_a(t,x,u)} - \overbrace{\varepsilon(u)u|u|^3}^{\mathcal{R}_e(u)} & x \in \mathcal{M}, t > 0, \\ u(0, x) = u^0(x) & x \in \mathcal{M}. \end{cases}$$

The Earth's surface is materialized by a sub-manifold \mathcal{M} of \mathbb{R}^3 which is assumed to be of dimension 2, compact, connected, oriented, and without boundary. The function u represents the mean annual or seasonal temperature, and $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on \mathcal{M} . The right hand side of the equation corresponds to

- the mean radiation flux depending on the solar radiation \mathcal{R}_a ,
- and the radiation \mathcal{R}_e emitted by the Earth.

For more details on the model, we refer the reader to [14, 15] and the references therein.

1.2 - Assumptions and main results

1.2.1 - Geometrical and regularity assumptions

Consider a sub-manifold \mathcal{M} of \mathbb{R}^3 which is assumed to be of dimension 2, compact, connected, oriented, and without boundary.

Throughout this paper, we make the following assumptions (that are compatible with the applications, see [39]):

Assumption 1.1.

- (2) $\beta \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\beta' \in L^\infty(\mathbb{R})$, β' is k -Lipschitz ($k > 0$),
- (3) $\exists \beta_{min} > 0, \forall u \in \mathbb{R}, \beta(u) \geq \beta_{min}$,
- (4) $q \in L^\infty(\mathcal{M})$, $q \geq 0$,
- (5) $r \in \mathcal{C}^1(\mathbb{R})$ is τ -periodic ($\tau > 0$),
- (6) $\exists r_{min} > 0, \forall t \in \mathbb{R}, r(t) \geq r_{min}$,
- (7) $\varepsilon \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, ε' is K -Lipschitz ($K > 0$),
- (8) $\exists \varepsilon_{min} > 0, \forall u \in \mathbb{R}, \varepsilon(u) > \varepsilon_{min}$.

We also make the following geometrical assumption:

Assumption 1.2. *Let ω be a non empty open subset of \mathcal{M} . We assume that there exists a weight function $\psi \in C^\infty(\mathcal{M})$ that satisfies:*

$$(9) \quad \nabla \psi(m) = 0 \quad \implies \quad m \in \omega.$$

(Here ∇ stands for the usual gradient associated to the Riemannian structure, see section 2.)

1.2.2 - Main results

As in [39], our aim is to prove some Lipschitz stability result for the inverse problem that consists in recovering the insolation function q in (1) from partial measurements. We introduce

- the set of admissible initial conditions: given $A > 0$, we consider \mathcal{U}_A :

$$(10) \quad \mathcal{U}_A := \{u^0 \in D(\Delta_{\mathcal{M}}) \cap L^\infty(\mathcal{M}) : \Delta_{\mathcal{M}}u^0 \in L^\infty(\mathcal{M}), \\ \|u_0\|_{L^\infty(\mathcal{M})} + \|\Delta_{\mathcal{M}}u_0\|_{L^\infty(\mathcal{M})} \leq A\},$$

where $D(\Delta_{\mathcal{M}})$ is the domain of the Laplace-Beltrami operator in $L^2(\mathcal{M})$ (we will recall the definition of $\Delta_{\mathcal{M}}$ and $D(\Delta_{\mathcal{M}})$ in section 2),

- and the set of admissible coefficients: given $B > 0$, we consider

$$(11) \quad \mathcal{Q}_B := \{q \in L^\infty(\mathcal{M}) : \|q\|_{L^\infty(\mathcal{M})} \leq B\}.$$

The main result of this paper is the following one:

Theorem 1.1. *Consider*

- $t_0 \in [0, T)$ and $T' \in (t_0, T)$,
- $A > 0$ and $u_1^0, u_2^0 \in \mathcal{U}_A$ (defined in (10)),
- $B > 0$ and $q_1, q_2 \in \mathcal{Q}_B$ (defined in (11)),
- u_1 the solution of (1) associated to q_1 and the initial condition u_1^0 , and u_2 the solution of (1) associated to q_2 and the initial condition u_2^0 ,
- $\omega \subset \mathcal{M}$ such that Assumption 1.2 holds.

Then there exists $C(t_0, T', T, A, B) > 0$ such that, for all $u_1^0, u_2^0 \in \mathcal{U}_A$, for all $q_1, q_2 \in \mathcal{Q}_B$, the corresponding solutions u_1, u_2 of problem (1) satisfy

$$(12) \quad \|q_1 - q_2\|_{L^2(\mathcal{M})}^2 \leq C(\|u_1(T') - u_2(T')\|_{D(\Delta_{\mathcal{M}})}^2 + \|u_{1,t} - u_{2,t}\|_{L^2((t_0, T) \times \omega)}^2).$$

We complete Theorem 1.1 by the following remarks:

- the geometrical assumption 1.2 is satisfied when \mathcal{M} is simply connected (hence in particular for the sphere \mathbb{S}^2):

Proposition 1.1. *Additionally, assume that \mathcal{M} is simply connected. Consider any ω non empty open set of \mathcal{M} . Then Assumption 1.2 is full-filled: there exists some smooth function ψ that satisfies (9).*

- as a consequence of the stability estimate (12) and of the Carleman estimate that we will prove in Theorem 3.1, we obtain a weighted stability estimate for the difference $u_1 - u_2$: there exists $C'(t_0, T', T, A, B) > 0$ such that,

$$(13) \quad \|e^{-R\sigma}(u_1 - u_2)\|_{L^2((0, T) \times \mathcal{M})}^2 \\ \leq C'(\|u_1(T') - u_2(T')\|_{D(\Delta_{\mathcal{M}})}^2 + \|u_{1,t} - u_{2,t}\|_{L^2((t_0, T) \times \omega)}^2),$$

where σ is the weight function defined in (35).

The proof of Theorem 1.1 is based on

- global Carleman estimates for the heat equation (see Theorem 3.1),
- maximum principles, useful to study this nonlinear problem (see Theorem 5.2 and Corollary 5.1),
- and Riemannian geometry tools, since we are in the manifold setting.

The proof of Proposition 1.1 is based on

- a direct construction when \mathcal{M} is the sphere \mathbb{S}^2 , using the stereographic projection,
- the celebrated uniformisation theorem ([1, 40]) when \mathcal{M} is simply connected.

(Remark: we do not know if the result remains true if $T' = T$.)

1.3 - Relation to literature

A similar problem is considered in [39], where stability estimates for the insolation function are obtained combining Carleman estimates with maximum principles, the main difference with the present paper being that the problem in [39] is stated and studied in the interval $(-1, 1)$ and with a degenerate diffusion coefficient.

Global Carleman estimates have proved their usefulness in the context of null controllability, unique continuation properties, we refer in particular to [25] for the seminal paper on the null controllability of the heat equation on compact manifolds, to [21, 18] for Carleman estimates in a general setting, to [29] for unique continuation properties for the heat equation on non compact manifolds, to [31, 32] for uniqueness results for manifolds with poles, to [6] for stabilization results of the wave equation on manifolds.

Concerning inverse problems, Isakov [23] provided many results for elliptic, hyperbolic and parabolic problems. Imanuvilov-Yamamoto [22] developed a general method to solve some standard inverse source problem for the linear heat equation, using global Carleman estimates. In the context of semilinear parabolic equations in bounded domains of \mathbb{R}^n , we can also mention in particular [33, 34], where uniqueness results are obtained under analyticity assumptions, [11], that combines also Carleman estimates with maximum principles to obtain stability estimates (for two coefficients but under rather strong assumptions on the time interval of observation).

1.4 - Contents of the paper

Let us now precise the organization of the paper.

- First of all, since the equation is stated on a surface, the operators needed for the definitions and the computations (Laplacian, divergence, gradient) are defined through a Riemannian metric associated to the surface. So, in order to fix the ideas, we begin in section 2 by introducing all the notations and recalling all the definitions and the properties useful for computations on manifolds.
- Next, in section 3, we state and prove some global Carleman estimate for the heat operator on a compact manifold without boundary. This will be a crucial tool in order to study our inverse problem.
- In section 4, we prove Proposition 1.1, studying first the case of the sphere \mathbb{S}^2 , and then the general case of a simply connected manifold.
- In section 5, we make some preliminary studies concerning the 2-D Sellers model on the manifold \mathcal{M} (well-posedness of course but also regularity results and maximum principles that will also be essential in the proof of the stability result for the inverse problem).
- Finally, in section 6, we prove Theorem 1.1.

2 - Notations, computations and heat operator on manifolds

In this section, we fix the notations and recall some classical definitions and results on manifolds. We refer in particular to [9, 19].

2.1 - Notions on topological and Riemannian manifolds

Charts, atlas, smooth manifolds. A topological manifold \mathcal{M} of dimension n is a separated topological space such that every point $m \in \mathcal{M}$ has a neighbourhood U which is homeomorphic to some connected open subset of \mathbb{R}^n . For any neighbourhood U and any homeomorphism $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$, we say that (U, ϕ) is a coordinate chart on U . A set $(U_i, \phi_i)_{i \in I}$ such that the set of neighbourhoods U_i covers \mathcal{M} is called an atlas on \mathcal{M} .

When two coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2) have overlapping domains U_1 and U_2 , there is a transition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \rightarrow \phi_2(U_1 \cap U_2)$ which is a homeomorphism between two open subsets of \mathbb{R}^n . A smooth manifold (or a \mathcal{C}^∞ -manifold) is a manifold for which all the transition maps are \mathcal{C}^∞ -diffeomorphisms. *In the following, \mathcal{M} always denotes a smooth manifold.*

Tangent vectors, tangent spaces, basis. A tangent vector at $m \in \mathcal{M}$ is an equivalence class $[c]$ of differentiable curves $c : I \rightarrow \mathcal{M}$ with I sub-interval of \mathbb{R} such that $0 \in I$ and $c(0) = m$, modulo the equivalence relation of first order contact between curves i.e.

$$c_1 \equiv c_2 \Leftrightarrow c_1(0) = c_2(0) = m \text{ and } (\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$$

for every coordinate chart (U, ϕ) such that $m \in U$.

The tangent space to \mathcal{M} at m , denoted by $T_m\mathcal{M}$, is the collection of all tangent vectors at m . Let (U, ϕ) be a chart such that $m \in U$ and define the map θ_ϕ :

$$\begin{aligned} \theta_\phi : T_m\mathcal{M} &\longrightarrow \mathbb{R}^n \\ [c] &\longmapsto (\phi \circ c)'(0). \end{aligned}$$

Then $\theta_\phi : T_m\mathcal{M} \rightarrow \mathbb{R}^n$ is a bijection (see [24, p. 64]). Therefore $T_m\mathcal{M}$ can be endowed with a structure of a vector space. It is possible to exhibit a basis $(\partial_i(m))_{1 \leq i \leq n}$ of $T_m\mathcal{M}$ in the following way. Let $m \in \mathcal{M}$ and (U, ϕ) be a chart of \mathcal{M} such that $m \in U$. In $\phi(U) \subset \mathbb{R}^n$, we have n coordinate fields:

$$\forall 1 \leq i \leq n, \quad \frac{\partial}{\partial x_i} : \begin{cases} \phi(U) & \rightarrow \mathbb{R}^n \\ x & \mapsto (0, 0, \dots, 1, 0, \dots, 0) \end{cases}$$

where 1 is at position i . Then we set

$$\forall 1 \leq i \leq n, \quad \partial_i(m) = \theta_\phi^{-1} \left(\frac{\partial}{\partial x_i}(\phi(m)) \right).$$

Regularity, derivatives. A continuous function $f : \mathcal{M} \rightarrow \mathbb{R}$ is of class C^k if, for any $m \in \mathcal{M}$ and for any chart (U, ϕ) with $m \in U$, $f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is of class C^k .

Assume $f : \mathcal{M} \rightarrow \mathbb{R}$ is of class C^1 and $m \in \mathcal{M}$. For any vector $\xi \in T_m\mathcal{M}$, the directional derivative of f at m along ξ , denoted by $\xi.f_m$ or $(\xi.f)(m)$, is:

$$\xi.f_m := (f \circ \omega)'(0),$$

where $\omega : I \rightarrow \mathcal{M}$ satisfies $\omega(0) = m$ and $\omega'(0) = \xi$. For all $m \in \mathcal{M}$, the map $\alpha_m : \xi \mapsto \xi.f_m$ is a linear form on $T_m\mathcal{M}$.

Let us explicit now the derivatives of f along each vector of the basis of the tangent space. Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be regular, $m \in \mathcal{M}$ and (U, ϕ) be a chart of \mathcal{M} containing m . Then $\partial_i(m).f_m = (f \circ \omega_i)'(0)$ where $\omega_i : t \mapsto \phi^{-1}(\phi(m) + t(0, \dots, 1, 0, \dots, 0))$. Moreover $(f \circ \omega_i)(t) = (f \circ \phi^{-1})(\phi(m) + t(0, \dots, 1, 0, \dots, 0))$. Hence $\partial_i(m).f_m = \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(m))$.

Tangent bundle, vector fields. The tangent bundle of a differentiable manifold \mathcal{M} is a manifold $T\mathcal{M}$, which assembles all the tangent vectors at \mathcal{M} , that is $T\mathcal{M} = \cup_{m \in \mathcal{M}} T_m\mathcal{M} = \cup_{m \in \mathcal{M}} \{m\} \times T_m\mathcal{M}$. We denote by $\Pi : (m, \xi) \in T\mathcal{M} \rightarrow m \in \mathcal{M}$ the canonical projection.

Vector fields, derivative along a vector field. A vector field X on a manifold \mathcal{M} is a regular map $X : \mathcal{M} \rightarrow T\mathcal{M}$ such that $\Pi \circ X = Id_{\mathcal{M}}$ (i.e. $X(m) \in T_m\mathcal{M}$ for any $m \in \mathcal{M}$).

Let $X : \mathcal{M} \rightarrow T\mathcal{M}$ be a vector field on \mathcal{M} and $f : \mathcal{M} \rightarrow \mathbb{R}$ regular. We define $X.f : \mathcal{M} \rightarrow \mathbb{R}$ the derivative of f along X in the following way: for all $m \in \mathcal{M}$, for any chart (U, ϕ) with $m \in U$,

$$(X.f)(m) = (f \circ \omega)'(0),$$

where $\omega : I \rightarrow \mathcal{M}$ satisfies $\omega(0) = m$ and $\omega'(0) = X(m)$.

Lie bracket of two vector fields. The Lie bracket of two vector fields X and Y is a third vector field $[X, Y]$ defined by

$$\forall f : \mathcal{M} \longrightarrow \mathbb{R}, \quad [X, Y].f := X.(Y.f) - Y.(X.f).$$

For the computations of Carleman inequalities, we will need the following result (see e.g. the proof in [38]): for all $1 \leq i, j \leq n$, then $[\partial_i, \partial_j] = 0$.

Riemannian manifolds. Let \mathcal{M} be a smooth manifold. A Riemannian metric on \mathcal{M} is a family $g = (g_m)_{m \in \mathcal{M}}$ of (positive definite) inner products $g_m := \langle \cdot, \cdot \rangle_m$ on $T_m \mathcal{M}$ for all $m \in \mathcal{M}$. Moreover the map $m \mapsto g_m$ is assumed to be regular. Then we say that (\mathcal{M}, g) is a Riemannian manifold.

Let $m \in \mathcal{M}$ and (U, ϕ) be a chart containing m , the matrix $G = (g_{j,k}) \in \mathcal{M}(n, \mathbb{R})$ of the scalar product $g_m := \langle \cdot, \cdot \rangle_m$ in the basis of $T_m \mathcal{M}$ is given by:

$$(14) \quad g_{j,k} := \langle \partial_j, \partial_k \rangle_m.$$

As $\langle \cdot, \cdot \rangle_m$ is a scalar product, G is invertible. We also denote

$$(15) \quad g := \det(G) \neq 0 \quad \text{and} \quad G^{-1} := (g^{i,l}).$$

Connexion on a manifold. A connexion on a manifold \mathcal{M} is an operator D which associates to any vectors fields X and Y a third vector field $D_X Y$ on \mathcal{M} such that, for all X, Y, Z vector fields and for all regular function $f : \mathcal{M} \longrightarrow \mathbb{R}$,

$$(16) \quad D_X(Y + Z) = D_X Y + D_X Z,$$

$$(17) \quad D_X(fY) = fD_X Y + (X.f)Y,$$

$$(18) \quad \xi \mapsto D_\xi Y \text{ is linear on } T_m \mathcal{M} \text{ for all } m \in \mathcal{M}.$$

Levi-Civita connexion. From the fundamental theorem of Riemannian geometry, there is a unique connection Γ , called Levi-Civita connection, on the tangent bundle of a Riemannian manifold (\mathcal{M}, g) such that:

- Γ is torsion-free, i.e. for all vectors fields X and Y on \mathcal{M} , then

$$(19) \quad \Gamma_X Y - \Gamma_Y X = [X, Y];$$

- and Γ preserves the Riemannian metric g , i.e., for all vector fields X, Y, Z ,

$$(20) \quad X.g(Y, Z) = g(\Gamma_X Y, Z) + g(Y, \Gamma_X Z).$$

Gradient. Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be a regular function. The gradient of f , denoted by $\text{grad}(f)$ or ∇f , is the vector field on \mathcal{M} defined for any $m \in \mathcal{M}$ as the unique vector $\text{grad}(f)_m$ such that

$$\forall \xi \in T_m \mathcal{M}, \quad \langle \text{grad}(f)_m, \xi \rangle_m = (\xi.f)(m).$$

where $(\xi.f)(m)$ is the derivative of f at m in the direction ξ .

Divergence. For X vector field on \mathcal{M} , we define the function $\text{div}(X)$ on \mathcal{M} by

$$\forall m \in \mathcal{M}, \quad \text{div}(X)(m) := \text{Tr}(\xi \mapsto \Gamma_\xi X), \quad \text{where } \xi \text{ belongs to } T_m \mathcal{M}.$$

Laplacian. Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a regular function. The Laplacian of f is the function Δf defined by:

$$(21) \quad \forall m \in \mathcal{M}, \quad \Delta f_m := \operatorname{div}(\operatorname{grad}(f)_m)(m).$$

Hessian. Let f be a regular function on \mathcal{M} . Then, for all $m \in \mathcal{M}$, the Hessian of f at m is the bilinear form defined by:

$$(22) \quad \forall (\xi_1, \xi_2) \in (T_m \mathcal{M})^2, \quad (\operatorname{Hess}(f)_m)(\xi_1, \xi_2) := \langle \Gamma_{\xi_1} \nabla f_m, \xi_2 \rangle.$$

Rules for computations.

$$(23) \quad \operatorname{grad}(fh) = f \operatorname{grad}(h) + h \operatorname{grad}(f),$$

$$(24) \quad \operatorname{div}(X + Y) = \operatorname{div}(X) + \operatorname{div}(Y),$$

$$(25) \quad \operatorname{div}(fX) = f \operatorname{div}(X) + \langle \operatorname{grad}(f), X \rangle.$$

Expressions in local coordinates. It can be proved (see [9] p. 4-5), that for $f : \mathcal{M} \rightarrow \mathbb{R}$ regular, X regular vector field on \mathcal{M} and for all $m \in \mathcal{M}$, then

$$(26) \quad \operatorname{grad}(f)_m := \sum_{k=1}^n \sum_{l=1}^n g^{k,l} \partial_l \cdot f \partial_k.$$

$$(27) \quad \operatorname{div}(X(m))_m = \frac{1}{\sqrt{g}} \sum_{i=1}^n \partial_i \cdot (\eta^i \sqrt{g}) \quad \text{if } X = \sum_{i=1}^n \eta^i \partial_i.$$

$$(28) \quad \Delta f = \frac{1}{\sqrt{g}} \sum_{i=1}^n \sum_{l=1}^n \partial_i \cdot (g^{i,l} \sqrt{g} \partial_l \cdot f).$$

2.2 - Integration on a compact manifold and Sobolev spaces

In the following, \mathcal{M} is a compact connected oriented Riemannian manifold without boundary. With the Riemann metric is associated an integration theory, the measure $d\mathcal{M}$ being defined globally on \mathcal{M} with the help of a partition of unity (see [9], p. 5-6).

Then we have ([9] p. 6):

Proposition 2.1.

$$(29) \quad \forall X : \mathcal{M} \rightarrow T\mathcal{M} \text{ regular}, \quad \int_{\mathcal{M}} \operatorname{div}(X) d\mathcal{M} = 0,$$

and

$$(30) \quad \forall h, f : \mathcal{M} \rightarrow \mathbb{R} \text{ regular}, \quad \int_{\mathcal{M}} h \Delta f + \langle \operatorname{grad}(h), \operatorname{grad}(f) \rangle d\mathcal{M} = 0.$$

L^2 -spaces. A function $f : \mathcal{M} \rightarrow \mathbb{R}$ is measurable if, for any chart (U, Φ) , $f \circ \Phi^{-1}$ is measurable. The space $L^2(\mathcal{M})$, constituted of the measurable functions

$f : \mathcal{M} \rightarrow \mathbb{R}$ such that $\int_{\mathcal{M}} |f|^2 d\mathcal{M}$ is finite, is a Hilbert space for the scalar product

$$(f, h)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} fhd\mathcal{M}.$$

Let X and Y be two regular vector fields. We define their scalar product by

$$(31) \quad (X, Y)_{L^2(T\mathcal{M})} := \int_{\mathcal{M}} \langle X, Y \rangle d\mathcal{M}.$$

Then $L^2(T\mathcal{M})$ is defined as the completion for the associated norm of the set of regular vector fields. It is a Hilbert space constituted of the vector fields whose components in the local basis of the tangent space are measurable and such that the integral $\int_{\mathcal{M}} |X|^2 d\mathcal{M}$ is finite.

Sobolev space $H^1(\mathcal{M})$. Let \mathcal{M} be a compact Riemannian manifold of dimension n without boundary. If $f \in C(\mathcal{M})$ then $f \in L^2(\mathcal{M})$. As \mathcal{M} is compact, the set of compactly supported C^∞ -functions on \mathcal{M} is simply the set of C^∞ -functions on \mathcal{M} and it is dense in $L^2(\mathcal{M})$ ([2] p. 79).

We define on $C^\infty(\mathcal{M})$ the scalar product $(\cdot, \cdot)_1$ in the following way:

$$\forall f, \tilde{f} \in C^\infty(\mathcal{M}), \quad (f, \tilde{f})_1 := (f, \tilde{f})_{L^2(\mathcal{M})} + (\nabla f, \nabla \tilde{f})_{L^2(T\mathcal{M})}.$$

$H^1(\mathcal{M})$ is defined as the completion of $C^\infty(\mathcal{M})$ for the norm associated to $(\cdot, \cdot)_1$.

Weak derivative. Let $f \in L^2(\mathcal{M})$ be given. f admits a weak derivative in $L^2(T\mathcal{M})$ if there exists a vector field $\varsigma \in L^2(T\mathcal{M})$ such that, for any regular vector field X ,

$$(32) \quad \int_{\mathcal{M}} f \operatorname{div}(X) d\mathcal{M} = - \int_{\mathcal{M}} \langle \varsigma, X \rangle d\mathcal{M}.$$

Then we denote $\varsigma = \nabla f$. Of course, if $f \in C^1(\mathcal{M})$, then it coincides with the classical gradient of f . $H^1(\mathcal{M})$ is also the set of functions in $L^2(\mathcal{M})$ having a weak derivative in $L^2(T\mathcal{M})$. It is endowed with the scalar product $(\cdot, \cdot)_1$.

Let us end this subsection by a general result (see [35] for its proof), that will be useful for the proofs of maximum principles:

Proposition 2.2. *Let $(U_i, \Phi_i)_{1 \leq i \leq N}$ be an atlas of \mathcal{M} . Then $f \in H^1(\mathcal{M})$ if and only if, for all $1 \leq i \leq N$, $f \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$.*

2.3 - The heat equation on a Riemannian manifold

The Laplace Beltrami operator in $L^2(\mathcal{M})$. $f \in L^2(\mathcal{M})$ admits a weak Laplacian in $L^2(\mathcal{M})$ if there exists $F \in L^2(\mathcal{M})$ such that, for any $\Phi \in C^\infty(\mathcal{M})$,

$$(F, \Phi)_{L^2(\mathcal{M})} = (f, \Delta \Phi)_{L^2(\mathcal{M})}.$$

Then we denote $F = \Delta f$. Of course, if $f \in C^2(\mathcal{M})$, the weak Laplacian of f coincides with the classical one.

Proposition 2.3. *Let $f \in H^1(\mathcal{M})$ admitting a weak Laplacian in $L^2(\mathcal{M})$. Then, for all $\Phi \in H^1(\mathcal{M})$, $(\Delta f, \Phi)_{L^2(\mathcal{M})} = -(\nabla f, \nabla \Phi)_{L^2(T\mathcal{M})}$.*

The Laplace Beltrami operator is the unbounded operator in $L^2(\mathcal{M})$ defined by the domain $D(\Delta) := \{u \in H^1(\mathcal{M}) \text{ having a weak Laplacian in } L^2(\mathcal{M})\}$ and the weak Laplacian. Note that, as $C^\infty(\mathcal{M}) \subset D(\Delta)$, $D(\Delta)$ is dense in $L^2(\mathcal{M})$. For all $u, v \in H^1(\mathcal{M})$, we define $a(u, v) := \int_{\mathcal{M}} \langle \nabla u, \nabla v \rangle d\mathcal{M}$. Then we define an unbounded operator in $L^2(\mathcal{M})$ by:

$$D(A) := \{u \in H^1(\mathcal{M}) : w \in H^1(\mathcal{M}) \mapsto a(u, w) \text{ is } C^0 \text{ for the norm } \|\cdot\|_{L^2(\mathcal{M})}\},$$

and for all $u \in D(A)$, $v \in H^1(\mathcal{M})$, $(Au, v)_{L^2(\mathcal{M})} = -a(u, v)$. The operator $(A, D(A))$ coincides with the Laplace-Beltrami operator $(\Delta, D(\Delta))$. Moreover, $(\Delta, D(\Delta))$ is the infinitesimal generator of an analytical semigroup.

The heat equation on a compact Riemannian manifold. We consider

$$(33) \quad \begin{cases} u_t - \Delta u = f & (0, T) \times \mathcal{M}, \\ u(0) = u_0 & \mathcal{M}. \end{cases}$$

The interpolation space $[D(\Delta), L^2(\mathcal{M})]_{\frac{1}{2}}$ is $H^1(\mathcal{M})$, (see [27, Prop. 21 p. 22]).

Theorem 2.1. *If $u_0 \in D(\Delta)$ et $f \in H^1(0, T; L^2(\mathcal{M}))$, (33) has a unique classical solution $u \in C([0, T], D(\Delta)) \cap C^1([0, T]; L^2(\mathcal{M}))$.*

If $u_0 \in H^1(\mathcal{M})$ et $f \in L^2(0, T; L^2(\mathcal{M}))$, (33) has a unique solution such that $u \in L^2(0, T; D(\Delta)) \cap H^1(0, T; L^2(\mathcal{M}))$.

If $u_0 \in L^2(\mathcal{M})$ et $f \in L^2(0, T; L^2(\mathcal{M}))$, (33) has a unique weak solution such that $u \in C([0, T]; L^2(\mathcal{M})) \cap L^2(0, T; H^1(\mathcal{M}))$, i.e. for any $v \in H^1(\mathcal{M})$,

$$(34) \quad \begin{cases} \frac{d}{dt} (u(t), v)_{L^2(\mathcal{M})} + \int_{\mathcal{M}} \langle \nabla u(t), \nabla v \rangle d\mathcal{M} = (f(t), v)_{L^2(\mathcal{M})}, \\ u(0) = u_0. \end{cases}$$

Moreover, for all $\epsilon > 0$, $u \in L^2(\epsilon, T; D(\Delta)) \cap H^1(\epsilon, T; L^2(\mathcal{M}))$.

Proof. : apply Prop. 3.3 p. 68, Theorem 3.1 p. 80 and Prop. 3.8 of [3]. \square

In order to treat later the questions of inverse problems, we will need some more regularity results for the time derivative of the solution:

Proposition 2.4. *Let $u_0 \in D(\Delta)$ and $f \in H^1(0, T; L^2(\mathcal{M}))$ be given. Let u be the classical solution of (33) associated to u_0 and f . Then $z := u_t \in L^2(0, T; H^1(\mathcal{M}))$ and z is the weak solution of*

$$\begin{cases} z_t - \Delta z = f_t & (0, T) \times \mathcal{M}, \\ z(0) = \Delta u_0 + f(0) & \mathcal{M}. \end{cases}$$

For the proof, we refer for example to [38, Proposition 2.5]. Finally, we end this section with a result concerning regular solutions (see [10] p. 139):

Theorem 2.2. *Let $u_0 \in C^\infty(\mathcal{M})$ and $f \in C^\infty((0, T) \times \mathcal{M})$ be given. Then (33) has a unique regular solution.*

3 - Global Carleman estimates for the heat operator on a compact manifold without boundary

In this section, we state and prove some global Carleman estimate for the heat operator on a compact Riemannian manifold without boundary \mathcal{M} with a locally distributed observation in some non empty open set ω of \mathcal{M} .

3.1 - Global Carleman estimate

We define the heat operator on \mathcal{M} :

$$\forall z \in C([0, T]; D(\Delta_{\mathcal{M}})) \cap C^1([0, T]; L^2(\mathcal{M})), \quad Pz := z_t - \Delta_{\mathcal{M}}z.$$

We denote $Q_{\mathcal{M}}^{0, T} := (0, T) \times \mathcal{M}$, $Q_{\omega}^{0, T} := (0, T) \times \omega$ and we consider $R > 0$, $S > 0$, ψ satisfying Assumption 1.2. Then we introduce first $0 < T_0 < T_1 < T$ and $\theta : (0, T) \rightarrow \mathbb{R}_+^*$ smooth, convex, such that

$$\theta(t) = \begin{cases} \frac{1}{t}, & t \in (0, T_0) \\ \frac{1}{T-t}, & t \in (T_1, T), \end{cases}$$

next

$$\forall (t, x) \in Q_{\mathcal{M}}^{0, T}, \quad p(x) := e^{2S\|\psi\|_{\infty}} - e^{S\psi(x)}, \quad \rho(t, x) := RS\theta e^{S\psi},$$

and finally

$$(35) \quad \forall (t, x) \in Q_{\mathcal{M}}^{0, T}, \quad \sigma(t, x) := \theta(t)p(x).$$

And we prove the following

Theorem 3.1. *Let ω be such that Assumption 1.2 holds. There exists constants $C = C(T, T_0, T_1, \omega) > 0$, $R_0 = R_0(T, T_0, T_1, \omega) > 0$, $S_0 = S_0(T, T_0, T_1, \omega) > 0$ such that, for all $S \geq S_0$ and all $R \geq R_0 e^{2S\|\psi\|_{\infty}}$, we have for all $z \in C([0, T]; D(\Delta_{\mathcal{M}})) \cap C^1([0, T]; L^2(\mathcal{M}))$*

$$(36) \quad \iint_{Q_{\mathcal{M}}^{0, T}} \rho^3 e^{-2R\sigma} z^2 + \iint_{Q_{\mathcal{M}}^{0, T}} \rho e^{-2R\sigma} |\nabla z|^2 + \iint_{Q_{\mathcal{M}}^{0, T}} \frac{1}{\rho} e^{-2R\sigma} z_t^2 \\ \leq C \left(\|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0, T})}^2 + \iint_{Q_{\omega}^{0, T}} \rho^3 e^{-2R\sigma} z^2 \right).$$

The proof of Theorem 3.1 is classical. It follows combining the proof of the Carleman estimate for the heat operator in a bounded domain of \mathbb{R}^n with the properties of the operators divergence, gradient, laplacian on the manifold \mathcal{M} . We refer to [38] for detailed proofs, and we mention here the main properties and steps:

3.2 - The basic properties

The following property are basic:

Lemma 3.1. *For any regular function h on \mathcal{M} , one has:*

$$\begin{aligned}
 (37) \quad & \nabla(h^2) = 2h\nabla h, \\
 (38) \quad & \nabla e^h = e^h\nabla h, \\
 (39) \quad & \Delta(h^2) = 2h\Delta h + 2|\nabla h|^2. \\
 (40) \quad & \langle \nabla(|\nabla h|^2), \nabla h \rangle = 2\text{Hess}(h)(\nabla h, \nabla h).
 \end{aligned}$$

Lemma 3.2. *For any $w \in C^\infty((0, T) \times \mathcal{M})$, one has:*

$$\begin{aligned}
 (41) \quad & \nabla(w_t) = (\nabla w)_t. \\
 (42) \quad & \text{Hess}(w)(\nabla w, \nabla p) = \text{Hess}(w)(\nabla p, \nabla w).
 \end{aligned}$$

Proof of Lemmas 3.1 and 3.2. The proofs are classical and derive from the basic material of Chavel [9], and can be found in [38], lemmas 3.3.4-3.3.7, p. 128-132. As an exercise, we prove (38): let $m \in \mathcal{M}$, (U, ϕ) be a chart such that $m \in U$ and $\xi \in T_m\mathcal{M}$. Consider $\omega : I \rightarrow \mathcal{M}$ a smooth curve with $0 \in I$, $\omega(0) = m$ and $\omega'(0) = \xi$. Then, if we set $f = e^h$, we have (using the definition of the gradient):

$$\begin{aligned}
 \langle \nabla f(m), \xi \rangle_m &= (\xi.f)(m) = (f \circ \omega)'(0) = (e^{h \circ \omega})'(0) \\
 &= (e^{h \circ \omega})(0)(h \circ \omega)'(0) = e^{h(m)}(h \circ \omega)'(0)
 \end{aligned}$$

and, on the other side, $\langle \nabla h(m), \xi \rangle_m = (\xi.h)(m) = (h \circ \omega)'(0)$. So, identifying the two expressions, we get $\nabla e^h = e^h\nabla h$, hence (38). The other proofs are in the same spirit. \square

3.3 - The main steps to prove Theorem 3.1

First we note that it is sufficient to prove (36) for regular functions. Indeed we have the following result (see the proof in [38]):

Lemma 3.3. *Let $u \in C([0, T]; D(\Delta)) \cap C^1([0, T]; L^2(\mathcal{M}))$ be given. Consider $(f_n)_n \subset \mathcal{D}((0, T) \times \mathcal{M})$ converging to Pu in $L^2((0, T) \times \mathcal{M})$ and $(u_{0,n})_n \subset C^\infty(\mathcal{M})$ converging to $u_0 \in H^1(\mathcal{M})$. We denote by u_n the regular solution (given in Theorem 2.2) of (33) associated to $u_{0,n}$ and f_n . Then we have*

$$\begin{aligned}
 u_n &\longrightarrow u \text{ in } L^2(0, T; L^2(\mathcal{M})), & \nabla u_n &\longrightarrow \nabla u \text{ in } L^2(0, T; L^2(T\mathcal{M})), \\
 \text{and} & & (u_n)_t &\longrightarrow u_t \text{ in } L^2(0, T; L^2(\mathcal{M})).
 \end{aligned}$$

3.3.1 - The decomposition of the weighted heat operator

So let $z \in C^\infty((0, T) \times \mathcal{M}) \cap C([0, T] \times \mathcal{M})$ be given and let us prove that z satisfies (36). We set $w := ze^{-R\sigma}$. Then we have

$$(43) \quad (we^{R\sigma})_t - \Delta(we^{R\sigma}) = P(we^{R\sigma}) = Pz.$$

We have $(we^{R\sigma})_t = w_t e^{R\sigma} + R\theta_t p w e^{R\sigma}$ and

$$\begin{aligned} \Delta(we^{R\sigma}) &= \operatorname{div}(\nabla(we^{R\sigma})) = \operatorname{div}(\nabla w e^{R\sigma}) + \operatorname{div}(w \nabla(e^{R\sigma})) \\ &= e^{R\sigma} \Delta w + 2\langle \nabla(w), R e^{R\sigma} \nabla \sigma \rangle + \Delta(e^{R\sigma})w. \end{aligned}$$

Of course $\nabla \sigma = \theta(t) \nabla p$. And $\Delta(e^{R\sigma}) = \operatorname{div}(\nabla(e^{R\sigma})) = \operatorname{div}(R\theta \nabla p e^{R\sigma})$. Hence

$$\Delta(e^{R\sigma}) = R\theta(e^{R\sigma} \Delta p + \langle \nabla p, \nabla(e^{R\sigma}) \rangle) = R\theta \Delta p e^{R\sigma} + R^2 \theta^2 |\nabla p|^2 e^{R\sigma}.$$

This allows us to consider P_R^+ and P_R^- as follows:

$$(44) \quad P_R^+ w = R\theta_t p w - R^2 \theta^2 |\nabla p|^2 w - \Delta w,$$

$$(45) \quad P_R^- w = w_t - R\theta \Delta p w - 2R\theta \langle \nabla w, \nabla p \rangle,$$

so that

$$P_R^+ w + P_R^- w = e^{-R\sigma} Pz.$$

This implies that

$$(46) \quad \begin{aligned} &\|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\ &\quad + 2\langle P_R^+ w, P_R^- w \rangle_{L^2(Q_{\mathcal{M}}^{0,T})} = \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2. \end{aligned}$$

3.3.2 - The expression of the scalar product

With some integrations by parts (see [38]), using Proposition 2.1 and the properties stated in Lemmas 3.1 and 3.2, we obtain

$$(47) \quad \begin{aligned} &2\langle P_R^+ w, P_R^- w \rangle_{L^2(Q_{\mathcal{M}}^{0,T})} \\ &= \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2 \theta \theta_t |\nabla p|^2 + R\theta \Delta(\Delta p) - R p \theta_{tt}) w^2 \\ &\quad - 4 \iint_{Q_{\mathcal{M}}^{0,T}} R^3 \theta^3 \operatorname{Hess}(p)(\nabla p, \nabla p) w^2 - 4 \iint_{Q_{\mathcal{M}}^{0,T}} R\theta \operatorname{Hess}(p)(\nabla w, \nabla w). \end{aligned}$$

The proof of Theorem 3.1 follows from suitable lower bounds of the terms appearing in (47).

3.3.3 - A bound from below of the zero order term of the scalar product

The main property is the following:

Lemma 3.4. *There exists $C > 0$ independent of R and S such that*

$$(48) \quad -4R^3\theta^3 \text{Hess}(p)(\nabla p, \nabla p) \geq -CR^3S^3\theta^3 e^{3S\psi} + R^3S^4\theta^3 e^{3S\psi} |\nabla\psi|^4.$$

Proof of Lemma 3.4. Since $\nabla p = -Se^{S\psi}\nabla\psi$, we have

$$\begin{aligned} -\text{Hess}(p)(\nabla p, \nabla p) &= -\langle \Gamma_{\nabla p} \nabla p, \nabla p \rangle \\ &= -\langle \Gamma_{-Se^{S\psi}\nabla\psi} (-Se^{S\psi}\nabla\psi), -Se^{S\psi}\nabla\psi \rangle \\ &= -\langle -Se^{S\psi}\Gamma_{\nabla\psi} (-Se^{S\psi}\nabla\psi), -Se^{S\psi}\nabla\psi \rangle \\ &= -S^2e^{2S\psi} \langle \Gamma_{\nabla\psi} (-Se^{S\psi}\nabla\psi), \nabla\psi \rangle \\ &= -S^2e^{2S\psi} \langle -Se^{S\psi}\Gamma_{\nabla\psi}(\nabla\psi) + \nabla\psi.(-Se^{S\psi})\nabla\psi, \nabla\psi \rangle \\ &= -S^2e^{2S\psi} \left(-Se^{S\psi} \langle \Gamma_{\nabla\psi}(\nabla\psi), \nabla\psi \rangle + \nabla\psi.(-Se^{S\psi}) \langle \nabla\psi, \nabla\psi \rangle \right). \end{aligned}$$

Now choose $m \in \mathcal{M}$, ω a smooth curve such that $\omega(0) = m$, $\omega'(0) = \nabla\psi$. Then

$$\begin{aligned} \nabla\psi.(-Se^{S\psi}) &= \frac{d}{dt} \Big|_{t=0} (-Se^{S\psi(\omega(t))}) = -S^2e^{S\psi(m)} \frac{d}{dt} \Big|_{t=0} (\psi(\omega(t))) \\ &= -S^2e^{S\psi(m)} \nabla\psi.\psi = -S^2e^{S\psi(m)} \langle \nabla\psi, \nabla\psi \rangle. \end{aligned}$$

Hence

$$-\text{Hess}(p)(\nabla p, \nabla p) = S^2e^{2S\psi} \left(Se^{S\psi} \langle \Gamma_{\nabla\psi}(\nabla\psi), \nabla\psi \rangle + S^2e^{S\psi} |\nabla\psi|^4 \right).$$

Hence

$$-R^3\theta^3 \text{Hess}(p)(\nabla p, \nabla p) = R^3S^3\theta^3 e^{3S\psi} \left(\langle \Gamma_{\nabla\psi}(\nabla\psi), \nabla\psi \rangle + S |\nabla\psi|^4 \right).$$

Therefore, there exists $C > 0$ independent of R and S such that

$$-4R^3\theta^3 \text{Hess}(p)(\nabla p, \nabla p) \geq -CR^3S^3\theta^3 e^{3S\psi} + R^3S^4\theta^3 e^{3S\psi} |\nabla\psi|^4.$$

Hence (48) is proved. \square

3.3.4 - A bound from below of the first order term of the scalar product

Now we turn to the last term of (47), and we prove the following

Lemma 3.5. *There exists $C > 0$ independent of R and S such that*

$$(49) \quad -4 \iint_{Q_{\mathcal{M}}^{0,T}} R\theta \text{Hess}(p)(\nabla w, \nabla w) \geq \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 - \frac{C}{S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 - C \iint_{Q_{\mathcal{M}}^{0,T}} R^3S^3\theta^3 e^{3S\psi} w^2.$$

Proof of Lemma 3.5. We have

$$\begin{aligned}
\text{Hess}(p)(\xi, \xi) &= \langle \Gamma_\xi \nabla p, \xi \rangle \\
&= \langle \Gamma_\xi (-S e^{S\psi} \nabla \psi), \xi \rangle = \langle -S e^{S\psi} \Gamma_\xi (\nabla \psi) + \xi \cdot (-S e^{S\psi}) \nabla \psi, \xi \rangle \\
&= -S e^{S\psi} \langle \Gamma_\xi (\nabla \psi), \xi \rangle + \langle -S^2 e^{S\psi} \langle \nabla \psi, \xi \rangle \nabla \psi, \xi \rangle \\
&= -S e^{S\psi} \langle \Gamma_\xi (\nabla \psi), \xi \rangle - S^2 e^{S\psi} \langle \nabla \psi, \xi \rangle^2.
\end{aligned}$$

Hence, there exists c_1 such that

$$\begin{aligned}
-R\theta \text{Hess}(p)(\nabla w, \nabla w) &= RS\theta e^{S\psi} \langle \Gamma_{\nabla w} (\nabla \psi), \nabla w \rangle + RS^2\theta e^{S\psi} \langle \nabla \psi, \nabla w \rangle^2 \\
&\geq -c_1 RS\theta e^{S\psi} |\nabla w|^2 + RS^2\theta e^{S\psi} \langle \nabla \psi, \nabla w \rangle^2,
\end{aligned}$$

hence

$$(50) \quad -R\theta \text{Hess}(p)(\nabla w, \nabla w) \geq -c_1 RS\theta e^{S\psi} |\nabla w|^2.$$

On the other hand,

$$\begin{aligned}
\langle RS\theta e^{S\psi} w, P_R^+ w \rangle &= \langle RS\theta e^{S\psi} w, R\theta_t p w - R^2\theta^2 |\nabla p|^2 w - \Delta w \rangle \\
&= \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} (R\theta_t p - R^2 S^2 \theta^2 e^{2S\psi} |\nabla \psi|^2) w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \langle \nabla (RS\theta e^{S\psi} w), \nabla w \rangle \\
&= \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} (R\theta_t p - R^2 S^2 \theta^2 e^{2S\psi} |\nabla \psi|^2) w^2 \\
&\quad + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 + RS\theta e^{S\psi} w \langle \nabla \psi, \nabla w \rangle,
\end{aligned}$$

hence

$$\begin{aligned}
\iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 &= \langle RS\theta e^{S\psi} w, P_R^+ w \rangle \\
&\quad - \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} (R\theta_t p - R^2 S^2 \theta^2 e^{2S\psi} |\nabla \psi|^2) w^2 \\
&\quad - \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} w \langle \nabla \psi, \nabla w \rangle \\
&\leq \frac{1}{2S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \frac{1}{2} RS\theta e^{S\psi} |\nabla w|^2.
\end{aligned}$$

Hence

$$(51) \quad \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \leq \frac{1}{S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + 2C \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.$$

From (50) and (51), we deduce that

$$\begin{aligned}
& - \iint_{Q_{\mathcal{M}}^{0,T}} R\theta \text{Hess}(p)(\nabla w, \nabla w) \\
& \geq \frac{-c_1}{S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 - 2C c_1 \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2 \\
& \geq \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 - \frac{1+c_1}{S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \quad - 2C(1+c_1) \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2,
\end{aligned}$$

hence (49) is proved. \square

3.3.5 - A first Carleman estimate

Now we are in position to obtain a first Carleman estimate: using (46), (47), (48), (49), and classical estimates of the type $|\theta_t| \leq C\theta^2$, $|\theta_{tt}| \leq C\theta^3$, we obtain that

$$\begin{aligned}
& \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& = \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + 2\langle P_R^+ w, P_R^- w \rangle_{L^2(Q_{\mathcal{M}}^{0,T})} \\
& \geq \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \quad + \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2\theta\theta_t |\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt}) w^2 \\
& \quad - 4 \iint_{Q_{\mathcal{M}}^{0,T}} R^3\theta^3 \text{Hess}(p)(\nabla p, \nabla p) w^2 \\
& \quad - 4 \iint_{Q_{\mathcal{M}}^{0,T}} R\theta \text{Hess}(p)(\nabla w, \nabla w) \\
& \geq \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \quad + \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2\theta\theta_t |\nabla p|^2 + R\theta\Delta(\Delta p) - Rp\theta_{tt}) w^2 \\
& \quad + \iint_{Q_{\mathcal{M}}^{0,T}} (-CR^3 S^3 \theta^3 e^{3S\psi} + R^3 S^4 \theta^3 e^{3S\psi} |\nabla \psi|^4) w^2 \\
& \quad + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 - \frac{C}{S} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 - C \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Hence, for S large enough,

$$\begin{aligned}
& \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^4 \theta^3 e^{3S\psi} |\nabla \psi|^4 w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \\
& \quad + \frac{1}{2} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& + \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2 \theta \theta_t |\nabla p|^2 + R\theta \Delta(\Delta p) - Rp\theta_{tt}) w^2 - C \int_0^T \int_{\mathcal{M} \setminus \omega} R^3 S^3 \theta^3 e^{3S\psi} w^2 \\
& \leq \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Moreover, assuming that Assumption 1.2 is satisfied, there exists $C_0 > 0$ such that $|\nabla(m)\psi| > C_0$ for all $m \in \mathcal{M} \setminus \omega$. Thus

$$\int_0^T \int_{\mathcal{M} \setminus \omega} R^3 S^3 \theta^3 e^{3S\psi} w^2 \leq \frac{C}{S} \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^4 \theta^3 e^{3S\psi} |\nabla \psi|^4 w^2.$$

We deduce, for S large enough,

$$\begin{aligned}
& \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} (1 + \frac{S}{2} |\nabla \psi|^4) w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \\
& \quad + \frac{1}{2} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \quad + \iint_{Q_{\mathcal{M}}^{0,T}} (4R^2 \theta \theta_t |\nabla p|^2 + R\theta \Delta(\Delta p) - Rp\theta_{tt}) w^2 \\
& \leq \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Finally, using the properties of the function θ and $R \geq R_0 e^{2S\|\psi\|_\infty}$, we get

$$\begin{aligned}
(52) \quad & \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} (1 + \frac{S}{4} |\nabla \psi|^4) w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} |\nabla w|^2 \\
& \quad + \frac{1}{2} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \leq \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2.
\end{aligned}$$

Going back to $z = e^{R\sigma} w$, we have

$$\begin{aligned}
(53) \quad & \iint_{Q_{\mathcal{M}}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} (1 + \frac{S}{4} |\nabla \psi|^4) e^{-2R\sigma} z^2 + \iint_{Q_{\mathcal{M}}^{0,T}} RS\theta e^{S\psi} e^{-2R\sigma} |\nabla z|^2 \\
& \quad + \frac{1}{2} \|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\
& \leq C' \|e^{-R\sigma} Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C' \iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} e^{-2R\sigma} z^2.
\end{aligned}$$

3.3.6 - End of the proof of Theorem 3.1

To complete the proof of Theorem 3.1, we only need to estimate z_t . First we estimate w_t , using $P_R^- w$: we have

$$\begin{aligned} w_t &= P_R^- w + R\theta\Delta p w + 2R\theta\langle\nabla w, \nabla p\rangle \\ &= P_R^- w - \rho(\Delta\psi + S|\nabla\psi|^2)w - 2\rho\langle\nabla w, \nabla\psi\rangle, \end{aligned}$$

Hence

$$\left\|\frac{w_t}{\sqrt{\rho}}\right\| \leq C\left\|\frac{P_R^- w}{\sqrt{\rho}}\right\| + CS\|\sqrt{\rho}w\| + C\|\sqrt{\rho}\nabla w\|.$$

Using (52), we obtain that

$$\begin{aligned} (54) \quad & \iint_{Q_{\mathcal{M}}^{0,T}} \left(1 + \frac{S}{4}|\nabla\psi|^4\right)\rho^3 w^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \rho|\nabla w|^2 + \iint_{Q_{\mathcal{M}}^{0,T}} \frac{1}{\rho} w_t^2 \\ & + \frac{1}{2}\|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \frac{1}{2}\|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\ & \leq C\|e^{-R\sigma}Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C\iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} w^2. \end{aligned}$$

Finally, going back to $z = e^{R\sigma}w$, we have

$$\begin{aligned} (55) \quad & \iint_{Q_{\mathcal{M}}^{0,T}} \left(1 + \frac{S}{4}|\nabla\psi|^4\right)e^{-2R\sigma}\rho^3 z^2 + \iint_{Q_{\mathcal{M}}^{0,T}} e^{-2R\sigma}\rho|\nabla z|^2 + \iint_{Q_{\mathcal{M}}^{0,T}} e^{-2R\sigma}\frac{1}{\rho} z_t^2 \\ & + \frac{1}{2}\|P_R^+ w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + \frac{1}{2}\|P_R^- w\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 \\ & \leq C\|e^{-R\sigma}Pz\|_{L^2(Q_{\mathcal{M}}^{0,T})}^2 + C\iint_{Q_{\omega}^{0,T}} R^3 S^3 \theta^3 e^{3S\psi} e^{-2R\sigma} z^2. \end{aligned}$$

This gives (36) and completes the proof of Theorem 3.1. \square

4 - Proof of Proposition 1.1

In this section, we study the validity of the geometrical assumption 1.2.

4.1 - The case of the sphere \mathbb{S}^2

Let us prove that Assumption 1.2 is satisfied in the case of the sphere \mathbb{S}^2 . Consider $\omega_{\mathbb{S}^2}$ a non-empty open domain of the sphere. Choose $N \in \omega_{\mathbb{S}^2}$, that will play the role of the North pole. Choose $S \in \omega_{\mathbb{S}^2}, S \neq N$. Consider a small neighborhood ω_N of N included in $\omega_{\mathbb{S}^2}$, and a small neighborhood ω_S of S included in $\omega_{\mathbb{S}^2}$ such that $\omega_N \cap \omega_S = \emptyset$.

Now consider π the stereographic projection of pole N :

$$\pi : \mathbb{S}^2 \setminus \{N\} \rightarrow \mathbb{R}^2.$$

Then $\Omega_\pi := \pi(\mathbb{S}^2 \setminus \omega_N)$ is a bounded domain of \mathbb{R}^2 , $\pi(\omega_S)$ is an open subdomain of Ω_π . The classical geometrical lemma of Fursikov-Imanuvilov [18] (see also [7]) ensures that there exists

$$\psi_\pi : \Omega_\pi \rightarrow \mathbb{R}, \quad y \mapsto \psi_\pi(y)$$

smooth such that

$$\nabla \psi_\pi(y) = 0 \quad \implies \quad y \in \pi(\omega_S).$$

Then consider

$$\psi_{\mathbb{S}^2} : \mathbb{S}^2 \setminus \omega_N \rightarrow \mathbb{R}, \quad \psi_{\mathbb{S}^2}(x) := \psi_\pi(\pi(x)).$$

Let us prove that

$$\nabla \psi_{\mathbb{S}^2}(x) = 0 \quad \implies \quad x \in \omega_S.$$

Indeed, fix $x \in \mathbb{S}^2 \setminus \omega_N$ and consider any $\xi \in T_x \mathbb{S}^2$, and take a smooth curve $\gamma : I \rightarrow \mathbb{S}^2$, $\gamma(0) = x$, $\gamma'(0) = \xi$. Then

$$\langle \nabla \psi_{\mathbb{S}^2}(x), \xi \rangle = (\xi \cdot \psi_{\mathbb{S}^2})(x) = \frac{d}{dt} \Big|_{t=0} (\psi_{\mathbb{S}^2}(\gamma(t))) = \frac{d}{dt} \Big|_{t=0} (\psi_\pi(\pi(\gamma(t)))).$$

Denote

$$\gamma_\pi : I \rightarrow \mathbb{R}^2, \quad \gamma_\pi(t) := \pi(\gamma(t)).$$

Then

$$\langle \nabla \psi_{\mathbb{S}^2}(x), \xi \rangle = \frac{d}{dt} \Big|_{t=0} (\psi_\pi(\gamma_\pi(t))) = \nabla \psi_\pi(\pi(x)) \cdot \gamma'_\pi(0).$$

Since $\gamma'_\pi(0)$ can be taken arbitrary in \mathbb{R}^2 , we obtain that

$$\nabla \psi_{\mathbb{S}^2}(x) = 0 \quad \implies \quad \nabla \psi_\pi(\pi(x)) = 0,$$

which implies $\pi(x) \in \pi(\omega_S)$, hence $x \in \omega_S$. Then it is sufficient to extend $\psi_{\mathbb{S}^2}$ to \mathbb{S}^2 . This can be done, it can bring new zeros of $\nabla \psi_{\mathbb{S}^2}$, but inside ω_N , hence inside $\omega_{\mathbb{S}^2}$. This proves that Assumption 1.2 is satisfied in the case of the sphere \mathbb{S}^2 . \square

4.2 - The case of a simply connected oriented manifold of dimension 2

Assume that \mathcal{M} is simply connected, and still compact, oriented, of dimension 2 and without boundary. Then the celebrated theorem of uniformisation of Riemann [1, 40] implies that there exists a C^1 -diffeomorphism between \mathcal{M} and the sphere \mathbb{S}^2 . We denote it

$$\Phi : \mathcal{M} \rightarrow \mathbb{S}^2, \quad m \mapsto \Phi(m).$$

Consider also a (small) non-empty open subdomain $\omega_{\mathcal{M}}$ of \mathcal{M} , and denote

$$\omega_{\mathbb{S}^2} := \Phi(\omega_{\mathcal{M}}).$$

Then consider $\psi_{\mathbb{S}^2}$ constructed in the previous section, that satisfies

$$\nabla \psi_{\mathbb{S}^2}(x) = 0 \quad \implies \quad x \in \omega_{\mathbb{S}^2},$$

and

$$\psi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{R}, \quad \psi_{\mathcal{M}}(m) := \psi_{\mathbb{S}^2}(\Phi(m)).$$

Then let us prove that

$$\nabla \psi_{\mathcal{M}}(m) = 0 \quad \implies \quad m \in \omega_{\mathcal{M}}.$$

Indeed, fix $m \in \mathcal{M}$ and consider any $\xi \in T_m \mathcal{M}$, $\gamma : I \rightarrow \mathcal{M}$ such that $\gamma(0) = m$, $\gamma'(0) = \xi$. Then

$$\langle \nabla \psi_{\mathcal{M}}(m), \xi \rangle_{\mathcal{M}} = (\xi \cdot \psi_{\mathcal{M}})'(m) = \frac{d}{dt} \Big|_{t=0} (\psi_{\mathcal{M}}(\gamma(t))) = \frac{d}{dt} \Big|_{t=0} (\psi_{\mathbb{S}^2}(\Phi(\gamma(t)))).$$

Denote

$$\gamma_{\mathbb{S}^2} : I \rightarrow \mathbb{S}^2, \quad \gamma_{\mathbb{S}^2}(t) := \Phi(\gamma(t)).$$

Then

$$\langle \nabla \psi_{\mathcal{M}}(m), \xi \rangle_{\mathcal{M}} = \frac{d}{dt} \Big|_{t=0} (\psi_{\mathbb{S}^2}(\gamma_{\mathbb{S}^2}(t))) = \langle \nabla \psi_{\mathbb{S}^2}(\Phi(m)), \gamma'_{\mathbb{S}^2}(0) \rangle_{\mathbb{S}^2}.$$

Since $\gamma'_{\mathbb{S}^2}(0)$ may describe all the tangent directions at $\Phi(m)$, we obtain that

$$\nabla \psi_{\mathcal{M}}(m) = 0 \quad \implies \quad \nabla \psi_{\mathbb{S}^2}(\Phi(m)) = 0,$$

which implies $\Phi(m) \in \omega_{\mathbb{S}^2} = \Phi(\omega_{\mathcal{M}})$, hence $m \in \omega_{\mathcal{M}}$. This completes the proof of Proposition 1.1. \square

5 - Preliminary study of the Sellers model on a manifold

5.1 - Local existence of classical solutions

In order to apply the theory in [28], we need to rewrite (1) as an evolution equation in $L^2(\mathcal{M})$. We recall that $(\Delta, D(\Delta)) = (A, D(A))$ defined in subsection 2.3. The natural energy space is $H^1(\mathcal{M})$ and the bilinear form a is $H^1(\mathcal{M})$ - $L^2(\mathcal{M})$ coercive, i.e.

$$\exists \alpha > 0, \exists \beta \in \mathbb{R}, \forall v \in H^1(\mathcal{M}), \quad a(v, v) + \beta \|v\|_{L^2(\mathcal{M})}^2 \geq \alpha \|v\|_{H^1(\mathcal{M})}^2.$$

To rewrite (1) as a evolution equation in $L^2(\mathcal{M})$, it remains to check that the second member of the equation takes its values in $L^2(\mathcal{M})$. So we define G by

$$G : \begin{cases} [0, T] \times H^1(\mathcal{M}) & \longrightarrow L^2(\mathcal{M}) \\ (t, u) & \longmapsto r(t)q\beta(u) - \varepsilon(u)u|u|^3. \end{cases}$$

If G is well-defined, then problem (1) on $[0, T]$ is equivalent to the evolution equation in $L^2(\mathcal{M})$

$$(56) \quad \begin{cases} u_t(t) + Au(t) = G(t, u(t)), & t \in [0, T], \\ u(0) = u_0. \end{cases}$$

We prove

Lemma 5.1. *G is well defined on $[0, T] \times H^1(\mathcal{M})$ with values in $L^2(\mathcal{M})$. Moreover, G satisfies*

$$\bullet \quad \forall t \in [0, T], \forall R > 0, \exists C > 0, \forall u_1, u_2 \in B_{H^1(\mathcal{M})}(0, R),$$

$$(57) \quad \|G(t, u_1) - G(t, u_2)\|_{L^2(\mathcal{M})} \leq C \|u_1 - u_2\|_{H^1(\mathcal{M})}.$$

$$\bullet \quad \forall R > 0, \exists \theta \in (0, 1), \exists C > 0, \forall u \in B_{H^1(\mathcal{M})}(0, R), \forall s, t \in [0, T],$$

$$(58) \quad \|G(t, u) - G(s, u)\|_{L^2(\mathcal{M})} \leq C |t - s|^\theta.$$

Proof. For the proof, we will need the following result (see [37, p. 14]):

Lemma 5.2. *For all $q \in [1, +\infty)$, $H^1(\mathcal{M}) \subset L^q(\mathcal{M})$ with continuous embedding .*

Let us first prove that G is well defined on $[0, T] \times H^1(\mathcal{M})$, with values in $L^2(\mathcal{M})$. We set $Q = rq$ and $Q_1 = \|Q\|_{L^\infty(\mathbb{R} \times I)}$. For $t \in [0, T]$, $u \in H^1(\mathcal{M})$, we write

$$\begin{aligned} \|G(t, u)\|_{L^2(\mathcal{M})}^2 &= \int_I |\mathcal{R}_a(t, u) - \mathcal{R}_e(u)|^2 \leq 2 \int_{\mathcal{M}} Q(t, x)^2 \beta(u)^2 + 2 \int_{\mathcal{M}} \varepsilon(u)^2 u^8 \\ &\leq 2Q_1^2 \|\beta\|_{L^\infty(\mathbb{R})}^2 + 2\|\varepsilon\|_{L^\infty(\mathbb{R})}^2 \int_{\mathcal{M}} u^8 \leq 2Q_1^2 \bar{C} \|\beta\|_{L^\infty(\mathbb{R})}^2 + C \|u\|_{H^1(\mathcal{M})}^8, \end{aligned}$$

where we used Lemma 5.2 (with $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M} < +\infty$).

Next, we prove that (57) is satisfied. Let $t \in [0, T]$, $R > 0$ and u_1, u_2 in $B_{H^1(\mathcal{M})}(0, R)$. Then

$$\begin{aligned} \|G(t, u_1) - G(t, u_2)\|_{L^2(\mathcal{M})}^2 &= \int_{\mathcal{M}} \left| Q(t, x)(\beta(u_1) - \beta(u_2)) + \mathcal{R}_e(u_1) - \mathcal{R}_e(u_2) \right|^2 \\ &\leq 2Q_1^2 \|\beta'\|_{L^\infty(\mathbb{R})}^2 \int_{\mathcal{M}} |u_1 - u_2|^2 + 2 \int_{\mathcal{M}} |\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)|^2 \\ &\leq 2Q_1^2 \|\beta'\|_{L^\infty(\mathbb{R})}^2 \|u_1 - u_2\|_{H^1(\mathcal{M})}^2 + 2 \int_{\mathcal{M}} |\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)|^2. \end{aligned}$$

To conclude the proof of (57), it remains to show

$$(59) \quad \int_{\mathcal{M}} |\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)|^2 d\mathcal{M} \leq C \|u_1 - u_2\|_{H^1(\mathcal{M})}^2,$$

for some $C > 0$. We compute

$$(60) \quad \int_{\mathcal{M}} |\mathcal{R}_e(u_1) - \mathcal{R}_e(u_2)|^2 \leq 3 \int_{\mathcal{M}} |\varepsilon(u_1) - \varepsilon(u_2)|^2 |u_1|^8 \\ + 3 \int_{\mathcal{M}} \varepsilon(u_2)^2 |u_1 - u_2|^2 |u_1|^6 + 3 \int_{\mathcal{M}} \varepsilon(u_2)^2 |u_2|^2 (|u_1|^3 - |u_2|^3)^2.$$

So it remains to estimate the three terms in the right hand side of the above inequality. From the assumptions on ε (Assumption 1.1), we have:

$$\int_{\mathcal{M}} |\varepsilon(u_1) - \varepsilon(u_2)|^2 |u_1|^8 \leq \|\varepsilon'\|_{L^\infty(\mathbb{R})}^2 \|u_1 - u_2\|_{L^4(\mathcal{M})}^2 \|u_1\|_{L^{16}(\mathcal{M})}^8, \\ \int_{\mathcal{M}} \varepsilon(u_2)^2 |u_1 - u_2|^2 |u_1|^6 \leq \|\varepsilon\|_{L^\infty(\mathbb{R})}^2 \|u_1 - u_2\|_{L^4(\mathcal{M})}^2 \|u_1\|_{L^{12}(\mathcal{M})}^6,$$

and

$$\int_{\mathcal{M}} \varepsilon(u_2)^2 |u_2|^2 (|u_1|^3 - |u_2|^3)^2 \\ \leq \|\varepsilon\|_{L^\infty(\mathbb{R})}^2 \int_{\mathcal{M}} |u_2|^2 (|u_1| - |u_2|)^2 (|u_1|^2 + |u_1||u_2| + |u_2|^2)^2 \\ \leq \|\varepsilon\|_{L^\infty(\mathbb{R})}^2 \|u_1 - u_2\|_{L^4(\mathcal{M})}^2 \left(\int_{\mathcal{M}} |u_2|^4 (|u_1|^2 + |u_1||u_2| + |u_2|^2)^4 \right)^{1/2}.$$

Using Lemma 5.2 and $u_1, u_2 \in B_{H^1(\mathcal{M})}(0, R)$, we end the proof of (57).

Finally, we prove condition (58): for all $t, s \in [0, T]$,

$$\|G(t, u) - G(s, u)\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} |r(t) - r(s)|^2 q(x)^2 \beta(u(x))^2 \\ \leq \bar{C} \|r'\|_{L^\infty(\mathbb{R})}^2 \|q\|_{L^\infty(\mathcal{M})}^2 \|\beta\|_{L^\infty(\mathbb{R})}^2 |t - s|^2,$$

where $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M} < +\infty$. This implies (58). \square

We are now ready to deduce a result of local existence:

Theorem 5.1. *For all $u^0 \in D(\Delta)$, there exists $T^*(u^0) \in (0, +\infty]$ such that, for all $0 < T < T^*(u^0)$, problem (56) has a unique solution $u \in \mathcal{C}([0, T], D(\Delta)) \cap \mathcal{C}^1([0, T], L^2(\mathcal{M}))$. Moreover, if $T^*(u^0) < +\infty$, then $\|u(t)\|_{H^1(\mathcal{M})} \rightarrow +\infty$ as $t \rightarrow T^*(u^0)$.*

Proof. Since $(\Delta, D(\Delta))$ generates an analytical semigroup and since the interpolation space $[D(\Delta), L^2(\mathcal{M})]_{1/2}$ is $H^1(\mathcal{M})$, Lemma 5.1 allows to apply [28, Theorem 7.1.2] to (56). So there exists a unique weak solution defined until a maximal time $T^*(u^0)$. Then [28, Proposition 7.1.8] implies that, if $T^*(u^0) < +\infty$ then $\|u(t)\|_{H^1(\mathcal{M})} \rightarrow +\infty$ as $t \rightarrow T^*(u^0)$. Moreover, since $Au^0 + G(0, u^0) \in L^2(\mathcal{M})$, [28, Proposition 7.1.10] ensures that, for all $T < T^*(u^0)$, $u \in \mathcal{C}([0, T], D(\Delta)) \cap \mathcal{C}^1([0, T], L^2(\mathcal{M}))$. \square

5.2 - Weak maximum principle

First we prove

Lemma 5.3. *Let $v \in H^1(\mathcal{M})$ and $M \geq 0$. Then $(u - M)^+ := \sup(u - M, 0) \in H^1(\mathcal{M})$ and $(u + M)^- := \sup(-(u + M), 0) \in H^1(\mathcal{M})$. Moreover*

$$(61) \quad \text{grad}(u - M)^+(m) = \begin{cases} \text{grad}(u)(m) & \text{if } u(m) \geq M \\ 0 & \text{otherwise} \end{cases}$$

$$(62) \quad \text{grad}(u + M)^-(m) = \begin{cases} -\text{grad}(u)(m) & \text{if } u(m) \leq -M \\ 0 & \text{otherwise} \end{cases}$$

Proof. In the context of a manifold, Lemma 5.3 replaces [39, Lemma 6.1] that is the classical result when working in an open subset of \mathbb{R}^n . Consider $(U_i, \Phi_i)_{1 \leq i \leq N}$ an atlas of \mathcal{M} . Let us first prove that: $\forall 1 \leq i \leq N, \forall f \in L^2(\mathcal{M})$,

$$(63) \quad \sup(f, 0) \circ \Phi_i^{-1} = \sup(f \circ \Phi_i^{-1}, 0) \text{ on } \Phi_i(U_i).$$

Indeed, let $y \in \Phi_i(U_i)$ be such that $(f \circ \Phi_i^{-1})(y) \geq 0$. Then $f(x) \geq 0$ with $x = \Phi_i^{-1}(y) \in U_i$. Consequently,

$$\sup(f \circ \Phi_i^{-1}, 0)(y) = (f \circ \Phi_i^{-1})(y) = f(x) = (\sup(f, 0) \circ \Phi_i^{-1})(y).$$

The reasoning is similar when $(f \circ \Phi_i^{-1})(y) \leq 0$. This proves (63).

Let us now prove Lemma 5.3. From Proposition 2.2, it suffices to show that, for all $1 \leq i \leq N$, $(u - M)^+ \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$. But $u - M \in H^1(\mathcal{M})$, so, for all $1 \leq i \leq N$, $(u - M) \circ \Phi_i^{-1} \in H^1(\Phi_i(U_i))$. Using [12, Proposition 6 p. 934], $((u - M) \circ \Phi_i^{-1})^+ \in H^1(\Phi_i(U_i))$. But, from (63),

$$(u - M)^+ \circ \Phi_i^{-1} = ((u - M) \circ \Phi_i^{-1})^+.$$

So we proved that $(u - M)^+ := \sup(u - M, 0) \in H^1(\mathcal{M})$. Moreover, from [12, Proposition 6 p. 934], we know that

$$\nabla((u - M)^+ \circ \Phi_i^{-1})(y) = \begin{cases} \nabla(u \circ \Phi_i^{-1})(y) & \text{if } u(\Phi_i^{-1}(y)) \geq M, \\ 0 & \text{otherwise.} \end{cases}$$

From the local definition of the weak gradient (see the proof of Proposition 2.2),

$$\text{grad}(u - M)^+(m) = \begin{cases} \sum_{l=1}^n \sum_{j=1}^n g^{l,j} \frac{\partial}{\partial x_j} (((u - M) \circ \Phi_i^{-1}) \circ \Phi_i) \partial_l & \text{if } u(m) \geq M, \\ 0 & \text{otherwise.} \end{cases}$$

We immediately deduce (61). The proof of (62) is similar. \square

Then we prove the following maximum principle:

Theorem 5.2. *Let $u^0 \in D(\Delta) \cap L^\infty(\mathcal{M})$ and $T^*(u^0)$ defined by Theorem 5.1. We denote*

$$(64) \quad M := \max \left\{ \|u^0\|_{L^\infty(\mathcal{M})}, \left(\frac{\|q\|_{L^\infty(\mathcal{M})} \|r\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})}}{\varepsilon_{min}} \right)^{1/4} \right\}.$$

Then the solution u of problem (1) satisfies $\|u\|_{L^\infty((0, T^(u^0)) \times \mathcal{M})} \leq M$.*

Proof. Theorem 5.2 replaces [39, theorem 3.3] obtained in case of the 1-dimensional Sellers model. The proof (based on Lemma 5.3) is similar so we omit it here. It can also be found in [38]. \square

From Theorem 5.2, we deduce that, for all $u^0 \in D(\Delta) \cap L^\infty(\mathcal{M})$, $\|u\|_{L^2(\mathcal{M})}$ does not blow up as $t \rightarrow T^*(u^0)$. However, this is not sufficient to ensure the existence of a *global* classical solution since we did not prove that $\|u\|_{H^1(\mathcal{M})}$ does not blow up. Before showing this, we begin by proving some regularity result on the time derivative of the solution.

5.3 - Regularity of the time derivative of the solution of (1)

We work with initial conditions defined in

$$(65) \quad \mathcal{U} := \{u^0 \in D(\Delta_{\mathcal{M}}) \cap L^\infty(\mathcal{M}) : Au^0 \in L^\infty(\mathcal{M})\}.$$

We denote:

$$W(0, T; H^1(\mathcal{M}), (H^1(\mathcal{M}))') := \{v \in L^2(0, T; H^1(\mathcal{M})) : v_t \in L^2(0, T; (H^1(\mathcal{M}))')\}.$$

Then we prove

Theorem 5.3. *Let $u^0 \in \mathcal{U}$ and u the corresponding solution of (1). Let T be such that $0 < T < T^*(u^0)$ (where $T^*(u^0)$ is defined in Theorem 5.1). Then $z := u_t$ belongs to $L^2(0, T; H^1(\mathcal{M}))$ and is solution of the following variational problem:*

$$(66) \quad \begin{cases} z \in W(0, T; H^1(\mathcal{M}), (H^1(\mathcal{M}))'), \\ \forall w \in H^1(\mathcal{M}), \quad \langle z_t(t), w \rangle + b(t, z(t), w) = \left(r'(t)q\beta(u(t)), w \right)_{L^2(\mathcal{M})}, \\ z(0) = -Au^0 + G(0, u^0), \end{cases}$$

where $b : [0, T] \times H^1(\mathcal{M}) \times H^1(\mathcal{M}) \rightarrow \mathbb{R}$ is the time-dependent bilinear form:

$$b(t, v, w) = \int_{\mathcal{M}} \langle \text{grad}(v), \text{grad}(w) \rangle d\mathcal{M} + \int_{\mathcal{M}} \tilde{\pi}(t, x) v w d\mathcal{M},$$

with $\tilde{\pi}(t, x) := \mathcal{R}'_e(u(t, x)) - r(t)q(x)\beta'(u(t, x))$.

Proof. Consider $u^0 \in \mathcal{U}$. Multiplying the equation satisfied by u by $w \in H^1(\mathcal{M})$, we obtain, thanks to Proposition 2.3: $\forall t \in [0, T]$,

$$\left(z(t), w \right)_{L^2(\mathcal{M})} + \left(\nabla u(t), \nabla w \right)_{L^2(T\mathcal{M})} = \left(r(t)q\beta(u(t)) - \varepsilon(u(t))u(t)|u(t)|^3, w \right)_{L^2(\mathcal{M})}.$$

In order to prove that $z \in L^2(0, T; H^1(\mathcal{M}))$, we use the method of differential quotients (see e.g. [26]). Let $0 < \delta < \frac{T}{2}$, $t \in (\delta, T - \delta)$ and $-\delta < s < \delta$. We observe that

$$(67) \quad \begin{cases} u_t(t+s) - \Delta u(t+s) &= Q(t+s)\beta(u(t+s)) - \mathcal{R}_e(u(t+s)), \\ u_t(t) - \Delta u(t) &= Q(t)\beta(u(t)) - \mathcal{R}_e(u(t)). \end{cases}$$

Then we define, for all $t \in (\delta, T - \delta)$,

$$u^{(s)}(t) := \frac{u(t+s) - u(t)}{s}.$$

For all $t \in (\delta, T - \delta)$, $u^{(s)}(t) \in H^1(\mathcal{M})$ and (67) implies

$$(68) \quad \begin{aligned} \frac{\partial u^{(s)}}{\partial t}(t) - \Delta u^{(s)}(t) &= \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s}. \end{aligned}$$

Multiplying (68) by $u^{(s)}(t)$, using Proposition 2.3 and integrating over $(\delta, T - \delta)$, we get

$$(69) \quad \begin{aligned} &\frac{1}{2} \|u^{(s)}(T-\delta)\|_{L^2(\mathcal{M})}^2 + \int_{\delta}^{T-\delta} \left(\nabla u^{(s)}(t), \nabla u^{(s)}(t) \right)_{L^2(T\mathcal{M})} dt = \frac{1}{2} \|u^{(s)}(\delta)\|_{L^2(\mathcal{M})}^2 \\ &+ \int_{\delta}^{T-\delta} \int_{\mathcal{M}} \left[\frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} + \frac{\mathcal{R}_e(u(t)) - \mathcal{R}_e(u(t+s))}{s} \right] u^{(s)}(t). \end{aligned}$$

With computations identical to [39, equations (6.11) and (6.12), p. 697], we have

$$(70) \quad \begin{aligned} &\int_{\delta}^{T-\delta} \int_{\mathcal{M}} \frac{Q(t+s)\beta(u(t+s)) - Q(t)\beta(u(t))}{s} u^{(s)}(t) \\ &\leq \bar{C}T \|\beta\|_{L^\infty(\mathbb{R})} \|q\|_{L^\infty(\mathcal{M})}^2 \|r'\|_{L^\infty(\mathbb{R})}^2 \\ &\quad + \left(\frac{1}{2} \|\beta\|_{L^\infty(\mathbb{R})} + \|Q\|_{L^\infty(\mathbb{R} \times \mathcal{M})} \|\beta'\|_{L^\infty(\mathbb{R})} \right) \int_{\delta}^{T-\delta} \int_{\mathcal{M}} |u^{(s)}(t)|^2 \end{aligned}$$

where $\bar{C} = \int_{\mathcal{M}} 1 d\mathcal{M}$ and

$$(71) \quad \int_{\delta}^{T-\delta} \int_{\mathcal{M}} \frac{\mathcal{R}_e(u(t, x)) - \mathcal{R}_e(u(t+s, x))}{s} u^{(s)}(t) \leq C \int_{\delta}^{T-\delta} \int_{\mathcal{M}} |u^{(s)}(t, x)|^2.$$

Thanks to (70) and (71), (69) becomes

$$\int_{\delta}^{T-\delta} \|\nabla u^{(s)}(t)\|_{L^2(T\mathcal{M})}^2 dt \leq \frac{1}{2} \|u^{(s)}(\delta)\|_{L^2(\mathcal{M})}^2 + C + C \int_{\delta}^{T-\delta} \int_{\mathcal{M}} |u^{(s)}(t,x)|^2 d\mathcal{M} dt.$$

As $u \in \mathcal{C}^1([0, T]; L^2(\mathcal{M}))$, we obtain

$$\int_{\delta}^{T-\delta} \|\nabla u^{(s)}(t)\|_{L^2(T\mathcal{M})}^2 dt \leq \frac{1}{2} \sup_{t \in [0, T]} \|u_t\|_{L^2(\mathcal{M})}^2 + C + CT \sup_{t \in [0, T]} \|u_t\|_{L^2(\mathcal{M})}^2.$$

Consequently, the quantity $\int_{\delta}^{T-\delta} \|\nabla u^{(s)}(t)\|_{L^2(T\mathcal{M})}^2 dt$ is bounded by a constant independent of s . So, there exists a sub-sequence, still denoted by $(u^{(s)})_s$, that weakly converges to some $v \in L^2(\delta, T - \delta; H^1(\mathcal{M}))$ as $s \rightarrow 0$. But $L^2(\delta, T - \delta; H^1(\mathcal{M}))$ is continuously embedded in $L^2(\delta, T - \delta; L^2(\mathcal{M}))$. So the sub-sequence $(u^{(s)})_s$ weakly converges to v in $L^2(\delta, T - \delta; L^2(\mathcal{M}))$, (see e.g. [5, Theorem III.9 p. 39]). But, from [8, Corollary 1.4.39 p. 15], $(u^{(s)})_s$ strongly converges to u_t in $L^2(\delta, T - \delta; L^2(\mathcal{M}))$. Hence $u_t = v \in L^2(\delta, T - \delta; H^1(\mathcal{M}))$. Moreover,

$$\begin{aligned} \|u_t\|_{L^2(\delta, T-\delta; H^1(\mathcal{M}))} &\leq \limsup_{s \rightarrow 0} \|u^{(s)}\|_{L^2(\delta, T-\delta; H^1(\mathcal{M}))} \\ &\leq \frac{1}{2} \sup_{t \in [0, T]} \|u_t\|_{L^2(\mathcal{M})}^2 + C + CT \sup_{t \in [0, T]} \|u_t\|_{L^2(\mathcal{M})}^2. \end{aligned}$$

As the right hand side above does not depend on δ , we may let δ tends to 0 and we obtain that $z \in L^2(0, T; H^1(\mathcal{M}))$. \square

Corollary 5.1. *Let $u^0 \in \mathcal{U}$ and $0 < T < T^*(u^0)$ with $T^*(u^0)$ defined by Theorem 5.1. Then the solution z of (66) satisfies*

$$\|z\|_{L^\infty((0, T) \times \mathcal{M})} \leq e^{(\|\bar{\pi}\|_{L^\infty((0, T) \times \mathcal{M})} + 1)T} N,$$

with $N := \max \{ \| -Au^0 + G(0, u^0) \|_{L^\infty(\mathcal{M})}, \|r'\|_{L^\infty(\mathbb{R})} \|q\|_{L^\infty(\mathcal{M})} \|\beta\|_{L^\infty(\mathbb{R})} \}$.

Proof. This result replaces [39, Corollary 3.1] obtained in the case of the 1-dimensional Sellers model. The proof (that uses Lemma 5.3) is similar to the proof of [39, Corollary 3.1] for dimension 1. The main difficulty in the proof relies on the lack of coercivity of the bilinear form b so one has to introduce some auxiliary variational problem associated to some coercive bilinear form b_1 . We omit the proof here. It can also be found in [38]. \square

5.4 - Global existence of the solutions of (1)

Theorem 5.4. *Let $u^0 \in \mathcal{U}$. Then the solution u of (1) is defined on $[0, +\infty)$, i.e. $T^*(u^0) = +\infty$. Consequently, Theorem 5.2 and Corollary 5.1 hold true with $T^*(u^0) = +\infty$.*

Proof. Theorem 5.4 replaces [39, Theorem 3.5] obtained in the 1-dimensional case and it can be proved in a similar way (except the fact that computations are now on a manifold). So the proof (that can be found in [38]) is omitted. \square

6 - Proof of Theorem 1.1

STEP 1: Reduction to some non standard linear inverse source problem. Let $T > 0$, $u_1, u_2 \in \mathcal{C}([0, T]; D(\Delta)) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M}))$ the solutions of (1) corresponding respectively to q_1 with the initial condition u_1^0 , and to q_2 with the initial condition u_2^0 . We introduce $w := u_1 - u_2$. Then one can prove that $w \in \mathcal{C}([0, T]; D(\Delta)) \cap \mathcal{C}^1([0, T]; L^2(\mathcal{M}))$ solves

$$(72) \quad \begin{cases} w_t - \Delta w = H^* + H + \tilde{H} & (t, x) \in (0, T) \times \mathcal{M}, \\ w(0, x) = u_1^0 - u_2^0 & x \in \mathcal{M}, \end{cases}$$

with

$$(73) \quad H^* := r(q_1 - q_2)\beta(u_1),$$

$$(74) \quad H := r q_2(\beta(u_1) - \beta(u_2)),$$

$$(75) \quad \tilde{H} := \varepsilon(u_2)u_2|u_2|^3 - \varepsilon(u_1)u_1|u_1|^3.$$

As r and β are bounded from below (see Assumption 1.1), it suffices to estimate H^* to deduce an estimate of $q_1 - q_2$ in $L^2(\mathcal{M})$. So we reduced the problem to the determination of H^* in the above *linear* problem (72).

STEP 2: Condition satisfied by h_1 . Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false. Motivated by [22], we introduce the following condition: given $C_0 > 0$, we consider the condition

$$(76) \quad \left| \frac{\partial h}{\partial t}(t, x) \right| \leq C_0 |h(T', x)| \text{ for almost all } (t, x) \in (0, T) \times \mathcal{M},$$

and we define the set of C_0 -admissible source terms:

$$\mathcal{G}(C_0) := \{h \in H^1(0, T; L^2(\mathcal{M})) \mid h \text{ satisfies (76)}\}.$$

Coming back to (72), we prove that the part H^* defined in (73) (and which is the part we wish to identify) is admissible (with some explicit C_0):

Lemma 6.1. *The function $H^* = r(q_1 - q_2)\beta(u_1)$ belongs to $\mathcal{G}(C_0)$ with $C_0 > 0$ defined by*

$$C_0 := \frac{\|r'\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})} + \|r\|_{L^\infty(\mathbb{R})} \|\beta'\|_{L^\infty(\mathbb{R})} e^{(\|\tilde{\pi}_1\|_{L^\infty((0, T) \times \mathcal{M}^{+1})})^T N_1}}{\beta_{\min} r(T')},$$

where $\tilde{\pi}_1$ is given in Theorem 5.3 with $u^0 = u_1^0$ and N_1 is given in Corollary 5.1 with $u^0 = u_1^0$.

Proof. The proof is based on Corollary 5.1. As it is identical to the similar result established in [39, Lemma 7.1], we omit it. \square

STEP 3: Application of global Carleman estimates and link with some more standard inverse source problem. In the following computations, C stands for generic constant depending on T, t_0, T', B, ω and the parameters in Assumption 1.1. Let us introduce $Z := w_t = u_{1,t} - u_{2,t}$ where w solves (72). Using Proposition 2.4, $Z \in L^2(t_0, T; D(\Delta)) \cap H^1(t_0, T; L^2(\mathcal{M}))$ and satisfies

$$(77) \quad Z_t - \Delta Z = H_t^* + H_t + \tilde{H}_t \quad (t, x) \in (t_0, T) \times \mathcal{M}.$$

Then we apply the Carleman estimate (36) to Z on the time interval (t_0, T) , with $\theta : (t_0, T) \rightarrow \mathbb{R}_+^*$ smooth, convex, such that

$$\theta(t) = \begin{cases} \frac{1}{t-t_0} & t \in (t_0, \frac{t_0+T'}{2}) \\ \frac{1}{T-t} & t \in (\frac{T'+T}{2}, T), \end{cases}$$

and θ attains its global minimum at T' . And we obtain

$$(78) \quad I_0 := \int_{t_0}^T \int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma} + \int_{t_0}^T \int_{\mathcal{M}} \rho |\nabla Z|^2 e^{-2R\sigma} + \int_{t_0}^T \int_{\mathcal{M}} \frac{1}{\rho} Z_t^2 e^{-2R\sigma} \\ \leq C \left(\|e^{-R\sigma} PZ\|_{L^2((t_0, T) \times \mathcal{M})}^2 + \int_{t_0}^T \int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \right).$$

Inequality (78) is the first step when dealing with standard inverse source problem, see [22]. Here the problem consists is retrieving only the part H^* in the source term $H^* + H + \tilde{H}$. First we estimate $\int_{t_0}^T \int_{\mathcal{M}} (H_t^2 + \tilde{H}_t^2) e^{-2R\sigma} d\mathcal{M} dt$ in the left hand side of (78):

Lemma 6.2. *There exists a constant $C > 0$ such that*

$$(79) \quad \int_{t_0}^T \int_{\mathcal{M}} (H_t^2 + \tilde{H}_t^2) e^{-2R\sigma} \leq C \left(\int_{t_0}^T \int_{\mathcal{M}} Z^2 e^{-2R\sigma} + \int_{\mathcal{M}} w(T')^2 \right).$$

Proof. The proof is similar to the proof of [39, lemma 5.2] (using Theorem 5.2 and Corollary 5.1 instead of their analogous 1-dimensional forms) and can also be found in [38]. \square

Coming back to (78), we deduce:

$$(80) \quad I_0 \leq C \left(\int_{t_0}^T \int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} + \int_{t_0}^T \int_{\mathcal{M}} Z^2 e^{-2R\sigma} \right. \\ \left. + \int_{\mathcal{M}} w(T')^2 + \int_{t_0}^T \int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \right).$$

For all $t \in (t_0, T)$, $1 \leq C\theta(t)$, so that, for R large,

$$C \int_{t_0}^T \int_{\mathcal{M}} Z^2 e^{-2R\sigma} \leq \frac{1}{2} \int_{t_0}^T \int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma}.$$

Hence, there exists $R_1 > 0$ and $C > 0$ such that: $\forall R \geq R_1$,

$$(81) \quad I_0 \leq C \left(\underbrace{\int_{t_0}^T \int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} + \int_{\mathcal{M}} w(T')^2 + \int_{t_0}^T \int_{\omega} \rho^3 Z^2 e^{-2R\sigma}}_{:=I_1} \right).$$

Let us note that, without the term $\int_{\mathcal{M}} w(T')^2 d\mathcal{M}$, inequality (81) would be the kind of inequality that one would obtain when dealing with the standard inverse source problem that consists in retrieving H^* in the equation $w_t - \Delta w = H^*$. Let us observe that this extra term satisfies

$$\int_{\mathcal{M}} w(T')^2 = \|(u_1 - u_2)(T', \cdot)\|_{L^2(\mathcal{M})}^2 \leq \|(u_1 - u_2)(T', \cdot)\|_{D(\Delta)}^2.$$

Consequently, it can easily be estimated by the right hand side of (12).

STEP 4: Estimate from above of I_1 . Let us prove that there exists $C > 0$ such that

$$(82) \quad I_1 \leq C \left[\frac{1}{\sqrt{R}} \int_{\mathcal{M}} (H^*(T'))^2 e^{-2R\sigma(T')} + \|w(T')\|_{L^2(\mathcal{M})}^2 + \|w_t\|_{L^2((t_0, T) \times \omega)}^2 \right].$$

Indeed, there exists $p_{min} > 0$ such that $p(x) \geq p_{min}$ for all $x \in \mathcal{M}$, hence $\rho^3 e^{-2R\sigma(t, x)} \leq R^3 S^3 e^{3S\|\psi\|_{\infty}} \theta(t)^3 e^{-2Rp_{min}\theta(t)}$, and since $\theta(t)^3 e^{-2Rp_{min}\theta(t)} \rightarrow 0$ as $t \rightarrow t_0$ and as $t \rightarrow T$, there exists C such that

$$\int_{t_0}^T \int_{\omega} \rho^3 Z^2 e^{-2R\sigma} \leq C \|Z\|_{L^2((t_0, T) \times \omega)}^2 = C \|w_t\|_{L^2((t_0, T) \times \omega)}^2.$$

Finally, the proof of (82) follows from

Lemma 6.3. *There exists $C > 0$ such that*

$$(83) \quad \int_{t_0}^T \int_{\mathcal{M}} (H_t^*)^2 e^{-2R\sigma} d\mathcal{M} dt \leq C \frac{1}{\sqrt{R}} \int_{\mathcal{M}} (H^*(T'))^2 e^{-2R\sigma(T')} d\mathcal{M}.$$

Proof. Lemma 6.3 is classical in inverse source problems. We refer to [22] for its proof. Indeed, the fact that one works on a manifold does not change the reasoning. The key point is the form of the weight function θ . \square

STEP 5: Estimate from below of I_0 . Let us show that there exists $C = C(t_0, T) > 0$ such that

$$(84) \quad \int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} \leq C I_0.$$

Indeed, since $Z(t, x)^2 e^{-2R\sigma(t, x)} \rightarrow 0$ as $t \rightarrow t_0$ for a.a. $x \in \mathcal{M}$, we can write

$$(85) \quad \int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} = \int_{t_0}^{T'} \frac{\partial}{\partial t} \left(\int_{\mathcal{M}} Z(t, x)^2 e^{-2R\sigma(t, x)} \right) \\ = \int_{t_0}^{T'} \int_{\mathcal{M}} \left[2Z Z_t - 2R\sigma_t Z^2 \right] e^{-2R\sigma}.$$

First, we estimate

$$(86) \quad \int_{t_0}^{T'} \int_{\mathcal{M}} 2Z Z_t e^{-2R\sigma} = \int_{t_0}^{T'} \int_{\mathcal{M}} 2\sqrt{\rho} Z e^{-R\sigma} \frac{Z_t e^{-R\sigma}}{\sqrt{\rho}} \\ \leq \int_{t_0}^{T'} \int_{\mathcal{M}} \left(\rho Z^2 e^{-2R\sigma} + \frac{Z_t^2 e^{-2R\sigma}}{\rho} \right) \leq CI_0.$$

Next we estimate the other term of (85): since $|\theta_t(t)| \leq C\theta(t)^3$, we have

$$(87) \quad \int_{t_0}^{T'} \int_{\mathcal{M}} 2R|\sigma_t| Z^2 e^{-2R\sigma} \leq C \int_{t_0}^{T'} \int_{\mathcal{M}} \rho^3 Z^2 e^{-2R\sigma} \leq CI_0.$$

Finally, (85), (86) and (87) imply (84).

STEP 6: Conclusion. Using (84), (81) and next (82), there exists $C > 0$ such that

$$(88) \quad \int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} \leq \frac{C}{\sqrt{R}} \int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \\ + C \|w(T')\|_{L^2(\mathcal{M})}^2 + C \|w_t\|_{L^2((t_0, T) \times \omega)}^2.$$

Let us recall that

$$Z(T') = w_t(T') = \Delta w(T') + H^*(T') + H(T') + \tilde{H}(T'),$$

hence

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \\ \leq C \left(\int_{\mathcal{M}} Z(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} |\Delta w(T')|^2 e^{-2R\sigma(T')} \right. \\ \left. + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')} \right).$$

Applying (88), we get

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \\ \leq C \left(\frac{1}{\sqrt{R}} \int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} + \|w_t\|_{L^2((t_0, T) \times \omega)}^2 + \|w(T')\|_{D(\Delta)}^2 \right. \\ \left. + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')} \right).$$

Choosing R large enough so that $C/\sqrt{R} = 1/2$, we get

$$(89) \quad \frac{1}{2} \int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \leq C \left(\|w_t\|_{L^2((t_0, T) \times \omega)}^2 + \|w(T')\|_{D(\Delta)}^2 \right) \\ + \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} + \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')}.$$

Let us now estimate the two last terms of the right hand side of (89). First, we recall that $|H| = |rq_2(\beta(u_1) - \beta(u_2))| \leq \|r\|_{L^\infty(\mathbb{R})} B \|\beta'\|_{L^\infty(\mathbb{R})} |u_1 - u_2|$. Therefore

$$(90) \quad \int_{\mathcal{M}} H(T')^2 e^{-2R\sigma(T')} \leq C \int_{\mathcal{M}} w(T')^2 e^{-2R\sigma(T')} \leq C \|w(T')\|_{L^2(\mathcal{M})}^2.$$

Next, we write

$$|\tilde{H}| = \left| \left(\varepsilon(u_2) - \varepsilon(u_1) \right) u_2 |u_2|^3 + \varepsilon(u_1) (u_2 - u_1) |u_2|^3 + \varepsilon(u_1) u_1 \left(|u_2|^3 - |u_1|^3 \right) \right| \\ \leq \|\varepsilon'\|_{L^\infty(\mathbb{R})} |u_2 - u_1| |u_2|^4 + \|\varepsilon\|_{L^\infty(\mathbb{R})} |u_2 - u_1| |u_2|^3 \\ + \|\varepsilon\|_{L^\infty(\mathbb{R})} |u_1| \left| |u_2| - |u_1| \right| \left(|u_2|^2 + |u_2 u_1| + |u_1|^2 \right).$$

By Theorem 5.2, for $i = 1, 2$, $\|u_i\|_{L^\infty((0, T) \times \mathcal{M})} \leq C$. Hence,

$$|\tilde{H}| \leq C |u_2 - u_1| + C \left| |u_2| - |u_1| \right| \leq C |u_2 - u_1|.$$

We deduce

$$(91) \quad \int_{\mathcal{M}} \tilde{H}(T')^2 e^{-2R\sigma(T')} \leq C \|w(T')\|_{L^2(\mathcal{M})}^2.$$

Finally, putting (90) and (91) into (89), we get

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} \leq C \left[\|w_t\|_{L^2((t_0, T) \times \omega)}^2 + \|w(T')\|_{D(\Delta)}^2 \right].$$

On the other hand, R being now fixed, there exists some $C_{min} > 0$ such that $e^{-2R\sigma(T')} \geq C_{min} > 0$. Hence

$$\int_{\mathcal{M}} H^*(T')^2 e^{-2R\sigma(T')} d\mathcal{M} \\ = \int_{\mathcal{M}} r(t)^2 |q_1(x) - q_2(x)|^2 \beta(u_1(T'))^2 e^{-2R\sigma(T')} d\mathcal{M} \\ \geq C_{min} r_{min}^2 \beta_{min}^2 \|q_1 - q_2\|_{L^2(\mathcal{M})}^2.$$

It follows

$$\|q_1 - q_2\|_{L^2(\mathcal{M})}^2 \leq C \left[\|w_t\|_{L^2((t_0, T) \times \omega)}^2 + \|w(T')\|_{D(\Delta)}^2 \right].$$

This concludes the proof of Theorem 1.1. \square

And (13) follows then immediately from the Carleman estimate of Theorem 3.1 and the stability estimate (12).

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