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From topological rigidity to topological semisimplicity

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Abstract

A topological (commutative) ring is said to be *rigid* when for every set X, the topological dual of the X-fold topological product of the ring is isomorphic to the free module over X. Examples are fields with a ring topology, discrete rings, and normed algebras. Rigidity translates into a dual equivalence between categories of free modules and of "topologically-free" modules and, with a suitable topological tensor product for the latter, it lifts to an equivalence between monoids in this category (some suitably generalized topological algebras) and coalgebras. Several notions of topological semisimplicity for these (commutative) monoids are introduced, and over an algebraically closed field $\Bbbk$, with a ring topology, all these notions, in addition to cosemisimplicity for coalgebras and classical Jacobson semisimplicity, are proven to coincide. Accordingly this leads to the new rigidity result that the function algebras of $\Bbbk$-valued maps defined on a set, with the topology of simple convergence, are the only topologically semisimple monoids.

Keywords: Topological dual space, topological basis, topological Jacobson radical, topological semisimplicity, coalgebras, finite duality.
MSC classification: 13J99, 54H13, 46A20, 16D60, 16N20, 19D23.

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1 Introduction

In this presentation a (Hausdorff[1]) topological field $(\Bbbk, \tau)$ is said to be *rigid* when for every set X, the topological dual $((\Bbbk, \tau)^X)'$ of the X-fold topological product $(\Bbbk, \tau)^X$ is isomorphic to the vector space $\Bbbk^{(X)}$ of finitely-supported $\Bbbk$-valued maps defined on X (i.e., those maps $X \xrightarrow{f} \Bbbk$ such that for all but finitely many members of X, $f(x) = 0$). Roughly speaking, this means that the topological dual of $(\Bbbk, \tau)^X$ does not depend on the field topology τ and which only really matters, is the fact that it has a product topology. This is essentially the main result of [18] (where it is established for

[1] All topologies will be assumed separated.

division rings instead of fields) of which this contribution is a continuation, in the commutative setting.

Actually this topological property of rigidity is shared by more general topological (commutative unital) rings[2] than only topological fields (a fact not noticed in [18]). For instance any discrete ring is rigid in the above sense (see Lemma 31, p. 21). And even if not all topological rings are rigid (see Section 3.6 for a counter-example), many of them still are (e.g., every real or complex normed commutative algebra).

It is our intention to study in more details some consequences of the property of rigidity for arbitrary commutative rings in particular for some of their topological algebras. This is logically performed in two steps. In the first step, which is carried out from Section 2 to Section 5, the basic ingredients, needed for the second step, are set up. The second step – Sections 6 and 7 – concerns some issues of (topological) semisimplicity for topological algebras over a commutative rigid ring.

The content of the first step is as follows. So far, for a topological ring (R, τ), rigidity reads as $((R, \tau)^X)' \simeq R^{(X)}$ (here, and everywhere else, R stands for the canonical left R-module structure on the underlying abelian group of R) for each set X. Suitably topologized (see Section 2.5.1), the algebraic dual $(R^{(X)})^*$ turns out to be isomorphic to $(R, \tau)^X$. Accordingly rigidity leads to a one-one correspondence between free R-modules $R^{(X)}$ and topological (R, τ)-modules of the form $(R, \tau)^X$, said to be *topologically-free*. In this contribution one observes that topologically-free modules are particularly nice since they have topological bases which play a rôle similar to linear bases of free modules (or vector spaces) and make them quite easy to handle.

More appropriately the above correspondence may be upgraded into a dual equivalence of categories[3] between free and topologically-free modules (Theorem 79, p. 36, in Section 4) under the algebraic and topological dual functors. (A similar interpretation of the main result [18, Theorem 5, p. 156] was given in the quoted reference as a dual equivalence between some categories of (topological) bimodules over a topological division ring. At this point what is new in this contribution is a similar interpretation for arbitrary (commutative) rigid rings.)

It is also worth mentioning at this stage that as a consequence not only one recovers a result of J. Dieudonné [8] namely that for a field $\Bbbk$, the category

[2] In this contribution, every ring is assumed non-trivial, commutative and unital (see Conventions 1).

[3] By *dual* equivalence is meant an equivalence between a category $\mathbf{C}$ and the opposite $\mathbf{D}^{\mathsf{op}}$ of another.

of vector spaces over $\Bbbk$ is equivalent to the dual of that of linearly compact vector spaces since these are nothing but topologically-free vector spaces over $\Bbbk$ with its discrete topology (this was already noticed in [18]), but also that the category of linearly compact $\Bbbk$-vector spaces is equivalent to any category of topologically-free $(\Bbbk, \tau)$-vector spaces, with τ a ring topology on $\Bbbk$ (Section 4.4.3, p. 37). We take advantage several times of this opportunity we have to replace a ring topology τ on a field by the discrete topology (in particular when we study the relations with dual coalgebras and the notions of topological semisimplicity, see below).

Under the rigidity assumption, the aforementioned dual equivalence enables to provide a topological tensor product $\circledast_{(\mathsf{R},\tau)}$ for topologically-free (R, τ)-modules by transporting the algebraic tensor product $\otimes_{\mathsf{R}}$ along the dual equivalence. It turns out that $\circledast_{(\mathsf{R},\tau)}$ is (coherently) associative, commutative and unital, i.e., makes monoidal the category of topologically-free modules. Not surprisingly the above dual equivalence remains well-behaved, i.e., monoidal, with respect to the (algebraic and topological) tensor products (Proposition 98, p. 47, and Theorem 102, p. 48, in Section 5.3).

The reason for this monoidal approach is twofold: On the one hand it makes it possible to consider monoids in these tensor categories and on the other hand such a category-theoretic treatment makes it easy and plainly accessible relationship with some dual objects, namely the comonoids or coalgebras, just by "reversing the arrows".

Let's take the time to be a little more specific here. Monoids in the tensor category of topologically-free modules form a particular kind of suitably generalized topological algebras or, more generally algebras with a separately continuous multiplication[4] (see Section 5.3.2.3, p. 53) to which one simply refers to as "monoids" in the continuation of the Introduction, while monoids in the dual category of free modules are coalgebras (with a free underlying module[5]), i.e., modules C together with a "comultiplication" $C \xrightarrow{\delta} C \otimes_{\mathsf{R}} C$ and a "counit" $C \xrightarrow{\epsilon} R$ satisfying axioms which are formally dual to associativity (for the multiplication) and to the defining property of a (two-sided) unit. Under the rigidity assumption, the dual equivalence between topologically-free and free modules, because it is monoidal (see above), has the nice feature to induce a dual equivalence between monoids and coalgebras (Corollary 104, p. 49). Thus one has two constructions: a *topological dual coalgebra* of a monoid (in the tensor category of topologically-free modules) and an *alge-*

[4]For the exposition, in this Introduction one does not make any more distinction between topological algebras and algebras with a separately continuous multiplication.

[5]Of course, this is automatically fulfilled when the base ring R is a field.

braic dual monoid of a coalgebra, and these constructions are inverse one from the other (up to isomorphism).

There already exists a standard duality theory between algebras and coalgebras, over a field, known as *finite duality* (see Appendix D.4, p. 116) but contrary to our "topological duality" it is merely an adjunction, not an equivalence. One discusses how these dualities interact (see Section 5.3.2, p. 50) and in particular one proves that the algebraic dual monoid of a coalgebra essentially corresponds to its dual in the finite duality sense (Section 5.3.2.2, p. 52), that over a discrete field, the topological dual coalgebra of a monoid always is a subcoalgebra of the finite dual coalgebra of its underlying algebra (see Section 5.3.2.1, p. 51), and furthermore that they are equal exactly when finite duality provides an equivalence of categories (Theorem 123, p. 57).

This is pretty much what the first step of our study contains. It is now time to address the second step. Again over a discrete field, our topological duality between monoids and coalgebras induces a one-one correspondence between the closed ideals of the former and the subcoalgebras of the latter, which in particular restricts to a bijection between closed maximal ideals and simple subcoalgebras (see Section 6, p. 61). Because of the discrete base field it seems to be quite restrictive, but due to the above dual equivalence, still we have the opportunity to replace the discrete topology by a ring topology on the base field.

This nice relation is involved in the last part of the paper (Section 7, p. 67) which deals with different notions of (topological) semisimplicity over a field $\Bbbk$ with a ring topology. Roughly speaking ring-theoretic (Jacobson) semisimplicity depicts the function algebra behaviour of a commutative algebra. Indeed, any commutative algebra A admits a canonical "representation" as a function algebra on its "spectrum" $spec(\mathsf{A})$, i.e., a conveniently chosen set of ideals (for instance all maximal ideals, or all closed maximal ideals for Banach algebras), namely $\mathsf{A} \xrightarrow{can_{\mathsf{A}}} \prod_{I \in spec(\mathsf{A})} \mathsf{A}/I$. An algebra is semisimple precisely when this representation is one-to-one.

For a commutative monoid over topologically-free $(\Bbbk, \tau)$-vector spaces (recall that it corresponds to a certain kind of topological commutative $(\Bbbk, \tau)$-algebras), there are several possible approaches for semisimplicity. The most direct consists of declaring semisimple those monoids whose underlying algebra (essentially obtained by forgetting the topology) is semisimple. Considering the topology too, by taking all closed maximal ideals for the spectrum, leads to the notion of *topological semisimplicity* (see Section 7.1.1.1, p. 68). Still more specialized, inspired by Gelfand's work on C^*-algebras, one may only consider the kernels of continuous characters (i.e., continuous

algebra maps from the monoid to the base topological field) as a spectrum, and this provides the notion of *Gelfand semisimplicity* (see Section 7.1.2.1, p. 71). Finally using our dual equivalence one has access to the concept of *cosemisimplicity* of the topological dual coalgebra (Appendix D.3, p. 114). These different notions are not unrelated since one immediately observes that Gelfand semisimplicity implies topological semisimplicity which in turn entails semisimplicity of the underlying ordinary algebra.

By the Fundamental Theorem (Appendix D.3, p. 114), coalgebras essentially are, at least locally, finite-dimensional because any finite subset spans a finite-dimensional subcoalgebra. This property should be reflected in some way, along our dual equivalence, for monoids in the tensor category of topologically-free $(\Bbbk, \tau)$-vector spaces. In fact over a discrete field, the closed maximal two-sided ideals of a monoid are finite-codimensional, and furthermore when the field is algebraically closed and the monoid is commutative, they even all have codimension one (Theorem 135, p. 66). This observation is directly related to Gelfand semisimplicity through the kernels of continuous characters (Proposition 154, p. 74), and has for consequence the equivalence between Gelfand semisimplicity and topological semisimplicity (Corollary 156, p. 76) for a discrete algebraically closed field.

Because the categories of topologically-free vector spaces are equivalent for all ring topologies on the same field, and in particular for the discrete topology, all of this interact in Theorem 167 (p. 81) where, for an algebraically closed field $\Bbbk$ with a ring topology τ, is established the equivalence between all the above notions of semisimplicity. As a last and direct consequence this leads to a new *topological rigidity* result: there is one, and only one, kind of (Gelfand or topological) semisimple commutative monoids in the tensor category of topologically-free $(\Bbbk, \tau)$-vector spaces, namely the X-fold topological products $(\Bbbk, \tau)^X$ of the beginning of the Introduction, for varying set X, together with the point-wise multiplication, i.e., the algebra of all functions from X to $\Bbbk$ with the topology of simple convergence. This extends the result from algebra that any Jacobson semisimple finite-dimensional commutative algebra over an algebraically closed field $\Bbbk$, is canonically isomorphic to an algebra of $\Bbbk$-valued functions defined on a finite set.

Some remarks on the presentation

As illustrated above, this contribution about topology and its applications to algebra essentially is at the confluence of two domains, topological algebras and coalgebras. Moreover the language of category theory is adopted, not only for the exposition, but more soundly for the theory of monoidal

categories which brings the fundamental tools and results we need to deal with (co)algebras.

The intention of the author is to make this presentation accessible to readers who, even familiar with categorical language, are not necessarily specialists in the theory of monoidal categories or in that of coalgebras. It follows that category-theoretic notions beyond the standard ones, as well as definitions and results on coalgebras, will be put together in the Appendix. In this way, the reader is not obliged to refer to the specific literature. Furthermore many proofs concerning monoidal categories basically consist in checking that some diagrams commute. They are sometimes tedious and provide only a minimal understanding. There are thereby all postponed into a specific Appendix to which the reader may refer if desired.

Of course, it results a long paper and the author's belief is that it is the reasonable price to make the presentation directly accessible for a large audience.

2 Conventions, notations and basic definitions

This section introduces the conventions adopted in the text as well as the most basic ingredients (e.g., function algebras, algebraic and topological dual functors) needed for the presentation. It also gives the opportunity to recall that other notations and definitions (e.g., names of some categories such as $\mathbf{Set}$, $\mathbf{Mod}_{\mathsf{R}}$, $\mathbf{Alg}_{\mathsf{R}}$, etc.) are provided in the Appendix A (p. 85). The reader should refer to this Appendix for the standard categorical notions used in the presentation. An index of notations and notions is also available at the very end of this contribution.

2.1 Conventions

Let us mention the conventions which are applied hereafter.

1 Conventions *1. Except as otherwise stipulated, all topologies are Hausdorff, and every ring is assumed non-trivial (i.e., distinct from the trivial ring* (0)*), unital and commutative.* (To the contrary an algebra over a ring won't be assumed commutative nor unital or even non-trivial, but associative.)

2. For a ring R*,* R *denotes both its underlying set and the canonical left* R*-module structure on its underlying additive group. Likewise if* A *is an* R*-algebra, then* A *is both its underlying set and its underlying*

R-*module. The unit of a ring* R *(resp., a unital algebra* A*) is either denoted by* 1_{R} *(resp.* 1_{A}*) or simply by* 1*. The bilinear multiplication will be denoted* $m_{\mathsf{R}}(x,y)$*,* $m_{\mathsf{A}}(x,y)$*,* $m(x,y)$*, by juxtaposition* xy *or even using an* ad hoc *notation.*

3. *A ring map (or morphism of rings) is assumed to preserve the units.*

4. *A module* M *over a ring* R *is actually a unital*[6] *left-*R*-module, and when* R *is a field – usually denoted* $\Bbbk$ *– it is called a vector space.*

5. *Let* X *be a set, let* (E_x,τ_x)*,* $x \in X$*, and* (F,σ) *be topological spaces. That a map* $\prod_{x\in X} E_x \xrightarrow{f} F$ *is continuous means that it is continuous from the product topological space* $\prod_{x\in X}(E_x,\tau_x)$*, i.e.,* $\prod_x E_x$ *with the product topology, to* (F,σ)*.*

6. *When all* (E_x,τ_x)*'s are equal to the same topological space* (E,τ)*, then the* X*-fold topological product* $\prod_{x\in X}(E_x,\tau_x)$ *is canonically identified with the set* E^X *of all maps from* X *to* E *equipped with the topology of simple convergence, and is denoted by* $(E,\tau)^X$*. Under this identification, the canonical projections* $(E,\tau)^X \xrightarrow{\pi_x} (E,\tau)$ *are given by* $\pi_x(f) = f(x)$*,* $x \in X$*,* $f \in E^X$*.*

7. *Given a topological space* (E,τ) *and* $x \in E$*,* $\mathfrak{V}_{(E,\tau)}(x)$ *is the set of all neighborhoods of* x*.*

2.2 Topological rings, modules and algebras

2 Definition *Let* R *be a ring, and let* τ *be a (Hausdorff, following our conventions) topology on (the carrier set of) the ring. It is said to be a* ring topology *when the structure maps of the ring, namely addition* $(x,y) \mapsto x+y$*, multiplication* $(x,y) \mapsto m(x,y)$*, and opposite* $x \mapsto -x$*, are continuous. By* topological ring (R,τ) *is meant a ring together with a ring topology* τ *on it. (In view of Convention 1.5, the multiplication of a topological ring is jointly continuous.) By a* field with a ring topology*, denoted* $(\Bbbk,\tau)$*, is meant a topological ring* $(\Bbbk,\tau)$ *with* $\Bbbk$ *a field. A* topological field *is a field with a ring topology* $(\Bbbk,\tau)$ *such that the inversion* $\alpha \mapsto \alpha^{-1}$ *is continuous from* $\Bbbk \smallsetminus \{0\}$ *to itself with the subspace topology.*

3 Example *A ring* R *with the discrete topology* d *is a topological ring. Actually the multiplication* $R \times R \xrightarrow{m_{\mathsf{R}}} R$ *is even uniformly continuous. (Indeed,* $m_{\mathsf{R}}(\{0\} \times R) + m_{\mathsf{R}}(R \times \{0\}) = \{0\}$*.)*

[6] This means that $1_R x = x$ for all $x \in M$.

Let (R,τ) be a topological ring.

1. A pair (M,σ) consisting of a (left and unital) R-module M and a topology σ on M which makes continuous the addition, opposite and scalar multiplication $R\times M\to M$, is called a *topological* (R,τ)*-module*. Such a topology is referred to as a (R,τ)*-module topology*. In particular, when R is a field $\Bbbk$, then this provides *topological* $(\Bbbk,\tau)$*-vector spaces*.

2. Given topological (R,τ)-modules $(M,\sigma),(N,\gamma)$, a *continuous* (R,τ)*-linear map* $(M,\sigma)\xrightarrow{f}(N,\gamma)$ is a R-linear map $M\xrightarrow{f}N$ which is continuous. Topological (R,τ)-modules and these morphisms form a category $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, which is denoted $\mathbf{TopVect}_{(\Bbbk,\tau)}$, when $\Bbbk$ is a field.

3. A pair (A,σ), with A an R-algebra, and σ a topology on A, is a *topological* (R,τ)*-algebra*, when σ is a module topology for the underlying R-module A, and the multiplication of A is a bilinear (jointly[7]) continuous map. A topological (unital, commutative and non-trivial) $(\mathbb{Z},\mathsf{d})$ -algebra thus is a topological ring.

4. Given topological (R,τ)-algebras $(\mathsf{A},\sigma),(\mathsf{B},\gamma)$, a *continuous* (R,τ)*-algebra map* $(\mathsf{A},\sigma)\xrightarrow{f}(\mathsf{B},\gamma)$ is a R-algebra map $\mathsf{A}\xrightarrow{f}\mathsf{B}$ which is also continuous. Topological (R,τ)-algebras with these morphisms form a category $\mathbf{TopAlg}_{(\mathsf{R},\tau)}$. One also has various full subcategories (cf. Appendix A, p. 85) such as e.g., ${}_1\mathbf{TopAlg}_{(\mathsf{R},\tau)}$, ${}_c\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ and ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ of topological unital, resp. commutative, resp. unital and commutative algebras. (In the categories of unital algebras, the morphisms are assumed *unital*, i.e., they preserve the units.)

4 Remark *Following Convention 1.1, it is clear from the above definitions, that any non-trivial commutative and unital (topological) algebra has an underlying (topological) ring (obtained by forgetting the scalar action).*

5 Remark *For each topological ring* (R,τ)*, one has obvious forgetful and*

[7] At some point (see Section 5.3.2.3, p. 53) one notwithstanding derogates from our conventions by considering "topological algebras" with a separately continuous multiplication, called "separately continuous algebras". See also Appendix E, p. 118.

embedding functors which make commute the following diagram.

$$
\begin{tikzcd}
{}_1\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[rrr] \arrow[dd] & & & {}_1\mathbf{Alg}_{\mathsf{R}} \arrow[dd] \\
& {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[ul, hook] \arrow[r] \arrow[d] & {}_{1,c}\mathbf{Alg}_{\mathsf{R}} \arrow[ur, hook] \arrow[d] & \\
& {}_c\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[r] \arrow[dl, hook] & {}_c\mathbf{Alg}_{\mathsf{R}} \arrow[dr, hook] & \\
\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[rrr] \arrow[dr] & & & \mathbf{Alg}_{\mathsf{R}} \arrow[dl] \\
& \mathbf{TopMod}_{(\mathsf{R},\tau)} \arrow[r] & \mathbf{Mod}_R &
\end{tikzcd}
\tag{1}
$$

6 Example (R,τ) *is a topological* (R,τ)*-module and the topological ring* (R,τ) *with the previous module structure, is a topological* (R,τ)*-algebra.*

7 Remark *A* R*-module (resp.* R*-algebra) with the discrete topology* d *is a topological* (R,d)*-module (resp.* (R,d)*-algebra).*

The following construction (see e.g.,[6, 23]) will be useful in what follows.

8 Definition *Let* (R,τ) *be a topological ring, and let* M *be a* R*-module (resp.* R*-algebra). Let* $F = (M \xrightarrow{f} M_f)_{f\in F}$ *be a family of* R*-linear (resp.* R*-algebra) maps, and for each* $f \in F$*,* (M_f,σ_f) *is a Hausdorff topological* (R,τ)*-module (resp.* (R,τ)*-algebra). The coarsest topology* $!_F$ *on* M *that makes the maps of* F *continuous is called the* initial topology induced by F*. It is a* not necessarily Hausdorff (R,τ)*-module (resp.* (R,τ)*-algebra) topology, and is Hausdorff when* F *separates the points of* M*, i.e., when for each* $x \neq y$ *in* M*, there is a map* $f \in F$ *such that* $f(x) \neq f(y)$*.*

A fundamental system of neighborhood of zero for the above topology $!_F$ consists in the sets $\bigcap_{f\in A} f^{-1}(U_f)$, where $A \subseteq F$ is a finite set, and U_f is an open neighborhood of zero in (M_f,σ_f). Moreover a R-linear map (resp. R-algebra map) $(N,\sigma) \xrightarrow{g} (M,!_F)$ is continuous if, and only if, for each $f \in F$, $(N,\sigma) \xrightarrow{f\circ g} (M_f,\sigma_f)$ is continuous.

2.3 Restriction of scalars

Let (R,τ) and (S,σ) be topological rings, and let $(\mathsf{R},\tau) \xrightarrow{f} (\mathsf{S},\sigma)$ be a continuous ring map. It may be used to transform a topological (S,σ)-module into a topological (R,τ)-module by restriction of scalars along f.

In details, let (M,γ) be a topological (S,σ)-module. There is a scalar action of R on M given by $\alpha\cdot v := f(\alpha)v$, $\alpha \in R$, $v \in M$ (where by juxtaposition is denoted the scalar action $\mathsf{S}\times M \to M$). Furthermore this action is again continuous (by composition of continuous maps).

Let $f^*(M,\gamma)$ be the topological (R,τ)-module just obtained. At present let $(M,\gamma) \xrightarrow{g} (N,\pi)$ be a continuous (S,σ)-linear map. g is also R-linear because of $g(\alpha\cdot v) = g(f(\alpha)v) = f(\alpha)g(v) = \alpha\cdot g(v)$, and thus provides a continuous (R,τ)-linear map $f^*(M,\gamma) \xrightarrow{g} f^*(N,\pi)$. All this results in a functor $\mathbf{TopMod}_{(\mathsf{S},\sigma)} \xrightarrow{f^*} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ of *restriction of scalars along f* (here f^* should not be confused with the image of f by the algebraic dual functor (see below)).

2.4 Example: X-fold product and finitely-supported maps

More than just examples, the algebras of all functions from a set to a topological base ring, and the free modules play a fundamental rôle in this contribution and so deserve some attention.

2.4.1 (Topological) module structures

Let R be a ring, and let X be a set. It is possible to consider the R-module R^X of all maps from X to R. It may be equivalently defined as the X-fold power of R in the category $\mathbf{Mod}_{\mathsf{R}}$. Therefore it is the object component of a functor $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ whose action on maps is as follows: given $X \xrightarrow{f} Y$ and $g \in R^Y$, $P_{\mathsf{R}}(f)(g) = g\circ f$.

Actually when the base ring is topological, then R^X itself may be given a somewhat natural topology (see Example 9 below).

9 Example *Let (R,τ) be a topological ring and let X be a set. The X-fold topological product $(R,\tau)^X$ of the topological (R,τ)-module (R,τ) is actually an instance of the more general construction given in Definition 8, obtained by taking for F the family of all projections $(R^X \xrightarrow{\pi_x} R)_{x\in X}$, $\pi_x(f) = f(x)$, $f \in R^X$.*

Let R be a ring and let X be a set. Let $x \in X$. Then, δ_x^{R}, or simply δ_x, is the member of R^X given for $y \in X$, by

$$\delta_x^{\mathsf{R}}(y) = \begin{cases} 1_{\mathsf{R}} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} .$$

10 Remark *Because* R *is assumed non-trivial, the map* $X \xrightarrow{\delta_X^{\mathsf{R}}} R^X$, $x \mapsto \delta_x$, *is one-to-one.*

Let $f \in R^X$. The *support* of f is the set $supp(f) := \{ x \in X : f(x) \neq 0 \}$. Let $R^{(X)}$ be the sub-R-module of R^X consisting of all *finitely-supported maps* (or maps with *finite support*), i.e., the maps f such that $supp(f)$ is finite.

$R^{(X)}$ is actually the free R-module over X, and a basis is given by $\{ \delta_x^{\mathsf{R}} : x \in X \}$.

11 Notation *Let* R *be a ring and let* X *be a set. Let* $x \in X$, *and define* $p_x := R^{(X)} \hookrightarrow R^X \xrightarrow{\pi_x} R$, *where* $R^{(X)} \hookrightarrow R^X$ *is the canonical inclusion.*

12 Remark 1. *The construction* $R^{(X)}$ *is functorial in the set* X. *Indeed, let* $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ *be the functor given by* $F_{\mathsf{R}}(X) := R^{(X)}$, *and for* $X \xrightarrow{f} Y$, $p \in R^{(X)}$,

$$F_{\mathsf{R}}(f)(p) := \sum_{y \in Y} \left(\sum_{x \in f^{-1}(\{ y \})} p(x) \right) \delta_y^{\mathsf{R}}.$$

In other words, $F_{\mathsf{R}}(f)(\delta_x^{\mathsf{R}}) = \delta_{f(x)}^{\mathsf{R}}$, $x \in X$.

$R^{(X)}$ *being free means in category-theoretic terms, that* F_{R} *is a left adjoint of the obvious forgetful functor* $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$. *The map* $X \xrightarrow{\delta_X^{\mathsf{R}}} |R^{(X)}|$ *(Remark 10) is the component at* X *of the unit of the adjunction* $F_{\mathsf{R}} \dashv |\cdot| : \mathbf{Set} \to \mathbf{Mod}_{\mathsf{R}}$.

2. *The construction of* $(R, \tau)^X$ *is also (contravariently) functorial in* X. *Let* $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ *be given by* $P_{(\mathsf{R},\tau)}(X) := (R, \tau)^X$, *and* $P_{(\mathsf{R},\tau)}(f)(g) := g \circ f$, $X \xrightarrow{f} Y$, $g \in R^Y$. *In other words, for each* $x \in X$, $\pi_x \circ P_{(\mathsf{R},\tau)}(f) = \pi_{f(x)}$ *(which also ensures continuity of* $P_{(\mathsf{R},\tau)}(f)$*). Of course, the following diagram commutes (where the unnamed arrow is the obvious forgetful functor).*

$$\begin{tikzcd} \mathbf{Set}^{\mathsf{op}} \arrow[r, "P_{(\mathsf{R},\tau)}"] \arrow[dr, "P_{\mathsf{R}}"'] & \mathbf{TopMod}_{(\mathsf{R},\tau)} \arrow[d] \\ & \mathbf{Mod}_{\mathsf{R}} \end{tikzcd} \tag{2}$$

2.4.2 Algebra structure

Given a ring R, and a set X, R^X is merely not just a R-module but, under point-wise multiplication $R^X \times R^X \xrightarrow{M_X} R^X$, a commutative R-algebra, the function algebra on X, denoted $A_{\mathsf{R}}(X)$, with unit $1_{A_{\mathsf{R}}(X)} := \sum_{x \in X} \delta_x^{\mathsf{R}}$.

13 Remark *$A_{\mathsf{R}}(X)$ is functorial in X because for $X \xrightarrow{f} Y$, $g, h \in R^Y$, $x \in X$, $(P_{\mathsf{R}}(f)(gh))(x) = ((gh) \circ f)(x) = g(f(x))h(f(x)) = (P_{\mathsf{R}}(f)(g)P_{\mathsf{R}}(f)(h))(x)$. This provides a functor $\mathbf{Set}^{\mathsf{op}} \xrightarrow{A_{\mathsf{R}}} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$.*

14 Notation *Let $X \neq \varnothing$ (in order to consider non-trivial rings). The underlying ring of $A_{\mathsf{R}}(X)$ is denoted R^X. Of course, $1_{\mathsf{R}^X} = \sum_{x \in X} \delta_x^{\mathsf{R}}$.*

15 Lemma *$(R, \tau)^X \times (R, \tau)^X \xrightarrow{M_X} (R, \tau)^X$ is continuous. In consequence $A_{(\mathsf{R},\tau)}(X) := ((R, \tau)^X, M_X, 1_{\mathsf{R}^X})$ is an object of ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$.*

Proof: The multiplication $R^X \times R^X \xrightarrow{M_X} R^X$ is continuous since for each $x \in X$, $\pi_x \circ M_X = m_{\mathsf{R}} \circ (\pi_x \times \pi_x)$. Indeed, this equation directly implies separate continuity in each variable of M_X (because so is m_{R}), and given a finite subset $A \subseteq X$ and for each $x \in X$, an open neighborhood U_x of zero in (R, τ), by continuity of m_{R} at zero, there are neighborhoods of zero V_x, W_x such that $m_{\mathsf{R}}(V_x, W_x) \subseteq U_x$, and thus $M_X(\bigcap_{x \in A} \pi_x^{-1}(V_x), \bigcap_{x \in A} \pi_x^{-1}(W_x)) \subseteq \bigcap_{x \in A} \pi_x^{-1}(U_x)$ which ensures continuity of M_X at zero, and thus by [23, Theorem 2.14, p. 16], M_X is continuous. □

16 Remark *Here again, $A_{(\mathsf{R},\tau)}(X)$ is functorial in the set X as it readily follows from Remarks 12 and 13. Besides Diag. (2) may be completed into the following commutative diagram with unnamed arrows the obvious forgetful functors.*

$$\begin{tikzcd}
{}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[rr] \arrow[d] & & {}_{1,c}\mathbf{Alg}_{\mathsf{R}} \arrow[d] \\
& \mathbf{Set}^{\mathsf{op}} \arrow[ul, "A_{(\mathsf{R},\tau)}"'] \arrow[ur, "A_{\mathsf{R}}"] \arrow[dl, "P_{(\mathsf{R},\tau)}"] \arrow[dr, "P_{\mathsf{R}}"'] & \\
\mathbf{TopMod}_{(\mathsf{R},\tau)} \arrow[rr] & & \mathbf{Mod}_{\mathsf{R}}
\end{tikzcd} \tag{3}$$

17 Notation *Let $X \neq \varnothing$. The underlying topological ring of $A_{(\mathsf{R},\tau)}(X)$ is denoted $(\mathsf{R}, \tau)^X$. (It of course coincides with the X-fold product of (R, τ) in the category of topological rings.)*

2.5 Recollection of results about algebraic and topological duals

2.5.1 Algebraic dual functor

Let R be a ring. Let M be a R-module. Let $M^* := \mathbf{Mod}_{\mathsf{R}}(M, R)$ be the *algebraic* (or *linear*) *dual* of M. This is readily a R-module on its own.

When (R, τ) is a topological ring, then M^* may be topologized with the initial topology $w^*_{(\mathsf{R},\tau)}$, called the *weak-$*$ topology*, induced by the family $(M^* \xrightarrow{\Lambda_M(v)} R)_{v \in M}$ of *evaluations* at some points, where $(\Lambda_M(v))(\ell) := \ell(v)$. This provides a structure of topological (R, τ)-module on M^*, which is even Hausdorff (since if $\ell(v) = 0$ for all $v \in M$, then $\ell = 0$).

Moreover given a linear map $M \xrightarrow{f} N$, $N^* \xrightarrow{f^*} M^*$, $\ell \mapsto f^*(\ell) := \ell \circ f$, is continuous for the above topologies. Consequently, this provides a functor $\mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ called the *algebraic dual functor*.

18 Remark *M^* or f^* stand for $Alg_{(\mathsf{R},\tau)}(M)$ or $Alg_{(\mathsf{R},\tau)}(f)$.*

Up to isomorphism, one recovers the module of all R-valued maps on a set X, with its product topology, as the algebraic dual of the module of all finitely-supported maps duly topologized as above.

19 Lemma *For each set X, $(R, \tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$ (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$) under the map*

$$\rho_X : (R, \tau)^X \to ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$$

given by $(\rho_X(f)(p)) := \sum_{x \in X} p(x) f(x)$, $f \in R^X$, $p \in R^{(X)}$.

Proof: Let $\ell \in (R^{(X)})^*$. Let us define $X \xrightarrow{\hat{\ell}} R$ by $\hat{\ell}(x) := \ell(\delta_x)$, $x \in X$. That the two constructions are linear and inverse one from the other is clear.

It remains to make sure that there are also continuous. Let $\ell \in (R^{(X)})^*$, and let $x \in X$. Then, $\pi_x(\hat{\ell}) = \hat{\ell}(x) = \ell(\delta_x) = (\Lambda_{R^{(X)}}(\delta_x))(\ell)$, which ensures continuity of $((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{\rho_X^{-1}} (R, \tau)^X$. Let $f \in R^X$, and $p \in R^{(X)}$. As $(\Lambda_{R^{(X)}}(p))(\rho_X(f)) = (\rho_X(f))(p) = \sum_{x \in X} p(x) f(x) = \sum_{x \in X} \pi_x(p) f(x) = \sum_{x \in X} \pi_x(p) \pi_x(f)$, $\Lambda_{R^{(X)}}(p) \circ \rho_X$ is a finite linear combination of projections, whence is continuous for the product topology. In conclusion, $(R, \tau)^X \xrightarrow{\rho_X} ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$ also is continuous. $\square$

20 Definition *Let M be a free R-module. Let B be a basis of M. This defines a family of R-linear maps, the* coefficient maps $(M \xrightarrow{b^*} R)_{b\in B}$ *such that each $v \in M$ is uniquely represented as a finite linear combination $v = \sum_{b\in B} b^*(v)b$. One denotes* $\mathbf{FreeMod}_{\mathsf{R}}$ *the full subcategory of* $\mathbf{Mod}_{\mathsf{R}}$ *spanned by the free modules.*

21 Remark *When $\Bbbk$ is a field,* $\mathbf{FreeMod}_{\Bbbk}$ *is just* $\mathbf{Vect}_{\Bbbk}$ *itself.*

22 Example *For each set X, $\{\delta_x^{\mathsf{R}} : x \in X\}$ is a free basis of $R^{(X)}$, and $p_x = \delta_x^*$, $x \in X$ (see Notation 11, p. 14).*

23 Remark *Given $b, d \in B$, $b^*(d) = \delta_b(d)$. In particular, $B \xrightarrow{(-)^*} B^* := \{b^* : b \in B\}$ is a bijection.*

Given a free R-module, any choice of a basis makes it possible to define a module topology on the algebraic dual: given a basis B, let Π_B^τ be the initial topology on M^* induced by the evaluations $(M^* \xrightarrow{\Lambda_M(b)} R)_{b\in B}$. (In view of Remark 23, this is clearly a Hausdorff topology.)

24 Lemma *Let M be a free R-module. The topology Π_B^τ is independent of the choice of the basis B of M since it is equal to $w^*_{(\mathsf{R},\tau)}$. Moreover, for each basis B of M, $(M^*, w^*_{(\mathsf{R},\tau)}) \simeq (R,\tau)^B$ (in* $\mathbf{TopMod}_{(\mathsf{R},\tau)}$*).*

Proof: Let us assume that B is a basis of M. By construction it is clear that the initial topology induced by $(\Lambda_M(b))_{b\in B}$ is contained into $w^*_{(\mathsf{R},\tau)}$.

Conversely for each $v \in M$, $v = \sum_{x\in X} b^*(v)b$ uniquely, where all but finitely many $b^*(v)$'s are zero. Given $\ell \in M^*$, $(\Lambda_M(v))(\ell) = \ell(\sum_{b\in B} b^*(v)b) = \sum_{b\in B} b^*(v)\ell(b) = \sum_{b\in B} b^*(v)(\Lambda_M(b))(\ell)$ so that $\Lambda_M(v)$ is a finite linear combination of some $\Lambda_M(b)$, whence is continuous from (M^*, Π_B^τ) to (R,τ), and therefore this topology contains $w^*_{(\mathsf{R},\tau)}$.

The last assertion is due to Lemma 19, by functoriality of $Alg_{(\mathsf{R},\tau)}$. $\square$

2.5.2 Topological dual functor

Let (R,τ) be a topological ring, and let (M,σ) be a topological (R,τ)-module. Let $(M,\sigma)' := \mathbf{TopMod}_{(\mathsf{R},\tau)}((M,\sigma),(R,\tau))$ be the *topological dual* of (M,σ). It is itself a R-submodule of M^*.

Let $(M,\sigma) \xrightarrow{f} (N,\gamma)$ be a continuous (R,τ)-linear map between topological modules. Let $(N,\gamma)' \xrightarrow{f'} (M,\sigma)'$ be the R-linear map given by $f'(\ell) := \ell \circ f$.

All of this evidently forms a functor $\mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{Mod}_{\mathsf{R}}$.

Let (R,τ) be a topological ring, and let X be a set. Let $R^{(X)} \xrightarrow{\lambda_X} (R^X)^*$ be given by $(\lambda_X(p))(f) \coloneqq \sum_{x\in X} p(x)f(x)$, $p \in R^{(X)}$, $f \in R^X$.

Let ρ_X be the map introduced in Lemma 19. Then, for each $p \in R^{(X)}$, $\lambda_X(p) = \Lambda_{R^{(X)}}(p) \circ \rho_X$, which ensures continuity of $\lambda_X(p)$, i.e., $\lambda_X(p) \in ((R,\tau)^X)'$.

25 Lemma *Let (R,τ) be a topological ring, and let X be a set. Then $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)'$ is one-to-one.*

Proof: Assume $p \in \ker \lambda_X$, then in particular for each $x \in X$, $p(x) = (\lambda_X(p)(\delta_x)) = 0$. Whence $p = 0$. □

3 Rigid rings: definitions and (counter-)examples

The term of *rigidity* to which the title of the present contribution refers, was originally introduced in [18] as a part of the title of the quoted reference. The corresponding notion of *rigidity*, recalled at the beginning of the Introduction, was originally but only implicitly introduced in [18, Theorem 5, p. 156] as the main result therein. Nevertheless the possibility that the conclusion of this result could remain valid for more general topological rings than topological division rings was not noticed. This is the reason why no explicit definition of a rigid ring was provided then. Since a large part of this presentation is given for arbitrary rigid rings (Definition 27 below), one here provides a stock of basic examples.

3.1 Duality pairing

Let R be a ring, and let X be a set. Let us define a R-bilinear map, called the *duality pairing*, $R^{(X)} \times R^X \xrightarrow{\langle\cdot,\cdot\rangle_X} R$ by

$$\langle p, f\rangle_X \coloneqq \sum_{x\in X} p(x)f(x)$$

(which is a sum with finitely many non-zero terms since p is finitely-supported).

This R-bilinear map is *non-degenetate*, i.e., $\langle p, f\rangle_X = 0$ for each $p \in R^{(X)}$ implies that $f = 0$, and $\langle p, f\rangle_X = 0$ for each $f \in R^X$ implies $p = 0$.

1. The one-to-one linear map $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)'$ from Lemma 25 is related to the pairing by $(\lambda_X(p))(f) = \langle p, f\rangle_X$, which implies that

$\langle\cdot,\cdot\rangle_X$ is separately continuous in its second variable, when its first variable is fixed.

2. The isomorphism $\rho_X\colon (R,\tau)^X \simeq ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$ from Lemma 19, p. 16, is also connected to the pairing by $(\rho_X(f))(p) = \langle p, f\rangle_X$.

3.2 Rigid rings: definition

26 Lemma *Let (R,τ) be a topological ring, and let X be a set. For each $f \in R^X$, $(f(x)\delta_x)_{x\in X}$ is summable in $(R,\tau)^X$ with sum f.*

Proof: According to [23, Theorem 10.10, p. 76] it suffices to check that for each $z \in X$, $(\pi_z(f(x)\delta_x))_{x\in X}$ is summable with sum $\pi_z(f) = f(z)$ in (R,τ). But since $\pi_z(f(x)\delta_x) = f(x)\delta_x(z)$ this is obvious. □

27 Definition *Let (R,τ) be a topological ring. It is said to be* rigid *when for each set X, $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)'$ is an isomorphism in $\mathbf{Mod}_R$, i.e., λ_X is onto. In this situation, one sometimes also called* rigid *a ring topology τ such that (R,τ) is rigid.*

By a rigid field *$(\Bbbk,\tau)$ is meant a field $\Bbbk$ with a ring topology τ, which is rigid as a topological ring.*

28 Lemma *Let $\ell \in ((R,\tau)^X)'$. $\ell \in im(\lambda_X)$ if, and only if, $\hat{\ell}\colon X \to R$ given by $\hat{\ell}(x) := \ell(\delta_x)$, belongs to $R^{(X)}$. Moreover, $im(\lambda_X) \xrightarrow{(\hat{-})} R^{(X)}$, $\ell \mapsto \hat{\ell} = \sum_{x\in X} \ell(\delta_x)\delta_x$, is the inverse of λ_X.*

Proof: Let $\ell \in ((R,\tau)^X)'$ such that $\hat{\ell} \in R^{(X)}$. Then, $\lambda_X(\hat{\ell}) = \ell$, and thus $\ell \in im(\lambda_X)$ since for $f \in R^X$, $(\lambda_X(\hat{\ell}))(f) = \sum_{x\in X} \ell(\delta_x)f(x) = \ell(\sum_{x\in X} f(x)\delta_x) = \ell(f)$ (Lemma 26).

Let $\ell \in ((R,\tau)^X)'$ such that $\ell \in im(\lambda_X)$. Let $p_\ell \in R^{(X)}$ with $\lambda_X(p_\ell) = \ell$. Then, for each $x \in X$, $\ell(\delta_x) = (\lambda_X(p_\ell))(\delta_x) = \sum_{z\in X} p_\ell(z)\delta_x(z) = p_\ell(x)$. Whence $\hat{\ell} = p_\ell \in R^{(X)}$.

The last assertion is obvious since λ_X is known to be one-to-one. □

3.3 Rigidity and finite powers

Let (R,τ) be a topological ring. Remember from Section 2.4.2 (p. 15) that for each set I, the I-fold topological product $(R,\tau)^I$ of the underlying topological module of (R,τ) inherits from the latter a structure of a topological

(R,τ)-algebra $A_{(\mathsf{R},\tau)}(I)$ under component-wise multiplication and with unit $1_{A_{(\mathsf{R},\tau)}(I)} := \sum_{i\in I} \delta_i^{\mathsf{R}}$.

Let us assume that $I \neq \varnothing$ so that the underlying ring $(\mathsf{R},\tau)^I$ (Notation 17, p. 15) of $A_{(\mathsf{R},\tau)}(I)$ is non-trivial (Convention 1.1, p. 9).

Any topological $(\mathsf{R},\tau)^I$-module is also a topological (R,τ)-module under restriction of scalars (see Section 2.3) along the unit map $(\mathsf{R},\tau) \xrightarrow{\eta_I} (\mathsf{R},\tau)^I$, $\eta_I(1_{\mathsf{R}}) = 1_{\mathsf{R}^I}$, which of course is a ring map, and is continuous (because $\eta_I(\alpha) = m_{\mathsf{R}^I}(\eta_I(\alpha), 1_{\mathsf{R}^I})$, $\alpha \in R$.)

Let X be a set, and let $\ell \in (((R,\tau)^I)^X)'$, i.e., $((R,\tau)^I)^X \xrightarrow{\ell} (R,\tau)^I$ is continuous and $(\mathsf{R},\tau)^I$-linear, and by restriction of scalar along η_I it is also continuous and (R,τ)-linear. Therefore for each $i \in I$, $((R,\tau)^I)^X \xrightarrow{\ell} (R,\tau)^I \xrightarrow{\pi_i} (R,\tau)$ belongs to the topological dual space of $((R,\tau)^I)^X$ seen as a (R,τ)-module.

Let us assume that (R,τ) is rigid. Then, by Lemma 28, $supp(\widehat{\pi_i \circ \ell})$ is finite for each $i \in I$. One may also define $X \xrightarrow{\hat{\ell}} R^I$ by $\hat{\ell}(x) := \ell(\delta_x^{\mathsf{R}^I})$, $x \in X$, and thus $supp(\hat{\ell}) = \{\, x \in X : \ell(\delta_x^{\mathsf{R}^I}) \neq 0 \,\} = \{\, x \in X : \exists i \in I,\ \ell(\delta_x^{\mathsf{R}^I})(i) \neq 0 \,\} = \{\, x \in X : \exists i \in I,\ \pi_i(\ell(\delta_x^{\mathsf{R}^I})) \neq 0 \,\} = \bigcup_{i\in I} supp(\widehat{\pi_i \circ \ell})$. Whence if I is finite, then $supp(\hat{\ell})$ is finite too, and $(\mathsf{R},\tau)^I$ is rigid.

29 Proposition *Let (R,τ) be a rigid ring. If I is finite and non-void, then $(\mathsf{R},\tau)^I$ is rigid too.*

When I is infinite, then $(\mathsf{R},\tau)^I$ never can be rigid as shown in Section 3.6 below (p. 25).

3.4 Basic stock of examples

Several examples of rigid rings are now presented.

3.4.1 A field with a ring topology

The first assertion of the following result is a slight generalization of the main theorem in [18] (which is precisely the second assertion below).

30 Lemma *Let $(\Bbbk,\tau)$ be a field with a ring topology (see Definition 2, p. 10). Then, $(\Bbbk,\tau)$ is rigid. In particular, any topological field is rigid.*

Proof: Let $\ell \in ((\Bbbk,\tau)^X)'$. Let $f \in \Bbbk^X$. By Lemma 26, $(f(x)\delta_x)_{x\in X}$ is summable in $(\Bbbk,\tau)^X$ with sum f, and by (linearity and) continuity of ℓ, $(f(x)\ell(\delta_x))_x$ is summable in $(\Bbbk,\tau)$ with sum $\ell(f)$. Let $f_\ell \in \Bbbk^X$ be given

by $f_\ell(x) := \ell(\delta_x)^{-1}$, when $\ell(\delta_x) \neq 0$, and $f_\ell(x) := 0$, when $\ell(\delta_x) = 0$. In this situation, $f_\ell(x)\ell(\delta_x) = 1_{\Bbbk}$ for all $x \in supp(\hat{\ell})$ (see Lemma 28 for the definition of $\hat{\ell}$). Now let $U \in \mathfrak{V}_{(\Bbbk,\tau)}(0)$ such that $1_{\Bbbk} \notin U$. By [23, Theorem 10.5, p. 73], since $(f_\ell(x)\ell(\delta_x))_x$ is summable, for all but finitely many x's, $f_\ell(x)\ell(\delta_x) \in U$. By definition of f_ℓ, there are thus only finitely many x's with $f_\ell(x)\ell(\delta_x) = 1_{\Bbbk}$ which means that $\hat{\ell} \in \Bbbk^{(X)}$, so that $(\Bbbk, \tau)$ is rigid by Lemma 28. □

3.4.2 Discrete rings

There are also rings that are not fields, but still they are rigid. The most obvious is given below.

31 Lemma *For each ring* R*, the discretely topologized ring* (R, d) *is rigid.*

Proof: Let $\ell \in ((R, \mathsf{d})^X)'$. For each $f \in R^X$, according to Lemma 26 (p. 19), $(f(x)\ell(\delta_x))_{x\in X}$ is summable in (R, d), with sum $\ell(f)$. Since $\{0\}$ is an open neighborhood of zero in (R, d), $f(x)\ell(\delta_x) = 0$ for all but finitely many $x \in X$ ([23, Theorem 10.5, p. 73]). One may apply this result to $f := \sum_{x\in X} \delta_x \in R^X$, which implies that $\ell(\delta_x) = 0$ for all but finitely many $x \in X$ so that by Lemma 28 (p. 19), λ_X is onto. □

3.4.3 Normed algebras

Every normed, complex or real, commutative and unital algebra is rigid. Thus so are any (unital and commutative) Banach and C^*-algebras for instance.

32 Lemma *Let* $\Bbbk = \mathbb{R}, \mathbb{C}$*. Let* $(\mathsf{A}, \|\cdot\|)$ *be a commutative normed* $\Bbbk$*-algebra*[8] *with a unit, assumed non-trivial* ($1_{\mathsf{A}} \neq 0$)*. Then, as a topological ring under the topology induced by the norm, it is rigid.*

Proof: Let $\tau_{\|-\|}$ be the topology on A induced by the norm of A, where A is the underlying $\Bbbk$-vector space of A. Let X be a set. Let $\ell \in ((A, \tau_{\|-\|})^X)'$. Let $f \in A^X$. Then, by Lemma 26 (p. 19), $(f(x)\delta_x)_{x\in X}$ is summable with sum f, whence $(f(x)\ell(\delta_x))_{x\in X}$ is summable in $(A, \tau_{\|-\|})$ with sum $\ell(f)$. So according to [23, Theorem 10.5, p. 73], for $\epsilon > 0$, there exists a finite set $F_\epsilon \subseteq X$ such that $\|f(x)\ell(\delta_x)\| < \epsilon$ for all $x \in X \setminus F_\epsilon$. Let $f\colon X \to A$ be given

[8] In a normed algebra $(\mathsf{A}, \|\cdot\|)$, unital or not, commutative or not, trivial or not, the norm is assumed *sub-multiplicative*, i.e., $\|xy\| \leq \|x\|\|y\|$, which ensures that the multiplication of A is jointly continuous with respect to the topology induced by the norm.

by $f(x) = \frac{1}{\|\ell(\delta_x)\|}1_{\mathsf{A}}$ if $x \in supp(\hat{\ell})$ and $f(x) = 0$ for $x \notin supp(\hat{\ell})$. Then, $1 = \|f(x)\ell(\delta_x)\|$ for all $x \in supp(\hat{\ell})$. Let us assume that $0 < \epsilon < 1$. Thus according to the above, $supp(\hat{\ell})$ is finite, so λ_X is onto by Lemma 28. □

3.5 A supplementary example: von Neumann regular rings

3.5.1 Basic results and examples

Let R be a ring. It is said to be *von Neumann regular* if for each $x \in R$, there exists $y \in R$ such that $x = xyx$ [14, Theorem 4.23, p. 65].

Following our Conventions 1 (p. 9), any von Neumann regular ring is assumed commutative, unital and non-trivial. It is worth noticing that commutativity forces uniqueness of a "weak inverse".

33 Lemma *Let us assume that* R *is a (commutative) von Neumann regular ring. For each* $x \in R$*, there is a unique* $y \in R$ *such that* $x = xyx$ *and* $y = yxy$*.*

Proof: For each x, there is at least one y which meets the requirement of the statement of the lemma. Indeed, let y such that $x = xyx$. Let $z := yxy$. So $xzx = x(yxy)x = (xyx)yx = xyx = x$ and $zxz = (yxy)x(yxy) = yxzxy = yxy = z$. It remains to make sure of uniqueness. Assume y, z both satisfy the statement. Thus, one has $x = x^2y = x^2z$, $y = y^2x$ and $z = z^2x$. Therefore, $z = z^2x = z^2x^2y = (x^2z)zy = xzy = (x^2y)zy = (x^2z)y^2 = xy^2 = y$. □

34 Definition *Let* R *be a von Neumann regular ring. For each* $x \in R$*, the member* y *of* R *such that* $x = xyx$ *and* $y = yxy$*, whose existence and uniqueness are guaranteed by Lemma 33, is referred to as the* weak inverse *of* x*, and is denoted* $x^\dagger$*.*

35 Examples

1. *Every field is a von Neumann regular ring with* $x^\dagger := x^{-1}$*,* $x \neq 0$*, and* $0^\dagger = 0$*.*
2. *Every non-void direct product of fields is von Neumann regular: let* $\Bbbk_i$ *be a field for each* $i \in I$*, let* $(x_i)_{i\in I} \in \prod_{i\in I}\Bbbk_i$*, then* $(x_i)^\dagger_{i\in I} := (x_i^\dagger)_{i\in I}$ *provides the required weak inverse.*

The following folklore result provides an important source of examples of von Neumann regular rings.

36 Lemma *Let* $(\Bbbk_i)_{i\in I}$ *be a non-void family of fields. Let* R *be a ring, and let* $\jmath\colon \mathsf{R} \hookrightarrow \prod_{i\in I}\Bbbk_i$ *be an embedding (i.e., a ring map which is one-to-one). Let*

us assume that for each $x \in R$, $\jmath(x)^\dagger \in im(\jmath)$, where $(x_i)_i^\dagger$ is defined as in Example 35.2. Then, R is a von Neumann regular ring.

Proof: For each $x \in R$. Let $x^\star$ be the unique member of R such that $\jmath(x^\star) = \jmath(x)^\dagger$. Because $\jmath(xx^\star x) = \jmath(x)\jmath(x^\star)\jmath(x) = \jmath(x)\jmath(x)^\dagger\jmath(x) = \jmath(x)$ and $\jmath(x^\star xx^\star) = \jmath(x^\star)\jmath(x)\jmath(x^\star) = \jmath(x)^\dagger\jmath(x)\jmath(x)^\dagger = \jmath(x)^\dagger = \jmath(x^\star)$, $xx^\star x = x$ and $x^\star xx^\star = x^\star$ in such a way that R is a von Neumann regular ring. □

One also provides the following definition for later.

37 Definition *Let $\Bbbk$ be a field. Let $\mathsf{A} = (A, m_{\mathsf{A}}, 1_{\mathsf{A}})$ be a unital $\Bbbk$-algebra. It is said to be* von Neumann regular *when for each $x \in A$, there exists $y \in A$ such that $x = xyx$.*

38 Remark *The proof of Lemma 33 works as well after substituting the von Neumann regular ring R by a commutative von Neumann regular (unital) $\Bbbk$-algebra $\mathsf{A} = (A, m_{\mathsf{A}}, 1_{\mathsf{A}})$, and thus asserts the existence of a uniquely defined weak inverse $x^\dagger$ for each member $x \in A$ (i.e., $x^\dagger xx^\dagger = x^\dagger$ and $xx^\dagger x = x$).*

39 Example *Let $\Bbbk$ be a field, and let X be a set. Then, $A_{\Bbbk}(X)$ (see Section 2.4.2, p. 15) is a commutative von Neumann regular $\Bbbk$-algebra with $f^\dagger := \sum_{x\in X}(f(x))^\dagger\delta_x^{\Bbbk}$, $f \in \Bbbk^X$, where $\alpha^\dagger$, for $\alpha \in \Bbbk$, is as in Example 35.*

3.5.2 A sufficient condition for rigidity

40 Remark *Let R be a von Neumann regular ring. For each $x \in R \smallsetminus \{0\}$, $xx^\dagger \neq 0$ since $0 \neq x = xx^\dagger x$. Moreover, for each $x \in R$, x belongs to the set $E(\mathsf{R})$ of all idempotents ($e^2 = e$) of R since $xx^\dagger xx^\dagger = xx^\dagger$.*

41 Proposition *Let (R, τ) be a topological ring such that R is von Neumann regular. If $0 \notin \overline{E(\mathsf{R}) \smallsetminus \{0\}}$, then (R, τ) is rigid. In particular, if $E(\mathsf{R})$ is finite, then (R, τ) is rigid.*

Proof: That the second assertion follows from the first is immediate.

Let X be a set. Let us assume that $0 \notin \overline{E(\mathsf{R}) \smallsetminus \{0\}}$. Let $\ell \in ((R,\tau)^X)'$. In particular, for each $f \in R^X$, $(f(x)\ell(\delta_x))_{x\in X}$ is summable in (R,τ) with sum $\ell(f)$. Let $V \in \mathfrak{V}_{(R,\tau)}(0)$ such that $V \cap (E(R) \smallsetminus \{0\}) = \varnothing$. By Cauchy's condition [23, Definition 10.3, p. 72], there exists a finite set $A_{f,V} \subseteq X$ such that for all $x \notin A_{f,V}$, $f(x)\ell(\delta_x) \in V$. Now, let us consider the case where f is given by $f(x) := \ell(\delta_x)^\dagger$ for each $x \in X$. Then, for each $x \in X$, $f(x)\ell(\delta_x) = \ell(\delta_x)^\dagger\ell(\delta_x) \in E(\mathsf{R})$. Whence, in view of Remark 40, for all but finitely many x's, $f(x)\ell(\delta_x) = 0$, i.e., $\ell(\delta_x) = 0$. □

42 Remark *One recovers Lemma 30, p. 20, as a consequence of Proposition 41 by taking* (R,τ) *to be a field* $(\Bbbk,\tau)$ *with a ring topology since* $E(\Bbbk) = \{0, 1_{\Bbbk}\}$.

3.5.3 An application: Boolean rings

A *Boolean ring* is a ring B in which $x^2 = x$ for all $x \in B$. In other words, $E(\mathsf{B}) = B$. (According to our Conventions 1 (p. 9) one excludes the trivial ring, even if it is usually considered as a perfectly correct Boolean ring.) It is apparent that any Boolean ring is von Neumann regular (with $x^\dagger = x$, for each x).

43 Proposition *Let* (B,τ) *be a topological ring such that* B *is a Boolean ring.* $0 \notin \overline{E(\mathsf{B}) \smallsetminus \{0\}}$ *if, and only if,* $\tau = \mathsf{d}$. *In other terms, a topological Boolean ring is rigid if, and only if, its topology is discrete.*

Proof: According to Lemma 31 (p. 21), (B,d) is rigid.

Conversely, by contraposition, let us assume that $\tau \neq \mathsf{d}$. Thus $\{0\}$ is not open in τ, so that for each $V \in \mathfrak{V}_{(\mathsf{B},\mathsf{d})}(0)$, there exists $x \in B \smallsetminus \{0\}$ with $x \in V$. Therefore, $0 \in \overline{B \smallsetminus \{0\}} = \overline{E(\mathsf{B}) \smallsetminus \{0\}}$. □

3.5.4 An application: the box topology

Let $(E_i,\tau_i)_{i\in I}$ be a family of topological spaces. On $\prod_{i\in I} E_i$ is defined the *box topology* [12, p. 107] a basis of open sets of which is given by the "box" $\prod_{i\in I} V_i$, where each $V_i \in \tau_i$, $i \in I$. The product $\prod_{i\in I} E_i$ together with the box topology is denoted by $\sqcap_{i\in I}(E_i,\tau_i)$. (This topology is Hausdorff as soon as all the (E_i,τ_i)'s are.)

It is not difficult to see that given a non-void family $(\mathsf{R}_i,\tau_i)_{i\in I}$ of topological rings, then $\sqcap_{i\in I}(\mathsf{R}_i,\tau_i)$ still is a topological ring (under component-wise operations).

44 Proposition *Let* $(\Bbbk_i)_{i\in I}$ *be a non-void family of fields, and for each* $i \in I$, *let* τ_i *be a ring topology on* $\Bbbk_i$. *Let* R *be a ring with an embedding* $\jmath\colon \mathsf{R} \hookrightarrow \prod_{i\in I} \Bbbk_i$. *Let us assume that for each* $x \in R$, $\jmath(x)^\dagger \in im(\jmath)$, *where* $(x_i)_i^\dagger$ *is defined as in Example 35.2. Let* R *be topologized with the subspace topology* $\tau_\jmath$ *inherited from* $\sqcap_{i\in I}(\Bbbk_i,\tau_i)$. *Then,* $(\mathsf{R},\tau_\jmath)$ *is rigid.*

Proof: Naturally $(x_i)_{i\in I} \in E(\prod_{i\in I} \Bbbk_i)$ if, and only if, $x_i \in \{0, 1_{\Bbbk_i}\}$ for each $i \in I$. Now, for each $i \in I$, let U_i be an open neighborhood of zero in $(\Bbbk_i,\tau_i)$

such that $1_{\Bbbk_i} \notin U_i$. Then, $\prod_i U_i$ is an open neighborhood of zero in $\prod_{i\in I}(\Bbbk_i, \tau_i)$ whose only idempotent member is 0. Therefore, $0 \notin \overline{E(\prod_{i\in I} \Bbbk_i) \smallsetminus \{0\}}$.

Under the assumptions of the statement, an application of Lemma 36 states that R is a (commutative) von Neumann regular ring. It is also of course a topological ring under $\tau_\jmath$ (since $\jmath$ is a one-to-one ring map). It is also clear that $E(R) \simeq E(\jmath(R)) \subseteq E(\prod_i \Bbbk_i)$. Furthermore, $\jmath(\overline{E(R) \smallsetminus \{0\}}) = \overline{E(\jmath(R)) \smallsetminus \{0\}} \cap \jmath(R) \subseteq \overline{E(\prod_i \Bbbk_i) \smallsetminus \{0\}}$, and thus $0 \notin \overline{E(R) \smallsetminus \{0\}}$ according to the above discussion. Therefore, by Proposition 41, $(\mathsf{R}, \tau_\jmath)$ is rigid. □

3.6 A counter-example

Let (R, τ) be a topological ring, and let us consider the topological $(\mathsf{R}, \tau)^X$-module $((R, \tau)^X)^X$ for a given non-empty[9] set X. In order to avoid being confused by the use of canonical projections of both $(R^X)^X$ and R^X, the former are momentarily denoted by $\Pi_x\colon (R^X)^X \to R^X$, while one retains the denotation $\pi_x\colon R^X \to R$ for the latter.

Let us define a linear map $(R^X)^X \xrightarrow{\ell} (R, \tau)^X$ by setting $\ell(f)\colon x \mapsto (f(x))(x)$, $f \in (R^X)^X$.

ℓ is continuous. To see this, by definition of the topology on $(R, \tau)^X$ it suffices to check that for each $x \in X$, $(R^X)^X \xrightarrow{\ell} R^X \xrightarrow{\pi_x} (R, \tau)$ is continuous. But given $f \in (R^X)^X$, $\pi_x(\ell(f)) = (f(x))(x) = \pi_x(\Pi_x(f))$. In other words, $\pi_x \circ \ell = \pi_x \circ \Pi_x$ which certainly is continuous. It follows that $\ell \in (((R, \tau)^X)^X)'$.

Now, for each $x \in X$, $(\ell(\delta_x^{\mathsf{R}^X}))(x) = \delta_x^{\mathsf{R}^X}(x) = 1_{\mathsf{R}^X}$, and in particular, $\hat{\ell}(x) = \ell(\delta_x^{\mathsf{R}^X}) \neq 0$, which means that $supp(\hat{\ell}) = X$.

In consequence the following is obtained.

45 Proposition *Let (R, τ) be topological ring, and let X be a non-empty set. If X is infinite, then $(\mathsf{R}, \tau)^X$ is not rigid.*

4 Recasting as an equivalence of categories

The purpose of this section consists to translate the rigidity condition (Definition 27) on a topological ring into a dual equivalence between some categories of (topological) modules. In a way similar to [18], the equivalence of categories will be given by the algebraic dual functor with equivalence

[9] X is assumed non-void to ensure that R^X is a non-trivial ring as requires our Convention 1.1, p. 9.

inverse the (opposite of the) topological dual functor (see Section 2.5), with both functors conveniently co-restricted. Therefore, a first step in this direction is to introduce the categories to prove equivalent, and actually just one, namely the category of topologically-free modules (see below), since the other is the category of free modules.

4.1 Topologically-free modules

4.1.1 Definition and characterization

46 Definition *Let* (R,τ) *be a topological ring. Let* (M,σ) *be a topological* (R,τ)*-module. It is said to be a* topologically-free (R,τ)-module *if* $(M,\sigma)\simeq(R,\tau)^X$*, in* $\mathbf{TopMod}_{(\mathsf{R},\tau)}$*, for some set* X*. Let* $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *be the full subcategory of* $\mathbf{TopMod}_{(\mathsf{R},\tau)}$ *spanned by the topologically-free* (R,τ)*-modules. Likewise when* $(\Bbbk,\tau)$ *is a field with a ring topology, then one defines correspondingly the* topologically-free $(\Bbbk,\tau)$-vector spaces *and their category* $\mathbf{TopFreeVect}_{(\Bbbk,\tau)}\hookrightarrow\mathbf{TopVect}_{(\Bbbk,\tau)}$.

47 Remark *Of course, for each set* X*,* $(R,\tau)^X$ *itself is a topologically-free* (R,τ)*-module. It follows that the functor* $\mathbf{Set}^{\mathsf{op}}\xrightarrow{P_{(\mathsf{R},\tau)}}\mathbf{TopMod}_{(\mathsf{R},\tau)}$ *from Remark 12 (p. 14) factors as follows.*

$$\begin{tikzcd}
\mathbf{Set}^{\mathsf{op}} \arrow[r, "P_{(\mathsf{R},\tau)}"] \arrow[dr] & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\
& \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \arrow[u, hook]
\end{tikzcd} \tag{4}$$

The co-restriction obtained is still denoted $\mathbf{Set}^{\mathsf{op}}\xrightarrow{P_{(\mathsf{R},\tau)}}\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$.

Topologically-free modules are characterized by the fact of possessing "topological bases" (see Corollary 51 below) which makes easier a number of calculations and proofs, once such a basis is chosen.

48 Definition *Let* (M,σ) *be a topological* (R,τ)*-module. Let* $B\subseteq M$*. It is said to be a* topological basis *of* (M,σ) *if the following hold.*

1. *For each* $v\in M$*, there exists a unique family* $(b'(v))_{b\in B}$*, with* $b'(v)\in R$ *for each* $b\in B$*, such that* $(b'(v)b)_b$ *is summable in* (M,σ) *with sum* v*.* $b'(v)$ *is referred to as the* coefficient *of* v *at* $b\in B$*.*

2. *For each family* $(\alpha_b)_{b\in B}$ *of elements of* R*, there is a member* v *of* M *such that* $b'(v)=\alpha_b$*,* $b\in B$*. (By the above point such* v *is unique.)*

3. *σ is equal to the initial topology induced by the* (topological) coefficient maps $(M \xrightarrow{b'} (R,\tau))_{b\in B}$. *(According to the two above points, each b' is* R*-linear.)*

49 Remark *It is an immediate consequence of the definition that for a topological basis B of some topological module, $0 \notin B$ and $b'(d) = \delta_b(d)$, $b, d \in B$ (since $\sum_{b\in B} \delta_b(d)b = d = \sum_{b\in B} b'(d)b$). In particular, $B \xrightarrow{(-)'} B' := \{ b' : b \in B \}$ is a bijection.*

The next two results are obvious.

50 Lemma *Let (M,σ) and (N,γ) be isomorphic topological (R,τ)-modules. Let $\Theta : (M,\sigma) \simeq (N,\gamma)$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$). Let B be a topological basis of (M,σ). Then, $\Theta(B) = \{ \Theta(b) : b \in B \}$ is a topological basis of (N,γ).*

51 Corollary *Let (M,σ) be a (Hausdorff) topological (R,τ)-module. It admits a topological basis if, and only if, it is topologically-free.*

52 Example *Let (R,τ) be a topological ring. For each set X, $\{ \delta_x : x \in X \}$ is a topological basis of $(R,\tau)^X$. Moreover $\pi_x = \delta'_x$, $x \in X$.*

4.1.2 Properties of topological bases

In this section one takes the time to establish a certain number of properties of topological bases since these will be useful hereafter (in particular in Appendix C).

53 Lemma *Let (M,σ) be a topologically-free (R,τ)-module with topological basis B. Then, B is R-linearly independent and the linear span $\langle B \rangle$ of B is dense in (M,σ).*

Proof: Let b_i, $i = 1,\cdots,n$ be pairwise distinct elements of B. Let $\alpha_i \in R$, $i = 1,\cdots,n$. Let us assume that $\sum_{i=1}^n \alpha_i b_i = 0$. Since $0 = \sum_{b\in B} 0b$ is the sum of the summable family $(z_b b)_b$, with $z_b = 0$, $b \in B$, and 0 is also the sum of the summable family $(a_b b)_b$ with $a_{b_i} = \alpha_i$, $i = 1,\cdots,n$, and $a_b = 0$, $b \neq b_i$, $i = 1,\cdots,n$, by uniqueness it follows that $\alpha_i = 0$, $i = 1,\cdots,n$.

Now, let us check that $\langle B \rangle$ is dense in (M,σ). Let $u \in M$ and let $V := \{ v \in M : b'(v) \in U_b,\ b \in A \}$ be a neighborhood of zero, where A is a finite subset of B and U_b is a neighborhood of zero in (R,τ) for each $b \in A$. One has to check that $u + V$ meets $\langle B \rangle$. Let $u_A := \sum_{b\in A}(b'(u) + b'(v))b \in \langle B \rangle$

for some $v = \sum_{b\in B} b'(v)b \in V$ (whence $b'(v) \in U_b$, $b \in A$). Then, $u_A - u = \sum_{b\in A} b'(v)b - \sum_{b\in B\smallsetminus A} b'(u)b$ so that for each $b \in A$, $b'(u_A - u)_b = b'(v) \in U_b$, whence $u_A - u \in V$. □

As a consequence of Lemma 53 one immediately gets the following.

54 Corollary *Let (M,σ) be a topologically-free (R,τ)-module, and let (N,γ) be a topological (R,τ)-module. Let $(M,\sigma) \xrightarrow{f,g} (N,\gamma)$ be two continuous (R,τ)-linear maps. $f = g$ if, and only if, for any topological basis B of (M,σ), $f(b) = g(b)$ for each $b \in B$.*

Topologically-free modules allow for the definition of changes of topological bases.

55 Lemma *Let (M,σ) and (N,γ) be two topologically-free (R,τ)-modules, and let B, D be respective topological bases. Let $f\colon B \to D$ be a bijection. Then, there is a unique isomorphism g in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$ such that $g(b) = f(b)$, $b \in B$.*

Proof: The question of uniqueness is settled by Corollary 54. If such an isomorphism g exists, then $g(v) = g\left(\sum_{b\in B} b'(v)b\right) = \sum_{b\in B} b'(v)g(b) = \sum_{b\in B} b'(v)f(b) = \sum_{d\in D}(f^{-1}(d))'(v)d$, $v \in M$. One observes that g as defined by the right hand-side of the last equality, is R-linear, whence one only needs to establish continuity. Let $d \in D$. Then, $d'(g(v)) = (f^{-1}(d))'(v)$, $v \in M$, so that $d' \circ g = (f^{-1}(d))'$ which ensures continuity. Finally, it is obvious from its definition that $g(b) = f(b)$, $b \in B$. □

56 Lemma *Let M be a free module with basis B. Then, $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically-free module with topological basis $B^* := \{\, b^*\colon b \in B \,\}$ (see Remark 23, p. 17).*

Proof: According to Lemma 24 (p. 17), $(M^*, w^*_{(\mathsf{R},\tau)})$ is a topologically-free module. It remains to check that B^* is a topological basis. Let $M \xrightarrow{\theta_B} R^{(B)}$ be the isomorphism given by $\theta_B(b) = \delta_b$, $b \in B$. Thus, $\theta_B^*\colon (R^{(B)})^* \simeq M^*$, and $\theta_B^* \circ \rho_B\colon R^B \simeq (R^{(B)})^* \simeq M^*$ is given by $\theta_B^*(\rho_B(\delta_b^{\mathsf{R}})) = \rho_B(\delta_b^{\mathsf{R}}) \circ \theta_B = p_b \circ \theta_B$ for $b \in B$. Whence for $v = \sum_{d\in B} d^*(v)d \in M$, $(\theta_B^*(\rho_B(\delta_b^{\mathsf{R}})))(v) = \sum_{d\in B} d^*(v)p_b(\theta_B(d)) = \sum_{d\in B} d^*(v)p_b(\delta_d) = b^*(v)$. Now, $\{\, \delta_b\colon b \in B \,\}$ being a topological basis of R^B, by Lemma 50, this shows that B^* is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)})$. □

57 Example *Let $M = R^{(X)}$. Then, $\{\delta_x^*: x \in X\}$ is a topological basis of $(R^{(X)})^*$. But, $\delta_x^*(p) = p(x)$, $x \in X$, $p \in R^{(X)}$. Therefore, $\delta_x^* = p_x$, $x \in X$.*

58 Remark *If B is a basis of a free module M, then $B \simeq B^*$ under $b \mapsto b^*$, because for each $b, d \in B$, $b^*(d) = \delta_b(d)$.*

59 Corollary *Let (R, τ) be a topological ring. The algebraic dual functors $\mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopMod}_{(\mathsf{R},\tau)}$ factors as illustrated in the diagram below. Moreover the resulting co-restriction of $Alg_{(\mathsf{R},\tau)}$ (the bottom arrow of the diagram) is essentially surjective.*

$$\begin{tikzcd} \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \arrow[r, "Alg_{(\mathsf{R},\tau)}"] & \mathbf{TopMod}_{(\mathsf{R},\tau)} \\ \mathbf{FreeMod}_{\mathsf{R}}^{\mathsf{op}} \arrow[u, hook] \arrow[r] & \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \arrow[u, hook] \end{tikzcd} \tag{5}$$

Proof: The first assertion is merely Lemma 56. Regarding the second assertion, let (M, σ) be a topologically-free module. So, for some set X, $(M, \sigma) \simeq (R, \tau)^X$. According to Lemma 19 (p. 16), $(R, \tau)^X \simeq Alg_{(\mathsf{R},\tau)}(R^{(X)})$. □

60 Remark *When $\Bbbk$ is a field with a ring topology τ, then one has the corresponding factorization of $\mathbf{Vect}_{\Bbbk}^{\mathsf{op}} \xrightarrow{Alg_{(\Bbbk,\tau)}} \mathbf{TopVect}_{(\Bbbk,\tau)}$.*

$$\begin{tikzcd} \mathbf{Vect}_{\Bbbk}^{\mathsf{op}} \arrow[r, "Alg_{(\Bbbk,\tau)}"] \arrow[dr] & \mathbf{TopVect}_{(\Bbbk,\tau)} \\ & \mathbf{TopFreeVect}_{(\Bbbk,\tau)} \arrow[u, hook] \end{tikzcd} \tag{6}$$

61 Corollary *Let M be a free module with basis B. Let $\ell \in M^*$. Then, for each $b \in B$, $\ell(b) = (b^*)'(\ell)$.*

Proof: Since by Lemma 56, B^* is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)})$, $\ell = \sum_{b \in B} (b^*)'(\ell) b^*$ (sum of a summable family). According to the definition of the topology $w^*_{(\mathsf{R},\tau)}$ this implies that for each $v \in M$, $\ell(v)$ is the sum of the summable family $((b^*)'(\ell)(\Lambda_M(v))(b^*))_{b \in B}$ in (R, τ). Whence $\ell(v) = \sum_{b \in B} (b^*)'(\ell) b^*(v)$ (sum of a summable family) and in particular, $\ell(b) = \sum_{d \in B} (d^*)'(\ell) d^*(b) = (b^*)'(\ell)$, $b \in B$. □

62 Lemma *Let (R, τ) be a rigid ring. Let (M, σ) be a topologically-free (R, τ)-module with topological basis B. Then, $(M, \sigma)'$ is free with basis $B' := \{b' : b \in B\}$.*

Proof: Let $\Theta_B\colon (M,\sigma) \simeq (R,\tau)^B$ be given by $\Theta_B(b) = \delta_b$. Therefore $\Theta'_B\colon ((R,\tau)^B)' \simeq (M,\sigma)'$, and thus one has an isomorphism $\Theta'_B \circ \lambda_B\colon R^{(B)} \simeq (M,\sigma)'$. Since a module isomorphic to a free module is free, $(M,\sigma)'$ is free. The previous isomorphism acts as: $\Theta'_B(\lambda_B(\delta_b)) = \Theta'_B(\pi_b) = \pi_b \circ \Theta_B$ for $b \in B$. It follows that $\{\, \pi_b \circ \Theta_B\colon b \in B \,\}$ is a basis of $(M,\sigma)'$. Now let $b \in B$ and let $v = \sum_{d\in B} d'(v)d \in M$ (sum of a summable family). One has by continuity and linearity, $\pi_b(\Theta_B(v)) = \pi_b(\sum_{d\in B} d'(v)\Theta_B(d)) = \pi_b(\sum_{d\in B} d'(v)\delta_b) = b'(v)$. □

63 Example *Let* (R,τ) *be a rigid ring. Let* $(M,\sigma) = (R,\tau)^X$*. By Example 52,* $\{\, \delta'_x\colon x \in X \,\} = \{\, \pi_x\colon x \in X \,\}$ *is a linear basis of* $((R,\tau)^X)'$*.*

64 Corollary *Let* (R,τ) *be a rigid ring. The functor* $\mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{Mod}_{\mathsf{R}}$ *factors as indicated by the diagram below.*

$$\begin{tikzcd} \mathbf{TopMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \arrow[r, "Top_{(\mathsf{R},\tau)}"] & \mathbf{Mod}_{\mathsf{R}} \\ \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \arrow[u, hook] \arrow[r] & \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \end{tikzcd} \tag{7}$$

65 Remark *Let* $(\Bbbk,\tau)$ *be a field with a ring topology. One has the corresponding factorization of* $\mathbf{TopVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \xrightarrow{Top_{(\Bbbk,\tau)}} \mathbf{Vect}_{\Bbbk}$*.*

$$\begin{tikzcd} \mathbf{TopVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \arrow[r, "Top_{(\Bbbk,\tau)}"] & \mathbf{Vect}_{\Bbbk} \\ \mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \arrow[u, hook] \arrow[ur] & \end{tikzcd} \tag{8}$$

66 Corollary *Under the assumptions of Lemma 62, for* $\ell \in (M,\sigma)'$ *and* $b \in B$*,* $\ell(b) = (b')^*(\ell)$*.*

Proof: Lemma 62 tells us that $\ell = \sum_{b\in B}(b')^*(\ell)b'$ (finite linear combination). For $v \in M$, one has $\ell(v) = \sum_{b\in B}(b')^*(\ell)b'(v)$. In particular, $\ell(b) = (b')^*(\ell)$ for each $b \in B$. □

4.2 The topological dual of the algebraic dual of a free module

Let (R,τ) be a topological ring. Let M be a R-module, and let us consider as in Section 2.5.1 (p. 16) the R-linear map

$$M \xrightarrow{\Lambda_M} (M^*, w^*_{(\mathsf{R},\tau)})'$$

$(\Lambda_M(v))(\ell) = \ell(v)$, $v \in M$, $\ell \in M^*$.

Our objective in this section is to prove that under the assumption that (R,τ) is rigid, Λ_M is an isomorphism, when M is free, and furthermore that it is natural in M. This will provide the first half of the dual equivalence of categories mentioned in the Introduction between free and topologically-free modules.

67 Lemma *Let M be a projective*[10] *R-module. Then, Λ_M is one-to-one.*

Proof: Let us consider a dual basis for M, i.e., sets $B \subseteq R$ and $\{\, e^* : e \in B \,\} \subseteq M^*$, such that for all $v \in M$, $e^*(v) = 0$ for all but finitely many $e^* \in B^*$ and $v = \sum_{e \in B} e^*(v) e$ ([14, p. 23]).

Let $v \in \ker \Lambda_M$, i.e., $(\Lambda_M(v))(\ell) = \ell(v) = 0$ for each $\ell \in M^*$. Then, in particular, $\Lambda_M(v)(e^*) = e^*(v) = 0$ for all $e \in B$, and thus $v = 0$. □

68 Corollary *If M is a free R-module, then Λ_M is one-to-one.*

69 Lemma $\Lambda^{(\mathsf{R},\tau)} := (\Lambda_M)_M : id \Rightarrow Top_{(\mathsf{R},\tau)} \circ Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} : \mathbf{Mod}_R \to \mathbf{Mod}_R$ *is a natural transformation.*

Proof: Let $M \xrightarrow{f} N$ be a R-linear map. Let $v \in M$ and $\ell \in N^*$. Then,

$$\begin{aligned} ((f^*)'(\Lambda_M(v)))(\ell) &= \Lambda_M(f^*(\ell)) \\ &= \ell(f(v)) \\ &= (\Lambda_N(f(v)))(\ell). \end{aligned} \tag{9}$$

This is equivalent to the commutativity of the following diagram, which implies naturality of $\Lambda^{(\mathsf{R},\tau)}$.

$$\begin{tikzcd} M \arrow[r, "\Lambda_M"] \arrow[d, "f"'] & (M^*, w^*_{(\mathsf{R},\tau)})' \arrow[d, "(f^*)'"] \\ N \arrow[r, "\Lambda_N"'] & (N^*, w^*_{(\mathsf{R},\tau)})' \end{tikzcd} \tag{10}$$

[10] A R-module P is *projective* if it is isomorphic to a direct summand of a free module, i.e., there exists some set X and modules M, N such that $R^{(X)} = M \oplus N$ and $M \simeq P$ [14, p. 22].

□

70 Lemma *Let M be a free module. For each free basis X of M, the following diagram commutes in $\mathbf{Mod}_{\mathsf{R}}$, where $M \xrightarrow{\theta_X} R^{(X)}$ is the canonical isomorphism given by $\theta_X(x) = \delta_x^{\mathsf{R}}$, $x \in X$.*

$$\begin{tikzcd}
M \arrow[r, "\Lambda_M"] \arrow[dd, bend right, "\theta_X"'] & (M^*, w^*_{(\mathsf{R},\tau)})' \arrow[dr, bend left, "(\theta_X^*)'"] & \\
& & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})' \arrow[dl, bend right, "\rho'_X"'] \\
R^{(X)} \arrow[r, "\lambda_X"'] & ((R,\tau)^X)' &
\end{tikzcd} \tag{11}$$

Proof: Let $x \in X$ and $f \in R^X$. Then,

$$\begin{aligned}
((\lambda_X \circ \theta_X)(x))(f) &= (\lambda_X(\delta_x))(f) \\
&= \langle \delta_x, f \rangle_X \\
&= f(x)
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
((\rho'_X \circ (\theta_X^*)' \circ \Lambda_M)(x))(f) &= \rho'_X((\Lambda_M(\eta_X(x)) \circ \phi_X^*)(f) \\
&= (\Lambda_M(x))(\theta_X^*(\rho_X(f))) \\
&= (\rho_X(f) \circ \theta_X)(x) \\
&= (\rho_X(f))(\delta_x) \\
&= \langle \delta_x, f \rangle_X \\
&= f(x).
\end{aligned} \tag{13}$$

Because M is free on X, this entails commutativity of the diagram (11). □

71 Remark *Let (R, τ) be rigid. According to the proof of Lemma 70, for each free module M and each basis X of M,*

$$\Lambda_M = ((\theta_X^{-1})^*)' \circ (\rho_X^{-1})' \circ \lambda_X \circ \theta_X.$$

Since a direct inspection shows that

$$(\rho_X^{-1})' \circ \lambda_X = \Lambda_{R^{(X)}} \tag{14}$$

it follows that

$$\Lambda_M = ((\theta_X^{-1})^*)' \circ \Lambda_{R^{(X)}} \circ \theta_X.$$

72 Corollary *Let us assume that* (R,τ) *is rigid. Then, for each free* R*-module,* $M \xrightarrow{\Lambda_M} (M^*, w^*_{(\mathsf{R},\tau)})'$ *is an isomorphism.*

Proof: This follows from the commutativity of Diagram (11) for any free basis X of M, because λ_X is an isomorphism too (since (R,τ) is rigid). □

Let (R,τ) (resp. $(\Bbbk,\tau)$) be a rigid ring (resp. field). Let us still denote by $\mathbf{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \xrightarrow{Alg_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ (resp. $\mathbf{Vect}^{\mathsf{op}}_{\Bbbk} \xrightarrow{Alg_{(\Bbbk,\tau)}} \mathbf{TopFreeVect}_{(\Bbbk,\tau)}$) and by $\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_{\mathsf{R}}$ (resp. $\mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \xrightarrow{Top_{(\Bbbk,\tau)}} \mathbf{Vect}_{\Bbbk}$) the functors provided by Corollaries 59 (p. 29) and 64 (p. 30)

73 Corollary *Let us assume that* (R,τ) *is rigid.* $\Lambda^{(\mathsf{R},\tau)} := (\Lambda_M)_M : id \Rightarrow Top_{(\mathsf{R},\tau)} \circ Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} : \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{FreeMod}_{\mathsf{R}}$ *is a natural isomorphism.*

Proof: It is a direct consequence of Lemma 69 and of Corollary 72. □

74 Corollary *Let us assume that* $(\Bbbk,\tau)$ *is a field with a ring topology. Then,* $\Lambda^{(\Bbbk,\tau)} := (\Lambda_M)_M : id \Rightarrow Top_{(\Bbbk,\tau)} \circ Alg^{\mathsf{op}}_{(\Bbbk,\tau)} : \mathbf{Vect}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ *is a natural isomorphism.*

4.3 The algebraic dual of the topological dual of a topologically-free module

Let (M,σ) be a topological (R,τ)-module. Let us consider the R-linear map $M \xrightarrow{\Gamma_{(M,\sigma)}} ((M,\sigma)')^*$ by setting $(\Gamma_{(M,\sigma)}(v))(\ell) := \ell(v)$.

In order to obtain the other half of the dual equivalence between free and topologically-free modules it remains to prove that when (R,τ) is rigid and (M,σ) topologically-free, $\Gamma_{(M,\sigma)}$ is an isomorphism, and is natural in (M,σ).

75 Lemma *Let us assume that* (R,τ) *is rigid. Let* (M,σ) *be a topologically-free* (R,τ)*-module.*

1. $(M,\sigma) \xrightarrow{\Gamma_{(M,\sigma)}} (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)})$ *is continuous.*
2. $\Gamma_{(M,\sigma)}$ *is a linear isomorphism.*
3. $\Gamma^{-1}_{(M,\sigma)}$ *is continuous.*

Proof: Let $\Theta\colon (M,\sigma) \simeq (R,\tau)^X$ be an isomorphism (in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$).

1. Since (R,τ) is rigid, $\lambda_X\colon R^{(X)} \simeq ((R,\tau)^X)'$ is an isomorphism. Therefore $R^{(X)} \xrightarrow{\lambda_X} ((R,\tau)^X)' \xrightarrow{\Theta'} (M,\sigma)'$ is an isomorphism too in $\mathbf{Mod}_R$. In particular, $(M,\sigma)'$ is a free with basis $\{\,\Theta'(\lambda_X(\delta_x^{\mathsf{R}}))\colon x \in X\,\}$. By Lemma 24 (p. 17), the weak-$*$ topology on $((M,\sigma)')^*$ is the same as the initial topology given by the maps $((M,\sigma)')^* \xrightarrow{\Lambda_{(M,\sigma)'}(\pi_x\circ\Theta)} (R,\tau)$, $x \in X$, because $\Theta'(\lambda_X(\delta_x^{\mathsf{R}})) = \Theta'(\pi_x) = \pi_x \circ \Theta$. Therefore, $\Gamma_{(M,\sigma)}$ is continuous if, and only if, for each $x \in X$, $\Lambda_{(M,\sigma)'}(\pi_x \circ \Theta) \circ \Gamma_{(M,\sigma)}$ is continuous. Let $v \in M$. One has $(\Lambda_{(M,\sigma)'}(\pi_x \circ \Theta))(\Gamma_{(M,\sigma)}(v)) = (\Gamma_{(M,\sigma)}(v))(\pi_x \circ \Theta) = \pi_x(\Theta(v)) = (\Theta(v))(x)$. Therefore, $\Lambda_{(M,\sigma)'}(\pi_x \circ \Theta) \circ \Gamma_{(M,\sigma)} = \pi_x \circ \Theta$. Since π_x and Θ are continuous, the expected result is obtained.

2. Let $v \in \ker \Gamma_{(M,\sigma)}$. Since for each $x \in X$, $\pi_x \in ((R,\tau)^X)'$, it follows that $\Theta'(\pi_x) \in (M,\sigma)'$, and thus $(\Theta(v))(x) = \pi_x(\Theta(v)) = (\Theta'(\pi_x))(v) = (\Gamma_{(M,\sigma)}(v))(\Theta'(\pi_x)) = 0$. Since this is so for an arbitrary $x \in X$, it follows that $\Theta(v) = 0$ whence $v = 0$.

   To prove that $\Gamma_{(M,\sigma)}$ is onto one proves that the following diagram commutes in $\mathbf{TopMod}_{(R,\tau)}$. This actually shows directly that $\Gamma_{(M,\sigma)}$ is an isomorphism and an homeomorphism (so the last above assertion will be already proved) by composition of isomorphisms and homeomorphisms.

$$\begin{tikzcd}
 & (M,\sigma) \arrow[r, "\Gamma_{(M,\sigma)}"] \arrow[dl, "\Theta"'] & (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \\
(R,\tau)^X \arrow[dr, "\rho_X"'] & & \\
 & ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \arrow[r, "(\lambda_X^{-1})^*"'] & ((((R,\tau)^X)')^*, w^*_{(\mathsf{R},\tau)}) \arrow[uu, "((\Theta')^*)^{-1}"']
\end{tikzcd} \tag{15}$$

   Let $v \in M$ and let $\ell \in (M,\sigma)'$. One has

$$\begin{array}{rl}
 & (((\lambda_X^{-1} \circ (\Theta^{-1})')^* \circ \rho_X \circ \Theta)(v))(\ell) \\
= & (\rho_X(\Theta(v)))(\lambda_X^{-1}((\Theta^{-1})'(\ell)) \\
= & (\rho_X(\Theta(v)))(\lambda_X^{-1}(\ell \circ \Theta^{-1})) \\
= & (\rho_X(\Theta(v)))(\sum\limits_{x\in X} \ell(\Theta^{-1}(\delta_x))\delta_x) \\
= & \sum_{x\in X} \ell(\phi^{-1}(\delta_x))(\Theta(v))(x) \\
= & \ell(\sum_{x\in X} \Theta^{-1}(\delta_x)(\Theta(v))(x)) \\
= & \ell(v) \\
= & (\Gamma_{(M,\sigma)}(v))(\ell).
\end{array} \tag{16}$$

Let us make explicit the penultimate equality: because $\Theta(v)$ is the sum $\sum_{x\in X}(\Theta(v))(x)\delta_x$ of a summable family, $v = \Theta^{-1}(\sum_{x\in X}(\Theta(v))(x)) = \sum_{x\in X}(\Theta(v))(x)\Theta^{-1}(\delta_x)$.

3. Follows from the above point.

□

76 Remark *Let* (R,τ) *be rigid. A direct computation shows that*

$$\Gamma_{(R,\tau)^X} = (\lambda_X^{-1})^* \circ \rho_X. \tag{17}$$

Together with Lemma 75 this implies that for each topologically-free (R,τ)*-module* (M,σ)*, and each* $\mathbf{TopMod}_{(\mathsf{R},\tau)}$*-isomorphism* $(M,\sigma) \xrightarrow{\Theta} (R,\tau)^X$

$$\Gamma_{(M,\sigma)} = ((\Theta^{-1})')^* \circ \Gamma_{(R,\tau)^X} \circ \Theta. \tag{18}$$

77 Lemma *Let* (R,τ) *be a rigid ring.* $\Gamma^{(\mathsf{R},\tau)} := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$ *provides a natural isomorphism* $\Gamma^{(\mathsf{R},\tau)}: id \Rightarrow Alg_{(\mathsf{R},\tau)} \circ Top^{\mathsf{op}}_{(\mathsf{R},\tau)}: \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$.

Proof: According to Lemma 75 it suffices to check naturality of $\Gamma^{(\mathsf{R},\tau)}$. Let $(M,\sigma),(N,\gamma)$ be topologicall-free (R,τ)-modules. Let $(M,\sigma) \xrightarrow{f} (N,\gamma)$ be a continuous (R,τ)-linear map. Let $v \in M$ and $\ell \in (N,\gamma)'$. One has

$$\begin{array}{rcl} ((f')^*((\Gamma_{(M,\sigma)}(v)))(\ell) & = & (\Gamma_{(M,\sigma)}(v))(f'(\ell)) \\ & = & (\Gamma_{(M,\sigma)}(v))(\ell \circ f) \\ & = & \ell(f(v)) \\ & = & (\Gamma_{(N,\gamma)}(f(v)))(\ell). \end{array} \tag{19}$$

This is equivalent to the commutativity of the following diagram, for each $f \in \mathbf{TopMod}_{(\mathsf{R},\tau)}((M,\sigma),(N,\gamma))$, which is naturality of $\Gamma^{(\mathsf{R},\tau)}$.

$$\begin{array}{ccc} (M,\sigma) & \xrightarrow{\Gamma_{(M,\sigma)}} & (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ (N,\gamma) & \xrightarrow[\Gamma_{(N,\gamma)}]{} & (((N,\gamma)')^*, w^*_{(\mathsf{R},\tau)}) \end{array} \tag{20}$$

□

78 Corollary *Let us assume that* $(\Bbbk,\tau)$ *is a field with a ring topology.* $\Gamma^{(\Bbbk,\tau)}: id \Rightarrow Alg_{(\Bbbk,\tau)} \circ Top^{\mathsf{op}}_{(\Bbbk,\tau)}: \mathbf{TopFreeVect}_{(\Bbbk,\tau)} \to \mathbf{TopFreeVect}_{(\Bbbk,\tau)}$ *with* $\Gamma^{(\Bbbk,\tau)} := (\Gamma_{(M,\sigma)})_{(M,\sigma)}$*, is a natural isomorphism.*

4.4 For a rigid ring (R,τ), $\mathbf{FreeMod}_{\mathsf{R}} \simeq \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$

The results of the two previous sections may be combined to provide the expected equivalence of categories.

4.4.1 The equivalence and some of its immediate consequences

Collecting Corollary 73 and Lemma 77, one immediately gets the following.

79 Theorem *Let us assume that (R,τ) is rigid.* $\mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_{\mathsf{R}}$ *is an equivalence of categories, with equivalence inverse the functor* $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{Alg^{\mathsf{op}}_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$.

80 Corollary *Let $(\Bbbk,\tau)$ be a rigid field.* $\mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)} \xrightarrow{Top_{(\Bbbk,\tau)}} \mathbf{Vect}_{\Bbbk}$ *is an equivalence of categories, and* $\mathbf{Vect}_{\Bbbk} \xrightarrow{Alg^{\mathsf{op}}_{(\Bbbk,\tau)}} \mathbf{TopFreeVect}^{\mathsf{op}}_{(\Bbbk,\tau)}$ *is its equivalence inverse.*

Section 4.1.2 about the properties of topological bases may be completed by the following remark.

81 Remark

1. *Let (M,σ) be a topologically-free (R,τ)-module. Then, $(M,\sigma)'$ is a free. Let B be a basis of $(M,\sigma)'$. Then, by Lemma 56 (p. 28), B^* is a topological basis of $((M,\sigma)')^*$. For each $b \in B$, let $b_* \in M$ be given by $b_* := \Gamma^{-1}_{(M,\sigma)}(b^*)$. Then, $B_* := \{\, b_* : b \in B \,\}$ is a topological basis of (M,σ). (This is so because $B_* = \Gamma^{-1}_{(M,\sigma)}(B^*)$.)*

   Moreover $(B_*)' = B$. (Indeed, let $b \in B$, and let $\ell \in (M,\sigma)'$. According to Corollary 66, $((b_*)')^*(\ell) = \ell(b_*) = \Gamma_{(M,\sigma)}(b_*)(\ell) = b^*(\ell)$. Whence $((b_*)')^* = b^*$ so that $(b_*)' = b$.)

2. *Let M be a free module. Then, M^* is topologically-free. Let B be a topological basis of M^*. Then, by Lemma 62, B' is a free basis of $(M^*)'$. For each $b \in B$, let $b_\prime \in M$ be given by $b_\prime := \Lambda^{-1}_M(b')$. Then, $B_\prime := \{\, b_\prime : b \in B \,\}$ is a free basis of M. (Because $B_\prime = \Lambda^{-1}_M(B')$.)*

   Moreover $(B_\prime)^* = B$. (Indeed, let $b \in B$, and let $\ell \in M^*$. According to Corollary 61, $((b_\prime)^*)'(\ell) = \ell(b_\prime) = \Lambda_M(b_\prime)(\ell) = b'(\ell)$. Whence $((b_\prime)^*)' = b'$ so that $(b_\prime)^* = b$.)

82 Corollary *Let* (R,τ) *be a rigid ring.*

1. *For each free topologically-free module* (M,σ)*,* $\Gamma'_{(M,\sigma)} = \Lambda^{-1}_{(M,\sigma)'}$*.*

2. *For each free module* M*,* $\Lambda^*_M = \Gamma^{-1}_{M^*}$*.*

Proof: This is a consequence of the triangular identities for an (adjoint) equivalence [17, p. 85] and of Theorem 79. □

83 Corollary *Let* (R,τ) *be a rigid ring.*

1. *For each free* R*-module* M*,* $\Lambda_{(M^*)'} = (\Lambda^*_M)'$*.*

2. *For each topologically-free* (R,τ)*-module* (M,σ)*,* $\Gamma_{(((M,\sigma)')^*,w^*_{(\mathsf{R},\tau)})} = (\Gamma'_{(M,\sigma)})^*$*.*

4.4.2 Finite-dimensional vector spaces

Let $\Bbbk$ be a field. Let (M,σ) be a topologically-free $(\Bbbk,\mathsf{d})$-vector space with M finite-dimensional. Then, σ is the discrete topology on M (indeed, let us consider $M = R^X$, X finite, then $\{0\} = \bigcap_{x\in X}\pi_x^{-1}(\{0\})$ is open in $(R,\mathsf{d})^X$). It follows that $(M,\sigma)' = M^*$, and the equivalence established in Corollary 80 coincides with the usual dual equivalence $\mathbf{FinDimVect}_\Bbbk \simeq \mathbf{FinDimVect}_\Bbbk^{\mathsf{op}}$ under the algebraic dual functor, where $\mathbf{FinDimVect}_\Bbbk$ is the category of finite-dimensional $\Bbbk$-vector spaces.

4.4.3 Linearly compact vector spaces

Let $\Bbbk$ be a field. A topological $(\Bbbk,\mathsf{d})$-vector space (M,σ) is said to be *linearly topologized* (or that σ is a *linear topology*) if σ admits a fundamental system of neighborhoods of zero consisting of open sub-vector spaces. Such a topological vector space (M,σ) is a *linearly compact* $\Bbbk$-vector space when $(M,\sigma) \simeq (\Bbbk,\mathsf{d})^X$ for some set X (see [4, Proposition 24.4, p. 105]). The full subcategory $\mathbf{LCpVect}_\Bbbk$ of $\mathbf{TopVect}_{(\Bbbk,\mathsf{d})}$ spanned by the linearly compact vector spaces thus is the same as $\mathbf{TopFreeVect}_{(\Bbbk,\mathsf{d})}$.

84 Corollary *(of Theorem 79) Let* R *be a ring. For each rigid topologies* τ,σ *on* R*, the categories* $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *and* $\mathbf{TopFreeMod}_{(\mathsf{R},\sigma)}$ *are equivalent. Moreover, for each rigid field* $(\Bbbk,\tau)$*,* $\mathbf{TopFreeVect}_{(\Bbbk,\tau)}$ *is equivalent to* $\mathbf{LCpVect}_\Bbbk$*.*

Proof: This follows from Theorem 79 because of transitivity of equivalence of categories (or more precisely, by composition of adjunctions, and in particular of adjoint equivalences). □

In particular, when $\Bbbk$ is a field, one recovers, as stated in the Introduction, the result of J. Dieudonné [8] that $\mathbf{Vect}_{\Bbbk}^{\mathsf{op}} \simeq \mathbf{LCpVect}_{\Bbbk}$.

4.4.4 The universal property of $(R, \tau)^X$

Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ be a functor. Let $\mathbf{EssIm}(F)$, called the *essential image of* F, be the full subcategory of $\mathbf{D}$ spanned by the $\mathbf{D}$-objects which are isomorphic to some $F(c)$, where c is a $\mathbf{C}$-object. Let $\mathbf{EssIm}(F) \xrightarrow{E} \mathbf{D}$ be the corresponding full embedding functor. Of course, by definition, F factors through E, i.e., there is a unique functor $\mathbf{C} \xrightarrow{F_0} \mathbf{EssIm}(F)$ such that $F = E \circ F_0$.

85 Lemma *Let us assume that F is a left adjoint of a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$. Then, F_0 is a left adjoint of $\mathbf{EssIm}(F) \xrightarrow{G \circ E} \mathbf{C}$.*

Proof: See Appendix A.2, p. 88. □

All this being said, for a ring R, the forgetful functor $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$ (see Remark 12, p. 14) may be restricted as indicated in the following commutative diagram, and the restriction still is denoted $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$.

$$\begin{tikzcd} \mathbf{Mod}_{\mathsf{R}} \arrow[r, "|\cdot|"] & \mathbf{Set} \\ \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \arrow[ur] & \end{tikzcd} \tag{21}$$

Likewise $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ (see again Remark 12) may be co-restricted as indicated by the commutative diagram below, and the co-restriction is given the same name $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{FreeMod}_{\mathsf{R}}$.

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "F_{\mathsf{R}}"] \arrow[dr] & \mathbf{Mod}_{\mathsf{R}} \\ & \mathbf{FreeMod}_{\mathsf{R}} \arrow[u, hook] \end{tikzcd} \tag{22}$$

(Of course, when R is a field $\Bbbk$, there is no need to consider the corresponding co-restrictions.)

Because $\mathbf{FreeMod}_{\mathsf{R}} = \mathbf{EssIm}(F_{\mathsf{R}})$, we are in a situation where Lemma 85 may be applied. Therefore, the adjunction $F_{\mathsf{R}} \dashv |\cdot| : \mathbf{Set} \to \mathbf{FreeMod}_{\mathsf{R}}$ is

obtained, and by composition, for each rigid ring (R,τ), there is also the following adjunction $Alg^{\mathsf{op}}_{(\mathsf{R},\tau)} \circ F_{\mathsf{R}} \dashv |\cdot| \circ Top_{(\mathsf{R},\tau)} \colon \mathbf{Set} \to \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$. Since $(R,\tau)^X \simeq ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)})$ (Lemma 19, p. 16), the previous adjunction may be translated into a *universal property* of $(R,\tau)^X$, as explained below, which somehow legitimates the terminology *topologically-free*.

86 Proposition *Let us assume that (R,τ) is rigid. Let X be a set. For each topologically-free module (M,σ) and any map $X \xrightarrow{f} |(M,\sigma)'|$, there is a unique continuous (R,τ)-linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X$ such that $|(f^\sharp)'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X = f$ (cf. Remark 10, p. 14, for the definition of δ^{R}_X).*

Proof: There is a unique R-linear map $R^{(X)} \xrightarrow{\tilde{f}} (M,\sigma)'$ such that $|\tilde{f}| \circ \delta^{\mathsf{R}}_X = f$. Let us define the continuous linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X := (M,\sigma) \xrightarrow{\Gamma_{(M,\sigma)}} (((M,\sigma)')^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{(\tilde{f})^*} ((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}) \xrightarrow{\rho_X^{-1}} (R,\tau)^X$. One has

$$\begin{array}{rcl} |(f^\sharp)'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X & = & |\Gamma'_{(M,\sigma)}| \circ |((\tilde{f})^*)'| \circ |(\rho_X^{-1})'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X \\ & = & |\Gamma'_{(M,\sigma)}| \circ |((\tilde{f})^*)'| \circ |\Lambda_{R^{(X)}}| \circ \delta^{\mathsf{R}}_X \\ & & \text{(by Eq. (14), p. 32)} \\ & = & |\Gamma'_{(M,\sigma)}| \circ |\Lambda_{(M,\sigma)'}| \circ |\tilde{f}| \circ \delta^{\mathsf{R}}_X \\ & & \text{(by naturality of } \Lambda) \\ & = & |\tilde{f}| \circ \delta^{\mathsf{R}}_X \\ & & \text{(by Corollary 82, p. 37)} \\ & = & f. \end{array} \tag{23}$$

It remains to check uniqueness of $f^\sharp$. Let $(M,\sigma) \xrightarrow{g} (R,\tau)^X$ be a continuous linear map such that $|g'| \circ |\lambda_X| \circ \delta^{\mathsf{R}}_X = f$. Then, $g' \circ \lambda_X = \tilde{f}$. Thus, $\lambda_X^* \circ (g')^* = \tilde{f}^* = \rho_X \circ f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$. So $\rho_X \circ \Gamma^{-1}_{(R,\tau)^X} \circ (g')^* = \rho_X \circ f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$ according to Eq. (17), p. 35, and thus $\Gamma^{-1}_{(R,\tau)^X} \circ (g')^* = f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$. Then, by naturality of Γ^{-1}, $g \circ \Gamma^{-1}_{(M,\sigma)} = f^\sharp \circ \Gamma^{-1}_{(M,\sigma)}$, i.e., $g = f^\sharp$. □

87 Remark *Let us make explicit the construction of the continuous (R,τ)-linear map $(M,\sigma) \xrightarrow{f^\sharp} (R,\tau)^X$ for $X \xrightarrow{f} |(M,\sigma)'|$. Let $v \in M$ and $x \in X$.*

Then, using the same notations as in the proof of Proposition 86, one has

$$\begin{array}{rcl}(f^\sharp(v))(x) & = & (\rho_X^{-1}((\tilde{f})^*(\Gamma_{(M,\sigma)}(v))))(x)\\ & = & (\rho_X^{-1}(\Gamma_{(M,\sigma)}(v)\circ\tilde{f}))(x)\\ & = & (\Gamma_{(M,\sigma)}(v))(\tilde{f}(\delta_x^{\mathsf{R}}))\\ & = & (\tilde{f}(\delta_x^{\mathsf{R}}))(v)\\ & = & (f(x))(v).\end{array} \tag{24}$$

Wherefore, $f^\sharp(v) = \sum_{x\in X}(f(x))(v)\delta_x^{\mathsf{R}}$, $v \in M$.

The definition of an isomorphism from a bijection between topological bases, provided by Lemma 55 (p. 28) is a particular instance of the above construction. Let indeed B *be a topological basis of* (M,σ)*, and let* $B \xrightarrow{f} X$ *be a bijection. Let* $h := X \xrightarrow{f^{-1}} B \xrightarrow{(-)'} (M,\sigma)'$*, where* $B \xrightarrow{(-)'} (M,\sigma)'$ *is the map* $b \mapsto b'$ *as in Remark 49 (p. 27). Then for* $v \in M$*,* $h^\sharp(v) = \sum_{x\in X}(f^{-1}(x))'(v)\delta_x^{\mathsf{R}}$*, and thus* $h^\sharp$ *corresponds to the isomorphism given in Lemma 55.*

88 Remark *Let* (R,τ) *be a rigid ring. The universal property raised in Proposition 86 may be turned into a functor* $\mathbf{Set}^{\mathsf{op}} \xrightarrow{TF_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *("TF" stands for "Topologically-Free") in a usual way as follows: let* $X \xrightarrow{f} Y$ *be a set-theoretic map. Then there is a unique continuous* (R,τ)*-linear map* $(R,\tau)^Y \xrightarrow{TF_{(\mathsf{R},\tau)}(f)} (R,\tau)^X$ *such that* $|TF_{(\mathsf{R},\tau)}(f)'|\circ|\lambda_X|\circ\delta_X^{\mathsf{R}} = |\lambda_Y|\circ\delta_Y^{\mathsf{R}}\circ f$.

In details, let $x \in X$*. Since* $\lambda_X(\delta_x^{\mathsf{R}}) = \pi_x$*,* $TF_{(\mathsf{R},\tau)}(f)'(\lambda_X(\delta_X^{\mathsf{R}}(x)) = TF_{(\mathsf{R},\tau)}(f)'(\lambda_X(\delta_x)) = TF_{(\mathsf{R},\tau)}(f)'(\pi_x) = \pi_x\circ TF_{(\mathsf{R},\tau)}(f)$*. But it is also equal to* $\lambda_Y(\delta_Y^{\mathsf{R}}(f(x))) = \lambda_Y(\delta_{f(x)}^{\mathsf{R}}) = \pi_{f(x)}$*. It follows that* $TF_{(\mathsf{R},\tau)} = \mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *(see Remark 47, p. 26).*

As a consequence of Remark 88, the following result is immediate, since $TF^{\mathsf{op}}_{(\mathsf{R},\tau)}$ by construction, is a left adjoint of $|-|\circ Top_{(\mathsf{R},\tau)}$ (this is basically the content of Proposition 86).

89 Proposition *Let* (R,τ) *be a rigid ring.* $\mathbf{Set} \xrightarrow{P^{\mathsf{op}}_{(\mathsf{R},\tau)}} \mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)}$ *is a left adjoint of* $\mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)} \xrightarrow{Top_{(\mathsf{R},\tau)}} \mathbf{FreeMod}_R \xrightarrow{|-|} \mathbf{Set}$*, and thus is naturally equivalent to* $\mathbf{Set} \xrightarrow{Alg^{\mathsf{op}}_{(\mathsf{R},\tau)}\circ F_{\mathsf{R}}} \mathbf{TopFreeMod}^{\mathsf{op}}_{(R,\tau)}$.

5 Tensor product of topologically-free modules

In this current section most of the ingredients so far introduced and developed fit together so as to lift the equivalence $\mathbf{FreeMod} \simeq \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ to a duality between some (suitably generalized) topological algebras and coalgebras. Let us briefly describe how this goal is achieved:

1. A topological tensor product $\circledast_{(\mathsf{R},\tau)}$ for topologically-free modules (over a rigid ring (R,τ)) is provided by transport of the algebraic tensor product $\otimes_{\mathsf{R}}$ of R-modules along the dual equivalence from Section 4.

2. A topological basis (Definition 48) for $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ is described in terms of topological bases of $(M,\sigma),(N,\gamma)$.

3. The aforementioned equivalence is proved to be compatible with $\otimes_{\mathsf{R}}$ and $\circledast_{(\mathsf{R},\tau)}$ (i.e., it is a monoidal equivalence) thanks to the above accurate description of a topological basis for the topological tensor product (the proofs are placed in Appendix C).

4. Accordingly, for category-theoretic reasons (see Appendix B), one obtains a dual equivalence between some topological algebras and coalgebras, still under the algebraic and topological dual functors.

5. Finally, one discusses how it interacts with the standard duality between algebras and coalgebras, known as *finite duality*.

5.1 Algebraic tensor product of modules

We do not claim to any originality in this first section, since the results are well-known (cf. [5] for instance) but they are recalled for the reader's convenience and because it gives us the opportunity to introduce some notations used hereafter.

Let R be a ring. For each R-modules M,N, there is a R-module $M\otimes_{\mathsf{R}}N$ and a R-bilinear map $M\times N\xrightarrow{\otimes} M\otimes_{\mathsf{R}}N$ *universal* with respect to all R-bilinear maps $M\times N\to P$ (P is an arbitrary R-module). In other words, each R-bilinear map $M\times N\xrightarrow{f}P$ uniquely factors as

$$\begin{tikzcd} M\times N \arrow[r, "f"] \arrow[d, "\otimes"'] & P \\ M\otimes_{\mathsf{R}} P \arrow[ur, "\tilde{f}"'] & \end{tikzcd} \tag{25}$$

where $\tilde{f}$ is R-linear. As usually the image of $(x,y) \in M \times N$ by $\otimes$ is denoted $x \otimes y$.

From this universality property, one deduces that for R-linear maps $M_i \xrightarrow{f_i} N_i$, $i = 1,2$, there is a unique R-linear map $M_1 \otimes_{\mathsf{R}} M_2 \xrightarrow{f_1 \otimes_{\mathsf{R}} f_2} N_1 \otimes_{\mathsf{R}} N_2$ such that $(f_1 \otimes_{\mathsf{R}} f_2)(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2)$, $x_i \in M_i$, $i = 1,2$.

Furthermore, with the following isomorphisms $(M \otimes_{\mathsf{R}} N) \otimes_{\mathsf{R}} P \xrightarrow{\alpha_{M,N,P}} M \otimes_{\mathsf{R}} (N \otimes_{\mathsf{R}} P)$, $\alpha_{M,N,P}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$, $M \otimes_{\mathsf{R}} R \xrightarrow{\rho_M} M$, $\rho_M(x \otimes 1_{\mathsf{R}}) = x$, $R \otimes_{\mathsf{R}} N \xrightarrow{\lambda_N} N$, $\lambda_N(1_{\mathsf{R}} \otimes y) = y$, $M \otimes_{\mathsf{R}} N \xrightarrow{\sigma_{M,N}} N \otimes_{\mathsf{R}} M$, $\sigma_{M,N}(x \otimes y) = y \otimes x$, $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}} = (\mathbf{Mod}_{\mathsf{R}}, \otimes_{\mathsf{R}}, R)$ is a symmetric monoidal category (see Appendix B, p. 89).

90 Lemma *Let M, N be two free modules. Then, $M \otimes_{\mathsf{R}} N$ is free.*

Proof: Let X be a basis of M and Y be a basis of N. $M \otimes_{\mathsf{R}} N \simeq R^{(X)} \otimes_{\mathsf{R}} R^{(Y)} \simeq R^{(X \times Y)}$ where the first isomorphism is due to functoriality of $\otimes_{\mathsf{R}}$, and where the second isomorphism is given as follows: let $R^{(X)} \times R^{(Y)} \xrightarrow{\phi_{X,Y}} R^{(X \times Y)}$, $\phi_{X,Y}(f,g) = \sum_{(x,y) \in X \times Y} f(x)g(y)\delta^{\mathsf{R}}_{(x,y)}$. It is R-bilinear so there is a unique R-linear map $R^{(X)} \otimes_{\mathsf{R}} R^{(Y)} \xrightarrow{\Phi_{X,Y}} R^{(X \times Y)}$, $\Phi_{X,Y}(f \otimes g) = \phi_{X,Y}(f,g)$. Conversely, let $R^{(X \times Y)} \xrightarrow{\Psi_{X,Y}} R^{(X)} \otimes_{\mathsf{R}} R^{(Y)}$ be the unique R-linear map given on the basis elements by $\Psi_{X,Y}(\delta^{\mathsf{R}}_{(x,y)}) = \delta^{\mathsf{R}}_x \otimes \delta^{\mathsf{R}}_y$. It is then easy to check that $\Phi_{X,Y}$ and $\Psi_{X,Y}$ are inverse one from the other. □

As a consequence of Lemma 90, it is a matter of simple verifications to see that $\mathbb{F}\mathrm{r}\mathrm{e}\mathrm{e}\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}} = (\mathbf{FreeMod}_{\mathsf{R}}, \otimes_{\mathsf{R}}, R)$ is a (symmetric) monoidal subcategory of $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$ (see again Appendix B).

5.2 Topological tensor product of topologically-free modules

We now wish to take advantage of the equivalence of categories $\mathbf{FreeMod}_{\mathsf{R}} \simeq \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ (Theorem 79, p. 36) for a rigid ring (R,τ), to introduce a topological tensor product of topologically-free modules.

From here to the end of Subsection 5.2, (R,τ) is a rigid ring.

5.2.1 The bifunctor $\circledast_{(\mathsf{R},\tau)}$

Let $(M,\sigma), (N,\gamma)$ be topologically-free (R,τ)-modules. One defines their *topological tensor product over* (R,τ) as

$$(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma) := Alg_{(\mathsf{R},\tau)}((M,\sigma)' \otimes_{\mathsf{R}} (N,\gamma)'). \tag{26}$$

One immediately observes that $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ still is a topologically-free (R,τ)-module as

1. $(M,\sigma)'$ and $(N,\gamma)'$ are free R-modules (Lemma 62, p. 29),
2. $(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'$ also is (Lemma 90),
3. The algebraic dual of a free module is topologically-free (Lemma 24, p. 17).

Actually, this definition is just the object component of a bifunctor (see Appendix A, p. 85)

$$\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\times\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\xrightarrow{-\circledast_{(\mathsf{R},\tau)}-}\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$$

namely $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\times\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\xrightarrow{Top^{\mathsf{op}}_{(\mathsf{R},\tau)}\times Top^{\mathsf{op}}_{(\mathsf{R},\tau)}}\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}}\times\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}}\xrightarrow{\otimes^{\mathsf{op}}_{\mathsf{R}}}\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}}\xrightarrow{Alg_{(\mathsf{R},\tau)}}\mathbf{TopMod}_{(\mathsf{R},\tau)}$. (Recall from Appendix A that $\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}}\times\mathbf{Mod}^{\mathsf{op}}_{\mathsf{R}}=(\mathbf{Mod}_{\mathsf{R}}\times\mathbf{Mod}_{\mathsf{R}})^{\mathsf{op}}$.)

91 Remark *Let X,Y be sets. One has*

$$\begin{array}{rcl}(R,\tau)^X\circledast_{(\mathsf{R},\tau)}(R,\tau)^Y & = & ((((R,\tau)^X)'\otimes_{\mathsf{R}}((R,\tau)^Y)')^*,w^*_{(\mathsf{R},\tau)})\\ & \simeq & ((R^{(X)}\otimes_{\mathsf{R}}R^{(Y)})^*,w^*_{(\mathsf{R},\tau)})\\ & & \textit{(under } (\lambda_X\otimes_{\mathsf{R}}\lambda_Y)^*\textit{)}\\ & \simeq & ((R^{(X\times Y)})^*,w^*_{(\mathsf{R},\tau)})\\ & & \textit{(under } \Psi^*_{X,Y}\textit{; see the proof of Lemma 90)}\\ & \simeq & (R,\tau)^{X\times Y}.\\ & & \textit{(under } \rho^{-1}_{X\times Y}\textit{)}\end{array}\tag{27}$$

Since $\circledast_{(\mathsf{R},\tau)}$ is a functor it is worth to make explicit its action on morphisms. Given $f_i\in\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}((M_i,\sigma_i),(N_i,\gamma_i))$, $i=1,2$, one has

$$f_1\circledast_{(\mathsf{R},\tau)}f_2:=(M_1,\sigma_1)\circledast_{(\mathsf{R},\tau)}(M_2,\sigma_2)\xrightarrow{(f_1'\otimes_{\mathsf{R}}f_2')^*}(N_1,\gamma_1)\circledast_{(\mathsf{R},\tau)}(N_2,\gamma_2).$$

In details, let $\ell\in((M_1,\sigma_1)'\otimes_{\mathsf{R}}(M_2,\sigma_2)')^*$, $\ell_1\in(N_1,\gamma_1)'$ and $\ell_2\in(N_2,\gamma_2)'$. Then,

$$\begin{array}{rcl}((f_1\circledast_{(\mathsf{R},\tau)}f_2)(\ell))(\ell_1\otimes\ell_2) & = & (((f'\otimes_{\mathsf{R}}g')^*)(\ell))(\ell_1\otimes\ell_2)\\ & = & \ell((f_1'\otimes_{\mathsf{R}}f_2')(\ell_1\otimes\ell_2))\\ & = & \ell((f_1'(\ell_1))\otimes(f_2'(\ell_2)))\\ & = & \ell((\ell_1\circ f_1)\otimes(\ell_2\circ f_2)).\end{array}\tag{28}$$

5.2.2 A topological basis of $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$

Our next goal will be to describe a topological basis (Definition 48) of $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ in terms of topological bases of (M,σ) and (N,γ). Once this goal achieved, we will take advantage of the use of topological bases in order to handle more appropriately the topological tensor products.

92 Definition *Given a ring S, for every S-modules M,N, one has a natural R-linear map $M^*\otimes_{\mathsf{S}}N^*\xrightarrow{\Theta_{M,N}}(M\otimes_{\mathsf{S}}M)^*$ given by $(\Theta_{M,N}(\ell_1\otimes\ell_2))(u\otimes v)=\ell_1(u)\ell_2(v)$, $\ell_1\in M^*$, $\ell_2\in N^*$, $u\in M$ and $v\in N$.*

Let $(M,\sigma),(N,\gamma)$ be two topologically-free (R,τ)-modules. Let $u\in M$ and $v\in N$. Let us define

$$u\circledast v:=\Theta_{(M,\sigma)',(N,\gamma)'}(\Gamma_{(M,\sigma)}(u)\otimes_{\mathsf{R}}\Gamma_{(N,\gamma)}(v))\in(M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma). \quad (29)$$

In details, given $\ell_1\in(M,\sigma)'$ and $\ell_2\in(N,\gamma)'$, $(u\circledast v)(\ell_1\otimes\ell_2)=\ell_1(u)\ell_2(v)$.

93 Lemma *Let (M,σ) and (N,γ) be both topologically-free (R,τ)-modules, with respective topological bases B,D. The map $B\times D\xrightarrow{-\circledast-}(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ given by $(b,d)\mapsto b\circledast d$, is one-to-one.*

Proof: Let $b_1,b_2\in B$ and $d_1,d_2\in D$, and let us assume that $b_1\circledast d_1=b_2\circledast d_2$. So $1_{\mathsf{R}}=b_1'(b_1)d_1'(d_1)=(b_1\circledast d_1)(b_1'\otimes d_1')=(b_2\circledast d_2)(b_1'\otimes d_1')=b_1'(b_2)d_1'(d_2)=\delta_{b_1}(b_2)\delta_{d_1}(d_2)$ by Remark 49 (p. 27). Thus $b_1=b_2$ and $d_1=d_2$. □

94 Lemma *Let (M,σ) and (N,γ) be both topologically-free (R,τ)-modules. The map $M\times N\xrightarrow{-\circledast-}(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ is R-bilinear and separately continuous in both variable. Moreover, if $\tau=\mathsf{d}$, then $\circledast$ is even jointly continuous.*

Proof: R-bilinearity is clear. The topology $w^*_{(\mathsf{R},\tau)}$ on $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)=((M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)')^*$ is the initial topology induced by evaluations $((M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)')^*\xrightarrow{\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(t)}(R,\tau)$, $\ell\mapsto\ell(t)$, $t\in(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'$. Since $(M,\sigma)'$ is free on say X and $(N,\gamma)'$ is free on Y, this topology is the same as the initial topology induced by $((M,\sigma)'\otimes_R((N,\mu)')^*\xrightarrow{\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(x\otimes y)}(R,\tau)$, $x\in X$, $y\in Y$ (Lemma 24, p. 17).

Let $x\in X$, $y\in Y$, $u\in M$ and $v\in N$. Then, $\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(x\otimes y)(u\circledast v)=(u\circledast v)(x\otimes y)=x(u)y(v)=m_{\mathsf{R}}(x(u),x(v))$, whence $\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(x\otimes y)\circ(-\circledast-)=m_{\mathsf{R}}\circ(x\times y)$. This automatically guarantees separate continuity in each variable of $\circledast$.

Let us assume that $\tau = \mathsf{d}$. According to the above general case, to see that $\circledast$ is continuous, by [23, Theorem 2.14, p. 17], it remains to check continuity at zero of $\circledast$. Let $A \subseteq X \times Y$ be a finite set, and for each $(x,y) \in A$, let $U_{(x,y)}$ be an open neighborhood of zero in (R, d). Let $A_1 := \{\, x \in X \colon \exists y \in Y,\ (x,y) \in A \,\}$ and $A_2 := \{\, y \in Y \colon \exists x \in X,\ (x,y) \in A \,\}$. A_1, A_2 are both finite and $A \subseteq A_1 \times A_2$. Let $u \in M$ such that $x(u) = 0$ for all $x \in A_1$, and $v \in N$ such that $y(v) = 0$ for all $y \in A_2$. Then, $(u \circledast v)(x \otimes y) = 0$ for all $(x,y) \in A_1 \times A_2$, and even more so $(u \circledast v)(x \otimes y) \in U_{(x,y)}$ for all $(x,y) \in A$. $\square$

95 Remark *Let X, Y be sets, and let $f \in R^X$, $g \in R^Y$. Then*[11]*, since $\circledast$ is separately continuous by Lemma 94,*

$$\begin{aligned} f \circledast g &= \left(\textstyle\sum_{x \in X} f(x)\delta_x^{\mathsf{R}}\right) \circledast \left(\textstyle\sum_{y \in Y} g(y)\delta_y^{\mathsf{R}}\right) \\ &= \textstyle\sum_{(x,y) \in X \times Y} f(x)g(y)\delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}}. \\ &\quad \textit{(as a sum of a summable family)} \end{aligned} \tag{30}$$

More generally, but for the same reason as above, if B is a topological basis of (M, σ) and D is a topological basis of (N, γ), then

$$u \circledast v = \sum_{(b,d) \in B \times D} b'(u)d'(v)b \circledast d,$$

$u \in M$, $v \in N$. In particular, one observes (in view of Proposition 96 below) that $(b \circledast d)'(u \circledast v) = b'(u)d'(v)$, $b \in B$, $u \in M$, $d \in D$, and $v \in N$.

96 Proposition *Let (M, σ) and (N, γ) be topologically-free (R, τ)-modules, with respective topological bases B, D. Then, $(b \circledast d)_{(b,d) \in B \times D}$ is a topological basis of $(M, \sigma) \circledast_{(\mathsf{R},\tau)} (N, \gamma)$.*

Proof: Let us first prove that $\{\, (\delta_x^{\mathsf{R}} \circledast \delta_y^{\mathsf{R}})_{(x,y) \in X \times Y} \colon (x,y) \in X \times Y \,\}$ is a topological basis of $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y$. By virtue of Lemma 50 (p. 27) and Remark 91, $\{\, ((\lambda_X^{-1} \otimes_{\mathsf{R}} \lambda_Y^{-1})^*(\Phi_{X,Y}^*(\rho_{X \times Y}(\delta_{(x,y)}^{\mathsf{R}})))) \colon (x,y) \in X \times Y \,\}$ is a topological basis of $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y$. Let $\ell_1 \in ((R,\tau)^X)'$ and $\ell_2 \in$

[11] The second equality in Eq. (30) follows from the proof of [23, Theorem 10.15, p. 78] which, by inspection, shows that the cited result still is valid more generally after the replacement of a jointly continuous bilinear map by a separately continuous bilinear map.

$((R,\tau)^Y)'$. Then,

$$\begin{aligned}
& ((\lambda_X^{-1} \otimes_{\mathsf{R}} \lambda_Y^{-1})^*(\Phi^*_{X,Y}(\rho_{X\times Y}(\delta^{\mathsf{R}}_{(x,y)}))))(\ell_1 \otimes \ell_2) \\
= \; & \rho_{X\times Y}(\delta^{\mathsf{R}}_{(x,y)})(\Phi_{X,Y}((\lambda_X^{-1}(\ell_1) \otimes \lambda_Y^{-1})(\ell_2))) \\
= \; & p_{(x,y)}(\Phi_{X,Y}(\hat{\ell}_1 \otimes \hat{\ell}_2)) \\
= \; & p_{(x,y)}\left(\textstyle\sum_{(s,t)\in X\times Y} \ell_1(\delta_s)\ell_2(\delta_t)\delta_{(s,t)}\right) \\
= \; & \ell_1(\delta_x)\ell_2(\delta_y) \\
= \; & (\delta_x \otimes \delta_y)(\ell_1 \otimes \ell_2).
\end{aligned} \tag{31}$$

Secondly, by assumption, $(M,\sigma) \simeq (R,\tau)^B$ and $(N,\gamma) \simeq (R,\tau)^D$ in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, and thus by functoriality of $\otimes_{(\mathsf{R},\tau)}$, $(M,\sigma) \otimes_{(\mathsf{R},\tau)} (N,\gamma) \simeq (R,\tau)^B \otimes_{(\mathsf{R},\tau)} (R,\tau)^D$. Let us be more specific by providing an isomorphism. Let $\Theta_B\colon (M,\sigma) \simeq (R,\tau)^B$, $\Theta_B(b) = \delta_b$, $b \in B$, and let $\Theta_D\colon (N,\gamma) \simeq (R,\tau)^D$, $\Theta_D(d) = \delta_d$, $d \in D$. Thus an isomorphism is given by $\Theta_B \otimes_{(\mathsf{R},\tau)} \Theta_D\colon (M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma) \simeq (R,\tau)^B \otimes_{(\mathsf{R},\tau)} (R,\tau)^D$. One knows by the first part of the proof that $(\delta_x \otimes \delta_y)_{(x,y)\in X\times Y}$ is a topological basis of $(R,\tau)^B \otimes_{(\mathsf{R},\tau)} (R,\tau)^D$ and one observes that $(\theta_B \otimes_{(\mathsf{R},\tau)} \theta_D)(b \otimes d) = \delta_b \otimes \delta_d$, which ensures that $(b \otimes d)_{(b,d)\in B\times D}$ is a topological basis of $(M,\sigma) \otimes_{(\mathsf{R},\tau)} (N,\gamma)$. □

97 Corollary *Let (M,σ) and (N,γ) be topologically-free (R,τ)-modules. Then, $\{\, u \otimes v\colon u \in M,\ v \in N \,\}$ spans a dense subspace in $(M,\sigma) \otimes_{(\mathsf{R},\tau)} (N,\gamma)$.*

Proof: Let B, D be topological bases of M, N respectively. By Proposition 96, $\{\, b\otimes d\colon b \in B,\ d \in D \,\}$ is a topological basis of $(M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma)$. According to Lemma 53, $\langle\{\, b\otimes d\colon b \in B,\ d \in D \,\}\rangle$ is dense in $(M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma)$. Since $\{\, b \otimes d\colon b \in B,\ d \in D \,\} \subseteq \{\, u \otimes v\colon u \in M,\ v \in N \,\}$ it follows even more that $\langle\{\, u \otimes v\colon u \in M,\ v \in N \,\}\rangle$ so is dense in $(M,\sigma) \otimes_{(\mathsf{R},\tau)} (N,\gamma)$. □

5.2.3 Application

Let (M_i,σ_i) and (N_i,γ_i), $i = 1,2$, be topologically-free (R,τ)-modules. Let $(M_i,\sigma_i) \xrightarrow{f_i} (N_i,\gamma_i)$, $i = 1,2$, be continuous (R,τ)-linear maps.

Let $(u,v) \in M_1 \times M_2$, and $\ell_i \in (N_i,\gamma_i)'$, $i = 1,2$. By Eq. (28), p. 43, $((f_1\otimes_{(\mathsf{R},\tau)}f_2)(u\otimes v))(\ell_1\otimes\ell_2) = (u\otimes v)((\ell_1\circ f_1)\otimes(\ell_2\circ f_2)) = \ell_1(f_1(u))\ell_2(f_2(v)) = (f_1(u) \otimes f_2(v))(\ell_1 \otimes \ell_2)$. Whence

$$(f_1 \otimes_{(\mathsf{R},\tau)} f_2)(u \otimes v) = f_1(u) \otimes f_2(v). \tag{32}$$

Let us assume that B is a topological basis of (M_1,σ_1) and D is a topological basis of (M_2,σ_2). Given $t \in (M_1,\sigma_1) \otimes_{(\mathsf{R},\tau)} (M_2,\sigma_2)$, by Proposition 96,

$t = \sum_{(b,d)\in B\times D}(b\otimes d)'(t)b\otimes d$ (sum of a summable family) uniquely, and thus using Eq. (32),

$$\begin{aligned}(f_1 \otimes_{(\mathsf{R},\tau)} f_2)(t) &= (f_1 \otimes_{(\mathsf{R},\tau)} f_2)\left(\sum_{(b,d)\in B\times D}(b\otimes d)'(t)b\otimes d\right) \\ &= \sum_{(b,d)\in B\times D}(b\otimes d)'(t)f_1(b)\otimes f_2(b).\end{aligned} \tag{33}$$

In particular, if $t = u\otimes v$, $u\in M_1$, $v\in M_2$, then in view of Remark 95,

$$f_1(u)\otimes f_2(v) = (f_1\otimes_{(\mathsf{R},\tau)} f_2)(u\otimes v) = \sum_{(b,d)\in B\times D} b'(u)d'(v)f_1(b)\otimes f_2(d). \tag{34}$$

5.3 Monoidality of $\otimes_{(\mathsf{R},\tau)}$ and its consequences

5.3.1 Monoidality

Most of the proofs of the results of this section are postponed to Appendix C (p. 98) because they mainly consist in rather tedious, but simple, inspections of commutativity of some diagrams. Moreover one freely uses notations and freely applies results about monoidal categories joined together in Appendix B (p. 89). The reader familiar with standard notions from this topic (monoidal categories and functors, and monoids in monoidal categories) may skip the specific appendices, unless a thorough understanding is wished.

98 Proposition *Let* (R,τ) *be a rigid ring.*

$$\mathbb{TopFreeMod}_{(\mathsf{R},\tau)} := (\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}, \otimes_{(\mathsf{R},\tau)}, (R,\tau))$$

is a symmetric monoidal category.

Proof: See Appendix C.1 (p. 98). □

99 Corollary *For each field* $(\Bbbk,\tau)$ *with a ring topology,*

$$\mathbb{TopFreeVect}_{(\Bbbk,\tau)} := (\mathbf{TopFreeVect}_{(\Bbbk,\tau)}, \otimes_{(\Bbbk,\tau)}, (\Bbbk,\tau))$$

is a symmetric monoidal category.

100 Example *Let* (R,τ) *be a rigid ring. Let* X *be a set. Let us define a commutative monoid* $M_{(\mathsf{R},\tau)}(X) := ((R,\tau)^X, \mu_X, \eta_X)$ *in* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *by* $\mu_X(f) := \sum_{x\in X}(\delta_x^{\mathsf{R}}\otimes\delta_x^{\mathsf{R}})'(f)\delta_x^{\mathsf{R}}$, $f\in(R,\tau)^X\otimes_{(\mathsf{R},\tau)}(R,\tau)^X$, *and* $\eta_X(1_{\mathsf{R}}) =$

$\sum_{x\in X}\delta^{\mathsf{R}}_X$. *(Under the isomorphism* $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^X \simeq (R,\tau)^{X\times X}$ *from Remark 91 (p. 43), one has* $\mu_X(f) = \sum_{x\in X} f(x,x)\delta^{\mathsf{R}}_X$*, for* $f \in R^{X\times X}$*, and thus given* $f,g \in R^X$*,* $\mu_X(f\circledast g) = \sum_{x\in X} f(x)g(x)\delta^{\mathsf{R}}_x$*.)*

101 Remark *The construction of* $M_{(\mathsf{R},\tau)}(X)$ *for a rigid ring* (R,τ)*, provided in Example 100, is the object component of a functor* $\mathbf{Set}^{\mathsf{op}} \xrightarrow{M_{(\mathsf{R},\tau)}} {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$*.*

Indeed, given a map $Y \xrightarrow{f} X$*,* $g \in R^X \mapsto g\circ f \in R^Y$ *is a monoid morphism from* $M_{(\mathsf{R},\tau)}(X)$ *to* $M_{(\mathsf{R},\tau)}(Y)$ *since for each* $g,h \in R^X$*,* $\mu_Y((g\circ f)\circledast(h\circ f)) = \sum_{y\in Y} g(f(y))h(f(y))\delta_y = \mu_X(g\circledast h)\circ f$*, and* $\eta_X(1_{\mathsf{R}})\circ f = \eta_Y(1_{\mathsf{R}})$*.*

Let (R,τ) be a rigid ring. For each free R-modules M,N, let us define $\Phi_{M,N} := (M^*, w^*_{(\mathsf{R},\tau)}) \circledast_{(\mathsf{R},\tau)} (N^*, w^*_{(\mathsf{R},\tau)}) = Alg_{(\mathsf{R},\tau)}((M^*, w^*_{(\mathsf{R},\tau)})' \otimes_{\mathsf{R}} (N^*, w^*_{(\mathsf{R},\tau)})') \xrightarrow{(\Lambda_M\otimes_{\mathsf{R}}\Lambda_N)^*} ((M\otimes_{\mathsf{R}} N)^*, w^*_{(\mathsf{R},\tau)})$. According to Corollary 72 (p. 33), $\Phi_{M,N}$ is an isomorphism in $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$. Naturality in M,N is clear, so this provides a natural isomorphism

$$\begin{array}{l}\Phi\colon Alg_{(\mathsf{R},\tau)}(-)\circledast_{(\mathsf{R},\tau)} Alg_{(\mathsf{R},\tau)}(-) \Rightarrow Alg_{(\mathsf{R},\tau)}(-\otimes_{\mathsf{R}} -)\colon \\ \mathbf{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \times \mathbf{FreeMod}^{\mathsf{op}}_{\mathsf{R}} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}.\end{array} \tag{35}$$

Furthermore, let us consider the isomorphism $(R,\tau) \xrightarrow{\phi} (R^*, w^*_{(\mathsf{R},\tau)})$ given by $\phi(1_{\mathsf{R}}) := id_R$, with inverse $\phi^{-1}(\ell) = \ell(1_{\mathsf{R}})$.

Let $(M,\sigma),(N,\gamma)$ be topologically-free (R,τ)-modules. One defines the map $\Psi_{(M,\sigma),(N,\gamma)} := (M,\sigma)'\otimes_{\mathsf{R}} (N,\gamma)' \xrightarrow{\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}} (Alg_{(\mathsf{R},\tau)}((M,\sigma)'\otimes_{\mathsf{R}} (N,\gamma)'))' = ((M,\sigma)\circledast_{(\mathsf{R},\tau)} (N,\gamma))'$. Still by Corollary 72, this gives rise to a natural isomorphism

$$\begin{array}{l}\Psi\colon Top_{(\mathsf{R},\tau)}(-)\otimes_{\mathsf{R}} Top_{(\mathsf{R},\tau)}(-) \Rightarrow Top_{(\mathsf{R},\tau)}(-\circledast_{(\mathsf{R},\tau)} -)\colon \\ \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)} \to \mathbf{FreeMod}_{\mathsf{R}}.\end{array} \tag{36}$$

Let also $R \xrightarrow{\psi} (R,\tau)'$ be given by $\psi(1_{\mathsf{R}}) = id_R$ and $\psi^{-1}(\ell) = \ell(1_{\mathsf{R}})$.

102 Theorem *Let* (R,τ) *be a rigid ring.*

1. $\mathbb{Alg}_{(\mathsf{R},\tau)} = (Alg_{(\mathsf{R},\tau)}, \Phi, \phi)\colon \mathbb{FreeMod}^{\mathbb{op}}_{\mathsf{R}} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ *is a strong symmetric monoidal functor.*

2. $\mathbb{Top}_{(\mathsf{R},\tau)} = (Top_{(\mathsf{R},\tau)}, \Psi, \psi)$ *is a strong symmetric monoidal functor from* $\mathbb{TopFreeMod}^{\mathbb{op}}_{(\mathsf{R},\tau)}$ *to* $\mathbb{FreeMod}_{\mathsf{R}}$*, so the opposite* $(\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}}$ *(Remark 178, p. 95, Appendix B.3) of its dual omonoidal functor* $\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}$

(see 179, Appendix B.3) is a strong symmetric monoidal functor from $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ *to* $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}^{\mathbb{op}}$.

3. $\Lambda^{\mathsf{op}}\colon (\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}} \circ \mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)} \Rightarrow \mathbb{id}\colon \mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}^{\mathbb{op}} \to \mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}^{\mathbb{op}}$ *is a monoidal isomorphism. (See Facts 172, p. 88, in Appendix A, for the definition of the opposite of a natural transformation.)*

4. $\Gamma\colon \mathbb{id} \Rightarrow \mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)} \circ (\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}}\colon \mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)} \to \mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ *is a monoidal isomorphism.*

In particular, $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}^{\mathbb{op}}$ *and* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ *are monoidally equivalent.*

Proof: See Appendix C.2, p. 103. □

103 Corollary *For each field* $(\Bbbk,\tau)$ *with a ring topology, the monoidal categories* $\mathbb{V}\mathrm{ect}_{\Bbbk}^{\mathbb{op}}$ *and* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\tau)}$ *are monoidally equivalent.*

104 Corollary *For each rigid ring* (R,τ)*, the induced natural transformations (see Remark 181 (p. 96) in Appendix B.3)*

- $\widetilde{\Lambda^{\mathsf{op}}}\colon (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}} \Rightarrow id_{{}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}}\colon {}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}} \to {}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}$,
- $\widetilde{\Lambda^{\mathsf{op}}}\colon (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}} \circ \widetilde{\mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}} \Rightarrow id_{{}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}}\colon {}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}} \to {}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}$,
- $\tilde{\Gamma}\colon id_{\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})} \Rightarrow \widetilde{\mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}\colon$ $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \to \mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$,
- $\tilde{\Gamma}\colon id_{{}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})} \Rightarrow \widetilde{\mathbb{A}\mathrm{lg}_{(\mathsf{R},\tau)}} \circ (\widetilde{\mathbb{T}\mathrm{op}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}\colon$ ${}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \to {}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$.

are all natural isomorphisms.

Thus ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}$ *(resp.,* ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}}$*) and* $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$ *(resp.,* ${}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$*) are equivalent for each rigid topology* τ *on* R.

In particular, $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \simeq \mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\sigma)})$ *(resp.,* ${}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \simeq {}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\sigma)})$*), for each rigid topologies* τ,σ *on* R.

Proof: Follows from Theorem 102 together with Remarks 181 (p. 96) and 182 (p. 97) in Appendix B.3 □

105 Example *Let us make explicit the domain and codomain of the natural isomorphism $\tilde{\Gamma}$ from Corollary 104.*

Let $((M,\sigma),m,e)$ be an object of $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$. *Let its* topological dual coalgebra *be*

$$\begin{aligned}((M,\sigma)',\delta,\epsilon) &:= (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}((M,\sigma),m,e)\\ &= ((M,\sigma)',\Lambda^{-1}_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'}\circ m',\psi^{-1}\circ e').\end{aligned} \tag{37}$$

In details, $\epsilon(\ell)=\psi^{-1}(e'(\ell))=\psi^{-1}(\ell\circ e)=\ell(e(1_{\mathsf{R}}))$, $\ell\in(M,\sigma)'$. Let again $\ell\in(M,\sigma)'$. Then, $\delta(\ell)=((\Lambda^{-1}_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'}\circ m')(\ell))=\sum_{i=1}^n \ell_i\otimes r_i$ for some $\ell_i,r_i\in(M,\sigma)'$. Thus, $\ell\circ m=\sum_{i=1}^n\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'}(\ell_i\otimes r_i)$. Whence given $u,v\in M$, $\ell(m(u\circledast v))=\sum_{i=1}^n\ell_i(u)r_i(v)$.

Now, $(((M,\sigma)')^*,M,E):=\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}((M,\sigma)',\delta,\epsilon)$ is given by $M:=\delta^*\circ(\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*$ and $E:=\epsilon^*\circ\phi$. Thus $E(1_{\mathsf{R}})=\epsilon^*(\phi(1_{\mathsf{R}}))=\epsilon^*(id_R)=\epsilon$, and given $L_1,L_2\in((M,\sigma)')^*$, and $\ell\in(M,\sigma)'$,

$$\begin{aligned}(M(L_1\circledast L_2))(\ell) &= (\delta^*((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*(L_1\circledast L_2)))(\ell)\\ &= ((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})^*(L_1\circledast L_2))(\delta(\ell))\\ &= (L_1\circledast L_2)((\Lambda_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda_{(M,\sigma)'})(\delta(\ell)))\\ &= (L_1\circledast L_2)(\textstyle\sum_{i=1}^n\Lambda_{(M,\sigma)'}(\ell_i)\otimes\Lambda_{(M,\sigma)'}(r_i))\\ &= \textstyle\sum_{i=1}^n\Lambda_{(M,\sigma)'}(\ell_i)(L_1)\Lambda_{(M,\sigma)'}(r_i)(L_2)\\ &= \textstyle\sum_{i=1}^n L_1(\ell_i)L_2(r_i).\end{aligned} \tag{38}$$

Finally, $\tilde{\Gamma}_{((M,\sigma),m,e)}=\Gamma_{(M,\sigma)}\colon((M,\sigma),m,e)\simeq(((M,\sigma)')^*,M,E)$ is a monoid isomorphism.

106 Corollary *The equivalence from Corollary 104 restricts to an equivalence between the category ${}_{\epsilon}\mathbf{FinDimCoalg}_{\Bbbk}$ (resp. ${}_{\epsilon,coc}\mathbf{FinDimCoalg}_{\Bbbk}$) of finite-dimensional (resp. cocommutative) coalgebras and* $\mathbf{Mon}(\mathbb{FinDimVect}_{\Bbbk})$ *(resp., ${}_{c}\mathbf{Mon}(\mathbb{FinDimVect}_{\Bbbk})$), where $\mathbb{FinDimVect}_{\Bbbk}=(\mathbf{FinDimVect}_{\Bbbk},\otimes_{\Bbbk},\Bbbk)$ (see Section 4.4.2).*

5.3.2 Relationship with finite duality

As already mentioned at the beginning of the current section, there is a standard and well-known notion of duality between algebras and coalgebras, known as the finite duality and briefly described in Appendix D.4 (p. 116). It is our intention to understand the relations if any, between the equivalence of categories from Corollary 104 and the finite duality.

In a few words, finite duality is implemented, over a field $\Bbbk$, by an adjunction between monoids in $\mathbb{Vect}_{\Bbbk}$ (i.e., up to a concrete isomorphism (see

Example 173, p. 92 in Appendix B.2), the ordinary algebras) and (the opposite of the category of) comonoids still in $\mathbb{V}\mathrm{ect}_{\Bbbk}$, i.e., coalgebras (see Appendix D, p. 111). So to make the comparison between the two approaches of duality possible, one first defines an underlying (ordinary) algebra of any monoid in $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\tau)}$, and this is sufficient to handle one half of the finite duality, namely the (contravariant) functorial relation from coalgebras to algebras since it is purely algebraic. Regarding the other direction, namely from algebras to coalgebras, topological aspects have to be taken into consideration and the relations between finite duality and our own duality are plainly described over a discrete field.

5.3.2.1 The underlying algebra

Let (R,τ) be a rigid ring. Let $(M,\sigma),(N,\gamma)$ be topologically-free (R,τ)-modules. According to Lemma 94 (p. 44), $M\times N\xrightarrow{-\circledast-}(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$ is R-bilinear. Denoting by $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}\xrightarrow{\|-\|}\mathbf{Mod}_{\mathsf{R}}$ the canonical forgetful functor, this means that there is a unique R-linear map $\|(M,\sigma)\|\otimes_{\mathsf{R}}\|(N,\gamma)\|\xrightarrow{\Xi_{(M,\sigma),(N,\gamma)}}\|(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)\|$ such that for each $u\in M$, $v\in N$, $\Xi_{(M,\sigma),(N,\gamma)}(u\otimes v)=u\circledast v$.

107 Lemma $\mathbb{A}:=(\|-\|,(\Xi_{(M,\sigma),(N,\gamma)})_{(M,\sigma),(N,\gamma)},id_R)$ *is a lax symmetric monoidal functor from* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ *to* $\mathbb{M}\mathrm{od}_{\mathsf{R}}$.

Proof: See Appendix C.3 (p. 109). □

Let $\tilde{\mathbb{A}}:\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})\to\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$ be the functor induced by $\mathbb{A}$ as introduced in Appendix B.3, p. 93. Using the functorial isomorphism $O:\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})\simeq{}_1\mathbf{Alg}_{\mathsf{R}}$ (Example 173, p. 92, Appendix B.2), to any monoid in $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ is associated an ordinary algebra.

108 Definition *Let us define* $UA:=\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})\xrightarrow{O\circ\tilde{\mathbb{A}}}{}_1\mathbf{Alg}_{\mathsf{R}}$. *Given a monoid* $((M,\sigma),\mu,\eta)$ *in* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$, *then* $UA((M,\sigma),\mu,\eta)=O(\tilde{\mathbb{A}}((M,\sigma),\mu,\eta))$ *is referred to as the* underlying (ordinary) algebra *of the monoid* $((M,\sigma),\mu,\eta)$.

In details, let $((A,\sigma),\mu,\eta)$ be an object of $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$. Then, $UA((A,\sigma),\mu,\eta)=(A,\mu_{bil},\eta(1_{\mathsf{R}}))$ with $\mu_{bil}:A\times A\to A$ given by $\mu_{bil}(u,v):=\mu(u\circledast v)$.

109 Remark *Since by Lemma 107, $\mathbb{A}$ is symmetric, one has an induced functor (see Remark 178, p. 95, Appendix B.3)* ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{\tilde{\mathbb{A}}} {}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}})$. *Because one has the co-restriction* ${}_c\mathbf{Mon}(\mathbb{M}\mathrm{od}_{\mathsf{R}}) \xrightarrow{O} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ *(Example 173, p. 92, in Appendix B.2), one may likewise consider the* underlying (ordinary) algebra *functor* $UA = {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{O\circ\tilde{\mathbb{A}}} {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$.

110 Example *(Continuation of Example 100, p. 47)* $UA(M_{(\mathsf{R},\tau)}(X)) = A_{\mathsf{R}}X$ *(see Section 2.4.2, p. 15).*

5.3.2.2 Relations with $\widetilde{\mathbb{D}_*}$

Let (R,τ) be a rigid ring. Let $\mathbf{FreeMod}_{\mathsf{R}} \xrightarrow{E} \mathbf{Mod}_{\mathsf{R}}$ be the canonical embedding functor. Since $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ is a symmetric monoidal subcategory of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ it follows that $\mathbb{E} = (E, id, id)$ is a strict monoidal functor from $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ to $\mathbb{M}\mathrm{od}_{\mathsf{R}}$ (see Appendix B.3, p. 93).

One claims that[12]

$$\mathbb{D}_* \circ \mathbb{E}^{\mathbb{op}} = \mathbb{A} \circ \mathbb{A}\mathbb{l}\mathbb{g}_{(\mathsf{R},\tau)}.$$

In particular, if $\Bbbk$ is a field (and τ is a ring topology on $\Bbbk$), then this reduces to $\mathbb{D}_* = \mathbb{A} \circ \mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}$.

To see this, using the notations from Appendix D.4 (p. 116), one has to check that $((-)^* \circ E^{\mathsf{op}}, \Theta, \theta) = (\|-\|, \Xi, id_R) \circ (Alg_{(\mathsf{R},\tau)}, \Phi, \phi) = (\|-\| \circ Alg_{(\mathsf{R},\tau)}, (\|\Phi_{M,N}\| \circ \Xi_{M^*,N^*})_{M,N}, \|\phi\|)$ (by the composition of monoidal functors recalled in Appendix B.3, p. 93).

1. That $\|-\| \circ Alg_{(\mathsf{R},\tau)} = (-)^* \circ E^{\mathsf{op}}$ is due to the very definition of $Alg_{(\mathsf{R},\tau)}$.
2. Of course, $\|\phi\| = \theta$.
3. It remains to check that for each free modules M, N, $\|(\Lambda_M \otimes \Lambda_N)^*\| \circ \Xi_{M^*,N^*} = \Theta_{M,N}$. Let $\ell_1 \in M^*$, $\ell_2 \in N^*$, $u \in M$ and $v \in N$. One has

$$\begin{array}{rl} & ((\Lambda_M \otimes_R \Lambda_N)^*(\Xi_{M^*,N^*}(\ell_1 \otimes \ell_2)))(u \otimes v) \\ = & (\ell_1 \circledast \ell_2)(\Lambda_M(u) \otimes \Lambda_N(v)) \\ = & (\Lambda_M(u))(\ell_1)(\Lambda_N(v))(\ell_2) \\ = & \ell_1(u)\ell_2(v) \\ = & (\Theta_{M,N}(\ell_1 \otimes \ell_2))(u \otimes v). \end{array} \tag{39}$$

It follows that the dual monoid[13] $\widetilde{\mathbb{D}_*}(\mathsf{C})$ of a $\Bbbk$-coalgebra C in finite duality,

[12] See Appendix D.4, p. 116, for the definition of the monoidal functor $\mathbb{D}_*$.
[13] In $\mathbb{V}\mathrm{ect}_{\Bbbk}$.

is equal to $\widetilde{\mathbb{A}}(\widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}}(\mathsf{C}))$ whatever the ring topology τ on the field $\Bbbk$, and thus as ordinary algebras, $O(\widetilde{\mathbb{D}_*}(\mathsf{C})) = UA(\widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}}(\mathsf{C}))$.

111 Proposition *Let $(\Bbbk,\tau)$ be a field with a ring topology. The functor $\mathbf{Mon}(\mathbb{FreeTopVect}_{(\Bbbk,\tau)}) \xrightarrow{O\circ\tilde{\mathbb{A}}} {}_1\mathbf{Alg}_\Bbbk$ has a left adjoint, namely $\widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}} \circ D_{fin}^{\mathsf{op}} \circ O^{-1}$, where $D_{fin}\colon \mathbf{Mon}(\mathbb{Vect}_\Bbbk)^{\mathsf{op}} \to {}_\epsilon\mathbf{Coalg}_\Bbbk$ is the finite dual functor (see Appendix D.4, p. 116).*

Proof: One has $\widetilde{\mathbb{D}_*} = \tilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}}$, whence $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}} = \tilde{\mathbb{A}} \circ \widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}} \simeq \tilde{\mathbb{A}}$ (natural isomorphism) by Theorem 102 (p. 48). Since $\widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}} \circ D_{fin}^{\mathsf{op}}$ is a left adjoint of $\widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}$ (Appendix D.4, p. 116), it follows from the lemma below that it is also the left adjoint of $\tilde{\mathbb{A}}$. □

Next lemma is rather easy so is provided without proof.

112 Lemma *Let F, G be two naturally isomorphic functors. If L is a left adjoint of F, then it is also a left adjoint of G.*

5.3.2.3 The underlying topological algebra

Let R be a ring. The underlying algebra functor $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}) \xrightarrow{UA} {}_1\mathbf{Alg}_\mathsf{R}$ "lifts" to a *topological algebra* functor $\mathbf{Mon}(\mathbb{TopFreeMod}_{(R,\mathsf{d})}) \xrightarrow{TA} {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$.

Indeed, let $((A,\sigma),\mu,\eta)$ be an object of $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$. One already knows that (A,σ) is an object of $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$ and $UA((A,\sigma),\mu,\eta) = (A,\mu_{bil},\eta(1_\mathsf{R}))$ is an object of ${}_1\mathbf{Alg}_\mathsf{R}$. So it suffices to check that $(A,\sigma)\times(A,\sigma) \xrightarrow{\mu_{bil}} (A,\sigma)$ is continuous. But this is clear since it is equal to the composition $(A,\sigma)\times(A,\sigma) \xrightarrow{-\circledast-} (A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(A,\sigma) \xrightarrow{\mu} (A,\sigma)$ of continuous maps (see Lemma 94, p. 44). Now, let $((A,\sigma),\mu,\eta) \xrightarrow{f} ((B,\gamma),\nu,\zeta)$ be a morphism in $\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})})$. In particular, $(A,\sigma) \xrightarrow{f} (B,\gamma)$ is

linear and continuous. Moreover, the following diagram commutes.

$$\begin{tikzcd}
(A,\sigma)\times(A,\sigma) \arrow[dr, "-\circledast-"'] \arrow[drr, bend left, "\mu_{bil}"] \arrow[ddd, "f\times f"'] & & \\
& (A,\sigma)\circledast_{(\mathsf{R},\mathsf{d})}(A,\sigma) \arrow[r, "\mu"] \arrow[d, "f\circledast_{(\mathsf{R},\mathsf{d})}f"'] & (A,\sigma) \arrow[d, "f"] \\
& (B,\gamma)\circledast_{(\mathsf{R},\mathsf{d})}(B,\gamma) \arrow[r, "\nu"] & (B,\gamma) \\
(B,\gamma)\times(B,\gamma) \arrow[ur, "-\circledast-"'] \arrow[urr, bend right, "\nu_{bil}"'] & &
\end{tikzcd} \tag{40}$$

Since by assumption, one also has $f \circ \eta = \zeta$, it follows that $f(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, and thus f is a continuous algebra map from $((A,\sigma),\mu_{bil},\eta(1_{\mathsf{R}}))$ to $((B,\nu_{bil},\zeta(1_{\mathsf{R}}))$.) One furthermore observes that TA is concrete over $\mathbf{TopMod}_{(\mathsf{R},\mathsf{d})}$, for the obvious forgetful functors. Therefore, TA is faithful. More precisely the following diagram commutes (the unnamed arrows are either the obvious forgetful functors or the evident embedding functor).

$$\begin{tikzcd}
& \mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}) \arrow[r, "TA"] \arrow[d] \arrow[dl, bend right, "\tilde{\mathbb{A}}"'] & {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})} \arrow[d] \arrow[ddl, bend left] \\
& \mathbf{TopFreeMod}_{(\mathsf{R},\mathsf{d})} \arrow[r, hook] \arrow[d, "\|\cdot\|"] & \mathbf{TopMod}_{(\mathsf{R},\mathsf{d})} \arrow[dl] \\
\mathbf{Mon}(\mathbb{Mod}_{\mathsf{R}}) \arrow[r] \arrow[rr, bend right, "O"'] & \mathbf{Mod}_{\mathsf{R}} & {}_1\mathbf{Alg}_{\mathsf{R}} \arrow[l]
\end{tikzcd} \tag{41}$$

113 Remark *When* A *is a commutative monoid in* $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$*, then* $TA(\mathsf{A})$ *is a commutative topological algebra.*

114 Example *(Continuation of Examples 100, p. 47, and 110, p. 52) For each set* X*,* $TA(M_{(\mathsf{R},\mathsf{d})}(X)) = A_{(\mathsf{R},\mathsf{d})}(X)$ *(see Section 2.4.2, p. 15).*

115 Lemma *TA is full.*

Proof: Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ and $\mathsf{B} = ((B,\gamma),\nu,\zeta)$ be two monoids in $\mathbb{TopFreeMod}_{(\mathsf{R},\mathsf{d})}$. Let $TA(\mathsf{A}) \xrightarrow{g} TA(\mathsf{B})$ be a morphism in ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$. In particular, $g \in {}_1\mathbf{Alg}_{\mathsf{R}}(UA(\mathsf{A}),UA(\mathsf{B})) \cap \mathbf{Top}((|A|,\sigma),(|B|,\gamma))$. (Recall

from Remark 12 (p. 14) that $\mathbf{Mod}_{\mathsf{R}} \xrightarrow{|\cdot|} \mathbf{Set}$ is the usual forgetful functor, and **Top** is the category of Hausdorff topological spaces.)

By assumption, for each $u, v \in A$, one has $g(\mu(u \otimes v)) = g(\mu_{bil}(u, v)) = \nu_{bil}(g(u), g(v)) = \nu(g(u) \otimes g(v))$. Thus, $g \circ \mu = \nu \circ (g \otimes_{(\mathsf{R},\mathsf{d})} g)$ on $\{ u \otimes v : u, v \in A \}$. Since this set spans a dense subset of $(A, \sigma) \otimes_{(R,\tau)} (A, \sigma)$ (according to Corollary 97, p. 46), by linearity and continuity, $g \circ \mu = \nu \circ (g \otimes_{(\mathsf{R},\mathsf{d})} g)$ on the whole of $(A, \sigma) \otimes_{(\mathsf{R},\mathsf{d})} (A, \sigma)$.

Moreover, $g(\eta(1_{\mathsf{R}})) = \zeta(1_{\mathsf{R}})$, then $g \circ \eta = \zeta$. Therefore, g may be seen as a morphism $\mathsf{A} \xrightarrow{f} \mathsf{B}$ in $\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})})$ with $TA(f) = g$. □

116 Lemma *TA is injective on objects.*

Proof: Let $\mathsf{A} = ((A, \sigma), \mu, \eta)$ and $\mathsf{B} = ((B, \gamma), \nu, \zeta)$ be two monoids in $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})}$ such that $TA(\mathsf{A}) = TA(\mathsf{B})$. In particular, $(A, \sigma) = (B, \gamma)$, and $\eta = \zeta$. By assumption $\mu_{bil} = \nu_{bil}$. Whence $\mu = \nu$ on $\{ u \otimes v : u \in A,\ v \in B \}$. Since this set is dense in $(A, \sigma) \otimes_{(\mathsf{R},\mathsf{d})} (B, \gamma)$ and μ, ν are continuous, they are equal on the whole tensor product $(A, \sigma) \otimes_{(\mathsf{R},\mathsf{d})} (B, \gamma)$. So $\mathsf{A} = \mathsf{B}$. □

As a consequence of Lemmas 115 and 116, $\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})}) \xrightarrow{TA} {}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ is a full embedding functor, because TA is faithful. Therefore, $\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})})$ is isomorphic to a full subcategory of ${}_1\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ ([2, Proposition 4.5, p. 49]). Accordingly a monoid in $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})}$ is essentially a topological algebra.

117 Remark *It is clear that the co-restriction ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\mathsf{d})}) \xrightarrow{TA} {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}$ of TA (see Remark 113) also is a full embedding functor.*

118 Remark *When (R, τ) is a rigid ring, not necessarily discrete, then the topology σ and the underlying algebra $UA(\mathsf{A}) = (A, \mu_{bil}, \eta(1_{\mathsf{R}}))$ of a monoid $\mathsf{A} = ((A, \sigma), \mu, \eta)$ in $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\tau)}$ still interact but not as nicely as in the discrete case because one cannot conclude to the joint continuity of $(A, \sigma) \times (A, \sigma) \xrightarrow{\mu_{bil}} (A, \sigma)$ but only to its separate continuity, by Lemma 94 (p. 44).*

However the correspondence $\mathsf{A} \mapsto ((A, \sigma), \mu_{bil}, \eta(1_{\mathsf{R}}))$ still is functorial, and still denoted TA by abuse of language, from $\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\mathsf{R},\tau)})$ to the category ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ of separately continuous algebras*, where for a topological ring (R, τ) (not necessarily rigid), ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ has objects $\mathsf{A} = ((A, \sigma), m_{\mathsf{A}}, 1_{\mathsf{A}})$ where $(A, m_{\mathsf{A}}, 1_{\mathsf{A}})$ is an ordinary unital R-algebra, (A, σ) is a topological (R, τ)-module and $(A, \sigma) \times (A, \sigma) \xrightarrow{m_{\mathsf{A}}} (A, \sigma)$ is separately continuous (the morphisms are the continuous unital algebra maps).*

Observe that if A *is a commutative monoid in* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$*, then* $TA(\mathsf{A})$ *is a commutative separately continuous algebra, i.e., an object of* ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *(see Appendix E, p. 118).*

It is clear that ${}_{1}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *is a full subcategory of* ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$*, and* $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{TA} {}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *remains faithful since concrete over* $\mathbf{TopMod}_{(\mathsf{R},\tau)}$*.*

Furthermore, the arguments given in the proofs of Lemmas 115 and 116 work as well when (R,d) *is replaced by an arbitrary rigid ring* (R,τ)*, and thus* $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{TA} {}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *is also faithful and injective on objects, in consequence of what* $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$ *is isomorphic to a full subcategory of* ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$*. Therefore a monoid in* $\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}$ *practically is a separately continuous algebra.*

The co-restriction ${}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \xrightarrow{TA} {}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *of* TA *has the corresponding properties.*

119 Example *(Continuation of Examples 100 (p. 47), 110 (p. 52), 114 (p. 54)) Let* (R,τ) *be rigid. Then,* $TA(M_{(\mathsf{R},\tau)}(X)) = A_{(\mathsf{R},\tau)}(X)$*. One also observes that in this particular case* $A_{(\mathsf{R},\tau)}(X)$ *is even an object of* ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *according to Lemma 15 (p. 15).*

120 Remark *Having in mind Example 119, Remark 101 (p. 48) may be completed by observing that the following diagram commutes.*

$$\begin{tikzcd} \mathbf{Set}^{\mathsf{op}} \arrow[r, "M_{(\mathsf{R},\tau)}"] \arrow[d, "A_{(\mathsf{R},\tau)}"'] & {}_{c}\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \arrow[d, "TA"] \\ {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[r, hook] & {}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \end{tikzcd} \tag{42}$$

5.3.2.4 Relations with D_{fin}

Finite duality between coalgebras and algebras (see Appendix D.4, p. 116) has some fundamental topological features that one now briefly discusses in order that afterwards relations with our topological duality can be brought.

More precisely, let V be any vector space on a field $\Bbbk$. Then, V^* has a somewhat natural topology called the *V-topology* ([1]) or the *finite topology* ([7]), with a fundamental system of neighborhoods of zero consisting of spaces

$$W^{\dagger} := \{\, \ell \in V^* \colon \forall w \in W,\ \ell(w) = 0 \,\} \tag{43}$$

where W runs over the finite-dimensional subspaces of V. This is definitively the same topology as our $w^*_{(\Bbbk,\mathsf{d})}$ (see Section 2.5.1, p. 16). Accordingly this turns V^* into a linearly compact $\Bbbk$-vector space (see Section 4.4.3, p. 37).

The closed subspace of $(V^*, w^*_{(\Bbbk,\mathsf{d})})$ are exactly the subspaces of the form $W^\dagger$, where W is any subspace of V ([4, Proposition 24.4, p. 105]).

121 Lemma *Let W be a subspace of V.* $\operatorname{codim}(W^\dagger)$ *is finite if, and only if,* $\dim(W)$ *is finite. In this case,* $\operatorname{codim}(W^\dagger) = \dim(W)$.

Proof: One observes that $V^*/W^\dagger \simeq W^*$. Indeed, let $W \xrightarrow{incl_W} V$ be the canonical inclusion. The map $V^* \xrightarrow{incl^*_W} W^*$ is onto since given $\ell_0 \in W^*$, there exists $\ell \in V^*$ with $(incl^*_W(\ell))(w) = \ell(incl_W(w)) = \ell(w) = \ell_0(w)$, $w \in W$. (It suffices to consider a $\Bbbk$-linear basis of W, and then an extension to a basis of the whole V.) Therefore, $V^*/\ker incl^*_W \simeq W^*$. But $\ker incl^*_W = W^\dagger$. Finally, since $V^*/W^\dagger \simeq W^*$, it follows that $\operatorname{codim}(W^\dagger) = \dim W^*$. In particular, $\operatorname{codim}(W^\dagger)$ is finite if, and only if, $\dim(W)$ is finite (since for finite-dimensional spaces W, $\dim(W) = \dim(W^*)$), and thus in this case $\operatorname{codim}(W^\dagger) = \dim(W)$. □

122 Remark *Given a $\Bbbk$-coalgebra $\mathsf{C} = (C, \delta_{\mathsf{C}}, \epsilon_{\mathsf{C}})$, there is a nice connection between its subcoalgebras and the two-sided ideals of $\widetilde{\mathbb{D}_*}(\mathsf{C})$ since D is a subcoalgebra of C if, and only if, $D^\dagger$ is a two-sided ideal of $\widetilde{\mathbb{D}_*}(\mathsf{C})$ ([1, Theorem 2.3.1, p. 78]).*

123 Theorem *Let $\Bbbk$ be a field. For each monoid A in $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\mathsf{d})}$, $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. Furthermore, the assertions below are equivalent.*

1. *In $TA(\mathsf{A})$ every finite-codimensional ideal is closed.*
2. *$\tilde{\mathbb{A}}(A)$ is reflexive*[14].
3. *The coalgebra $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is coreflexive.*
4. $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$.

Proof: Let $\mathsf{A} = ((A, \sigma), \mu, \eta)$ be a monoid in $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{(\Bbbk,\mathsf{d})}$. Whence its underlying topological vector space is a linearly compact vector space

[14] See Remark 202, p. 117, in Appendix D.4.

(see Section 4.4.3, p. 37). Let $\mathsf{C} := (\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$. Since $\tilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\mathsf{d})}} = \widetilde{\mathbb{D}_*}$ it follows that $\tilde{\mathbb{A}} \simeq \tilde{\mathbb{A}} \circ \widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\mathsf{d})}} \circ (\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}} \simeq \widetilde{\mathbb{D}_*} \circ (\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}$ (naturally isomorphic). In particular, $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(\mathsf{C})$. By construction, the underlying topological vector space of A, namely (A, σ), is also the underlying topological vector space of $TA(\mathsf{A})$. Also $\mathsf{A}, TA(\mathsf{A})$ and $\tilde{\mathbb{A}}(\mathsf{A})$ share the same underlying vector space A, which is isomorphic to C^*, where C is the underlying vector space of the coalgebra C. Of course, $(A, \sigma) \simeq Alg_{(\Bbbk,\mathsf{d})}(C) = (C^*, w^*_{(\Bbbk,\mathsf{d})})$. Therefore, up to such an isomorphism, (A, σ) has a fundamental system of neighborhoods of zero consisting of $V^\dagger = \{\, \ell \in A \colon \forall v \in V,\ \ell(v) = 0 \,\}$ where V is a finite-dimensional subspace of C (see the paragraph prior to the statement of this theorem).

Let $\ell \in (A, \sigma)'$, i.e., $(A, \sigma) \xrightarrow{\ell} (\Bbbk, \mathsf{d})$ is a continuous linear map. By continuity of ℓ, there exists a finite-dimensional subspace V of C such that $V^\dagger \subseteq \ker \ell$. Let B be a (finite) basis of V, and let D be the (necessarily finite-dimensional, by Corollary 192, p. 114, in Appendix D.3) subcoalgebra of C it generates. Then, $V \subseteq D$, which implies that $D^\dagger \subseteq V^\dagger \subseteq \ker \ell$. But $D^\dagger$ is a finite-codimensional ideal of $\tilde{\mathbb{A}}(\mathsf{A})$ (by Lemma 121 and Remark 122), whence $\ell \in A^0$, where A^0 denotes, similarly to Eq. (118), p. 116 (Appendix D.4), the underlying vector space of the finite dual coalgebra $D_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. Therefore, $(A, \sigma)' \subseteq A^0$.

Now let us check that the above inclusion $incl_{(A,\sigma)'}$ is a coalgebra map from $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ to $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, which is equivalent to $(A, \sigma)'$ being a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. One thus needs to make explicit the two coalgebra structures.

By construction the comultiplication of $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is given by the composition $\Lambda^{-1}_{(A,\sigma)' \otimes_\Bbbk (A,\sigma)'} \circ \mu'$. So for $\ell \in (A, \sigma)'$, $(\Lambda^{-1}_{(A,\sigma)' \otimes_\Bbbk (A,\sigma)'} \circ \mu')(\ell) = \sum_{i=1}^n \ell_i \otimes r_i$, for some $\ell_i, r_i \in (A, \sigma)'$. Therefore, given $\ell \in (A, \sigma)'$, $u, v \in A$,

$$\begin{array}{rcl} \ell(\mu(u \circledast v)) & = & (\mu'(\ell))(u \circledast v) \\ & = & (\Lambda_{(A,\sigma)' \otimes_\Bbbk (A,\sigma)'}(\sum_{i=1}^n \ell_i \otimes r_i))(u \circledast v) \\ & = & (u \circledast v)(\sum_{i=1}^n \ell_i \otimes r_i) \\ & = & \sum_{i=1}^n (u \circledast v)(\ell_i \otimes r_i) \\ & = & \sum_{i=1}^n \ell_i(u) r_i(v). \end{array} \tag{44}$$

The counit of $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is $(A, \sigma)' \xrightarrow{\eta'} (\Bbbk, \mathsf{d})' \xrightarrow{\psi^{-1}} \Bbbk$, i.e., $\ell \mapsto \ell(\eta(1_\Bbbk))$.

Let $\mathsf{B} = (B, \mu, \eta)$ be a monoid in $\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_\Bbbk$. Appendix D.4 (p. 116) tells us that the comultiplication $\Delta \colon B^0 \to B^0 \otimes_\Bbbk B^0$ of $D_{fin}(\mathsf{B})$ is equal to $(\Theta^0)^{-1} \circ \mu^0$ (where $\mu^0 \colon B^0 \to (B \otimes_\Bbbk B)^0$ is the co-restriction of μ^*, and

$\Theta^0\colon B^0 \otimes_{\Bbbk} B^0 \simeq (B \otimes_{\Bbbk} B)^0$ is the isomorphism obtained by co-restriction of the canonical injection $\Theta_{B,B}\colon B^* \otimes_{\Bbbk} B^* \hookrightarrow (B \otimes_{\Bbbk} B)^*$), and its counit $\epsilon\colon B^0 \to \Bbbk$ is $\epsilon(\ell) = \ell(\eta(1_{\Bbbk}))$. Now let $\mathsf{B} := \tilde{\mathbb{A}}(A) = (A, \mu \circ \Xi_{A,A}, \eta)$. Then, $\Delta(\ell) = (\Theta^0)^{-1}((\mu \circ \Xi)^0(\ell)) = \sum_{i=1}^n \ell_i \otimes r_i$ for some $\ell_i, r_i \in A^0$. Thus, $\ell \circ (\mu \circ \Xi_{A,A}) = (\mu \circ \Xi)^0(\ell) = \sum_{i=1}^n \Theta^0(\ell_i \otimes r_i)$, and thus for $u, v \in A$, $\ell(\mu(u \circledast v)) = \ell(\mu(\Xi_{A,A}(u \otimes v))) = \sum_{i=1}^n \ell_i(u) r_i(v)$. Whence the comultiplication from Eq. (44) coincides with this one for $\ell \in (A,\sigma)'$. Since it is furthermore patent that the counit of $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is the restriction of that of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, it follows that $(A,\sigma)'$ is a subcoalgebra of $D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$.

It remains to prove the equivalence of the four assertions given in the statement.

That 2⇔3 follows from Remark 202 (p. 117), Appendix D.4 and the fact that $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}((\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A}))$.

The coalgebra $\mathsf{C} := (\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ is coreflexive if, and only if, every finite-codimensional ideal of $\widetilde{\mathbb{D}_*}(\mathsf{C}) \simeq \tilde{\mathbb{A}}(\mathsf{A})$ is closed in the finite topology of C^* ([1, Lemma 2.2.15, p. 76]), which coincides with our topology $w^*_{(\Bbbk,\mathsf{d})}$, and thus it turns out that $(\widetilde{\mathbb{D}_*}(\mathsf{C}), w^*_{(\Bbbk,\mathsf{d})}) \simeq TA(\mathsf{A})$ (since C^* under the finite topology is equal to $Alg_{(\Bbbk,\mathsf{d})}(C) \simeq (A,\sigma)$ by functoriality). Whence 3⇔1.

Let us assume that in $TA(\mathsf{A})$ every finite-codimensional ideal is closed. Let $\ell \in D_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$. By definition $\ker \ell$ contains a finite-codimensional ideal say I of $\tilde{\mathbb{A}}(\mathsf{A})$. Since I is closed, there exists a finite-dimensional subspace D of C such that $D^\dagger = I$ (since the closed subspaces are of the form $D^\dagger$ for a subspace D of C and by Lemma 121, $\operatorname{codim}(I) = \operatorname{codim}(D^\dagger) = \dim(D)$), whence I is open, which shows that $(A,\sigma) \xrightarrow{\ell} (\Bbbk, \mathsf{d})$ is continuous, i.e., $\ell \in (A,\sigma)'$, because $\ker \ell = \ell^{-1}(\{0\})$ and $\{0\}$ is open in $(\Bbbk, \mathsf{d})$. Whence 1⇒4.

Let $\mathsf{C} := (\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, so that $\tilde{\mathbb{A}}(\mathsf{A}) \simeq \widetilde{\mathbb{D}_*}(\mathsf{C})$, as above. Whence $\mathsf{C} = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A})) \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(\mathsf{C}))$. This is not sufficient to ensure coreflexivity of C, since there is at this stage no guaranty that the above isomorphism corresponds to the unit of the adjunction given in Theorem 201 (p. 117, in Appendix D.4) namely $A^0 = (A,\sigma)' = C \hookrightarrow (C^*)^0 = (((A,\sigma)')^*)^0 = ((A^0)^*)^0$, $u \mapsto (\ell \mapsto \ell(u))$.

One knows from the beginning of the proof that

$$\tilde{\mathbb{A}}(\tilde{\Gamma}_{\mathsf{A}})\colon \mathbb{A}(\mathsf{A}) \simeq \tilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\mathsf{d})}}((\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})))$$

which, in this case where $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A}) = D^{\mathsf{op}}_{fin}(\tilde{\mathbb{A}}(\mathsf{A}))$, is the isomorphism $\|\Gamma_{(A,\sigma)}\|\colon A \simeq ((A,\sigma)')^* = (A^0)^*$, $u \mapsto (\ell \mapsto \ell(u))$ (refer to Remark 181,

p. 96, in Appendix B.3 for the definition of the induced natural transformation $\tilde{\Gamma}$). This shows at once that $\tilde{\mathbb{A}}(\mathsf{A})$ is reflexive (Remark 202, p. 117, Appendix D.4), and thus its finite dual coalgebra, namely C is coreflexive (again by Remark 202 since the adjunction from Theorem 201 reduces to an equivalence between coreflexive coalgebras and reflexive algebras). Thus item 4⇒3. □

124 Example *Let* (R,τ) *be rigid. Let us use Example 105 (p. 50) to prove that* $(\widetilde{\mathbb{TOP}^{\mathsf{d}}_{(\mathsf{R},\tau)}})^{\mathbb{OP}}(M_{(\mathsf{R},\tau)}(X)) \simeq C_{\mathsf{R}}X$*, i.e., that the topological dual coalgebra of* $M_{(\mathsf{R},\tau)}(X)$ *is the group-like coalgebra* $C_{\mathsf{R}}X$ *on* X *(Appendix D.2, p. 112).*

Let us check that the following diagram commutes.

$$\begin{tikzcd}
((R,\tau)^X)' \arrow[rr, "\mu'_X"] \arrow[d, "\lambda_X"'] & & ((R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^X)' \\
R^{(X)} \arrow[r, "d_X"'] & R^{(X)} \otimes_{\mathsf{R}} R^{(X)} \arrow[r, "\lambda_X^{-1}\otimes_{\mathsf{R}}\lambda_X^{-1}"'] & ((R,\tau)^X)' \otimes_{\mathsf{R}} ((R,\tau)^X)' \arrow[u, "\Lambda_{((R,\tau)^X)'\otimes_{\mathsf{R}}((R,\tau)^X)'}"']
\end{tikzcd} \tag{45}$$

Let $\ell \in ((R,\tau)^X)'$*,* $x, y \in X$*. Then,*

$$\begin{array}{rl}
 & (\Lambda_{((R,\tau)^X)'\otimes_{\mathsf{R}}((R,\tau)^X)'}((\lambda_X^{-1}\otimes_{\mathsf{R}}\lambda_X^{-1})(d_X(\lambda_X(\ell)))))(\delta_x \circledast \delta_y) \\
= & (\Lambda_{((R,\tau)^X)'\otimes_{\mathsf{R}}((R,\tau)^X)'}((\lambda_X^{-1}\otimes_{\mathsf{R}}\lambda_X^{-1})(\sum_{z\in Z}\ell(\delta_z)\delta_z\otimes\delta_z)))(\delta_x \circledast \delta_y) \\
= & (\delta_x \circledast \delta_y)(\sum_{z\in Z}\ell(\delta_z)\pi_z\otimes\pi_z) \\
= & \sum_{z\in Z}\ell(\delta_z)\pi_z(\delta_x)\pi_z(\delta_y) \\
= & \ell(\delta_x)\delta_x(y) \\
= & \ell(\mu_X(\delta_x\circledast\delta_y)) \\
= & (\mu'_X(\ell))(\delta_x\circledast\delta_y).
\end{array} \tag{46}$$

Moreover $\eta'_X(\ell) = \psi(e_X(\lambda_X(\ell)))$ *for each* $\ell \in ((R,\tau)^X)'$ *since* $(\eta'_X(\ell))(1_{\mathsf{R}}) = \ell(\eta_X(1_{\mathsf{R}})) = \sum_{x\in X}\ell(\delta_x)$ *and*

$$\begin{array}{rcl}
\psi(e_X(\lambda_X(\ell))) & = & \psi(\sum_{x\in X}\ell(\delta_x)) \\
 & = & (\sum_{x\in X}\ell(\delta_x))id_R.
\end{array} \tag{47}$$

All of this shows that $\lambda_X\colon(\widetilde{\mathbb{TOP}^{\mathsf{d}}_{(\mathsf{R},\tau)}})^{\mathbb{OP}}(M_{(\mathsf{R},\tau)}(X)) \simeq C_{\mathsf{R}}X$ *is the required isomorphism of coalgebras.*

Let $\Bbbk$ *be a field. It follows from Theorem 123 and Example 114 (p. 54) that in* $A_{(\Bbbk,\mathsf{d})}(X)$ *every finite-codimensional ideal is closed if, and only if* $C_{\Bbbk}X$ *is coreflexive if, and only if,* $\{\pi_x : x \in X\} = {}_1\mathbf{Alg}_{\Bbbk}(A_{\Bbbk}(X),\Bbbk)$ *([21,*

Corollary 3.2, p. 528]). This holds in particular if $|X| \leq |\Bbbk|$ (see [21, Corollary 3.6, p. 529]). If $\Bbbk$ is a finite field, then $C_{\Bbbk}(X)$ is coreflexive if, and only if, X is finite (see [21, Remark 3.7, p. 530]).

6 Duality between closed ideals and subcoalgebras

One now shows that the equivalence $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\mathsf{d})}) \simeq {}_{\epsilon}\mathbf{Coalg}_{\Bbbk}^{\mathsf{op}}$ (Corollary 104, p. 49) induces a one-one correspondence between closed ideals of a topological algebra and subcoalgebras of its topological dual coalgebra.

Finite duality also induces a rather nice correspondence between ideals and subcoalgebras since to an ideal of an algebra corresponds a subcoalgebra of its finite dual coalgebra. But the converse does not necessarily hold (see [1, Remark, p. 79]), contrary to the correspondence provided hereafter (see Corollary 134 below).

As the main theorem of this section, Theorem 135, it is proven that closed maximal ideals are automatically finite-codimensional and that when the base field is algebraically closed they even have a codimension one. This reflects, by means of the equivalence $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\mathsf{d})}) \simeq {}_{\epsilon}\mathbf{Coalg}_{\Bbbk}^{\mathsf{op}}$, the local finite-dimensional behaviour of coalgebras (see the Fundamental theorem 191, p. 114, in Appendix D.3) to the topological algebras that belong to the image of the TA functor (Section 5.3.2.3).

This bijection, between closed ideals and subcoalgebras, will be used in the following in particular to obtain a logical relationship between topological semisimplicity of some topological algebras and cosemisimplicity of their dual coalgebras (see Section 7).

6.1 Orthogonal spaces

6.1.1 Duality pairing revisited

Let (R, τ) be a rigid ring. Let (M, σ) be an object of $\mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$. Let us define the R-bilinear map

$$\langle \cdot, \cdot \rangle_{(M,\sigma)} \colon (M,\sigma)' \times M \to R$$

by $\langle \ell, u \rangle_{(M,\sigma)} := \ell(u) = \Gamma_{(M,\sigma)}(u)(\ell)$.

This provides a *perfect* pairing, i.e., it has the following two properties.

1. The map $(M,\sigma) \xrightarrow{\Gamma_{(M,\sigma)}} ((M,\sigma)')^*$, $u \mapsto \langle \cdot, u \rangle_{(M,\sigma)}$, is an isomorphism.
2. The map $(M,\sigma)' \to (M,\sigma)'$, $\ell \mapsto \langle \ell, \cdot \rangle_{(M,\sigma)}$ is an isomorphism.

(The second property is tautological while the first follows from Lemma 77, p. 35.)

125 Remark *According to Eq. (17), p. 35,* $\Gamma_{(R,\tau)^X} = (\lambda_X^{-1})^* \circ \rho_X$. *Then,* $\langle\cdot,\cdot\rangle_{(R,\tau)^X}$ *is related with* $\langle\cdot,\cdot\rangle_X$ *(Section 3.1, p. 18) by the equality* $\langle\cdot,\cdot\rangle_{(R,\tau)^X} = \langle\cdot,\cdot\rangle_X \circ (\lambda_X^{-1} \times id_{R^X})$.

The above duality pairing makes it possible to define *orthogonal sets* in a rather usual way, as described below. Let S be a subset of $(M,\sigma)'$, and T be a subset of M. Then define the following sets.

1. $S^\perp := \{\, u \in M : \ell(u) = 0,\ \forall \ell \in S \,\} = \bigcap_{\ell \in S} \ker \ell$, which is closed in (M,σ).
2. ${}^\perp T := \{\, \ell \in (M,\sigma)' : \ell(u) = 0,\ \forall u \in T \,\} \subseteq (M,\sigma)'$.

The following hold.

1. For each $S \subseteq (M,\sigma)'$, $S \subseteq {}^\perp(S^\perp)$.
2. $S_1 \subseteq S_2 \subseteq (M,\sigma)'$ implies $S_2^\perp \subseteq S_1^\perp$.
3. For each $T \subseteq M$, $T \subseteq ({}^\perp T)^\perp$.
4. $T_1 \subseteq T_2 \subseteq M$ implies ${}^\perp T_2 \subseteq {}^\perp T_1$.

126 Lemma *Let* $(\Bbbk,\tau)$ *be a field with a ring topology. Let* (M,σ) *be an object of* $\mathbf{TopFreeVect}_{(\Bbbk,\tau)}$. *Let* V *be a subspace of* $(M,\sigma)'$. *Then,* ${}^\perp(V^\perp) = V$.

Proof: Since $V \subseteq {}^\perp(V^\perp)$, only the converse assertion has to be checked. Let B be a basis of V and let D be a basis of $(M,\sigma)'$ which extends B. Let as in Remark 81 (p. 36), $D_* = \{\, \Gamma_{(M,\sigma)}^{-1}(d),\ d \in D \,\}$, which is a topological basis of (M,σ). Let also $B_* := \{\, \Gamma_{(M,\sigma)}^{-1}(b) : b \in B \,\} \subseteq D_*$. Let $d \in D \smallsetminus B$. Then for each $b \in B$, $b(d_*) = \Gamma_{(M,\sigma)}(d_*)(b) = d^*(b) = 0$. Whence $D_* \smallsetminus B_* \subseteq V^\perp$.

Let $\ell \in {}^\perp(V^\perp)$. In particular, for each $d \in D \smallsetminus B$, $\ell(d_*) = d^*(\ell) = 0$ so that $\ell \in \langle B \rangle = V$. □

Let $\Bbbk$ be a field. Let (M,σ) be an object of $\mathbf{TopFreeVect}_{(\Bbbk,\mathsf{d})}$. According to Section 4.4.3 (p. 37), (M,σ) is a linearly compact $\Bbbk$-vector space, and as such admits a fundamental system of neighborhoods of zero consisting of subvector spaces (see [4]).

127 Lemma *Let* $\Bbbk$ *be a field. Let* (M,σ) *be an object of* $\mathbf{TopFreeVect}_{(\Bbbk,\mathsf{d})}$. *Let* V *be a closed subspace of* (M,σ). *Let* $u \in M$ *such that* $u \notin V$. *Then, there exists* $\ell \in (M,\sigma)'$ *such that* $\ell(u) = 1$ *and* $\ell_{|_V} = 0$.

Proof: Let W be an open subspace of (M,σ) such that $(u+W)\cap V=\varnothing$ (this is possible since σ has a fundamental system of neighborhoods of zero consisting of subvector spaces). Then, $V+W$ is a subspace of M which does not contain u. Let $\ell\in M^*$ such that $\ell(u)=1$ and $\ell_{|_{V+W}}=0$. In particular, $W\subseteq\ker\ell$ so that $\ell\in(M,\sigma)'$. □

128 Corollary *Under the same assumptions as Lemma 127, if V is a closed subspace of (M,σ), then, $({}^\perp V)^\perp=V$.*

Proof: Since $V\subseteq({}^\perp V)^\perp$, it suffices to prove that $({}^\perp V)^\perp\subseteq V$. Let $u\in({}^\perp V)^\perp\smallsetminus V$. Since $({}^\perp V)^\perp$ is closed, there exists by Lemma 127, $\ell\in(M,\sigma)'$ such that $\ell(u)=1$ and $\ell_{|_{({}^\perp V)^\perp}}=0$. In particular, $\ell\in{}^\perp(({}^\perp V)^\perp)={}^\perp V$ (by Lemma 126). But then $\ell(u)$ must be equal to zero, which is a contradiction. □

It follows from Lemma 126 and Corollary 128 that given an object (M,σ) of $\mathbf{TopFreeVect}_{(\Bbbk,\mathsf{d})}$ for a field $\Bbbk$, the operation ${}^\perp(-)$ is an order isomorphism, with inverse $(-)^\perp$, between the poset of all closed subspaces of (M,σ) and the dual[15] of the poset of all subspaces of $(M,\sigma)'$.

Actually ${}^\perp(-)$ and $(-)^\perp$ even transform infima into suprema as is shown by the following result.

129 Proposition *Let $\Bbbk$ be a field. Let (M,σ) be a topologically-free $(\Bbbk,\mathsf{d})$-vector space.*

1. *Let F be a family of closed subspaces of (M,σ). Then, ${}^\perp(\bigcap F)=\sum_{V\in F}{}^\perp V$. (In particular, if F is the empty family, then ${}^\perp(\bigcap\varnothing)={}^\perp M=(0)$.)*

2. *Given a family G of subspaces of $(M,\sigma)'$, $(\sum_{V\in G}V)^\perp=\bigcap_{V\in G}V^\perp$.*

Proof:

A. Let F be a family of closed subspaces of (M,σ). Since for each $V\in F$, $\bigcap F\subseteq V$ it follows that ${}^\perp V\subseteq{}^\perp(\bigcap F)$. Then, $\sum_{V\in F}{}^\perp V\subseteq{}^\perp(\bigcap F)$.

B. Now, let G be a family of subspaces of $(M,\sigma)'$. Since for each $V\in G$, $V\subseteq\sum_{V\in G}V$, it follows that $(\sum_{V\in G}V)^\perp\subseteq V^\perp$, and thus $(\sum_{V\in G}V)^\perp\subseteq\bigcap_{V\in G}V^\perp$.

[15] The *dual* of a poset $(P,\leq)$ is the poset consisting of the same set P with the opposite order $\leq^{\mathsf{op}}$ defined by $x\leq^{\mathsf{op}}y$ if, and only if, $y\leq x$.

1. An Application of item B. with $G := \{ {}^{\perp}V : V \in F \}$ leads to $(\sum_{V\in F} {}^{\perp}V)^{\perp} \subseteq \bigcap_{V\in F}({}^{\perp}V)^{\perp} = \bigcap_{V\in F} V$, so that ${}^{\perp}(\bigcap_{V\in F} V) \subseteq {}^{\perp}((\sum_{V\in F} {}^{\perp}V)^{\perp}) = \sum_{V\in F} {}^{\perp}V$. Thus by A., ${}^{\perp}(\bigcap F) = \sum_{V\in F} {}^{\perp}V$ as desired.

2. An application of item A. with $F := \{ V^{\perp} : V \in G \}$ leads to $\sum_{V\in G} V = \sum_{V\in G} {}^{\perp}(V^{\perp}) \subseteq {}^{\perp}(\bigcap_{V\in G} V^{\perp})$ so $\bigcap_{V\in G} V^{\perp} = ({}^{\perp}(\bigcap_{V\in G} V^{\perp}))^{\perp} \subseteq (\sum_{V\in G} V)^{\perp}$. Whence by B., $\bigcap_{V\in G} V^{\perp} = (\sum_{V\in G} V)^{\perp}$ as required.

□

130 Lemma *Let (V,σ) be a topologically-free $(\Bbbk, \mathsf{d})$-vector space. Let W be a closed subspace of (V,σ) with ${}^{\perp}W$ finite-dimensional. Then,* $\operatorname{codim}(W) = \dim({}^{\perp}W)$.

Proof: First, let W be a subspace of $(V,\sigma)'$. Then, one claims that $W^{\perp} \simeq W^{\dagger}$ (see Eq. (43), p. 56) under $\Gamma_{(V,\sigma)} : (V,\sigma) \simeq ((V,\sigma)')^{*}$. Indeed, let $u \in W^{\perp}$, then $\Gamma_{(V,\sigma)}(u)(\ell) = 0$ for all $\ell \in W$. Conversely, let $L \in W^{\dagger}$. Then, $L = \Gamma_{(V,\sigma)}(u)$ for a unique $u \in V$, and $\Gamma_{(V,\sigma)}(u)(\ell) = L(\ell) = 0$ for all $\ell \in W$ so that $u \in W^{\perp}$.

It follows in particular, that $\operatorname{codim}(W^{\perp}) = \operatorname{codim}(W^{\dagger})$. (Indeed, $(V,\sigma) \simeq ((V,\sigma)')^{*}$, whence $V/W^{\perp} \simeq ((V,\sigma)')^{*}/W^{\dagger}$.)

Secondly, let W be a closed subspace of (V,σ) with ${}^{\perp}W$ finite-dimensional. Since $({}^{\perp}W)^{\perp} = W$ by Corollary 128, then $\operatorname{codim}(W) = \operatorname{codim}(({}^{\perp}W)^{\perp}) = \operatorname{codim}(({}^{\perp}W)^{\dagger})$ (by the above) $= \dim({}^{\perp}W)$ (according to Lemma 121, p. 57).

□

6.1.2 The second copy

Let (R,τ) be a rigid ring. Let (M,σ) be a topologically-free (R,τ)-module. Let $\langle\cdot,\cdot\rangle^{\otimes 2}_{(M,\sigma)} := ((M,\sigma)' \otimes_{\mathsf{R}} (M,\sigma)') \times ((M,\sigma) \circledast_{(\mathsf{R},\tau)} (M,\sigma)) \xrightarrow{\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'} \times id}$ $((M,\sigma) \circledast_{(\mathsf{R},\tau)} (M,\sigma))' \times ((M,\sigma) \circledast_{(\mathsf{R},\tau)} (M,\sigma)) \xrightarrow{\langle\cdot,\cdot\rangle_{(M,\sigma)\circledast_{(\mathsf{R},\tau)}(M,\sigma)}} R$, called the *second copy* of $\langle\cdot,\cdot\rangle_{(M,\sigma)}$. In other words, $\langle \ell_1 \otimes \ell_2, u \circledast v\rangle^{\otimes 2}_{(M,\sigma)} = \ell_1(u)\ell_2(v)$, $\ell_i \in (M,\sigma)'$, $i = 1,2$, $u,v \in M$.

Now let $((M,\sigma), m, e)$ be an object of $\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$. It follows directly from Example 105 (p. 50) that

$$\langle \ell, m(u \circledast v)\rangle_{(M,\sigma)} = \langle \delta(\ell), u \circledast v\rangle^{\otimes 2}_{(M,\sigma)}. \tag{48}$$

131 Remark *For each* $\ell \in (M,\sigma)' \otimes_{\mathsf{R}} (M,\sigma)'$, $(M,\sigma) \circledast_{(\mathsf{R},\tau)} (M,\sigma) \xrightarrow{\langle \ell,\cdot\rangle^{\otimes 2}_{(M,\sigma)}} (R,\tau)$ *is linear and continuous (continuous since* $\ell_0 := \Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(M,\sigma)'}(\ell) \in ((M,\sigma) \circledast_{\mathsf{R}} (M,\sigma))'$ *and* $\langle \ell,\cdot\rangle^{\otimes 2}_{(M,\sigma)} = \ell_0$*).*

Moreover, $\ell \mapsto \langle \ell,\cdot\rangle^{\otimes 2}_{(M,\sigma)}$ *is a one-to-one map from* $(M,\sigma)' \otimes_{\mathsf{R}} (M,\sigma)'$ *to* $((M,\sigma) \circledast_{(\mathsf{R},\tau)} (M,\sigma))'$.

More generally one may define the *nth copy* $\langle \cdot,\cdot\rangle^{\otimes n}_{(M,\sigma)}$ of $\langle \cdot,\cdot\rangle_{(M,\sigma)}$, for $n \geq 1$, as follows. Let $M^{\otimes 1} := M$, and $M^{\otimes n+1} := (M^{\otimes n}) \otimes_{\mathsf{R}} M$ and $(M,\sigma)^{\circledast 1} := (M,\sigma)$ and $(M,\sigma)^{\circledast n+1} := ((M,\sigma)^{\circledast n}) \circledast_{(\mathsf{R},\tau)} (M,\sigma)$, $n \geq 1$. Let $(M,\sigma)' \xrightarrow{\Lambda_1} (M,\sigma)' := id_{(M,\sigma)'}$ and $((M,\sigma)')^{\otimes n+1} \xrightarrow{\Lambda_{n+1}} ((M,\sigma)^{\circledast n+1})'$ be the composition $((M,\sigma)')^{\otimes n+1} \xrightarrow{(\Lambda_n \otimes_{\mathsf{R}} id_{(M,\sigma)'})} ((M,\sigma)^{\circledast n})' \otimes_{\mathsf{R}} (M,\sigma)' \xrightarrow{\Lambda_{((M,\sigma)^{\circledast n})'\otimes_{\mathsf{R}}(M,\sigma)'}} ((M,\sigma)^{\circledast n+1})'$, $n \geq 1$. Then, one defines $((M,\sigma)')^{\otimes n} \times (M,\sigma)^{\circledast n} \xrightarrow{\langle \cdot,\cdot\rangle^{\otimes n}_{(M,\sigma)}} R := \langle \cdot,\cdot\rangle_{(M,\sigma)^{\circledast n}} \circ (\Lambda_n \times id)$, $n \geq 1$.

6.2 The duality between closed ideals and subcoalgebras

In this section $\mathsf{A} = ((A,\sigma),\mu,\eta)$ is an object of $\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\mathsf{d})})$, where $\Bbbk$ is a field.

By a *closed (two-sided) ideal* of A is meant a closed two-sided ideal of $TA(\mathsf{A})$ (see Section 5.3.2.4, p. 56), i.e., an ideal of the underlying algebra $UA(\mathsf{A})$ of A (see Definition 108, p. 51) which is closed in (A,σ). The definition of subcoalgebra is recalled in Appendix D.3 (p. 114).

The operations ${}^{\perp}(-)$ and $(-)^{\perp}$ establish a nice connection between closed ideals of A and subcoalgebras of its topological dual coalgebra $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$ that we now explore.

132 Proposition *Let* I *be closed subspace of* (A,σ)*. If* I *is a closed two-sided ideal of* A*, then* ${}^{\perp}I$ *is a subcoalgebra of* $(\widetilde{\mathbb{T}\mathbb{o}\mathbb{p}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{o}\mathbb{p}}(\mathsf{A})$.

Proof: Let us assume that I is a (closed two-sided) ideal. One has to prove that $\delta({}^{\perp}I) \subseteq {}^{\perp}I \otimes_{\Bbbk} {}^{\perp}I$, where $\delta = \Lambda^{-1}_{(A,\sigma)'\otimes_{\mathsf{R}}(A,\sigma)'} \circ \mu'$ as in Example 105 (p. 50). Let $\ell \in (A,\sigma)'$. Then, $\delta(\ell) = \sum_{i=1}^{n} \ell_i \otimes r_i$ for some $\ell_i, r_i \in (A,\sigma)'$, and one assumes that the ℓ_i's are $\Bbbk$-linearly independent. Let $\{ u_i : i = 1,\cdots,n \}$ be a set of members of A such that $\ell_i(u_j) = \delta_{i,j}$, $i,j = 1,\cdots,n$ ([7, Lemma 1.5.8, p. 39]). Let $u \in I$. Then, for each $i = 1,\cdots,n$, $\mu(u_i \circledast u) \in I$ so that $\ell(\mu(u_i \circledast u)) = 0$. But $0 = \ell(\mu(u_i \circledast u)) = \langle \mu(u_i \circledast u), \ell\rangle_{(A,\sigma)} = \langle u \circledast v, \delta(\ell)\rangle^{\otimes 2}_{(A,\sigma)} =$

$\sum_{j=1}^{n}\langle u_i,\ell_j\rangle_{(A,\sigma)}\langle u,r_j\rangle_{(A,\sigma)} = \sum_{j=1}^{n}\ell_j(u_i)r_j(u) = r_i(u)$. Whence $r_i \in {}^{\perp}I$, and $\delta(\ell) \in (A,\sigma)' \otimes_{\Bbbk} {}^{\perp}I$. Similarly we obtain $\delta(\ell) \in {}^{\perp}I \otimes_{\Bbbk} {}^{\perp}I$. □

133 Proposition *Let D be a subspace of $(A,\sigma)'$. If D is a subcoalgebra of $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$, then $D^{\perp}$ is a closed two-sided ideal of A.*

Proof: Let $u \in A$, $v \in D^{\perp}$ and $\ell \in D$. We have $\ell(\mu(u \circledast v)) = \langle \mu(u \circledast v),\ell\rangle_{(A,\sigma)} = \langle u \circledast v, \delta(\ell)\rangle^{\otimes 2}_{(A,\sigma)} = \sum_{i=1}^{n}\langle u,\ell_i\rangle_{(A,\sigma)}\langle v,r_i\rangle_{(A,\sigma)} = 0$ since $\delta(\ell) = \sum_{i=1}^{n}\ell_i \otimes r_i \in D \otimes_{\Bbbk} D$, and $v \in D^{\perp}$. Therefore $\mu(u \circledast v) \in D^{\perp}$. Similarly, $\mu(v \circledast u) \in D^{\perp}$. Thus $D^{\perp}$ is a two-sided ideal. □

134 Corollary *1. Let I be a closed subspace of (A,σ). I is a two-sided ideal if, and only if, ${}^{\perp}I$ is a subcoalgebra of $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$.*

2. Let D be a subspace of $(A,\sigma)'$. D is a subcoalgebra of $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$ if, and only if, $D^{\perp}$ is a closed two-sided ideal of A.

It follows from Corollary 134 that given a monoid A in $\mathbb{TopFreeVect}_{(\Bbbk,\mathsf{d})}$, the operation ${}^{\perp}(-)$ is an order isomorphism, with inverse $(-)^{\perp}$, between the poset of all closed two-sided ideals of A and the dual of the poset of all subcoalgebras of $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$. In particular, there is a one-one correspondence between closed maximal two-sided ideals and simple subcoalgebras (see Appendix D.3, p. 114).

135 Theorem *Let A be a monoid in $\mathbb{TopFreeVect}_{(\Bbbk,\mathsf{d})}$. Every closed maximal two-sided ideal $\mathfrak{m}$ of A is finite-codimensional with* $\operatorname{codim}(\mathfrak{m}) = \dim({}^{\perp}\mathfrak{m})$. *In particular if $\Bbbk$ is an algebraically closed field and A is a commutative monoid, then every closed maximal ideal $\mathfrak{m}$ of A has codimension* 1.

Proof: ${}^{\perp}\mathfrak{m}$ is a simple subcoalgebra of $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$ (essentially) by Corollary 134. Whence ${}^{\perp}\mathfrak{m}$ is finite-dimensional.

Indeed, let D be a simple subcoalgebra of a coalgebra C. Let us assume that $\dim(D)$ is not finite. Each member of D generates a finite-dimensional subcoalgebra of D (by Theorem 191, p. 114, in Appendix D.3). In particular, for a non-zero member of D, this would provide a proper (i.e., $\neq (0)$, $\neq D$) subcoalgebra of D, which contradicts simplicity of D.

Then, it follows from Lemma 130 that $\operatorname{codim}(\mathfrak{m}) = \dim({}^{\perp}\mathfrak{m}) < \infty$.

Now, let us assume that $\Bbbk$ is an algebraically closed field and A is a commutative monoid. In this case, $(\widetilde{\mathbb{Top}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$ is a cocommutative coalgebra, and is pointed (see Appendix D.3, p. 114) by [1, Theorem 2.3.3, p. 80],

i.e., $\dim(D) = 1$ for each simple subcoalgebra D of $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\mathsf{d})}})^{\mathbb{op}}(\mathsf{A})$. Therefore by the first part of the proof $\operatorname{codim}(\mathfrak{m}) = 1$ as soon as $\mathfrak{m}$ is a closed maximal ideal of A. $\square$

7 Topological semisimplicity

In this final section of the presentation, all the notions introduced so far and many of the results obtained beforehand, are exploited so as to get a thorough understanding of the function algebra behaviour of a commutative monoid in $\mathbb{TopFreeVect}_{(\Bbbk,\tau)}$ for a field $\Bbbk$ with a ring topology τ. As a main result is provided a characterization of the function algebras among these monoids as suitably generalized von Neumann regular algebras, when $\Bbbk$ is algebraically closed (Theorem 167).

Let us take the time to succinctly review the content of this section. Among commutative algebras, function algebras $A_{\Bbbk}(X)$ (Section 2.4.2) or even ring direct products $\prod_{i\in I} \Bbbk_i$ of fields, provides the most direct and apparent access to their structure. Therefore a natural way to get information on more general algebras is to represent them as function algebras; this is precisely to what semisimplicity[16] leads to with the canonical representation[17] $\mathsf{A} \xrightarrow{can_{\mathsf{A}}} \prod_{\mathfrak{m}\in Max(\mathsf{A})} \mathsf{A}/\mathfrak{m}$ of an algebra A (where $Max(\mathsf{A})$ is the set of all maximal ideals of A). The kernel of this representation (i.e., the Jacobson radical) measures what is missing for the algebra to be a subalgebra of a function algebra.

Our topological setting gives access to several notions of semisimplicity for a commutative monoid A in the monoidal category $\mathbb{TopFreeVect}_{(\Bbbk,\tau)}$.

1. Cosemisimplicity of the topological dual coalgebra of A (Appendix D.3),
2. Jacobson semisimplicity of the underlying algebra $UA(\mathsf{A})$,
3. Topological semisimplicity of $TA(\mathsf{A})$ (Section 7.1.1.1), based on a representation of A to a topological product of fields (with a separately continuous multiplication),
4. Gelfand semisimplicity of A (Section 7.1.2), which measures how A differs from a "subalgebra" of some $M_{(\Bbbk,\tau)}(X)$ (Example 100).

[16] In this presentation, "semisimplicity" means what is sometimes called "Jacobson semisimplicity" or "semiprimitivity" elsewhere.

[17] Here by a "representation" is meant an algebra map $\mathsf{A} \xrightarrow{f} \prod_{i\in I} \Bbbk_i$.

Our objective is to understand the relations between the above notions beyond the trivial implications: A Gelfand semisimple $\Rightarrow TA(\mathsf{A})$ topologically semisimple $\Rightarrow UA(\mathsf{A})$ semisimple.

As a summary of the results obtained in this section:

1. Corollary 141 which provides the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ for a discrete field $(\Bbbk, \mathsf{d})$, using the duality between closed ideals and subcoalgebras developed in Section 6.

2. Corollary 156 which establishes the equivalence $3 \Leftrightarrow 4$ for a discrete algebraically closed field, thanks to Theorem 135 (p. 66) that states that all the closed maximal ideals of A are one codimensional.

3. Corollary 161 tells us that the only Gelfand semisimple commutative monoids are exactly those of the form $M_{(\Bbbk,\tau)}(X)$ for some set X.

4. Theorem 167 which definitively settles the question by providing the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ for an algebraically closed field $(\Bbbk, \tau)$ with a ring topology. The equivalence ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\mathsf{d})}) \simeq {}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$ from Corollary 104, p. 49, plays a decisive rôle as an argument of the proof.

7.1 Various notions of topological semisimplicity

7.1.1 Topological Jacobson radical and topological semisimplicity

7.1.1.1 Definitions

Let (R, τ) be a topological ring. Let us denote as illustrated in the commutative diagram below some obvious forgetful functors (which are faithful).

$$\begin{tikzcd}
\mathbf{Top} & {}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[l, "|\cdot|_{alg}"'] \arrow[r, "|\cdot|_{top}"] & {}_{1,c}\mathbf{Alg}_{\mathsf{R}} \\
& {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)} \arrow[ul, "|\cdot|_{alg}"] \arrow[u, hook] \arrow[ur, "|\cdot|_{top}"'] &
\end{tikzcd} \tag{49}$$

(The abbreviations in the index of the forgetful functors correspond to the part of the structure which is forgotten.) Whence for a separately continuous algebra $\mathsf{A} = ((A, \sigma), m_{\mathsf{A}}, 1_{\mathsf{A}})$, $|\mathsf{A}|_{top} = (A, m_{\mathsf{A}}, 1_{\mathsf{A}})$ and $|\mathsf{A}|_{alg} = (|A|, \sigma)$ (where $\mathbf{TopMod}_{(\mathsf{R},\tau)} \xrightarrow{|\cdot|} \mathbf{Top}$ is the forgetful functor).

136 Remark *Let (R, τ) be a rigid ring, and $\mathsf{A} = ((A, \sigma), m, e)$ be a commutative monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$. Then, $|TA(\mathsf{A})|_{alg} = (A, \sigma)$ while $|TA(\mathsf{A})|_{top} = UA(\mathsf{A})$ (see Section 5.3.2.3, p. 53, for the definition of TA).*

1. Let R be a ring. Let A be an object of ${}_{1,c}\mathbf{Alg}_{\mathsf{R}}$. Let $Max(\mathsf{A})$ be the set of all maximal ideals of A, and let $J(\mathsf{A}) := \bigcap_{\mathfrak{m}\in Max(\mathsf{A})} \mathfrak{m} = \bigcap Max(\mathsf{A})$ be the *Jacobson radical* of A. (In particular, if $Max(\mathsf{A}) = \varnothing$, then $J(\mathsf{A}) = \mathsf{A}$.) A is said to be *(Jacobson) semisimple* when $J(\mathsf{A}) = (0)$.

   Let $\mathsf{A} \xrightarrow{can_{\mathsf{A}}} \prod_{\mathfrak{m}\in Max(\mathsf{A})} \mathsf{A}/\mathfrak{m}$ be the canonical map $can_{\mathsf{A}}(x) = (x + \mathfrak{m})_{\mathfrak{m}\in Max(\mathsf{A})}$.

   Of course, $\ker can_{\mathsf{A}} = J(\mathsf{A})$ and thus A is semisimple if, and only if, can_{A} is one-to-one.

2. Let (R, τ) be a topological ring. Let A be an object of ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ (or ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$). Let $ClMax(\mathsf{A})$ be the set of all closed maximal ideals of A (i.e., maximal ideals of the algebra $|\mathsf{A}|_{top}$ which are closed as subset of the topological space $|\mathsf{A}|_{alg}$). The *topological Jacobson radical* $J_{top}(\mathsf{A})$ of A is the closed ideal of A defined as $\bigcap ClMax(\mathsf{A})$. (Again, if $ClMax(\mathsf{A}) = \varnothing$, then $J_{top}(\mathsf{A}) = \mathsf{A}$.) A is said to be *topologically semisimple* when $J_{top}(\mathsf{A}) = (0)$.

   Let $|\mathsf{A}|_{top} \xrightarrow{can_{\mathsf{A}}} \prod_{\mathfrak{m}\in ClMax(\mathsf{A})} |\mathsf{A}|_{top}/\mathfrak{m}$ be the canonical map, which is an algebra map. It is continuous from A to $\prod_{\mathfrak{m}\in ClMax(\mathsf{A})} \mathsf{A}/\mathfrak{m}$ when, as in Appendix E (p. 118), the product denotes a topological product space and each factor $\mathsf{A}/\mathfrak{m}$ corresponds to $|A|_{top}/\mathfrak{m}$ with the quotient topology, as for $\mathfrak{m} \in ClMax(\mathsf{A})$ and $x \in A$, $\pi_{\mathfrak{m}}(can_{\mathsf{A}}(x)) = x + \mathfrak{m} = \pi_{\mathfrak{m}}(x)$.

   Whence $\ker can_{\mathsf{A}} = J_{top}(\mathsf{A})$ and A is topologically semisimple if, and only if, can_{A} is one-to-one.

3. Let us assume that (R, τ) is rigid, and let A be a commutative monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$. One defines $ClMax(\mathsf{A}) := ClMax(TA(\mathsf{A}))$ and $J_{top}(\mathsf{A}) := J_{top}(TA(\mathsf{A}))$. A is said to be *topologically semisimple* if $TA(\mathsf{A})$ is so.

137 Remark *Let (R, τ) be a topological ring. Let A be an object of the category ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ (or ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$). As $ClMax(\mathsf{A}) \subseteq Max(|\mathsf{A}|_{top})$ it follows that $J(|\mathsf{A}|_{top}) \subseteq J_{top}(\mathsf{A})$ and thus if A is topologically semisimple, then $|\mathsf{A}|_{top}$ is semisimple.*

The following lemma will be used rather frequently in what follows, without further ado. (Its proof essentially follows from the fact that if $\mathsf{A} \xrightarrow{f} \mathsf{B}$ is an isomorphism, then it induces a bijection $ClMax(\mathsf{A}) \to ClMax(\mathsf{B})$, $\mathfrak{m} \mapsto f(\mathfrak{m})$.)

138 Lemma *Let* (R,τ) *be a topological ring. Let* A,B *be isomorphic objects of* ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *(or* ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$*).* A *is topologically semisimple if, and only if,* B *so is. In particular, when* (R,τ) *is rigid, topological semisimplicity of objects of* ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$ *is preserved by isomorphisms.*

7.1.1.2 Topological semisimplicity and cosemisimplicity

See Section D.3 (p. 114) for some recalls about cosemisimplicity of coalgebras.

139 Proposition *Let* $(\Bbbk,\tau)$ *be a field with a ring topology and let* $\mathsf{C} = (C,\delta_{\mathsf{C}},\epsilon_{\mathsf{C}})$ *be a counital cocommutative coalgebra.* C *is cosemisimple if, and only if,* $O(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ *is semisimple if, and only if,* $UA(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}}(\mathsf{C}))$ *is semisimple.*

Proof: Recall from Eq. (43), p. 56, that for a coalgebra C, $(\mathrm{corad}(\mathsf{C}))^\dagger := \{\,\ell\in C^*\colon \ell(x)=0,\ \forall x\in\mathrm{corad}(\mathsf{C})\,\}$. It is clear that $(\mathrm{corad}(\mathsf{C}))^\dagger=(0)$ if, and only if, $\mathrm{corad}(\mathsf{C})=C$.

Let us assume that C is a cosemisimple (cocommutative) coalgebra. One has $\widetilde{\mathbb{D}_*}(\mathsf{C}) = \tilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}}(\mathsf{C}))$ (see Section 5.3.2.2, p. 52), and according to [1, Theorem 2.3.9, p. 84], $J(O(\widetilde{\mathbb{D}_*}(\mathsf{C})))=\mathrm{corad}(\mathsf{C})^\dagger$. But $O(\widetilde{\mathbb{D}_*}(\mathsf{C})) = UA(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}}(\mathsf{C}))$ so that the algebra $UA(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}}(\mathsf{C}))$ is semisimple if, and only if, the algebra $O(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ is semisimple if, and only if, $(\mathrm{corad}(\mathsf{C}))^\dagger = J(O(\widetilde{\mathbb{D}_*}(\mathsf{C})))=(0)$ if, and only if, $\mathrm{corad}(\mathsf{C})=C$ if, and only if, C is cosemisimple. □

140 Corollary *Let* $(\Bbbk,\tau)$ *be a field with a ring topology, let* $\mathsf{A}=((A,\sigma),\mu,\eta)$ *be an object of* ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\tau)})$*. Let* $\mathsf{C}:=(\widetilde{\mathbb{T}\mathrm{op}^{\mathsf{d}}_{(\Bbbk,\tau)}})^{\mathrm{op}}(\mathsf{A})$*.* $UA(\mathsf{A})$ *is semisimple if, and only if,* $O(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ *is semisimple if, and only if,* C *is cosemisimple. In particular, if* A *is topologically semisimple, then* C *is cosemisimple.*

Proof: It suffices to apply Proposition 139 since $\widetilde{\mathbb{D}_*}(\mathsf{C}) = \tilde{\mathbb{A}}(\widetilde{\mathbb{A}\mathbb{l}\mathbb{g}_{(\Bbbk,\tau)}}(\mathsf{C}))\simeq\tilde{\mathbb{A}}(\mathsf{A})$ (Corollary 104, p. 49) and thus $O(\widetilde{\mathbb{D}_*}(\mathsf{C}))\simeq UA(\mathsf{A})$. The last assertion is due to Remark 137. □

141 Corollary *Let* $\Bbbk$ *be a field. Let* $\mathsf{A}=((A,\sigma),\mu,\eta)$ *be an object of* ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{V}\mathrm{ect}_{(\Bbbk,\mathsf{d})})$ *and let* $\mathsf{C}:=(\widetilde{\mathbb{T}\mathrm{op}^{\mathsf{d}}_{(\Bbbk,\mathsf{d})}})^{\mathrm{op}}(\mathsf{A})$*.* A *is topologically semisimple if, and only if,* $UA(\mathsf{A})$ *is semisimple if, and only if,* C *is cosemisimple.*

Proof: By Corollary 140 the last equivalence holds. It remains to prove the first one. One already knows by Remark 137 that A topologically semisimple implies $UA(\mathsf{A})$ semisimple.

Let us assume that C is cosemisimple. Let $simpl(\mathsf{C})$ be the collection of all simple subcoalgebras of C. One has $(A,\sigma)' = \mathrm{corad}(\mathsf{C}) = \sum_{D\in simpl(\mathsf{C})} D = \sum_{D\in simpl(\mathsf{C})} {}^{\perp}(D^{\perp}) = {}^{\perp}(\bigcap_{D\in simpl(\mathsf{C})} D^{\perp})$ (according to Proposition 129, p. 63) $= {}^{\perp}(\bigcap_{\mathfrak{m}\in ClMax(\mathsf{A})} \mathfrak{m})$ (since $(-)^{\perp}$ provides a one-one correspondence between closed maximal ideals of A and simple subcoalgebras of C; see Section 6.2, p. 65) $= {}^{\perp}J_{top}(\mathsf{A})$. Whence $J_{top}(\mathsf{A}) = ((A,\sigma)')^{\perp} = (0)$, i.e., A is topologically semisimple. □

7.1.2 Gelfand semisimplicity

7.1.2.1 Definition

Let (R,τ) be a rigid ring.

One observes easily that $((\mathsf{R},\tau),\mu_{(\mathsf{R},\tau)},id_R)$ is a commutative monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$, where $(R,\tau)\circledast_{(\mathsf{R},\tau)}(R,\tau)\xrightarrow{\mu_{(\mathsf{R},\tau)}}(R,\tau)$ is defined by $\mu_{(\mathsf{R},\tau)}(1_{\mathsf{R}}\circledast 1_{\mathsf{R}}) := 1_{\mathsf{R}}$ (or alternatively, $\mu_{(\mathsf{R},\tau)}(\alpha\circledast\beta) := m_{\mathsf{R}}(\alpha,\beta)$, $\alpha,\beta\in R$, and $R\times R\xrightarrow{m_{\mathsf{R}}} R$ is the bilinear multiplication of R). Moreover, it is clear that $TA((\mathsf{R},\tau),\mu_{(\mathsf{R},\tau)},id_R) = (\mathsf{R},\tau)$.

Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ be a commutative monoid in $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$. Let $\mathbf{X}_{(\mathsf{R},\tau)}\mathsf{A} := {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})(\mathsf{A},((\mathsf{R},\tau),\mu_{(\mathsf{R},\tau)},id_R))$ be the set of all *continuous characters* of A.

Since according to Remark 118 (p. 55), TA is fully faithful, its restriction $\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A}) \to {}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}(TA(\mathsf{A}),(\mathsf{R},\tau))$ is a bijection. In what follows both sets are treated as equal.

142 Remark *When $\tau = \mathsf{d}$, then $\mathbf{X}_{(\mathsf{R},\mathsf{d})}(\mathsf{A}) \simeq {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\mathsf{d})}(TA(\mathsf{A}),(\mathsf{R},\mathsf{d}))$ since in this case $TA(\mathsf{A})$ has a jointly continuous multiplication (see Section 5.3.2.3, p. 53).*

Let (R,τ) be a rigid ring. Let $\mathsf{A} = ((A,\sigma),\mu,\eta)$ be an object of the category ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$. One defines the morphism of separately continuous algebras $TA(\mathsf{A})\xrightarrow{G_{\mathsf{A}}} A_{(\mathsf{R},\tau)}(\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A}))$ by $G_{\mathsf{A}}(u)(\chi) := \chi(u)$, $u\in A$, $\chi\in\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$. (It is continuous since for each $\chi\in\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$, $\pi_\chi\circ G_{\mathsf{A}} = \chi$. Moreover, $((G_{\mathsf{A}}\circ\mu)(u\circledast v))(\chi) = \chi(\mu(u\circledast v)) = \chi(u)\chi(v) = (G_{\mathsf{A}}(u)(\chi))(G_{\mathsf{A}}(v)(\chi))$ for each $u,v\in A$ and $\chi\in\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$, and one also has $(G_{\mathsf{A}}(\eta(1_{\mathsf{R}})))(\chi) = \chi(\eta(1_{\mathsf{R}})) = 1_{\mathsf{R}}$.)

Since by Remark 118, TA is fully faithful, there is a unique monoid morphism $\mathsf{A} \xrightarrow{\mathsf{G}_{\mathsf{A}}} M_{(\mathsf{R},\tau)}(\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A}))$ such that $TA(\mathsf{G}_{\mathsf{A}}) = G_{\mathsf{A}}$. (See Example 100, p. 47, for the definition of $M_{(\mathsf{R},\tau)}(X)$.)

143 Definition *The monoid morphism* G_{A} *is referred to as the* Gelfand transform *of* A. *Moreover,* A *is said to be* Gelfand semisimple *when* G_{A} *is one-to-one.*

144 Remark $\ker \mathsf{G}_{\mathsf{A}} = \bigcap_{\chi \in \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})} \ker \chi$ *is a closed ideal of* A. *Of course,* $J_{top}(\mathsf{A}) \subseteq \ker \mathsf{G}_{\mathsf{A}}$, *and thus if* A *is Gelfand semisimple, then it is topologically semisimple.*

7.1.2.2 Functoriality and naturality

145 Lemma *Let* (R,τ) *be a rigid ring. Let* X *be a set. Then,* $\{\pi_x : x \in X\} \subseteq \mathbf{X}_{(\mathsf{R},\tau)}(M_{(\mathsf{R},\tau)}(X))$. *If moreover* R *has no non-trivial idempotents (i.e.,* $E(\mathsf{R}) = \{0, 1_{\mathsf{R}}\}$*), then* $\mathbf{X}_{(\mathsf{R},\tau)}(M_{(\mathsf{R},\tau)}(X)) = \{\pi_x : x \in X\}$. *In particular the last assertion holds for a field* $(\Bbbk,\tau)$ *with a ring topology.*

Proof: Clearly $\{\pi_x : x \in X\} \subseteq \mathbf{X}_{(\mathsf{R},\tau)}(M_{(\mathsf{R},\tau)}(X))$. Assume that the unique idempotents of R are 0 and 1_{R}. Let $\chi \in {}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}(A_{(\mathsf{R},\tau)}(X), (\mathsf{R},\tau))$. (Recall from Example 119, p. 56, that $TA(M_{(\mathsf{R},\tau)}(X)) = A_{(\mathsf{R},\tau)}(X)$.) In particular, $\chi(\delta_x)$ is an idempotent of R for each $x \in X$ since $\chi(\delta_x) = \chi(\delta_x^2) = \chi(\delta_x)\chi(\delta_x)$, whence $\chi(\delta_x) \in \{0, 1_{\mathsf{R}}\}$ for each x. Since $\chi(1_{A_{(\mathsf{R},\tau)}(X)}) = 1_{\mathsf{R}}$, and $1_{A_{(\mathsf{R},\tau)}(X)} = \sum_{x \in X} \delta_x$, it follows (since R is non-trivial by Conventions 1) that there is at least one $x_\chi \in X$ such that $\chi(\delta_{x_\chi}) = 1_{\mathsf{R}}$. Let $x \in X \smallsetminus \{x_\chi\}$ such that $\chi(\delta_x) = 1_{\mathsf{R}}$. Then, $0 = \chi(0) = \chi(\delta_x \delta_{x_\chi}) = \chi(\delta_x)\chi(\delta_{x_\chi}) = 1_{\mathsf{R}}$ which is a contradiction. Whence there is a unique $x_\chi \in X$ with $\chi(\delta_{x_\chi}) = 1_{\mathsf{R}}$. Now, $\chi(f) = \chi(\sum_{x \in X} f(x)\delta_x) = \sum_{x \in X} f(x)\chi(\delta_x) = f(x_\chi) = \pi_{x_\chi}(f)$, $f \in R^X$. □

146 Example *Let* $(\Bbbk,\tau)$ *be a field with a ring topology. Let* X *be a set. Then,* $M_{(\Bbbk,\tau)}(X)$ *is Gelfand semisimple. (It suffices to use Lemma 145 to see that* $\ker \mathsf{G}_{M_{(\Bbbk,\tau)}(X)} = (0)$.*)*

147 Lemma *Let* (R,τ) *be a rigid ring.* $\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$ *extends to a functor from* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})^{\mathsf{op}}$ *to* $\mathbf{Set}$.

Proof: Let $\mathsf{A} \xrightarrow{f} \mathsf{B}$ be a morphism in ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$. Then, with $\mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{B}) \xrightarrow{\mathbf{X}_{(R,\tau)}(f)} \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$, $\chi \mapsto \chi \circ f$, one obtains the expected functor. □

148 Lemma *Let* (R,τ) *be a rigid ring.* $\mathsf{G} := (\mathsf{G}_\mathsf{A})_\mathsf{A}\colon id_{{}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})} \Rightarrow M_{(\mathsf{R},\tau)} \circ \mathbf{X}^{\mathsf{op}}_{(\mathsf{R},\tau)}\colon {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)}) \to {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$. *(Recall from Remark 101, p. 48, that* $M_{(\mathsf{R},\tau)}$ *is functorial.)*

Proof: Let $\mathsf{A} \xrightarrow{f} \mathsf{B}$ be a morphism in ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$. One has to check that the following diagram commutes.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\mathsf{G}_\mathsf{A}"] \arrow[d, "f"'] & M_{(\mathsf{R},\tau)}(\mathbf{X}^{\mathsf{op}}_{(\mathsf{R},\tau)}(\mathsf{A})) \arrow[d, "M_{(\mathsf{R},\tau)}(\mathbf{X}^{\mathsf{op}}_{(R,\tau)}(f))"] \\ \mathsf{B} \arrow[r, "\mathsf{G}_\mathsf{B}"'] & M_{(\mathsf{R},\tau)}(\mathbf{X}^{\mathsf{op}}_{(\mathsf{R},\tau)}(\mathsf{B})) \end{tikzcd} \tag{50}$$

Let $\chi \in \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{B})$ and let $u \in A$. Then, $(\mathsf{G}_\mathsf{B}(f(u)))(\chi) = \chi(f(u))$, while $(M_{(\mathsf{R},\tau)}(\mathbf{X}^{\mathsf{op}}_{(\mathsf{R},\tau)}(f))(\mathsf{G}_\mathsf{A}(u)))(\chi) = (\mathsf{G}_\mathsf{A}(u) \circ \mathbf{X}^{\mathsf{op}}_{(\mathsf{R},\tau)}(f))(\chi) = \mathsf{G}_\mathsf{A}(u)(\chi \circ f) = \chi(f(u))$. □

149 Proposition *Let* (R,τ) *be a rigid ring. Then the functor* $\mathbf{Set} \xrightarrow{M^{\mathsf{op}}_{(\mathsf{R},\tau)}} {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})^{\mathsf{op}}$ *is a left adjoint of the continuous characters functor* ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})^{\mathsf{op}} \xrightarrow{\mathbf{X}_{(\mathsf{R},\tau)}} \mathbf{Set}$.

Proof: Let $X \xrightarrow{f} \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{B})$ be a set-theoretic map where $\mathsf{B} = ((B,\gamma),m,e)$ is an object of ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$. Let $(B,\gamma) \xrightarrow{\hat{f}} (R,\tau)^X$ be the unique continuous linear map such that for each $x \in X$, $\pi_x \circ \hat{f} = f(x)$. Let $u,v \in B$, and $x \in X$. Then, $(\hat{f}(m(u \circledast v)))(x) = \pi_x(\hat{f}(m(u \circledast v)) = f(x)(m(u \circledast v)) = f(x)(u)f(x)(v) = (\hat{f}(u))(x)(\hat{f}(v))(x)$, and moreover $(\hat{f}(e(1_\mathsf{R})))(x) = \pi_x(\hat{f}(e(1_\mathsf{R}))) = f(x)(e(1_\mathsf{R})) = 1_\mathsf{R}$. Therefore, $TA(\mathsf{B}) \xrightarrow{\hat{f}} A_{(\mathsf{R},\tau)}(X)$ is a morphism of separately continuous algebras, whence by Remarks 118 (p. 55) and 120 (p. 56), $\mathsf{B} \xrightarrow{\hat{f}} M_{(\mathsf{R},\tau)}(X)$ is a morphism in ${}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})$.

□

150 Remark *The unit of the adjunction* $M^{\mathsf{op}}_{(\mathsf{R},\tau)} \dashv \mathbf{X}_{(\mathsf{R},\tau)}$ *is given by* $X \xrightarrow{\pi_X} \mathbf{X}_{(R,\tau)}(M^{\mathsf{op}}_{(\mathsf{R},\tau)}(X))$, $\pi_X(x) := \pi_x$, *while the counit is the opposite* $\mathsf{G}^{\mathsf{op}}\colon M^{\mathsf{op}}_{(\mathsf{R},\tau)} \circ \mathbf{X}_{(\mathsf{R},\tau)} \Rightarrow id\colon {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})^{\mathsf{op}} \to {}_c\mathbf{Mon}(\mathbb{T}\mathrm{op}\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{(\mathsf{R},\tau)})^{\mathsf{op}}$ *of the Gelfand transform (see Facts 172, p. 88, in Appendix A for the definition of the opposite natural transformation).*

151 Remark *It is possible to compare the adjunctions* $P^{\mathsf{op}}_{(\mathsf{R},\tau)} \dashv Top_{(\mathsf{R},\tau)} \circ |\cdot|\colon \mathbf{Set} \to \mathbf{TopFreeMod}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ *(Proposition 89, p. 40) and* $M^{\mathsf{op}}_{(\mathsf{R},\tau)} \dashv \mathbf{X}_{(\mathsf{R},\tau)}$ *for a rigid ring* (R,τ)*. Let* A *be a commutative monoid in* $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$*, and* X *be a set. Let* $X \xrightarrow{g} |U_m(\mathsf{A})'|$ *be a map (where, as in Proposition 177, p. 95, Appendix B.3,* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \xrightarrow{U_m} \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$ *is the obvious forgetful functor). The unique continuous* (R,τ)*-linear map* $U_m(\mathsf{A}) \xrightarrow{g^\sharp} P_{(\mathsf{R},\tau)}(X) = U_m(M_{(\mathsf{R},\tau)}(X))$ *(Proposition 86, p. 39) lifts to a morphism of monoid* $\mathsf{A} \xrightarrow{g^\flat} M_{(\mathsf{R},\tau)}(X)$*, i.e.,* $U_m(g^\flat) = g^\sharp$*, if, and only if,* g *factors has* $X \to \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A}) \hookrightarrow |U_m(\mathsf{A})'|$*. (It suffices to compare the explicit definition of* $g^\sharp$ *given in Remark 87, p. 39, with the construction of a monoid morphism* $\hat{f}$ *given in the proof of Proposition 149.)*

152 Remark *The proof of Proposition 149 can be adapted to show that* $\mathbf{Set} \xrightarrow{A^{\mathsf{op}}_{(\mathsf{R},\tau)}} {}_{1,c}\mathbf{TopAlg}^{\mathsf{op}}_{(\mathsf{R},\tau)}$ *is a left adjoint to the contravariant hom-functor* ${}_{1,c}\mathbf{TopAlg}^{\mathsf{op}}_{(\mathsf{R},\tau)} \xrightarrow{{}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}(-,(\mathsf{R},\tau))} \mathbf{Set}$*, where for a category* $\mathbf{C}$*, and a* $\mathbf{C}$*-object* D*,* $\mathbf{C}^{\mathsf{op}} \xrightarrow{\mathbf{C}(-,D)} \mathbf{Set}$ *is given by* $C \mapsto \mathbf{C}(C,D)$ *([17, pp. 34–35]).*

7.1.2.3 Relations between different "maximal ideal spaces"

153 Definition *Let* $(\Bbbk,\tau)$ *be a field with a ring topology. Let* A *be a commutative monoid in* $\mathbb{TopFreeVect}_{(\Bbbk,\tau)}$*. One defines* $ClMax(\mathsf{A})_1$ *as the subset of* $ClMax(\mathsf{A})$ *consisting of closed maximal ideals of codimension one.*

154 Proposition *Let* (R,τ) *be a rigid ring. Let* $\mathsf{A} = ((A,\sigma),m,e)$ *be an object of* ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$*. Let* $\mathsf{C} := (\widehat{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}(\mathsf{A})$ *be its topological dual coalgebra.*

1. $\mathbf{X}_{(\mathsf{R},\tau)}\mathsf{A} = G(\mathsf{C})$ *(cf. Appendix D.2, p. 112, for the definition of the set* $G\mathsf{C}$ *of group-like elements of a coalgebra* C*).*

2. *If* $(\mathsf{R},\tau) = (\Bbbk,\tau)$ *is a field with a ring topology, then* $\mathbf{X}_{(\Bbbk,\tau)}\mathsf{A} \xrightarrow{\ker} ClMax(\mathsf{A})_1$ *is one-to-one.*

   (a) *Moreover, if* τ *is the discrete topology, then* $\mathbf{X}_{(\Bbbk,\mathsf{d})}\mathsf{A} \xrightarrow{\ker} ClMax(\mathsf{A})_1$ *is onto, whence* $\mathbf{X}_{(\Bbbk,\mathsf{d})}\mathsf{A} \simeq ClMax(\mathsf{A})_1$*.*

   (b) *If* $\Bbbk$ *is an algebraically closed field, then*

$$G\mathsf{C} = \mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A}) \simeq ClMax(\mathsf{A})_1 = ClMax(\mathsf{A}).$$

Proof:

1. Let $\chi \in \mathbf{X}_{(\mathsf{R},\tau)}\mathsf{A}$. Eq. (48), p. 64, shows that for each $u, v \in A$, $\langle \chi \otimes \chi, u \circledast v\rangle^{\otimes 2}_{(A,\sigma)} = \langle \chi, u\rangle_{(A,\sigma)}\langle \chi, v\rangle_{(A,\sigma)} = \chi(u)\chi(v) = \chi(m(u \circledast v)) = \langle \chi, m(u \circledast v)\rangle_{(A,\sigma)} = \langle \delta(\chi), u \circledast v\rangle^{\otimes 2}_{(A,\sigma)}$. Whence $\langle \delta(\ell) - \ell \otimes \ell, \cdot\rangle^{\otimes 2}_{(A,\sigma)} = 0$ on $\{ u \circledast v : u, v \in A \}$ which spans a dense subset in $(A,\sigma) \circledast_{(\mathsf{R},\tau)} (A,\sigma)$. By continuity and linearity, this implies that $\langle \delta(\ell) - \ell \otimes \ell, \cdot\rangle^{\otimes 2}_{(A,\sigma)} = 0$ on the whole $(A,\sigma) \circledast_{(\mathsf{R},\tau)} (A,\sigma)$, so that $\delta(\ell) = \ell \otimes \ell$ according to Remark 131, p. 65. Moreover, $\epsilon(\chi) = \langle \chi, e(1_{\mathsf{R}})\rangle = 1_{\mathsf{R}}$, whence $\chi \in G(\mathsf{C})$.

   Conversely, let us assume that $\chi \in G(\mathsf{C})$. Then, $\chi(e(1_{\mathsf{R}})) = \epsilon(\chi) = 1_{\mathsf{R}}$, and for each $u, v \in A$, $\langle \delta(\chi), u \circledast v\rangle^{\otimes 2}_{(A,\sigma)} = \langle \chi \otimes \chi, u \circledast v, \ell\rangle^{\otimes 2}_{(A,\sigma)} = \langle \chi, u\rangle_{(A,\sigma)}\langle \chi, v\rangle_{(A,\sigma)} = \chi(m(u \circledast v)) = \langle \chi, m(u \circledast v)\rangle_{(A,\sigma)}$, i.e., $\chi(m(u \circledast v)) = \chi(u)\chi(v)$. Whence $\chi \in {}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}(TA(\mathsf{A}), (\mathsf{R},\tau))$, so that $\chi \in \mathbf{X}_{(\mathsf{R},\tau)}(\mathsf{A})$.

2. Let us assume that $(\Bbbk, \tau)$ is a field with a ring topology. In this situation, given $\chi \in \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$, $\ker \chi$ is a closed ideal of A with codimension one, so $\ker \chi \in ClMax(\mathsf{A})_1$. Let $\chi_1 \in \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$ such that $\mathfrak{m} := \ker \chi = \ker \chi_1$. Since $\mathrm{codim}(\mathfrak{m}) = 1$, each member of A may be expressed uniquely as $u = \alpha e(1_{\mathsf{R}}) + v$, $\alpha \in R$ and $v \in \mathfrak{m}$. As $\chi(e(1_{\mathsf{R}})) = 1_{\mathsf{R}} = \chi_1(e(1_{\mathsf{R}}))$, $\chi(u) = \alpha + \chi(v) = \alpha = \alpha + \chi_1(v) = \chi_1(u)$. Whence $\chi = \chi_1$, and $\mathbf{X}_{(\Bbbk,\tau)}\mathsf{A} \xrightarrow{\ker} ClMax(\mathsf{A})_1$ is one-to-one.

   (a) Let us furthermore assume that τ is the discrete topology of $\Bbbk$. Let $\mathfrak{m} \in ClMax(\mathsf{A})_1$. Then, the quotient field $UA(\mathsf{A})/\mathfrak{m}$ is an extension field of $\Bbbk$. Since $\dim(UA(\mathsf{A})/\mathfrak{m}) = \mathrm{codim}(\mathfrak{m}) = 1$, it follows that $UA(\mathsf{A})/\mathfrak{m} \simeq \Bbbk$.

   Moreover, the quotient topological algebra $TA(\mathsf{A})/\mathfrak{m}$, i.e., the algebra $UA(\mathsf{A})/\mathfrak{m}$ with the final topology induced by the canonical epimorphism $UA(\mathsf{A}) \xrightarrow{\pi_{\mathfrak{m}}} UA(\mathsf{A})/\mathfrak{m}$, is isomorphic to $(\Bbbk, \mathsf{d})$. Indeed, (A,σ) is a topologically-free $(\Bbbk, \mathsf{d})$-vector space, whence a linearly compact $\Bbbk$-vector space. $\mathfrak{m}$ being both closed and finite-codimensional, it is also open ([13, p. 95]). Therefore, the quotient topology is the discrete topology by [23, Theorem 5.7, p. 34].

   All of this implies that $\pi_{\mathfrak{m}}$ may be identified with a member of $\mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A})$, and since $\ker \pi_{\mathfrak{m}} = \mathfrak{m}$, ker is onto.

   (b) For each field $\Bbbk$, one has already $G\mathsf{C} = \mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A}) \simeq ClMax(\mathsf{A})_1 \subseteq ClMax(\mathsf{A})$. Let us assume that $\Bbbk$ is algebraically closed. With Theorem 135 (p. 66) one also has $ClMax(\mathsf{A}) \subseteq ClMax(\mathsf{A})_1$.

□

155 Remark *From Proposition 154.1 one easily deduces that for each rigid ring* (R,τ), $\mathbf{X}_{(\mathsf{R},\tau)} = G \circ ((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}})^{\mathsf{op}}$. *Since* G *has for left adjoint* C_{R} *(Lemma 189, p. 113, in Appendix D.2) and* $((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}})^{\mathsf{op}}$ *also has a left adjoint, namely* $(\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}})^{\mathsf{op}}$, *it follows, in view of Proposition 149, that* $M^{\mathsf{op}}_{(\mathsf{R},\tau)}$ *is naturally equivalent to* $(\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}})^{\mathsf{op}} \circ C_{\mathsf{R}}$.

In fact $(\rho_X)_X : M_{(\mathsf{R},\tau)} \Rightarrow \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}} \circ C^{\mathsf{op}}_{\mathsf{R}} : \mathbf{Set}^{\mathsf{op}} \to {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$ *is a natural isomorphism.*

Naturality is due to the following equalities, with $Y \xrightarrow{f} X$, $g \in R^X$ *and* $p \in R^{(Y)}$.

$$\begin{array}{rcl} ((C_{\mathsf{R}}f)^*(\rho_X(g)))(p) & = & \rho_X(g)((C_{\mathsf{R}}f)(p)) \\ & = & \rho_X(g)\left(\sum_{x\in X}(\sum_{y\in f^{-1}(\{x\})} p(y))\delta_x\right) \\ & = & \sum_{x\in X}\left(\sum_{y\in f^{-1}(\{x\})} p(y)\right) g(x) \\ & = & \sum_{y\in Y} p(y) g(f(y)) \\ & = & (\rho_Y(g\circ f))(p) \\ & = & ((\rho_Y \circ M_{(\mathsf{R},\tau)}(f))(g))(p). \end{array} \tag{51}$$

That ρ_X *is a morphism of monoids from* $M_{(\mathsf{R},\tau)}(X) = ((R,\tau)^X, \mu_X, \eta_X)$ *(see Example 100, p. 47) to* $\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}(C^{\mathsf{op}}_{\mathsf{R}}(X)) = \widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}(R^{(X)}, d_X, e_X) = (((R^{(X)})^*, w^*_{(\mathsf{R},\tau)}), d^*_X \circ (\Lambda_{R^{(X)}} \otimes_{\mathsf{R}} \Lambda_{R^{(X)}})^*, e^*_X \circ \phi)$ *may be checked by hand.*

156 Corollary *Let* $\Bbbk$ *be an algebraically closed field. Let* $\mathsf{A} = ((A,\sigma),\mu,\eta)$ *be an object of* ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\mathsf{d})})$. A *is Gelfand semisimple if, and only if,* A *is topologically semisimple.*

Proof: One knows from Remark 144 that A Gelfand semisimple implies A topologically semisimple.

So let us assume that A is topologically semisimple. Whence the continuous algebra map $TA(\mathsf{A}) \xrightarrow{can_{TA(\mathsf{A})}} \prod_{\mathfrak{m}\in ClMax(\mathsf{A})} TA(\mathsf{A})/\mathfrak{m}$ is one-to-one. $|TA(\mathsf{A})/\mathfrak{m}|_{top} = UA(\mathsf{A})/\mathfrak{m}$ is a field extension of $\Bbbk$. By Theorem 135 (p. 66), $\mathfrak{m}$ has codimension one, and thus $UA(\mathsf{A})/\mathfrak{m} \simeq \Bbbk$, since $\Bbbk$ is algebraically closed. In details: by Proposition 154 (p. 74), $\ker : \mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A}) \simeq ClMax(\mathsf{A})$. So for each $\chi \in \mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A})$, each member u of A is written uniquely as $u = \chi(u)\eta(1_{\Bbbk}) + u_\chi$, $u_\chi := \chi(u)\eta(1_{\Bbbk}) - u \in \ker\chi$. Then, an isomorphism $\phi_\chi : UA(\mathsf{A})/\ker\chi \simeq \Bbbk$ is given by $\phi_\chi(u) := \chi(u)$ with inverse $\phi_\chi^{-1}(\alpha) := \alpha\eta(1_{\Bbbk}) + \ker\chi$, $\alpha \in \Bbbk$.

Since (A,σ) is a linearly compact vector space, and $\mathfrak{m}$ is closed and is of finite codimension, it is also open ([13, p. 95]). Whence $TA(\mathsf{A})/\mathfrak{m} \simeq (\Bbbk,\mathsf{d})$ (under $\phi_{\ker^{-1}\mathfrak{m}}$) according to [23, Theorem 5.7, p. 34]. Therefore, $\prod_{\mathfrak{m}\in ClMax(\mathsf{A})} TA(\mathsf{A})/\mathfrak{m} \simeq A_{(\Bbbk,\mathsf{d})}(ClMax(\mathsf{A}))$ in ${}_{1,c}\mathbf{TopAlg}_{(\Bbbk,\mathsf{d})}$, under $\Phi((u+\mathfrak{m})_{\mathfrak{m}})\colon \mathfrak{m}\in ClMax(\mathsf{A}) \mapsto \phi_{\ker^{-1}\mathfrak{m}}(u) = (\ker^{-1}\mathfrak{m})(u)$.

Because ${}_c\mathbf{Mon}(\mathbb{Top}\mathbb{Free}\mathbb{Vect}_{(\Bbbk,\mathsf{d})}) \xrightarrow{TA} {}_{1,c}\mathbf{TopAlg}_{(\Bbbk,\mathsf{d})}$ is fully faithful, there is a unique morphism of monoids $\mathsf{A} \xrightarrow{can_{\mathsf{A}}} M_{(\Bbbk,\mathsf{d})}(ClMax(\mathsf{A}))$ with $TA(can_{\mathsf{A}}) = \Phi\circ can_{TA(\mathsf{A})}$.

One observes that for each $u\in A$, $\chi\in \mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A})$, $(can_{\mathsf{A}}(u))(\ker\chi) = \chi(u) = \mathsf{G}_{\mathsf{A}}(u)(\chi)$. Therefore the following diagram commutes in the category ${}_c\mathbf{Mon}(\mathbb{Top}\mathbb{Free}\mathbb{Vect}_{(\Bbbk,\mathsf{d})})$.

$$\begin{tikzcd} \mathsf{A} \arrow[r, "can_{\mathsf{A}}"] \arrow[dr, "\mathsf{G}_{\mathsf{A}}"'] & M_{(\Bbbk,\mathsf{d})}(ClMax(\mathsf{A})) \arrow[d, "M_{(\Bbbk,\mathsf{d})}(\ker)"] \\ & M_{(\Bbbk,\mathsf{d})}(\mathbf{X}_{(\Bbbk,\mathsf{d})}(\mathsf{A})) \end{tikzcd} \tag{52}$$

Since A is assumed topologically semisimple, $can_{TA(\mathsf{A})}$ is one-to-one, and so is $\Phi\circ can_{TA(\mathsf{A})}$ and thus also can_{A}, whence according to Diag. (52), G_{A} is one-to-one, i.e., A is Gelfand semisimple. □

7.1.2.4 The Gelfand transform is onto

Let $(\Bbbk,\tau)$ be a field with a ring topology. Let $\mathsf{A} = ((A,\sigma),m,e)$ be an object of ${}_c\mathbf{Mon}(\mathbb{Top}\mathbb{Free}\mathbb{Vect}_{(\Bbbk,\tau)})$. Let us consider two algebra maps, $UA(\mathsf{A}) \xrightarrow{g_{\mathsf{A}}:=UA(\mathsf{G}_{\mathsf{A}})} UA(M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}\mathsf{A})) = A_{\Bbbk}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ (see Example 110, p. 52) and also $TA(\mathsf{A}) \xrightarrow{G_{\mathsf{A}}=TA(\mathsf{G}_{\mathsf{A}})} TA(M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}\mathsf{A})) = A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ (as already defined). Of course, $|G_{\mathsf{A}}|_{top} = g_{\mathsf{A}}$.

157 Lemma *The map g_{A} is onto.*

Proof: By Proposition 154, $\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}) = G(\mathsf{C})$ where $\mathsf{C} = (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}(\mathsf{A})$. By Fact 187 (p. 112, Appendix D.2), $G\mathsf{C}$ is $\Bbbk$-linearly independent. Let $C = (A,\sigma)'$ be the underlying vector space of C. Let B be a basis of C which extends $G\mathsf{C}$. Therefore B_* is a topological basis of (A,σ) (see Remark 81, p. 36). One defines a linear section of g_{A} as follows: let $f\in \Bbbk^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})}$, then let $s(f) := \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)\chi_* \in A$ (sum of a summable family). Since $TA(\mathsf{A}) \xrightarrow{G_{\mathsf{A}}=TA(\mathsf{G}_{\mathsf{A}})} TA(M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}\mathsf{A})) = A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ is

continuous, and because the underlying algebra map $|G_{\mathsf{A}}|_{top}$ of G_{A} is g_{A}, it follows that $g_{\mathsf{A}}(s(f)) = \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)g_{\mathsf{A}}(\chi_*) = \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)g_{\mathsf{A}}(\chi_*) = \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)\delta_\chi$ (since $g_{\mathsf{A}}(\chi_*)(\chi_1) = \chi_1(\chi_*) = \chi^*(\chi_1)$ (by Remark 81) $= \delta_\chi(\chi_1)$, for each $\chi_1 \in \mathbf{X}_{(\Bbbk,\tau)}\mathsf{A}$). Whence $g_{\mathsf{A}}(s(f)) = f$. □

According to Lemma 157, one may define a unique algebra isomorphism $UA(\mathsf{A})/\ker g_{\mathsf{A}} \xrightarrow{\tilde{g}_{\mathsf{A}}} A_{\Bbbk}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ such that $\tilde{g}_{\mathsf{A}} \circ \pi_{\ker g_{\mathsf{A}}} = g_{\mathsf{A}}$, where one denotes by $UA(\mathsf{A}) \xrightarrow{\pi_{\ker g_{\mathsf{A}}}} UA(\mathsf{A})/\ker g_{\mathsf{A}}$ the canonical epimorphism. Whence $TA(\mathsf{A})/\ker G_{\mathsf{A}}$ is a separately continuous algebra (under the quotient topology induced by $\pi_{\ker G_{\mathsf{A}}}$; see Appendix E, p. 118). Since $\tilde{g}_{\mathsf{A}} \circ \pi_{\ker G_{\mathsf{A}}} = g_{\mathsf{A}}$ and $|TA(\mathsf{A})/\ker G_{\mathsf{A}}|_{top} = UA(\mathsf{A})/\ker g_{\mathsf{A}}$, it follows that one has a continuous algebra map $\tilde{G}_{\mathsf{A}}{:}TA(\mathsf{A})/\ker G_{\mathsf{A}} \to A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ with $|\tilde{G}_{\mathsf{A}}|_{top} = \tilde{g}_{\mathsf{A}}$.

158 Lemma *$\tilde{g}_{\mathsf{A}}^{-1}$ is a continuous algebra map from $A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ to $TA(\mathsf{A})/\ker G_{\mathsf{A}}$. Whence $\tilde{G}_{\mathsf{A}}{:}TA(\mathsf{A})/\ker G_{\mathsf{A}} \to A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ is an isomorphism in ${}_{1,c,sep}\mathbf{TopAlg}_{(\Bbbk,\tau)}$.*

Proof: Let $\Bbbk^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} \xrightarrow{s} A$ be the $\Bbbk$-linear section of g_{A} as defined in the proof of Lemma 157. One observes that $\pi_{\ker G_{\mathsf{A}}} \circ s$ is the inverse $\tilde{g}_{\mathsf{A}}^{-1}$ of $\tilde{g}_{\mathsf{A}}$. (Indeed, $\tilde{g}_{\mathsf{A}}(\pi_{\ker G_{\mathsf{A}}}(s(f))) = g_{\mathsf{A}}(s(f)) = f$ for each $f \in \Bbbk^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})}$.)

The section s is actually a continuous map $(\Bbbk,\tau)^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} \xrightarrow{s} (A,\sigma)$. Indeed, first recall that B is a basis of $(A,\sigma)'$ which extends $G(\mathsf{C})$, with $\mathsf{C} = (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}(\mathsf{A}))$. So the topology σ on A is the same as the initial topology induced by $(A \xrightarrow{(b_*)'=b} (\Bbbk,\tau))_{b\in B}$ since B_* is a topological basis of (A,σ) (Remark 81, p. 36). So let $b \in B$, and let us consider the map $b \circ s{:}\Bbbk^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} \to \Bbbk$, given by $f \mapsto b(s(f)) = b(\sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)\chi_*)$. Thus, $b(s(f)) = \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)b(\chi_*) = \sum_{\chi\in\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} f(\chi)\chi^*(b) = 0$ if $b \notin \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$ and is equal to $f(b)$ if $b \in \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$. In other terms $b \circ s = 0$ whenever $b \notin \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$ and $b \circ s = \pi_b$ when $b \in \mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})$, which shows that $b \circ s$ is continuous for each b, and so is s.

Whence $(\Bbbk,\tau)^{\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})} \xrightarrow{\tilde{g}_{\mathsf{A}}^{-1}=\pi_{\ker G_{\mathsf{A}}}\circ s} (A,\sigma)/\ker G_{\mathsf{A}}$ is continuous too (where $(A,\sigma)/\ker G_{\mathsf{A}}$ has the quotient topology), and thus is a continuous algebra map $A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})) \xrightarrow{\tilde{g}_{\mathsf{A}}^{-1}} TA(\mathsf{A})/\ker G_{\mathsf{A}}$. □

159 Corollary *Let $(\Bbbk,\tau)$ be a field with a ring topology. Let A be an object of ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$. The Gelfand transform G_{A} is an isomorphism in ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$ if, and only if, A is Gelfand semisimple.*

Proof: Let us assume that $\ker \mathsf{G}_{\mathsf{A}} = (0)$. By Lemma 158, $TA(\mathsf{A}) \xrightarrow{G_{\mathsf{A}}} TA(M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))) = A_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ is an isomorphism in the category ${}_{1,c,sep}\mathbf{TopAlg}_{(\Bbbk,\tau)}$. Since TA is fully faithful, it follows that $\mathsf{A} \xrightarrow{\mathsf{G}_{\mathsf{A}}} M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ is an isomorphism in ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})$. □

160 Remark *The adjunction from Proposition 149 (p. 73)*

$$M^{\mathsf{op}}_{(\Bbbk,\tau)} \dashv \mathbf{X}_{(\Bbbk,\tau)} \colon \mathbf{Set} \to {}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})^{\mathsf{op}}$$

provides an equivalence of categories between **Set** *and the opposite of full subcategory of* ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})$ *spanned by the Gelfand semisimple commutative monoids (according to Corollary 159, Lemma 145 (p. 72) and Remark 150 (p. 73)).*

161 Corollary $(\Bbbk,\tau)$ *be a field with a ring topology. Let* A *be an object of* ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})$. A *is Gelfand semisimple if, and only if,* $A \simeq M_{(\Bbbk,\tau)}(X)$ *for some set* X.

Proof: According to Corollary 159, if A is Gelfand semisimple, then $\mathsf{A} \simeq M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$. Conversely, let us assume that $\mathsf{A} \simeq M_{(\Bbbk,\tau)}(X)$ in ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})$ for some set X. Let ϕ be such an isomorphism. Since the Gelfand transform is natural (Lemma 148, p. 73), the following diagram commutes.

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "\phi"] \arrow[d, "\mathsf{G}_{\mathsf{A}}"'] & M_{(\Bbbk,\tau)}(X) \arrow[d, "\mathsf{G}_{M_{(\Bbbk,\tau)}(X)}"] \\
M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})) \arrow[r, "M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\phi))"'] & M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(M_{(\Bbbk,\tau)}(X)))
\end{tikzcd} \tag{53}$$

Example 146 (p. 72) tells us that $M_{(\Bbbk,\tau)}(X)$ is Gelfand semisimple, and thus by Corollary 159, $\mathsf{G}_{M_{(\Bbbk,\tau)}(X)}$ is an isomorphism. Since by functoriality, $M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\phi))$ is an isomorphism too, it follows that G_{A} is an isomorphism, whence A is Gelfand semisimple. □

162 Remark *Corollary 161 states that, for a field* $(\Bbbk,\tau)$ *with a ring topology, the essential image (see Section 4.4.4, p. 38) of* $M_{(\Bbbk,\tau)}$ *is equivalent to the full subcategory of* ${}_c\mathbf{Mon}(\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{(\Bbbk,\tau)})$ *spanned by the Gelfand semisimple monoids and thus also to* $\mathbf{Set}^{\mathsf{op}}$ *by Remark 160.*

7.2 Von Neumann regular monoids and (topological) semisimplicity

When $\Bbbk$ is an algebraically closed field, a commutative finite-dimensional (unital) $\Bbbk$-algebra A is (Jacobson) semisimple if, and only if, it is of the form $A_{\Bbbk}(X)$ (see Section 2.4.2, p. 15) for a finite set X.

Indeed a finite-dimensional algebra A is artinian[18], whence has finitely many maximal ideals [11, Theorem 7.13, p. 425], and when it is also Jacobson semisimple the canonical map $\mathsf{A} \xrightarrow{can_{\mathsf{A}}} \prod_{\mathfrak{m}\in Max(\mathsf{A})} \mathsf{A}/\mathfrak{m}$, which thus is one-to-one, is also onto by [5] since maximal ideals are mutually coprime[19], and thus provides an isomorphism. Now, each $\mathfrak{m}$ being maximal, $\mathsf{A}/\mathfrak{m}$ is an extension field of $\Bbbk$. By finite-dimensionality of A, $\mathsf{A}/\mathfrak{m}$ certainly is finite-dimensional too, and $\Bbbk$ being assumed algebraically closed, $\mathsf{A}/\mathfrak{m}$ is isomorphic to $\Bbbk$ so that $\mathsf{A} \simeq A_{\Bbbk}(Max(\mathsf{A}))$.

Our next goal is to provide a similar characterization for topologically semisimple commutative monoids in $\mathbb{TopFreeVect}_{(\Bbbk,\tau)}$.

Let $(\Bbbk,\tau)$ be a field with a ring topology. Let us consider the full subcategory of ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$ spanned by those monoids A whose underlying algebra $UA(\mathsf{A})$ is a (commutative) regular von Neumann $\Bbbk$-algebra with a unit (Definition 37, p. 23), i.e., for each $u \in A$, there exists $v \in A$ with $u = uvu$. This category is denoted by ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$. Its objects are also referred to as *von Neumann regular monoids* (the adjective "commutative" being dropped).

163 Remark ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$ *is* isomorphism-closed*, i.e., given an isomorphism* $\mathsf{A} \xrightarrow{f} \mathsf{B}$ *in* ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$ *where either* A *or* B *is an object of* ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$*, then the other also is. (It suffices to transport the weak inverse map along the isomorphism.)*

164 Example *Let* $\Bbbk$ *be a field, and let* τ *be a ring topology on* $\Bbbk$*. Then, for each set* X*,* $M_{(\Bbbk,\tau)}X$ *is an object of* ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$*. (Indeed,* $UA(M_{(\Bbbk,\tau)}(X)) = A_{\Bbbk}(X)$ *and* $A_{\Bbbk}(X)$ *is von Neumann regular, cf. Example 39, p. 23)). Therefore,* $M_{(\Bbbk,\tau)}$ *co-restricts as a functor from* $\mathbf{Set}^{\mathsf{op}}$ *to* ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$*.*

165 Lemma *Let* (R,τ) *be a rigid ring. Let* C *be an object of* ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$ *(see Appendix D, p. 111). Then,* $\mathbf{X}_{(\mathsf{R},\tau)}(\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}(\mathsf{C})) \simeq G\mathsf{C}$*.*

[18] Equivalently, having the *descending chain condition* on ideals (i.e., there does not exist an infinite strictly descending chain of ideals).

[19] I.e., $\mathfrak{m}_1 + \mathfrak{m}_2 = A$ for each $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(\mathsf{A})$, $\mathfrak{m}_1 \neq \mathfrak{m}_2$.

Proof: $\mathbf{X}_{(\mathsf{R},\tau)}(\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}(\mathsf{C})) = G((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}(\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}}(\mathsf{C})))$ (by Remark 155, p. 76) $\simeq G\mathsf{C}$ by Theorem 102 (p. 48), since G is a functor. □

One already knows from Example 124 (p. 60) that for a rigid ring (R,τ), $C_{\mathsf{R}}X \simeq (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}}(M_{(\mathsf{R},\tau)}(X))$ for each set X. Actually this isomorphism relation is natural in the set X.

166 Lemma *Let (R,τ) be a rigid ring. Let X be a set. Then, $C_{\mathsf{R}} \simeq ((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}})^{\mathsf{op}} \circ M^{\mathsf{op}}_{(\mathsf{R},\tau)}$.*

Proof: By Remark 155 (p. 76), $M^{\mathsf{op}}_{(\mathsf{R},\tau)} \simeq (\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}})^{\mathsf{op}} \circ C_{\mathsf{R}}$, and thus $((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}})^{\mathsf{op}} \circ M^{\mathsf{op}}_{(\mathsf{R},\tau)} \simeq ((\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)}})^{\mathbb{op}})^{\mathsf{op}} \circ (\widetilde{\mathbb{Alg}_{(\mathsf{R},\tau)}})^{\mathsf{op}} \circ C_{\mathsf{R}} \simeq C_{\mathsf{R}}$. □

We are now in position to state the main result of this section.

167 Theorem *Let $(\Bbbk,\tau)$ be a field with a ring topology. Let A be an object of ${}_c\mathbf{Mon}(\mathbb{TopFreeVect}_{(\Bbbk,\tau)})$. Let us consider the following assertions.*

1. *A is Gelfand semisimple.*
2. *A is topologically semisimple.*
3. *$UA(\mathsf{A})$ is semisimple.*
4. *$(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(\mathsf{A})$ is cosemisimple.*
5. *A is an object of ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$.*

The following implications hold: 1 ⇒ 2 ⇒ 3 ⇔ 4 and 1 ⇒ 5 ⇒ 3. Under the additional assumption that $\Bbbk$ is algebraically closed, all the above assertions are equivalent.

Proof: Remark 144 (p. 72) states that 1⇒2.

Remark 137 (p. 69) provides 2⇒3.

Corollary 140 (p. 70) gives 3⇔4.

Corollary 161 (p. 79) establishes the implication 1⇒5 since by Example 164 (p. 80), $M_{(\Bbbk,\tau)}(X)$ is an object of ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$, and ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$ is an isomorphism-closed category (Remark 163, p. 80).

That 5 ⇒ 3 follows from the definition of ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$ and the fact that any von Neumann regular (unital) algebra (commutative or not) is Jacobson semisimple. (Indeed, let $\mathsf{A} = (A, m, 1_{\mathsf{A}})$ be a von Neumann regular algebra. Let $x \in J(\mathsf{A})$. Let $y \in A$ such that $x = xyx$. Then, $x(1_{\mathsf{A}} - yx) = 0$. But

$-yx \in J(\mathsf{A})$ (since $J(\mathsf{A})$ is a two-sided ideal). Since by [10, Proposition 2.9, p. 64], for each $z \in J(\mathsf{A})$, $1+z$ is invertible, it follows that $x=0$.)

It remains to prove that 3⇒1 for instance, under the assumption that $\Bbbk$ is algebraically closed. So let us assume that $UA(\mathsf{A})$ is semisimple. By Corollary 140 (p. 70), this is equivalent to $O(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ semisimple, with $\mathsf{C} := (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(\mathsf{A})$. But $\widetilde{\mathbb{D}_*}(\mathsf{C}) = \tilde{\mathbb{A}}(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C}))$ (section 5.3.2.2, p. 52) so that $O(\widetilde{\mathbb{D}_*}(\mathsf{C})) = O(\tilde{\mathbb{A}}(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C}))) = UA(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C}))$, and thus by Corollary 141 (p. 70), $\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C})$ is topologically semisimple. By Corollary 156, p. 76 (since $\Bbbk$ is assumed algebraically closed) this equivalently means that $\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C})$ is Gelfand semisimple. Therefore according to Corollary 159, p. 78, $\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C}) \simeq M_{(\Bbbk,\mathsf{d})}(\mathbf{X}_{(\Bbbk,\mathsf{d})}(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C})))$.

But

$$\begin{array}{rcl} M_{(\Bbbk,\mathsf{d})}(\mathbf{X}_{(\Bbbk,\mathsf{d})}(\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(\mathsf{C}))) & \simeq & M_{(\Bbbk,\mathsf{d})}(G(\mathsf{C})) \\ & & \text{(by Lemma 165)} \\ & = & M_{(\Bbbk,\mathsf{d})}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})) \\ & & \text{(according to Proposition 154, p. 74)} \\ & \simeq & \widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}(C_{\Bbbk}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))). \\ & & \text{(by Remark 155, p. 76)} \end{array} \tag{54}$$

Therefore, $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(\mathsf{A}) = \mathsf{C} \simeq C_{\Bbbk}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$ (since $\widetilde{\mathbb{Alg}_{(\Bbbk,\mathsf{d})}}$, as any equivalence, reflects isomorphisms) $\simeq (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A})))$ (by Lemma 166). Since $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}$ also reflects isomorphisms, it follows that $\mathsf{A} \simeq M_{(\Bbbk,\tau)}(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))$. Finally, by Corollary 161 (p. 79), A is Gelfand semisimple, i.e., 1 holds. □

As a first corollary one recovers the result recalled at the beginning of Section 7.2 by considering the discrete topology on the base field.

168 Corollary *Let $\Bbbk$ be an algebraically closed field. Let A be an object of ${}_c\mathbf{Mon}(\mathbb{FinDimVect}_{\Bbbk})$. The finite-dimensional commutative algebra $UA(\mathsf{A})$ is semisimple if, and only if, $O(\mathsf{A}) \simeq A_{\Bbbk}(Max(O(\mathsf{A})))$.*

169 Corollary *Let $(\Bbbk,\tau)$ be an algebraically closed field with a ring topology. Then, the essential image (see Section 4.4.4, p. 38) $\mathbf{EssIm}(M_{(\Bbbk,\tau)})$ of $M_{(\Bbbk,\tau)}$ is equal to ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$ and the equivalence ${}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\Bbbk,\tau)}) \simeq {}_{\epsilon,coc}\mathbf{Coalg}^{\mathsf{op}}_{\Bbbk}$ from Corollary 104 (p. 49) restricts to an equivalence ${}_c\mathbf{VN}_{(\Bbbk,\tau)} \simeq {}_{\epsilon,coc}\mathbf{CSCoalg}^{\mathsf{op}}_{\Bbbk}$ (see Definition 196, p. 115, in Appendix D.3). In particular, ${}_c\mathbf{VN}_{(\Bbbk,\tau)} \simeq \mathbf{Set}^{\mathsf{op}}$ for each ring topology τ on an algebraically closed field $\Bbbk$.*

Proof: Theorem 167, together with Remark 162, provides the equality $\mathbf{EssIm}(M_{(\Bbbk,\tau)}) = {}_c\mathbf{VN}_{(\Bbbk,\tau)}$.

Proposition 139 states that if C is a cosemisimple (cocommutative) coalgebra, then $\widetilde{\mathbb{Alg}_{(\Bbbk,\tau)}}(\mathsf{C})$ is semisimple, whence is an object of ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$ by Theorem 167. By Theorem 167 again, if A is an object of ${}_c\mathbf{VN}_{(\Bbbk,\tau)}$, then the topological dual coalgebra $(\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(\mathsf{A})$ is cosemisimple.

To conclude, the equivalence with $\mathbf{Set}^{\mathsf{op}}$ follows from Corollary 199, p. 116 (Appendix D.3). □

170 Remark *An equivalence of categories ${}_c\mathbf{VN}_{(\Bbbk,\tau)} \simeq \mathbf{Set}^{\mathsf{op}}$ for an algebraically closed field with a ring topology $(\Bbbk,\tau)$, whose existence is guaranteed by Corollary 169, is given by the (restricted) functor ${}_c\mathbf{VN}_{(\Bbbk,\tau)} \xrightarrow{\mathbf{X}^{\mathsf{op}}_{(\Bbbk,\tau)}} \mathbf{Set}^{\mathsf{op}}$ with equivalence inverse the (co-restricted) functor $\mathbf{Set}^{\mathsf{op}} \xrightarrow{M_{(\Bbbk,\tau)}} {}_c\mathbf{VN}_{(\Bbbk,\tau)}$.*

Moreover, one also notices that the underlying topologically-free $(\Bbbk,\tau)$-vector space $U_m(\mathsf{A})$ of a von Neumann regular monoid A has a canonical topological basis, namely $(\mathbf{X}_{(\Bbbk,\tau)}(\mathsf{A}))_* = (G\mathsf{C})_*$ (with $\mathsf{C} := (\widetilde{\mathbb{Top}^{\mathbb{d}}_{(\Bbbk,\tau)}})^{\mathbb{op}}(\mathsf{A})$). E.g., $(\mathbf{X}_{(\Bbbk,\tau)}(M_{(\Bbbk,\tau)}(X))_* = \{\, (\pi_x)_* : x \in X \,\} = \{\, \delta_x : x \in X \,\}$.

171 Remark *Another interpretation of Corollary 169 goes as follows. Let (R,τ) be a rigid ring. One notices that $\mathbf{Set}^{\mathsf{op}} \xrightarrow{P_{(\mathsf{R},\tau)}} \mathbf{FreeTopMod}_{(\mathsf{R},\tau)}$ (Remark 12, p. 14) provides a strong symmetric monoidal functor (see Appendix B.3, p. 93) $\mathbb{Set}^{\mathbb{op}} \xrightarrow{\mathbb{P}_{(\mathsf{R},\tau)}} \mathbb{FreeTopMod}_{(\mathsf{R},\tau)}$, with coherence components the isomorphism $(R,\tau)^X \circledast_{(\mathsf{R},\tau)} (R,\tau)^Y \xrightarrow{\Omega_{X,Y}} (R,\tau)^{X\times Y}$ from Remark 91 (p. 43) and $(R,\tau) \xrightarrow{\omega} M_{(\mathsf{R},\tau)}(1) = (R,\tau)^1$, $\omega(1_{\mathsf{R}})(\varnothing) := 1_{\mathsf{R}}$ (where, as in Remark 174, p. 92, in Appendix B.2, $\mathbb{Set} = (\mathbf{Set},\times,1)$ and $1 := \{\varnothing\}$). Therefore one has an induced functor ${}_{coc}\mathbf{Comon}(\mathbb{Set})^{\mathsf{op}} = {}_c\mathbf{Mon}(\mathbb{Set}^{\mathbb{op}}) \xrightarrow{\widetilde{\mathbb{P}_{(\mathsf{R},\tau)}}} {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)})$. This functor factors as indicated in the following diagram (see Remark 174 for the definition of Δ).*

$$\begin{tikzcd} {}_{coc}\mathbf{Comon}(\mathbb{Set}^{\mathbb{op}}) \arrow[r, "\widetilde{\mathbb{P}_{(\mathsf{R},\tau)}}"] & {}_c\mathbf{Mon}(\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}) \\ \mathbf{Set}^{\mathsf{op}} \arrow[u, hook, "\Delta^{\mathsf{op}}"] \arrow[ur, "M_{(\mathsf{R},\tau)}"'] & \end{tikzcd} \tag{55}$$

By Corollary 169, for an algebraically closed field $(\Bbbk,\tau)$ with a ring topology, there is no other structure of (Gelfand or topologically) semisimple

monoid than the one corresponding, by means of $\widetilde{\mathbb{P}_{(\Bbbk,\tau)}}$*, to the unique diagonal cosemigroup on a set (see Remark 174 in Appendix B.2).*

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A Category theory

This appendix is about basic and standard concepts from category theory. One does not worry about size issues and in this presentation "category" means a locally small category while "set" loosely means both small and large set.

A.1 Standard notions

One assumes that the reader is familiar with basic notions from category theory among subcategories, (full, faithful) functors, natural transformations,

natural isomorphisms, equivalence of categories, categorical product, terminal object, left/right adjoints, unit and counit of an adjunction. Only some notations and terminology about them are introduced. Of course [17] is a fundamental reference for this subject.

1. Given a category $\mathbf{C}$, an object C of $\mathbf{C}$ is also referred to as a $\mathbf{C}$-*object* and a morphism of $\mathbf{C}$ may be equally called a $\mathbf{C}$-*morphism*. The *identity at* C is usually denoted by id_C or just by id. Given $\mathbf{C}$-objects C, D, $\mathbf{C}(C, D)$ denotes the *hom-set*, i.e., the set of all $\mathbf{C}$-morphisms from C to D. The *identity functor* $id_{\mathbf{C}}$, or id, of $\mathbf{C}$ is the functor which is both the identity on objects and on morphisms. As usually by a $\mathbf{C}$-*isomorphism* or simply an *isomorphism* (in $\mathbf{C}$) is meant a $\mathbf{C}$-morphism $f \in \mathbf{C}(C, D)$ such that there is a $\mathbf{C}$-morphism $g \in \mathbf{C}(D, C)$ with $f \circ g = id_D$ and $g \circ f = id_C$. This uniquely defined g is denoted by $f^{-1} := g$. One writes $C \simeq D$ to denote isomorphic objects.

2. Some categories: Let R be a non-trivial commutative ring with a unit.

   (a) $\mathbf{Set}$ the category of sets with set-theoretic maps.

   (b) $\mathbf{Top}$ the category of Hausdorff topological spaces with continuous maps.

   (c) $\mathbf{Mod}_{\mathsf{R}}$ the category of unital left-R-modules with R-linear maps (where *unital* means that the scalar action of the unit of R is the identity on the module). When R is a field $\Bbbk$ one uses $\mathbf{Vect}_{\Bbbk}$ instead.

   (d) $\mathbf{Alg}_{\mathsf{R}}$ the category of (associative) R-algebras and with algebra maps. (The multiplication m_{A} of an algebra A thus is in particular a R-bilinear map $A \times A \xrightarrow{m_{\mathsf{A}}} A$.)

   (e) ${}_1\mathbf{Alg}_{\mathsf{R}}$, ${}_c\mathbf{Alg}_{\mathsf{R}}$, and ${}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ are the categories of unital algebras, commutative algebras, and unital and commutative algebras, where in the unital cases the algebra maps are assumed unit-preserving.

3. A *concrete category* $\mathbf{C}$ over $\mathbf{D}$ is a pair $(\mathbf{C}, \mathbf{C} \xrightarrow{U} \mathbf{D})$ with U a faithful functor. Given concrete categories $(\mathbf{C}_i, U_i)$, $i = 1, 2$, over $\mathbf{D}$, by a *concrete functor* $(\mathbf{C}_1, U_1) \xrightarrow{F} (\mathbf{C}_2, U_2)$ is meant an ordinary functor $\mathbf{C}_1 \xrightarrow{F} \mathbf{C}_2$ such that the following diagram commutes. Such a functor

is necessarily faithful.

$$\begin{tikzcd} \mathbf{C}_1 \arrow[rr, "F"] \arrow[dr, "U_1"'] & & \mathbf{C}_2 \arrow[dl, "U_2"] \\ & \mathbf{D} & \end{tikzcd} \tag{56}$$

4. A functor $\mathbf{C} \xrightarrow{F} \mathsf{D}$ is *essentially surjective* when each object D in $\mathbf{D}$ is isomorphic to an object of the form FC, for some object C in $\mathbf{C}$. Whence an equivalence of categories is a fully faithful and essentially surjective functor.

5. By an *embedding* (or *embedding functor*) $\mathbf{C} \xrightarrow{F} \mathbf{D}$ is meant a faithful functor F which is injective on objects. For instance, given a subcategory $\mathbf{C}$ of $\mathbf{D}$, the canonical inclusion functor of $\mathbf{C}$ in $\mathbf{D}$ is an embedding denoted by $\mathbf{C} \hookrightarrow \mathbf{D}$. By a *full subcategory* is meant a subcategory $\mathbf{C}$ of D such that its embedding functor $\mathbf{C} \hookrightarrow \mathbf{D}$ is full.

6. Observe that given a category $\mathbf{C}$, each subset X of the objects of $\mathbf{C}$ determines uniquely a full subcategory of $\mathbf{C}$ called the *full subcategory of* $\mathbf{C}$ *spanned by* X, namely the subcategory $\mathbf{C}_X$ whose set of objects is X and $\mathbf{C}_X(C, D) = \mathbf{C}(C, D)$, $C, D \in X$.

7. The notation $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ means that α is a *natural transformation* between two functors $\mathbf{C} \xrightarrow{F,G} \mathbf{D}$. Given a $\mathbf{C}$-object C, $\alpha_C \in \mathbf{D}(F(C), G(C))$ denotes the *component at* C of α. Thus $(\alpha_C)_C$ will be another notation for α. For each functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, there is an *identity* at F, $id_F\colon F \Rightarrow F\colon \mathbf{C} \to \mathbf{D}$ with $(id_F)_C := id_{F(C)}$.

8. A *natural isomorphism* $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ is a natural transformation the components of which are isomorphisms in $\mathbf{D}$. Two functors $F, G\colon \mathbf{C} \to \mathbf{D}$ are said to be *naturally isomorphic*, which is denoted $F \simeq G$, if there is a natural isomorphism $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$.

9. $\mathbf{C} \simeq \mathbf{D}$ means that the categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, i.e., that there is an equivalence of categories $\mathbf{C} \xrightarrow{F} \mathbf{D}$. In this situation an *equivalent inverse* (or *quasi-inverse*) of F is a functor $\mathbf{D} \xrightarrow{G} \mathbf{C}$ such that there are two natural isomorphisms $\eta\colon id_{\mathbf{C}} \Rightarrow G \circ F\colon \mathbf{C} \to \mathbf{C}$ and $\epsilon\colon F \circ G \Rightarrow id_{\mathbf{D}}\colon \mathbf{D} \to \mathbf{D}$.

   One observes that an equivalence of categories $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *reflects isomorphisms*, i.e., $F(C) \xrightarrow{g} F(C')$ is a $\mathbf{D}$-isomorphism if, and only if, the

unique $\mathbf{C}$-morphism $C \xrightarrow{f} C'$ such that $F(f) = g$ is a $\mathbf{C}$-isomorphism ([2, Proposition 3.31, p. 35]).

10. The *product* $\mathbf{C} \times \mathbf{D}$ of two categories $\mathbf{C}, \mathbf{D}$, is the category with objects (C, D), where C is a $\mathbf{C}$-object and D is $\mathbf{D}$-object, and with

$$\mathbf{C} \times \mathbf{D}((C_1, D_1), (C_2, D_2)) = \mathbf{C}(C_1, C_2) \times \mathbf{D}(D_1, D_2).$$

Composition is component-wise. By a *bifunctor* is meant a functor with domain a product of two categories.

11. The *opposite category* $\mathbf{C}^{\mathsf{op}}$ is the category with the same objects and morphisms as $\mathbf{C}$ but with opposite composition. In other words one has $\mathbf{C}^{\mathsf{op}}(D, C) = \mathbf{C}(C, D)$. f^{op} denotes the $\mathbf{C}$-morphism f considered as a $\mathbf{C}^{\mathsf{op}}$-morphism.

172 Facts *(a)* $(\mathbf{C} \times \mathbf{D})^{\mathsf{op}} = \mathbf{C}^{\mathsf{op}} \times \mathbf{D}^{\mathsf{op}}$.

(b) Let $\mathbf{C} \xrightarrow{F} \mathsf{D}$ *be a functor. Then,* $\mathbf{C}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathbf{D}^{\mathsf{op}}$ *is just the functor* F*, called the* opposite *of* F*. In details, for a* $\mathbf{C}$*-morphism* f*,* $F^{\mathsf{op}}(f^{\mathsf{op}}) = F(f)^{\mathsf{op}}$*. Actually the opposite construction provides a functor of the meta-category of categories. In particular,* $id_{\mathbf{C}}^{\mathsf{op}} = id_{\mathbf{C}^{\mathsf{op}}}$*, and* $(G \circ F)^{\mathsf{op}} = G^{\mathsf{op}} \circ F^{\mathsf{op}}$ *for composable functors* G, F*.*

(c) A natural transformation $\alpha: F \Rightarrow G: \mathbf{C} \to \mathbf{D}$ *may be also interpreted as a natural transformation* $\alpha^{\mathsf{op}}: G^{\mathsf{op}} \Rightarrow F^{\mathsf{op}}: \mathbf{C}^{\mathsf{op}} \to \mathbf{D}^{\mathsf{op}}$ *with the same components but considered as* $\mathbf{D}^{\mathsf{op}}$*-morphisms. In details,* $(\alpha^{\mathsf{op}})_C = \alpha_C^{\mathsf{op}}$ *for each* $\mathbf{C}$*-object* C*.* α^{op} *is the* opposite of α*. In particular,* $\mathbf{C} \simeq \mathbf{D}$ *if, and only if,* $\mathbf{C}^{\mathsf{op}} \simeq \mathbf{D}^{\mathsf{op}}$*.*

12. By $F \dashv G: \mathbf{C} \to \mathbf{D}$ is meant an adjunction with left adjoint $\mathbf{C} \xrightarrow{F} \mathbf{D}$ and right adjoint $\mathbf{D} \xrightarrow{G} \mathbf{C}$.

A.2 The Proof of Lemma 85, p. 38

Proof: Let c be a $\mathbf{C}$-object. Let $c \xrightarrow{\eta_c} G(F(c))$ in $\mathbf{C}$ be the unit of the adjunction $F \dashv G: \mathbf{C} \to \mathbf{D}$. One observes that $c \xrightarrow{\eta_c} G(F(c)) = G(E(F_0(c)))$. Now, let $c \xrightarrow{f} G(E(d))$ be a $\mathbf{C}$-morphism, with d a $\mathbf{EssIm}(F)$-object. Then, there exists a unique $\mathbf{D}$-morphism $F(c) = E(F_0(c)) \xrightarrow{f^\sharp} E(d)$ such that the

following diagram commutes.

$$\begin{tikzcd} G(F(c)) \arrow[r, "G(f^\sharp)"] & G(E(d)) \\ c \arrow[u, "\eta_c"] \arrow[ur, "f"'] & \end{tikzcd} \tag{57}$$

Since E is fully faithful, there is a unique $\mathbf{EssIm}(F)$-morphism $F_0(c) \xrightarrow{\tilde{f}} d$ such that $E(\tilde{f}) = f^\sharp$. Of course, the following diagram commutes.

$$\begin{tikzcd} G(F(c)) = G(E(F_0(c))) \arrow[r, "G(E(\tilde{f}))=G(f^\sharp)"] & G(E(d)) \\ c \arrow[u, "\eta_c"] \arrow[ur, "f"'] & \end{tikzcd} \tag{58}$$

Since $f^\sharp = E(\tilde{f})$ is unique with this property, it follows that $\tilde{f}$ is unique too with this property. In order to conclude, by applying [17, Theorem 2.(i), p. 83], it remains to check that $\eta\colon id_{\mathbf{C}} \Rightarrow G \circ E \circ F_0\colon \mathbf{C} \to \mathbf{C}$ is a natural transformation, but this is immediate because $G \circ E \circ F_0 = G \circ F$. □

B Monoidal categories and functors

This appendix contains basic facts about monoidal categories and monoidal functors, and a part of it is reprinted from [19]. See [17, Chap. VII] for more details.

B.1 Monoidal categories and coherence conditions

Throughout $\mathbb{C} = (\mathbf{C}, - \otimes -, I, \alpha, \lambda, \rho)$ denotes a monoidal category with α the associativity and λ and ρ the left and right unit constraints (they are referred to as *coherence constraints*). These constraints have to make commute some diagrams to ensure *coherence* of the monoidal category (see [17, Chap. VII, p. 165]). The *coherence conditions* are recalled below.

1. Coherence of associativity: for each $\mathbf{C}$-objects A, B, C, D, the following

diagram commutes.

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes id_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd} \tag{59}$$

2. Coherence of left and right units: for each $\mathbf{C}$-objects A, B, the following diagram commutes.

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd} \tag{60}$$

A monoidal category is usually abbreviated into $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ or even $\mathbb{C} = (\mathbf{C}, - \otimes -)$. If $\mathbb{C}$ even is symmetric monoidal, the symmetry will be denoted by $\sigma = (C \otimes D \xrightarrow{\sigma_{C,D}} D \otimes C)_{C,D}$. Symmetry requires the following supplementary coherence conditions.

1. Unit coherence for symmetry: for each $\mathbf{C}$-object A, the following diagram commutes.

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\sigma_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd} \tag{61}$$

2. Associativity coherence for symmetry: for each $\mathbf{C}$-objects A, B, C, the following diagram commutes.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\sigma_{A,B} \otimes id_C"] \arrow[d, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\
A \otimes (B \otimes C) \arrow[d, "\sigma_{A,B \otimes C}"'] & B \otimes (A \otimes C) \arrow[d, "id_B \otimes \sigma_{A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"'] & B \otimes (C \otimes A)
\end{tikzcd} \tag{62}$$

3. Inverse law for symmetry: for each $\mathbf{C}$-objects A, B, $\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B}$.

When $\mathbf{C}$ has finite products, then with the tensor product given by the categorical product and the unit given by the terminal object of $\mathbf{C}$, the above coherence conditions are automatically fulfilled, and the symmetric monoidal category thus provided is referred to as *cartesian*.

If $\mathbb{C}$ is a (symmetric) monoidal category, then so is $\mathbb{C}^{\mathbb{op}} := (\mathbf{C}^{\mathsf{op}}, - \otimes^{\mathsf{op}} -, I, (\alpha^{-1})^{\mathsf{op}}, (\varrho^{-1})^{\mathsf{op}}, (\lambda^{-1})^{\mathsf{op}})$, called the *dual* monoidal category of $\mathbb{C}$.

B.2 Monoids and comonoids

A *monoid in* $\mathbb{C}$ is a triple $(C, C \otimes C \xrightarrow{m} C, I \xrightarrow{e} C)$ such that the diagrams

$$\begin{tikzcd}
(C \otimes C) \otimes C \arrow[r, "m \otimes id_C"] \arrow[d, "\alpha_{C,C,C}"'] & C \otimes C \arrow[dd, "m"] \\
C \otimes (C \otimes C) \arrow[d, "id_C \otimes m"'] & \\
C \otimes C \arrow[r, "m"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes I \arrow[r, "id_C \otimes e"] \arrow[dr, "\rho_C"'] & C \otimes C \arrow[d, "m"] & I \otimes C \arrow[l, "e \otimes id_C"'] \arrow[dl, "\lambda_C"] \\
& C &
\end{tikzcd}$$

commute, while a *monoid morphism* $(C, m, e) \longrightarrow (C', m', e')$ is any $f{:}C \to C'$ making the diagrams

$$\begin{tikzcd}
C \otimes C \arrow[r, "m"] \arrow[d, "f \otimes f"'] & C \arrow[d, "f"] \\
C' \otimes C' \arrow[r, "m'"'] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, "e'"'] & C \arrow[d, "f"] \\
& C'
\end{tikzcd}$$

commutative. This defines the category $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$.
The category $\mathbf{Comon}\mathbb{C}$ of *comonoids* over $\mathbb{C}$ is defined to be $(\mathbf{Mon}\mathbb{C}^{\mathbb{op}})^{\mathsf{op}}$, the opposite of the category of monoids in $\mathbb{C}^{\mathbb{op}}$. *Comonoids* thus are triples $(C, C \xrightarrow{\mu} C \otimes C,\ C \xrightarrow{\epsilon} I)$ such that the diagrams

$$\begin{tikzcd}
C \arrow[r, "\mu"] \arrow[dd, "\mu"'] & C \otimes C \arrow[d, "id_C \otimes \mu"] \\
& C \otimes (C \otimes C) \arrow[d, "\alpha^{-1}_{C,C,C}"] \\
C \otimes C \arrow[r, "\mu \otimes id_C"'] & (C \otimes C) \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes I \arrow[dr, "\rho^{-1}_C"'] & C \otimes C \arrow[l, "id_C \otimes \epsilon"'] \arrow[r, "\epsilon \otimes id_C"] & I \otimes C \\
& C \arrow[u, "\mu"] \arrow[ur, "\lambda^{-1}_C"'] &
\end{tikzcd}$$

commute, while a *comonoid morphism* $(C, \mu, \epsilon) \longrightarrow (C', \mu', \epsilon')$ is any $f{:}C \to C'$ making the diagrams

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f \otimes f"] \\ C' \arrow[r, "\mu'"'] & C' \otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "f"'] & I \\ C' \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

commute.

A monoid (C, m, e) is called *commutative* if, and only if, $m = m \circ \sigma_{C,C}$ with $\sigma_{C,C} \colon C \otimes C \to C \otimes C$ the symmetry; dually, a comonoid (C, μ, ϵ) is called *cocommutative*, provided that $\mu = \sigma_{C,C} \circ \mu$. By ${}_c\mathbf{Mon}\mathbb{C}$ and ${}_{coc}\mathbf{Comon}\mathbb{C}$ we denote the categories of commutative monoids and cocommutative comonoids respectively, with all (co)monoid morphisms as morphisms. Of course, one has ${}_{coc}\mathbf{Comon}\mathbb{C} = ({}_c\mathbf{Mon}\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$.

The following are well-known examples of these concepts.

173 Example *1. If $\mathbb{C}$ is the (cartesian) monoidal category* **Set** *of sets and maps, then* $\mathbf{Mon}\mathbb{C}$ *is (isomorphic to) the category of ordinary monoids.*

2. If $\mathbb{C}$ is the monoidal category $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$ of R-modules and R-linear maps for a commutative unital ring R with its usual tensor product $\otimes_{\mathsf{R}}$, then:

(a) $\mathbf{Mon}\mathbb{C}$ *is isomorphic to the category ${}_1\mathbf{Alg}_{\mathsf{R}}$ of unital R-algebras. Indeed, a unital R-algebra $\mathsf{A} = (A, m_{\mathsf{A}}, 1_{\mathsf{A}})$ corresponds to the monoid $(A, \mu_{\mathsf{A}}, \eta_{\mathsf{A}})$, with $A \otimes_{\mathsf{R}} A \xrightarrow{\mu_{\mathsf{A}}} A$ the unique R-linear map such that $\mu_{\mathsf{A}}(x \otimes y) = m_{\mathsf{A}}(x, y)$, $x, y \in A$, and $R \xrightarrow{\eta_{\mathsf{A}}} A$ is given by $\eta_{\mathsf{A}}(1_{\mathsf{R}}) = 1_{\mathsf{A}}$. Conversely from a monoid (A, μ, η) one obtains a unital algebra $(A, \mu_{bil}, \eta(1_{\mathsf{R}}))$, where $A \times A \xrightarrow{\mu_{bil}} A$ is the unique R-bilinear map associated with $A \otimes_{\mathsf{R}} A \xrightarrow{\mu} A$.*
The correspondence $O \colon (A, \mu, \eta) \mapsto (A, \mu_{bil}, \eta(1_{\mathsf{R}}))$ defines a functorial isomorphism $\mathbf{Mon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}) \simeq {}_1\mathbf{Alg}_{\mathsf{R}}$, which is the identity on arrows, and is concrete over $\mathbf{Mod}_{\mathsf{R}}$ (for the obvious forgetful functors). ("O" stands for "ordinary" since from $O(\mathsf{A})$ is an ordinary unital algebra.)

(b) Likewise ${}_c\mathbf{Mon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}) \simeq {}_{1,c}\mathbf{Alg}_{\mathsf{R}}$ under the (co-)restriction of O as above.

(c) $\mathbf{Comon}\mathbb{C}$ *is the category ${}_\epsilon\mathbf{Coalg}_{\mathsf{R}}$ of counital R-coalgebras (see Appendix D, p. 111, for more details).*

174 Remark *Let $\mathbb{C} = (\mathbf{C}, - \times -, 1)$ be a cartesian monoidal category. Let $C \xrightarrow{\Delta_C} C \times C$ be the* diagonal *of the $\mathbf{C}$-object C, i.e., $\pi_i \circ \Delta_C = id_C$, where*

$C \times C \xrightarrow{\pi_i} C$ is the canonical projection, $i = 1, 2$. Let $C \xrightarrow{!_C} 1$ be the terminal arrow. Then, $(C, \Delta_C, !_C)$ is a cocommutative comonoid in $\mathbb{C}$, called the diagonal comonoid *on C. As a matter of fact $\mathbf{C} \simeq {}_{coc}\mathbf{Comon}(\mathbb{C})$ under the functor $\Delta\colon C \mapsto (C, \Delta_C, !_C)$, with inverse the forgetful functor ${}_{coc}\mathbf{Comon}(\mathbb{C}) \to \mathbf{C}$. (See [20, p. 5].)*

This happens in particular with $\mathbf{C} := \mathbf{Set}$, and in such case $\Delta(X) = (X, \Delta_X, !_X)$ with $\Delta_X(x) = (x, x)$, $!_X(x) = \varnothing$, $x \in X$ (with $1 := \{\varnothing\}$).

B.3 Monoidal functors and their induced functors

We briefly recall the following definitions and facts, too, which are fundamental for this note. See e.g. [3, 22] for a more detailed treatment and for the missing proofs.

175 Definition *Let $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ and $\mathbb{C}' = (\mathbf{C}', - \otimes' -, I')$ be monoidal categories. A* lax monoidal functor from $\mathbb{C}$ to $\mathbb{C}'$ *is a triple $\mathbb{F} := (F, \Phi, \phi)$, where $F\colon \mathbf{C} \to \mathbf{C}'$ is a functor, $\Phi_{C_1,C_2}\colon FC_1 \otimes' FC_2 \to F(C_1 \otimes C_2)$ is a natural transformation and $\phi\colon I' \to FI$ is a $\mathbf{C}$-morphism, subject to certain coherence conditions (see the diagrams below). A lax monoidal functor is called* strong monoidal *(resp.* strict monoidal *), if Φ and ϕ are isomorphisms (resp. identities). Φ, ϕ are the* coherence constraints *of $\mathbb{F}$.*

1. Associativity coherence: for each $\mathbf{C}$-objects A, B, C the following diagram commutes.

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "\alpha'_{FA,FB,FC}"] \arrow[d, "\Phi_{A,B} \otimes' id_{FC}"'] & FA \otimes' (FB \otimes' FC) \arrow[d, "id_A \otimes' \Phi_{B,C}"] \\
F(A \otimes B) \otimes' FC \arrow[d, "\Phi_{A \otimes B, C}"'] & FA \otimes' F(B \otimes C) \arrow[d, "\Phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(\alpha_{A,B,C})"'] & F(A \otimes (B \otimes C))
\end{tikzcd} \tag{63}$$

2. Left unit coherence: for each $\mathbf{C}$-object B, the following diagram commutes.

$$\begin{tikzcd}
I' \otimes' FB \arrow[r, "\phi \otimes' id_{FB}"] \arrow[d, "\lambda'_{FB}"'] & FI \otimes' FB \arrow[d, "\Phi_{I,B}"] \\
FB & F(I \otimes B) \arrow[l, "F\lambda_B"]
\end{tikzcd} \tag{64}$$

3. Right unit coherence: for each $\mathbf{C}$-object A, the following diagram commutes.

$$\begin{tikzcd} FA \otimes' I' \arrow[r, "id_{FA} \otimes' \phi"] \arrow[d, "\rho'_{FA}"'] & FA \otimes' FI \arrow[d, "\Phi_{A,I}"] \\ FB & F(B \otimes I) \arrow[l, "F\rho_A"] \end{tikzcd} \tag{65}$$

Let $\mathbb{C}, \mathbb{C}'$ be symmetric. A monoidal functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{C}'$ is said to be *symmetric* when furthermore the following diagram commutes for each $\mathbb{C}$-objects A, B.

$$\begin{tikzcd} FA \otimes' FB \arrow[r, "\sigma'_{FA,FB}"] \arrow[d, "\Phi_{A,B}"'] & FB \otimes' FA \arrow[d, "\Phi_{B,A}"] \\ F(A \otimes B) \arrow[r, "F\sigma_{A,B}"'] & F(B \otimes A) \end{tikzcd} \tag{66}$$

176 Facts *Let $\mathbb{C} := (\mathbf{C}, \otimes, I)$, $\mathbb{C}' := (\mathbf{C}', \otimes', I')$ and $\mathbb{C}'' := (\mathbf{C}'', \otimes'', I'')$ be (symmetric) monoidal categories.*

1. *$\mathbb{id}_{\mathbb{C}} := (id_{\mathbf{C}}, id_{-\otimes-}, id_I)$ is a monoidal functor from $\mathbb{C}$ to itself, and serves as a unit for the composition of monoidal functors given below. If no ambiguities arise, one let $\mathbb{id} := \mathbb{id}_{\mathbb{C}}$.*

2. *Given (symmetric) monoidal functors $\mathbb{F} = (F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{D}$ and $\mathbb{G} = (G, \Psi, \psi)\colon \mathbb{D} \to \mathbb{E}$, one defines a (symmetric) monoidal functor $\mathbb{G} \circ \mathbb{F} := \mathbb{H} = (H, \Theta, \theta)\colon \mathbb{C} \to \mathbb{E}$ with*

   (a) *$H = G \circ F$.*

   (b) *Given objects C_1, C_2 of $\mathbf{C}$, Θ_{C_1,C_2} is the composite $\mathbf{C}''$-morphism $G(F(C_1)) \otimes'' GF(C_2) \xrightarrow{\Psi_{F(C_1),F(C_2)}} G(F(C_1) \otimes' F(C_2)) \xrightarrow{G(\Phi_{C_1,C_2})} G(F(C_1 \otimes C_2))$.*

   (c) *$\theta := I'' \xrightarrow{\psi} G(I') \xrightarrow{G(\phi)} G(F(I))$.*

By a *(symmetric) monoidal subcategory* of a (symmetric) monoidal category $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ we mean a subcategory $\mathbf{C}'$ of $\mathbf{C}$, closed under tensor products, containing I, and the coherence constraints of $\mathbb{C}$ between $\mathbf{C}'$-objects. (The last condition is automatically fulfilled when $\mathbf{C}'$ is a full subcategory.) The embedding $E_{\mathbf{C}'}$ of $\mathbf{C}'$ into $\mathbf{C}$ then is a strict monoidal functor $\mathbb{E}_{\mathbb{C}'}$.

Let $\mathbb{F} = (F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{D}$ be a monoidal functor, and let $\mathbb{C}'$ and $\mathbb{D}'$ be monoidal subcategories of $\mathbb{C}$ and $\mathbb{D}$, respectively. By a *restriction* of $\mathbb{F}$ to these subcategories is meant a monoidal functor $\mathbb{F}' = (F', \Phi', \phi')\colon \mathbb{C}' \to \mathbb{D}'$ such that (a) $F'C = FC$ for all $\mathbf{C}'$-object C, (b) $\Phi'_{A,B} = \Phi_{A,B}$ for all $\mathbf{C}'$-objects A, B, and (c) $\phi' = \phi$. This is equivalent to saying that $\mathbb{F} \circ \mathbb{E}_{\mathbb{C}'} = \mathbb{E}_{\mathbb{D}'} \circ \mathbb{F}'$.

177 Proposition *Let $\mathbb{F} = (F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{C}'$ be a lax monoidal functor.*

$\tilde{\mathbb{F}}(M, m, e) = (FM, FM \otimes FM \xrightarrow{\Phi_{M,M}} F(M \otimes M) \xrightarrow{Fm} FM, I' \xrightarrow{\phi} FI \xrightarrow{Fe} FM)$ *and* $\tilde{\mathbb{F}} f = Ff$ *define an* induced functor $\tilde{\mathbb{F}}\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$, *such that the diagram*

$$\begin{tikzcd}
\mathbf{Mon}\mathbb{C} \arrow[r, "\tilde{\mathbb{F}}"] \arrow[d, "U_m"'] & \mathbf{Mon}\mathbb{C}' \arrow[d, "U'_m"] \\
\mathbf{C} \arrow[r, "F"'] & \mathbf{C}'
\end{tikzcd}$$

commutes (with forgetful functors U_m and U'_m).

178 Remark *1. When $\mathbb{F}$ is symmetric, then $\tilde{\mathbb{F}}$ also provides a functor ${}_c\mathbf{Mon}\mathbb{C}$ to ${}_c\mathbf{Mon}\mathbb{C}'$ with similar properties as above.*

2. $\widetilde{\mathbb{id}_{\mathbb{C}}} = id_{\mathbf{Mon}(\mathbb{C})}$.

3. $\widetilde{\mathbb{G} \circ \mathbb{F}} = \tilde{\mathbb{G}} \circ \tilde{\mathbb{F}}$.

The data $\mathbb{F} = (F, \Phi, \phi)$ consisting of a functor $\mathbf{C} \xrightarrow{F} \mathbf{C}'$, and, for each $\mathbf{C}$-objects C_1, C_2, $\mathbf{C}'$-morphisms

1. $\Phi_{C_1,C_2}\colon F(C_1 \otimes C_2) \to F(C_1) \otimes' F(C_2)$,

2. $F(I) \xrightarrow{\phi} I'$

such that $(F^{\mathsf{op}}, \Phi^{\mathsf{op}}, \phi^{\mathsf{op}})\colon \mathbb{C}^{\mathbb{op}} \Rightarrow \mathbb{C}'^{\mathbb{op}}$ is a monoidal functor, is referred to as an *opmonoidal functor* from $\mathbb{C}$ to $\mathbb{C}'$. The construction of the *opposite monoidal functor* $\mathbb{F} \mapsto \mathbb{F}^{\mathbb{op}} := (F^{\mathsf{op}}, \Phi^{\mathsf{op}}, \phi^{\mathsf{op}})$ of an opmonoidal functor clearly is a one-one correspondence between opmonoidal functors from $\mathbb{C}$ to $\mathbb{C}'$ and monoidal functors from $\mathbb{C}^{\mathbb{op}}$ to $\mathbb{C}'^{\mathbb{op}}$. The *opposite opmonoidal functor* $\mathbb{F}^{\mathbb{op}}$ of a monoidal functor $\mathbb{F}$ is defined similarly.

An opmonoidal functor $\mathbb{F}$ is said *lax, strong, strict, symmetric* if $\mathbb{F}^{\mathbb{op}}$ is so as a monoidal functor. Moreover opmonoidal functors compose in the dual way as monoidal functors do.

179 Remark *By duality an opmonoidal functor $\mathbb{F}$ from $\mathbb{C}$ to $\mathbb{C}'$ induces a functor $\mathbb{F}_m := \widetilde{\mathbb{F}^{\mathsf{op}}}^{\mathsf{op}}: \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$, commuting with the forgetful functors as in the diagram above.*

A strong monoidal functor $\mathbb{F} = (F, \Phi, \phi): \mathbb{C} \to \mathbb{C}'$ may also be considered as a strong opmonoidal functor $\mathbb{F}^{\mathsf{d}} := (F, \Phi^{-1}, \phi^{-1}): \mathbb{C} \to \mathbb{C}'$, the dual of $\mathbb{F}$. *Thus it also induces a functor $\mathbb{F}^{\mathsf{d}}_m: \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$.*

Of course, one may also define the dual *of a strong opmonoidal functor $\mathbb{F} = (F, \Phi, \phi): \mathbb{C} \to \mathbb{C}'$ as $\mathbb{F}^{\mathsf{d}} := (F, \Phi^{-1}, \phi^{-1}): \mathbb{C} \to \mathbb{C}'$ which is a strong monoidal functor.*

Observe that

1. $\mathbb{id}^{\mathsf{d}}_{\mathbb{C}} = \mathbb{id}_{\mathbb{C}}$.
2. *If $\mathbb{G}: \mathbb{C}' \to \mathbb{C}''$ is also a strong monoidal functor, then $\mathbb{G} \circ \mathbb{F}$ is strong, and $(\mathbb{G} \circ \mathbb{F})^{\mathsf{d}} = \mathbb{F}^{\mathsf{d}} \circ \mathbb{G}^{\mathsf{d}}$.*
3. $(\mathbb{F}^{\mathsf{d}})^{\mathsf{d}} = \mathbb{F}$.
4. $(\mathbb{F}^{\mathsf{op}})^{\mathsf{d}} = (\mathbb{F}^{\mathsf{d}})^{\mathsf{op}}$.

180 Definition *Let $\mathbb{F} = (F, \Phi, \phi)$ and $\mathbb{G} = (G, \Psi, \psi)$ be monoidal functors from $\mathbb{C}$ to $\mathbb{C}'$. A natural transformation $\alpha: F \Rightarrow G: \mathbf{C} \to \mathbf{C}'$ is a* monoidal transformation *$\alpha: \mathbb{F} \Rightarrow \mathbb{G}: \mathbb{C} \to \mathbb{C}'$ when the following diagrams commute, for every $\mathbf{C}$-objects C_1, C_2.*

$$\begin{tikzcd} FC_1 \otimes' FC_2 \arrow[r, "\alpha_{C_1} \otimes' \alpha_{C_2}"] \arrow[d, "\Phi_{C_1,C_2}"'] & GC_1 \otimes' GC_2 \arrow[d, "\Psi_{C_1,C_2}"] \\ F(C_1 \otimes C_2) \arrow[r, "\alpha_{C_1 \otimes C_2}"'] & G(C_1 \otimes C_2) \end{tikzcd} \tag{67}$$

and

$$\begin{tikzcd} I' \arrow[r, "\phi"] \arrow[dr, "\psi"'] & FI \arrow[d, "\alpha_I"] \\ & GI \end{tikzcd} \tag{68}$$

181 Remark 1. *Let $\alpha: \mathbb{F} \Rightarrow \mathbb{G}: \mathbb{C} \to \mathbb{C}'$ be a monoidal transformation. Let $\tilde{\alpha}_{(C,m,e)} := \alpha_C$ for a monoid (C, m, e) in $\mathbb{C}$. Then, $\tilde{\alpha}: \tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}}: \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$.*

2. *Let* $\alpha\colon \mathbb{F} \Rightarrow \mathbb{G}\colon \mathbb{C} \to \mathbb{C}'$ *be a monoidal transformation between symmetric monoidal functors. Then,* $\tilde{\alpha}$ *as above provides a natural transformation* $\tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}}\colon {}_c\mathbf{Mon}\mathbb{C} \to {}_c\mathbf{Mon}\mathbb{C}'$. *In other words, a monoidal transformation between symmetric monoidal functors, is automatically "symmetric".*

3. *By duality, when* $\alpha\colon \mathbb{F}^{\mathsf{op}} \Rightarrow \mathbb{G}^{\mathsf{op}}\colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}'^{\mathsf{op}}$ *is a monoidal transformation, between opmonoidal functors* $\mathbb{F}, \mathbb{G}$ *from* $\mathbb{C}$ *to* $\mathbb{C}'$, *one defines* $(\alpha_m)_{(C,\mu,\epsilon)} := \alpha_C^{\mathsf{op}}$ *for a comonoid* (C, μ, ϵ) *in* $\mathbb{C}$, *and then* $\alpha_m\colon \mathbb{G}_m \Rightarrow \mathbb{F}_m\colon \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$. *When furthermore* $\mathbb{F}$ *and* $\mathbb{G}$ *are symmetric,* $\alpha_m\colon \mathbb{G}_m \Rightarrow \mathbb{F}_m\colon {}_{coc}\mathbf{Comon}\mathbb{C} \to {}_{coc}\mathbf{Comon}\mathbb{C}'$.

4. *Let* $\alpha\colon \mathbb{F} \Rightarrow \mathbb{G}\colon \mathbb{C} \to \mathbb{C}'$ *be a monoidal transformation between strong monoidal functors, then* $\alpha^{\mathsf{op}}\colon G^{\mathsf{op}} \to F^{\mathsf{op}}\colon \mathbf{C}^{\mathsf{op}} \to \mathbf{C}'^{\mathsf{op}}$ *also provides a monoidal transformation* $\alpha^{\mathsf{d}}\colon (\mathbb{G}^{\mathsf{d}})^{\mathsf{op}} \to (\mathbb{F}^{\mathsf{d}})^{\mathsf{op}}\colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}'^{\mathsf{op}}$ *between strong monoidal functors, called the* dual *of* α.

A *monoidal isomorphism* is a monoidal transformation which is a natural isomorphism.

182 Remark *Let* $\alpha\colon \mathbb{F} \Rightarrow \mathbb{G}\colon \mathbb{C} \to \mathbb{C}'$ *be a monoidal isomorphism. Since for each* $\mathbf{C}$*-objects* C_1, C_2, $(\alpha_{C_1} \otimes' \alpha_{C_2})^{-1} = \alpha_{C_1}^{-1} \otimes' \alpha_{C_2}^{-1}$ *and* $(\alpha_{C_1 \otimes C_2})^{-1} = \alpha_{C_1 \otimes C_2}^{-1}$, *Diag. (67) and (68) still commute when* α *is replaced by* α^{-1}, *and thus* $\alpha^{-1}\colon \mathbb{G} \Rightarrow \mathbb{F}\colon \mathbb{C} \to \mathbb{C}'$ *is also a monoidal isomorphism.*

Accordingly, the induced natural transformation $\tilde{\alpha}\colon \tilde{\mathbb{F}} \Rightarrow \tilde{\mathbb{G}}\colon \mathbf{Mon}(\mathbb{C}) \to \mathbf{Mon}(\mathbb{C}')$ *is a natural isomorphism with inverse* $(\tilde{\alpha})^{-1} = \widetilde{(\alpha^{-1})}$.

A *monoidal equivalence* of monoidal categories is given by a monoidal functor $\mathbb{F}$ such that there are a monoidal functor $\mathbb{G}$ and monoidal isomorphisms $\eta\colon \mathbb{id} \Rightarrow \mathbb{G} \circ \mathbb{F}$ and $\epsilon\colon \mathbb{F} \circ \mathbb{G} \Rightarrow \mathbb{id}$. If $\mathbb{F}\colon \mathbb{C} \to \mathbb{C}'$ is a monoidal equivalence, then $\mathbb{C}$ and $\mathbb{C}'$ are *monoidally equivalent.*

183 Remark *If* $\mathbb{F}$ *is a monoidal equivalence, then* $\tilde{\mathbb{F}}$ *is an equivalence between the corresponding categories of monoids.*

184 Remark *Let* $\mathbb{F}, \mathbb{G}\colon \mathbb{C}' \to \mathbb{C}$ *be strong monoidal functors. Let* $\eta\colon \mathbb{id} \Rightarrow \mathbb{G} \circ \mathbb{F}$ *and* $\epsilon\colon \mathbb{F} \circ \mathbb{G} \Rightarrow \mathbb{id}$ *be monoidal isomorphisms. Whence* $\eta^{\mathsf{d}}\colon (\mathbb{F}^{\mathsf{d}})^{\mathsf{op}} \circ (\mathbb{G}^{\mathsf{d}})^{\mathsf{op}} = ((\mathbb{G} \circ \mathbb{F})^{\mathsf{d}})^{\mathsf{op}} \Rightarrow (\mathbb{id}_{\mathbb{C}}^{\mathsf{d}})^{\mathsf{op}} = \mathbb{id}_{\mathbb{C}^{\mathsf{op}}}$ *and* $\epsilon^{\mathsf{d}}\colon \mathbb{id}_{\mathbb{C}'^{\mathsf{op}}} = ((\mathbb{id}_{\mathbb{C}'})^{\mathsf{d}})^{\mathsf{op}} \Rightarrow (\mathbb{G}^{\mathsf{d}})^{\mathsf{op}} \circ (\mathbb{F}^{\mathsf{d}})^{\mathsf{op}}$ *are also monoidal isomorphisms, and this provides a monoidal equivalence between* $\mathbb{C}^{\mathsf{op}}$ *and* $\mathbb{C}'^{\mathsf{op}}$.

C Monoidality of $\circledast_{(\mathsf{R},\tau)}$ and its consequences: the proofs

In what follows (R,τ) is a rigid ring.

C.1 Proof of Proposition 98, p. 47

C.1.1 Notations

The coherence constraints of $\mathbb{F}\mathrm{ree}\mathbb{M}\mathrm{od}_{\mathsf{R}}$ are those of $\mathbb{M}\mathrm{od}_{\mathsf{R}}$, and are given by $(M\otimes_{\mathsf{R}}N)\otimes_{\mathsf{R}}P \xrightarrow{\alpha_{M,N,P}} M\otimes_{\mathsf{R}}(N\otimes_{\mathsf{R}}P)$, $(u\otimes v)\otimes w \mapsto u\otimes(v\otimes w)$ (associativity), $R\otimes_{\mathsf{R}} N \xrightarrow{\lambda_N} N$, $1_{\mathsf{R}}\otimes v \mapsto v$ (left unit), $M\otimes_{\mathsf{R}} R \xrightarrow{\rho_M} M$, $u\otimes 1_{\mathsf{R}} \mapsto u$ (right unit) and $M\otimes_{\mathsf{R}} N \xrightarrow{\sigma_{M,N}} N\otimes_{\mathsf{R}} M$, $u\otimes v \mapsto v\otimes u$ (symmetry).

C.1.2 Associativity constraint

C.1.2.1 Definition

Let (M,σ), (N,γ) and (P,μ) be topologically-free (R,τ)-modules. One defines an isomorphism $((M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma))\circledast_{(\mathsf{R},\tau)}(P,\mu) \xrightarrow{\alpha_{(M,\sigma),(N,\gamma),(P,\mu)}} (M,\sigma)\circledast_{(\mathsf{R},\tau)}((N,\gamma)\circledast_{(\mathsf{R},\tau)}(P,\mu))$ by $\alpha_{(M,\sigma),(N,\gamma),(P,\mu)} := ((\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}\otimes_{\mathsf{R}} id_{(P,\mu)'})\circ\alpha^{-1}_{(M,\sigma)',(N,\gamma)',(P,\mu)'}\circ(id_{(M,\sigma)'}\otimes_{\mathsf{R}}\Lambda^{-1}_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'}))^*$.

C.1.2.2 Explicit form

Let (M,σ), (N,γ) and (P,μ) be topologically-free (R,τ)-modules with respective topological bases, B,D,E. As already known by Proposition 96 (p. 45), $(b\circledast d)_{(b,d)\in B\times E}$ is a topological basis of $(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)$, and $((b\circledast d)\circledast e)_{(b,d,e)\in B\times D\times E}$ is a topological basis of $((M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma))\circledast_{(\mathsf{R},\tau)}(P,\mu)$.

Likewise $(b\circledast(d\circledast e))_{(b,d,e)\in B\times D\times E}$ is a topological basis for $(M,\sigma)\circledast_{(\mathsf{R},\tau)}((N,\gamma)\circledast_{(\mathsf{R},\tau)}(P,\mu))$.

According to Lemma 55 (p. 28), one has an isomorphism, and only one, in $\mathbf{TopMod}_{(\mathsf{R},\tau)}$, given by

$$\theta\colon (b\circledast d)\circledast e \mapsto b\circledast(d\circledast e),$$

$(b,d,e)\in B\times D\times E$.

185 Example *Let $u\in M$, $v\in N$, and $w\in P$. Then, $\theta((u\circledast v)\circledast w) = u\circledast(v\circledast w)$.*

One claims that

$$\theta = \alpha_{(M,\sigma),(N,\gamma),(P,\mu)}. \tag{69}$$

Eq. (69) is equivalent to

$$\begin{array}{rl} & (id_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'})^* \circ \theta \\ = & ((\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \otimes_{\mathsf{R}} id_{(P,\mu)'}) \circ \alpha^{-1}_{(M,\sigma)',(N,\gamma)',(P,\mu)'})^*. \end{array} \tag{70}$$

According to Corollary 54 (p. 28) it suffices to check the equality on a topological basis of $((M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)) \circledast_{(\mathsf{R},\tau)} (P,\mu)$.

Let us first compute $((id_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'})^* \circ \theta)((b \circledast d) \circledast e) = (b \circledast (d \circledast e)) \circ (id_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'})$ for $b \in B$, $d \in D$ and $e \in E$.

One has for $\ell_1 \otimes (\ell_2 \otimes \ell_3) \in (M,\sigma)' \otimes_{\mathsf{R}} ((N,\gamma)' \otimes_{\mathsf{R}} (P,\mu)')$,

$$\begin{array}{l} ((b \circledast (d \circledast e)) \circ (id_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'}))(\ell_1 \otimes (\ell_2 \otimes \ell_3)) \\ = (b \circledast (d \circledast e))(\ell_1 \otimes \Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'}(\ell_2 \otimes \ell_3)) \\ = \ell_1(b)(\Lambda_{(N,\gamma)'\otimes_{\mathsf{R}}(P,\mu)'}(\ell_2 \otimes \ell_3))(d \circledast e) \\ = \ell_1(b)((d \circledast e)(\ell_2 \otimes \ell_3)) \\ = \ell_1(b)(\ell_2(d)\ell_3(e)). \end{array} \tag{71}$$

Secondly, let us compute for $b \in B$, $d \in D$ and $e \in E$,

$$\begin{array}{rl} & ((\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \otimes_{\mathsf{R}} id_{(P,\mu)'}) \circ \alpha^{-1}_{(M,\sigma)',(N,\gamma)',(P,\mu)'})^*((b \circledast d) \circledast e) \\ = & ((b \circledast d) \circledast e) \circ (\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \otimes_{\mathsf{R}} id_{(P,\mu)'}) \circ \alpha^{-1}_{(M,\sigma)',(N,\gamma)',(P,\mu)'} \end{array} \tag{72}$$

One has for $(\ell_1 \otimes (\ell_2 \otimes \ell_3) \in (M,\sigma)' \otimes_{\mathsf{R}} ((N,\gamma)' \otimes_{\mathsf{R}} (P,\mu)')$,

$$\begin{array}{l} (((b \circledast d) \circledast e) \circ (\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \otimes_{\mathsf{R}} id_{(P,\mu)'}) \circ \alpha^{-1}_{(M,\sigma)',(N,\gamma)',(P,\mu)'})(\ell_1 \otimes (\ell_2 \otimes \ell_3)) \\ = (((b \circledast d) \circledast e) \circ (\Lambda_{(M,\sigma)'\otimes_R(N,\gamma)'} \otimes_{\mathsf{R}} id_{(P,\mu)'}))((\ell_1 \otimes \ell_2) \otimes \ell_3) \\ = ((b \circledast d) \circledast e)((\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(\ell_1 \otimes \ell_2)) \otimes \ell_3) \\ = (\Lambda_{(M,\sigma)'\otimes_R(N,\gamma)'}(\ell_1 \otimes \ell_2))(b \circledast d)\ell_3(e) \\ = ((b \circledast d)(\ell_1 \otimes \ell_2))\ell_3(e) \\ = \ell_1(b)\ell_2(d)\ell_3(e). \end{array} \tag{73}$$

C.1.2.3 Coherence of the associativity constraint

In view of Eq. (69), by definition of θ, coherence of the associativity constraint (i.e., that Diag. (59), p. 90, commutes) is easily checked: it suffices to compute both arrows of Diag. (59) on a topological basis of say $(((M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)) \circledast_{(\mathsf{R},\tau)} (P,\mu)) \circledast_{(\mathsf{R},\tau)} (Q,\omega)$, and then apply Corollary 54, p. 28.

C.1.3 Left unit constraint

Let $(R,\tau)' \xrightarrow{\psi} R$ be the isomorphism given by $\psi(\ell) = \ell(1_{\mathsf{R}})$ and $\psi^{-1}(1_{\mathsf{R}}) = id_R$.

The left unit constraint $\lambda_{(M,\sigma)}$ for (M,σ) a topologically-free (R,τ)-module, is defined as illustrated in the following diagram.

$$\begin{tikzcd}
(R,\tau) \circledast_{(\mathsf{R},\tau)} (M,\sigma) \arrow[r, "\lambda_{(M,\sigma)}"] \arrow[d, "(\psi^{-1}\otimes_{\mathsf{R}} id_{(M,\sigma)'})^*"'] & (M,\sigma) \\
(R \otimes_{\mathsf{R}} (M,\sigma)')^* \arrow[r, "(\lambda^{-1}_{(M,\sigma)'})^*"'] & ((M,\sigma)')^* \arrow[u, "\Gamma^{-1}_{(M,\sigma)}"']
\end{tikzcd} \tag{74}$$

Let B be a topological basis of (M,σ). Let $\vartheta\colon (R,\tau) \otimes_{(\mathsf{R},\tau)} (M,\sigma) \simeq (M,\sigma)$ be given by $\theta(1 \circledast b) = b$, $b \in B$. One claims that $\vartheta = \lambda_{(M,\sigma)}$. To see this it suffices to check that

$$\Gamma_{(M,\sigma)} \circ \vartheta = ((\psi^{-1} \otimes_{\mathsf{R}} id_{(M,\sigma)}) \circ \lambda^{-1}_{(M,\sigma)'})^*.$$

Let $b \in B$ and let $\ell \in (M,\sigma)'$. Then,

$$(\Gamma_{(M,\sigma)}(\theta(1_{\mathsf{R}} \circledast b)))(\ell) = (\Gamma_{(M,\sigma)}(b))(\ell) = \ell(b).$$

One also has

$$\begin{aligned}
& (((\psi^{-1} \otimes_{\mathsf{R}} id_{(M,\sigma)'}) \circ \lambda^{-1}_{(M,\sigma)'})^*(1_{\mathsf{R}} \circledast b))(\ell) \\
= \; & ((1_{\mathsf{R}} \circledast b) \circ (\psi^{-1} \otimes_{\mathsf{R}} id_{(M,\sigma)'}) \circ (\lambda^{-1}_{(M,\sigma)'}))(\ell) \\
= \; & (1_{\mathsf{R}} \circledast b)((\psi^{-1} \otimes_{\mathsf{R}} id_{(M,\sigma)'})(1_{\mathsf{R}} \otimes \ell)) \\
= \; & (1_{\mathsf{R}} \circledast b)(\psi^{-1}(1_{\mathsf{R}}) \otimes \ell) \\
= \; & (1_{\mathsf{R}} \circledast b)(id_R \otimes \ell) \\
= \; & id_R(1_{\mathsf{R}})\ell(b) \\
= \; & \ell(b).
\end{aligned} \tag{75}$$

C.1.4 Right unit constraint

The right unit constraint $\rho_{(M,\sigma)}$ for (M,σ) a topologically-free (R,τ)-module, is defined as illustrated in the following diagram.

$$\begin{tikzcd}
(M,\sigma) \circledast_{(\mathsf{R},\tau)} (R,\tau) \arrow[r, "\rho_{(M,\sigma)}"] \arrow[d, "(id_{(M,\sigma)'}\otimes_{\mathsf{R}} \psi^{-1})^*"'] & (M,\sigma) \\
((M,\sigma)' \otimes_{\mathsf{R}} R)^* \arrow[r, "(\rho^{-1}_{(M,\sigma)'})^*"'] & ((M,\sigma)')^* \arrow[u, "\Gamma^{-1}_{(M,\sigma)}"']
\end{tikzcd} \tag{76}$$

Let B be a topological basis of (M,σ). Let $\omega\colon (M,\sigma)\circledast_{(\mathsf{R},\tau)}(R,\tau)\simeq(M,\sigma)$ be defined by $\omega(b\circledast 1_{\mathsf{R}})=b$, $b\in B$. One claims that $\rho_{(M,\sigma)}=\omega$. To see this it suffices to check that $\Gamma_{(M,\sigma)}\circ\omega=(\rho^{-1}_{(M,\sigma)'})^*\circ(id_{(M,\sigma)'}\otimes_{\mathsf{R}}\psi^{-1})^*=((id_{(M,\sigma)'}\otimes_{\mathsf{R}}\psi^{-1})\circ\rho^{-1}_{(M,\sigma)'})^*$.

Let $b\in B$, and let $\ell\in(M,\sigma)'$. Then,

$$(\Gamma_{(M,\sigma)}(\omega(b\circledast 1_{\mathsf{R}})))(\ell)=(\Gamma_{(M,\sigma)}(b))(\ell)=\ell(b).$$

One also has

$$\begin{aligned}
& (((id_{(M,\sigma)'}\otimes_{\mathsf{R}}\psi^{-1})\circ\rho^{-1}_{(M,\sigma)'})^*(b\circledast 1_{\mathsf{R}}))(\ell)\\
= \;& ((b\circledast 1_{\mathsf{R}})\circ(id_{(M,\sigma)'}\otimes_{\mathsf{R}}\psi^{-1})\circ\rho^{-1}_{(M,\sigma)'})(\ell)\\
= \;& (b\circledast 1_{\mathsf{R}})(id_{(M,\sigma)'}\otimes_{\mathsf{R}}\psi^{-1})(\ell\otimes 1_{\mathsf{R}})\\
= \;& (b\circledast 1_{\mathsf{R}})(\ell\otimes\psi^{-1}(1_{\mathsf{R}}))\\
= \;& (b\circledast 1_{\mathsf{R}})(\ell\otimes id_R)\\
= \;& \ell(b)id_R(1_{\mathsf{R}})\\
= \;& \ell(b).
\end{aligned}\tag{77}$$

C.1.5 Coherence of the left and right unit constraints

One has to check that the following diagram commutes for every topologically-free (R,τ)-modules $(M,\sigma),(N,\gamma)$ (where $\circledast$ is an abbreviation for $\circledast_{(\mathsf{R},\tau)}$).

$$\begin{tikzcd}
((M,\sigma)\circledast(R,\tau))\circledast(N,\gamma) \arrow[rr, "\alpha_{(M,\sigma),(R,\tau),(N,\gamma)}"] \arrow[dr, "\rho_{(M,\sigma)}\circledast id_{(N,\gamma)}"'] & & (M,\sigma)\circledast((R,\tau)\circledast(N,\gamma)) \arrow[dl, "id_{(M,\sigma)}\circledast\lambda_{(N,\gamma)}"] \\
& (M,\sigma)\circledast(N,\gamma) &
\end{tikzcd}\tag{78}$$

Let B,D be respective topological bases of (M,σ) and (N,γ). Let $(b,d)\in B\times D$. One has

$$\begin{aligned}
& (id_{(M,\sigma)}\circledast_{(\mathsf{R},\tau)}\lambda_{(N,\gamma)})(\alpha_{(M,\sigma),(R,\tau),(N,\gamma)}((b\circledast 1_{\mathsf{R}})\circledast d))\\
= \;& (id_{(M,\sigma)}\circledast_{(\mathsf{R},\tau)}\lambda_{(N,\gamma)})(b\circledast(1\circledast d))\\
= \;& b\circledast d.
\end{aligned}\tag{79}$$

One also has $(\rho_{(M,\sigma)}\circledast_{(\mathsf{R},\tau)}id_{(N,\gamma)})((b\circledast 1)\circledast d)=b\circledast d$.

C.1.6 Symmetry constraint

C.1.6.1 Explicit form

Let $(M,\sigma),(N,\gamma)$ be two topologically-free (R,τ)-modules. The symmetry constraint $\sigma_{(M,\sigma),(N,\gamma)}\colon(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)\to(N,\gamma)\circledast_{(\mathsf{R},\tau)}(M,\sigma)$ is defined as $\sigma^*_{(N,\gamma)',(M,\sigma)'}$.

Let B (resp. D) be a topological basis of (M,σ) (resp. (N,γ)). Let $o\colon (M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma) = ((M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)')^* \to (N,\gamma)\otimes_{(\mathsf{R},\tau)}(M,\sigma) = ((N,\gamma)'\otimes_{\mathsf{R}}(M,\sigma)')^*$ be given by $o(b\otimes d) = d\otimes b$, $(b,d)\in B\times D$.

One claims that $\sigma_{(M,\sigma),(N,\gamma)} = o$. Let $\ell_1\in(M,\sigma)'$ and $\ell_2\in(N,\mu)'$. One has $(\sigma_{(M,\sigma),(N,\gamma)}(b\otimes d))(\ell_2\otimes\ell_1) = (\sigma^*_{(N,\gamma)',(M,\sigma)'}(b\otimes d))(\ell_2\otimes\ell_1) = (b\otimes d)(\sigma_{(N,\gamma)',(M,\sigma)'}(\ell_2\otimes\ell_1)) = (b\otimes d)(\ell_1\otimes\ell_2) = \ell_1(b)\ell_2(d) = \ell_2(d)\otimes\ell_1(b) = (d\otimes b)(\ell_2\otimes\ell_1) = (o(b\otimes d))(\ell_2\otimes\ell_1)$.

C.1.6.2 Coherence of the symmetry constraint

First $\sigma_{(N,\gamma)}\circ\sigma_{(M,\sigma)} = id_{(M,\sigma)\otimes_{(\mathsf{R},\tau)}(N,\gamma)}$ since one has $(\sigma_{(N,\gamma)}(\sigma_{(M,\sigma)}(b\otimes d)) = \sigma_{(N,\gamma)}(d\otimes b) = b\otimes d$, $(b,d)\in B\times D$.

Secondly, one has to check that the following diagram commutes for topologically-free (R,τ)-modules $(M,\sigma),(N,\gamma)$ and (P,μ) (where again $\otimes$ stands for an abbreviation of $\otimes_{(\mathsf{R},\tau)}$).

$$\begin{tikzcd}
((M,\sigma)\otimes(N,\gamma))\otimes_{(R,\tau)}(P,\mu) \arrow[r, "\sigma_{(M,\sigma),(N,\gamma)}\otimes id_{(P,\mu)}"] \arrow[d, "\alpha_{(M,\sigma),(N,\gamma),(P,\mu)}"'] & ((N,\gamma)\otimes(M,\sigma))\otimes(P,\mu) \arrow[d, "\alpha_{(N,\gamma),(M,\sigma),(P,\mu)}"] \\
(M,\sigma)\otimes((N,\gamma)\otimes(P,\mu)) \arrow[d, "\sigma_{(M,\sigma),(N,\gamma)\otimes(P,\mu)}"'] & (N,\gamma)\otimes((M,\sigma)\otimes(P,\mu)) \arrow[d, "id_{(N,\gamma)}\otimes\sigma_{(M,\sigma),(P,\mu)}"] \\
((N,\gamma)\otimes(P,\mu))\otimes(M,\sigma) \arrow[r, "\alpha_{(N,\gamma),(P,\mu),(M,\sigma)}"'] & (N,\gamma)\otimes((P,\mu)\otimes(M,\sigma))
\end{tikzcd} \tag{80}$$

Let B,D,E be respective topological bases of $(M,\sigma),(N,\gamma)$ and (P,μ). Let $(b,d,e)\in B\times D\times E$. The top-right sequence of arrows acts as follows: $(b\otimes d)\otimes e\mapsto(d\otimes b)\otimes e\mapsto d\otimes(b\otimes e)\mapsto d\otimes(e\otimes b)$, whereas the left-bottom sequence of arrows acts as follows: $(b\otimes d)\otimes e\mapsto b\otimes(d\otimes e)\mapsto(d\otimes e)\otimes b\mapsto d\otimes(e\otimes b)$.

Thirdly, it remains to check commutativity of the following diagram for each topologically-free (R,τ)-module (M,σ) which is almost immediate by computing the corresponding morphisms at the level of a topological basis $(b\otimes 1_{\mathsf{R}})_{b\in B}$ of $(M,\sigma)\otimes_{(\mathsf{R},\tau)}(R,\tau)$, where B is a topological basis of (M,σ).

$$\begin{tikzcd}
(M,\sigma)\otimes_{(\mathsf{R},\tau)}(R,\tau) \arrow[rr, "\sigma_{(M,\sigma),(R,\tau)}"] \arrow[dr, "\rho_{(M,\sigma)}"'] & & (R,\tau)\otimes_{(\mathsf{R},\tau)}(M,\sigma) \arrow[dl, "\lambda_{(M,\sigma)}"] \\
& (M,\sigma) &
\end{tikzcd} \tag{81}$$

C.2 Proof of Theorem 102, p. 48

C.2.1 $\mathbb{Alg}_{(\mathsf{R},\tau)}$ is a strong symmetric monoidal functor

Let M, N be free R-modules. Let B (resp. D) be a linear basis of M (resp. N). Then by Lemma 56 (p. 28), B^* (resp. D^*) is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)})$ (resp. $(N^*, w^*_{(\mathsf{R},\tau)})$). It follows that $\{\, b^* \circledast d^* \colon (b,d) \in B \times D \,\}$ is a topological basis of $(M^*, w^*_{(\mathsf{R},\tau)}) \circledast (N^*, w^*_{(\mathsf{R},\tau)})$.

Let $b \in B$, $d \in D$. Let $v \in M$ and $w \in N$. Then,

$$\begin{aligned}
\Phi_{M,N}(b^* \circledast d^*)(v \otimes w) &= (b^* \circledast d^*)((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)(v \otimes w)) \\
&= (b^* \circledast d^*)(\Lambda_M(v) \otimes \Lambda_N(w)) \\
&= (\Lambda_M(v)(b^*))(\Lambda_N(w))(d^*) \\
&= b^*(v) d^*(w) \\
&= (b \otimes d)^*(v \otimes w).
\end{aligned} \tag{82}$$

Whence

$$\Phi_{M,N}(b^* \circledast d^*) = (b \otimes d)^*, b \in B, d \in D. \tag{83}$$

Now to prove that

$$\mathbb{Alg}_{(\mathsf{R},\tau)} = (Alg_{(\mathsf{R},\tau)}, \Phi, \phi) \colon \mathbb{FreeMod}_{\mathsf{R}}^{\mathbb{op}} \to \mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$$

is symmetric monoidal, one has to prove that the following diagrams commute. (That it is strong will follow since all coherence maps are isomorphisms.) In the diagrams below, M^*, N^*, ... abbreviate $(M^*, w^*_{(\mathsf{R},\tau)})$, $(N^*, w^*_{(\mathsf{R},\tau)})$, ..., and $\circledast$ stands for $\circledast_{(\mathsf{R},\tau)}$.

$$\begin{tikzcd}
(M^* \circledast N^*) \circledast P^* \arrow[r, "\alpha_{M^*,N^*,P^*}"] \arrow[d, "\Phi_{M,N} \circledast id_{P^*}"'] & M^* \circledast (N^* \circledast P^*) \arrow[d, "id_{M^*} \circledast \Phi_{N,P}"] \\
(M \otimes_{\mathsf{R}} N)^* \circledast P^* \arrow[d, "\Phi_{M \otimes_{\mathsf{R}} N, P}"'] & M^* \circledast (N \otimes_{\mathsf{R}} P)^* \arrow[d, "\Phi_{M, N \otimes_{\mathsf{R}} P}"] \\
((M \otimes_{\mathsf{R}} N) \otimes_{\mathsf{R}} P)^* \arrow[r, "(\alpha^{-1}_{M,N,P})^*"'] & (M \otimes_{\mathsf{R}} (N \otimes_{\mathsf{R}} P))^*
\end{tikzcd} \tag{84}$$

$$\begin{tikzcd}
M^* \circledast (R,\tau) \arrow[r, "id_{M^*} \circledast \phi"] \arrow[d, "\rho_{M^*}"'] & M^* \circledast R^* \arrow[d, "\Phi_{M,R}"] \\
M^* & (M \otimes_{\mathsf{R}} R)^* \arrow[l, "(\rho^{-1}_M)^*"]
\end{tikzcd} \tag{85}$$

$$\begin{tikzcd}
(R,\tau)\circledast N^* \arrow[r, "\phi\circledast id_{N^*}"] \arrow[d, "\lambda_{N^*}"'] & R^*\circledast N^* \arrow[d, "\Phi_{R,N}"] \\
N^* & (R\otimes_{\mathsf{R}} N)^* \arrow[l, "(\lambda_N^{-1})^*"]
\end{tikzcd} \tag{86}$$

and

$$\begin{tikzcd}
M^*\circledast N^* \arrow[r, "\sigma_{M^*,N^*}"] \arrow[d, "\Phi_{M,N}"'] & N^*\circledast M^* \arrow[d, "\Phi_{N,M}"] \\
(M\otimes_{\mathsf{R}} N)^* \arrow[r, "\sigma^*_{N,M}"'] & (N\otimes_{\mathsf{R}} M)^*
\end{tikzcd} \tag{87}$$

Let E be a free basis of P.

1. Let $(b,d,e)\in B\times D\times E$. The top-right sequence of arrows acts as follows: $(b^*\circledast d^*)\circledast e^*\mapsto b^*\circledast(d^*\circledast e^*)\mapsto b^*\circledast(d\otimes e)^*$ (according to Eq.(83)) $\mapsto (b\otimes(d\otimes e))^*$. The right-bottom sequence of arrows acts as: $(b^*\circledast d^*)\circledast e^*\mapsto (b\otimes d)^*\circledast e^*\mapsto((b\otimes d)\otimes e)^*\mapsto(b\otimes(d\otimes e))^*$.

2. One has for $v\in M$ and $b\in B$, $((\rho_M^{-1})^*((\Lambda_M\otimes_{\mathsf{R}}\Lambda_R)^*((id_{M^*}\circledast\phi)(b^*\circledast 1_{\mathsf{R}}))))(v) = (((\Lambda_M\otimes_{\mathsf{R}}\Lambda_R)\circ\rho_M^{-1})^*((id_{M^*}\circledast\phi)(b^*\circledast 1_{\mathsf{R}})))(v) = ((id_{M^*}\circledast\phi)(b^*\circledast 1_{\mathsf{R}}))((\Lambda_M\otimes_{\mathsf{R}}\Lambda_R)\circ\rho_M^{-1})(v) = (b^*\circledast\phi(1_{\mathsf{R}}))(\Lambda_M(v)\otimes\Lambda_R(1_{\mathsf{R}})) = (b^*\circledast id_R)(\Lambda_M(v)\otimes\Lambda_R(1_{\mathsf{R}})) = (\Lambda_M(v))(b^*)(\Lambda_R(1_{\mathsf{R}}))(id_R) = b^*(v) = (\rho_{M^*}(b^*\circledast 1_{\mathsf{R}}))(v)$.

3. Given $w\in N$ and $d\in D$,

$$\begin{array}{rl}
 & (((\lambda_N^{-1})^*\circ(\Lambda_R\otimes_{\mathsf{R}}\Lambda_N)^*\circ(\phi\circledast id_{N^*}))(1_{\mathsf{R}}\otimes_{(R,\tau)} d^*))(w) \\
= & ((\phi\circledast id_{N^*})(1_{\mathsf{R}}\circledast d^*))((\Lambda_R\otimes_{\mathsf{R}}\Lambda_N)(\lambda_N^{-1}(w))) \\
= & (\phi(1_{\mathsf{R}})\circledast id_{N^*}(d^*))((\Lambda_R\otimes_{\mathsf{R}}\Lambda_N)(1\otimes w)) \\
= & (id_R\circledast d^*)(\Lambda_R(1_{\mathsf{R}})\otimes\Lambda_N(w)) \\
= & (\Lambda_R(1_{\mathsf{R}}))(id_R)(\Lambda_N(w))(d^*) \\
= & id_R(1_{\mathsf{R}})d^*(w) \\
= & d^*(w) \\
= & (\lambda_{N^*}(1_{\mathsf{R}}\circledast d^*))(w).
\end{array} \tag{88}$$

4. Let $(b,d)\in B\times D$ and let $(u,v)\in M\times N$. One has

$$\begin{array}{rl}
 & ((\Lambda_N\otimes_{\mathsf{R}}\Lambda_M)^*(\sigma_{M^*,N^*}(b^*\circledast d^*)))(v\otimes u) \\
= & (\sigma_{M^*,N^*}(b^*\circledast d^*))(\Lambda_N(v)\otimes\Lambda_M(u)) \\
= & (d^*\circledast b^*)(\Lambda_N(v)\otimes\Lambda_M(u)) \\
= & d^*(v)b^*(u).
\end{array} \tag{89}$$

One also has $(\sigma^*_{N,M}((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*(b^* \circledast d^*)))(v \otimes u) = ((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*(b^* \circledast d^*))(\sigma_{N,M}(v \otimes u)) = ((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*(b^* \circledast d^*))(u \otimes v) = (b^* \circledast d^*))(\Lambda_M(u) \otimes \Lambda_N(v)) = b^*(u)d^*(v)$.

C.2.2 $\mathbb{T}\mathbb{o}\mathbb{p}_{(\mathsf{R},\tau)}$ is a strong symmetric monoidal functor

One claims that $\mathbb{T}\mathbb{o}\mathbb{p}_{(\mathsf{R},\tau)} = (Top_{(\mathsf{R},\tau)}, \Psi, \psi)$ is a strong symmetric monoidal functor from $\mathbb{T}\mathbb{o}\mathbb{p}\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}^{\mathbb{o}\mathbb{p}}_{(\mathsf{R},\tau)}$ to $\mathbb{F}\mathbb{r}\mathbb{e}\mathbb{e}\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$.

To see this one needs to check that the following diagrams commute for each topologically-free (R,τ)-modules $(M,\sigma),(N,\gamma),(P,\mu)$. (Below $\circledast$ stands for $\circledast_{(\mathsf{R},\tau)}$ and $\otimes$ means $\otimes_{\mathsf{R}}$.)

$$\begin{tikzcd}
((M,\sigma)' \otimes (N,\gamma)') \otimes (P,\mu)' \arrow[r, "\alpha_{(M,\sigma)',(N,\gamma)',(P,\mu)'}"] \arrow[d, "\Psi_{(M,\sigma),(N,\gamma)} \otimes id_{(P,\mu)'}"'] & (M,\sigma)' \otimes ((N,\gamma)' \otimes (P,\mu)') \arrow[d, "id_{(M,\sigma)'} \otimes \Psi_{(N,\gamma),(P,\mu)}"] \\
((M,\sigma) \circledast (N,\gamma))' \otimes (P,\mu)' \arrow[d, "\Psi_{(M,\sigma)\circledast(N,\gamma),(P,\mu)}"'] & (M,\sigma)' \otimes ((N,\gamma) \circledast (P,\mu))' \arrow[d, "\Psi_{(M,\sigma),(N,\gamma)\circledast(P,\mu)}"] \\
(((M,\sigma) \circledast (N,\gamma)) \circledast (P,\mu))' \arrow[r, "(\alpha^{-1}_{(M,\sigma),(N,\gamma),(P,\mu)})'"'] & ((M,\sigma) \circledast ((N,\gamma) \circledast (P,\mu)))'
\end{tikzcd} \tag{90}$$

$$\begin{tikzcd}
(M,\sigma)' \otimes R \arrow[r, "id_{(M,\sigma)'} \otimes \psi"] \arrow[d, "\rho_{(M,\sigma)'}"'] & (M,\sigma)' \otimes (R,\tau)' \arrow[d, "\Psi_{(M,\sigma),(R,\tau)}"] \\
(M,\sigma)' & ((M,\sigma) \circledast (R,\tau))' \arrow[l, "(\rho^{-1}_{(M,\sigma)})'"]
\end{tikzcd} \tag{91}$$

$$\begin{tikzcd}
R \otimes (N,\gamma)' \arrow[r, "\psi \otimes id_{(N,\gamma)'}"] \arrow[d, "\lambda_{(N,\gamma)'}"'] & (R,\tau)' \otimes (N,\gamma)' \arrow[d, "\Psi_{(R,\tau),(N,\gamma)}"] \\
(N,\gamma)' & ((R,\tau) \circledast (N,\gamma))' \arrow[l, "(\lambda^{-1}_{(N,\gamma)})'"]
\end{tikzcd} \tag{92}$$

and

$$\begin{tikzcd}
(M,\sigma)' \otimes (N,\gamma)' \arrow[r, "\sigma_{(M,\sigma)',(N,\gamma)'}"] \arrow[d, "\Psi_{(M,\sigma),(N,\gamma)}"'] & (N,\gamma)' \otimes_R (M,\sigma)' \arrow[d, "\Psi_{(N,\gamma),(M,\sigma)}"] \\
((M,\sigma) \circledast (N,\gamma))' \arrow[r, "\sigma'_{(N,\gamma),(M,\sigma)}"'] & ((N,\gamma) \circledast (M,\sigma))'
\end{tikzcd} \tag{93}$$

Let B, D, E be respective topological bases of $(M,\sigma), (N,\gamma)$ and (P,μ). By Lemma 62 (p. 29), B', D' and E' are respective linear bases of $(M,\sigma)'$, $(N,\gamma)'$ and $(P,\mu)'$.

1. Let $b, b_1 \in B$, $d, d_1 \in D$ and $e, e_1 \in E$. One has

   $((\Psi_{(M,\sigma),(N,\gamma)\circledast(P,\mu)} \circ (id_{(M,\sigma)'} \otimes \Psi_{(N,\gamma),(P,\mu)})((b' \otimes d') \otimes e'))(b_1 \circledast (d_1 \circledast e_1)) = ((\Lambda_{(M,\sigma)'\otimes((N,\gamma)\circledast(P,\mu))'} \circ (id_{(M,\sigma)'} \otimes \Lambda_{(N,\gamma)'\otimes(P,\mu)'})(b' \otimes (d' \otimes e')))(b_1 \circledast (d_1 \circledast e_1)) = ((\Lambda_{(M,\sigma)'\otimes((N,\gamma)\circledast(P,\mu))'}(b' \otimes \Lambda_{(N,\gamma)'\otimes(P,\mu)'}(d' \otimes e')))(b_1 \circledast (d_1 \circledast e_1)) = (b_1 \circledast (d_1 \circledast e_1))(b' \otimes \Lambda_{(N,\gamma)'\otimes(P,\mu)'}(d' \otimes e')) = b'(b_1)((d_1 \circledast e_1)(\Lambda_{(N,\gamma)'\circledast(P,\mu)'}(d' \otimes e'))) = b'(b_1)d'(d_1)e'(e_1)$.

   Let $F := \Psi_{(M,\sigma)\circledast(N,\gamma),(P,\mu)} \circ (\Psi_{(M,\sigma),(N,\gamma)} \otimes id_{(P,\mu)'})$. One has

$$\begin{aligned} & (((\alpha^{-1}_{(M,\sigma),(N,\gamma),(P,\mu)})' \circ F)(b' \otimes d') \otimes e'))(b_1 \circledast (d_1 \circledast e_1)) \\ = \; & (F((b' \otimes d') \otimes e'))((b_1 \circledast d_1) \circledast e_1) \\ = \; & ((b_1 \circledast d_1) \circledast e_1)((\Lambda_{(M,\sigma)'\otimes(N,\gamma)'} \otimes id_{(P,\mu)'})((b' \otimes d') \otimes e')) \\ = \; & ((b_1 \circledast d_1) \circledast e_1)(\Lambda_{(M,\sigma)'\otimes(N,\gamma)'}(b' \otimes d') \otimes e') \\ = \; & b'(b_1)d'(d_1)e'(e_1). \end{aligned} \tag{94}$$

2. Let $b \in B$ and $v \in M$. One has $(\rho^{-1}_{(M,\sigma)})'(\Lambda_{(M,\sigma)'\otimes(N,\gamma)'}((id_{(R,\tau)'} \otimes \psi)(b' \otimes 1_{\mathsf{R}})))(u) = (\Lambda_{(M,\sigma)'\otimes(N,\gamma)'}(b' \otimes \psi(1_{\mathsf{R}})))(u \circledast 1_{\mathsf{R}}) = (u \circledast 1_{\mathsf{R}})(b' \otimes id_R) = b'(u)id_R(1_{\mathsf{R}}) = b'(u)$, and $(\rho_{(M,\sigma)'}(b' \otimes 1_{\mathsf{R}}))(u) = b'(u)$.

3. Let $d \in D$ and $v \in N$. One has

$$\begin{aligned} & ((\lambda^{-1}_{(N,\gamma)})'(\Lambda_{(R,\tau)'\otimes(N,\gamma)'}((\psi \otimes id_{(N,\gamma)'})(1_{\mathsf{R}} \otimes d')))))(v) \\ = \; & (\Lambda_{(R,\tau)'\otimes(N,\gamma)'}(\psi(1_{\mathsf{R}}) \otimes d'))(1_{\mathsf{R}} \circledast v) \\ = \; & (1_{\mathsf{R}} \circledast v)(id_R \otimes d') = id_R(1_{\mathsf{R}})d'(v) \\ = \; & d'(v) \end{aligned} \tag{95}$$

   and $(\lambda_{(N,\gamma)'}(1_{\mathsf{R}} \otimes d'))(v) = d'(v)$.

4. Let $b, b_1 \in B$ and $d, d_1 \in D$. One has

$$\begin{aligned} & (\Lambda_{(N,\gamma)'\otimes(M,\sigma)'}(\sigma_{(M,\sigma)',(N,\gamma)'}(b' \otimes d')))(d_1 \circledast b_1) \\ = \; & (\Lambda_{(N,\gamma)'\otimes(M,\sigma)'}(d' \otimes b'))(d_1 \circledast b_1) \\ = \; & (d_1 \circledast b_1)(d' \otimes b') \\ = \; & d'(d_1)b'(b_1), \end{aligned} \tag{96}$$

   and

$$\begin{aligned} & ((\sigma_{(N,\gamma),(M,\sigma)})'(\Lambda_{(M,\sigma)',(N,\gamma)'}(b' \otimes d')))(d_1 \circledast b_1) \\ = \; & (\Lambda_{(M,\sigma)',(N,\gamma)'}(b' \otimes d'))(b_1 \circledast d_1) \\ = \; & (b_1 \circledast d_1)(b' \otimes d') \\ = \; & b'(b_1)d'(d_1). \end{aligned} \tag{97}$$

C.2.3 $\Lambda^{\mathsf{op}}\colon (\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}} \circ \mathbb{Alg}_{(\mathsf{R},\tau)} \Rightarrow \mathbb{id}$ is a monoidal isomorphism

By Corollary 73 (p. 33),

$$\Lambda := \Lambda^{(\mathsf{R},\tau)}\colon id_{\mathbf{FreeMod}_{\mathsf{R}}} \Rightarrow Top_{(\mathsf{R},\tau)} \circ Alg^{\mathsf{op}}_{(\mathsf{R},\tau)}\colon \mathbf{FreeMod}_{\mathsf{R}} \to \mathbf{FreeMod}_{\mathsf{R}}$$

is a natural isomorphism, thus so is its opposite $\Lambda^{\mathsf{op}}\colon Top^{\mathsf{op}}_{(\mathsf{R},\tau)} \circ Alg_{(\mathsf{R},\tau)} \Rightarrow id_{\mathbf{FreeMod}^{\mathsf{op}}_{\mathsf{R}}}$. Since $(\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}} = (Top^{\mathsf{op}}_{(\mathsf{R},\tau)}, (\Psi^{-1})^{\mathsf{op}}, (\psi^{-1})^{\mathsf{op}})$ is a (strong) monoidal functor from $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ to $\mathbb{FreeMod}^{\mathbb{op}}_{\mathsf{R}}$ (Remark 179, p. 96), it suffices to check that Λ^{op} is a monoidal transformation (Definition 180, p. 96) from $(\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}} \circ \mathbb{Alg}_{(\mathsf{R},\tau)}$ to $\mathbb{id}_{\mathbb{FreeMod}^{\mathbb{op}}_{\mathsf{R}}}$. By definition of the composition of monoidal functors (Facts 176, p. 94), and according to the definitions of Φ and Ψ, this is equivalent to the commutativity of the following two diagrams for each free modules M, N.

$$\begin{tikzcd}
M \otimes_{\mathsf{R}} N \arrow[r, "\Lambda_M \otimes_{\mathsf{R}} \Lambda_N"] \arrow[dr, "\Lambda_{M \otimes_{\mathsf{R}} N}"'] & (M^*)' \otimes_{\mathsf{R}} (N^*)' \arrow[dr, "\Lambda_{(M^*)' \otimes_{\mathsf{R}} (N^*)'}"] & \\
& & (M^* \circledast_{(\mathsf{R},\tau)} N^*)' \arrow[dl, "((\Lambda_M^{-1} \otimes_{\mathsf{R}} \Lambda_N^{-1})^*)'"] \\
& ((M \otimes_{\mathsf{R}} N)^*)' &
\end{tikzcd} \tag{98}$$

and

$$\begin{tikzcd}
R \arrow[r, "id_R"] \arrow[dr, "(\phi^{-1})' \circ \Psi"'] & R \arrow[d, "\Lambda_R"] \\
& (R^*)'
\end{tikzcd} \tag{99}$$

1. Commutativity of the first above diagram is equivalent to the equality $((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*)' \circ \Lambda_{M \otimes_{\mathsf{R}} N} = \Lambda_{(M^*)' \otimes_{\mathsf{R}} (N^*)'} \circ (\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)$.

   Let $u \in M$, $v \in N$, $\ell_1 \in M^*$ and $\ell_2 \in N^*$. Then,

$$\begin{aligned}
& ((((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*)' \circ \Lambda_{M \otimes_{\mathsf{R}} N})(u \otimes v))(\ell_1 \circledast \ell_2) \\
= \; & ((\Lambda_{M \otimes_{\mathsf{R}} N})(u \otimes v))((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)^*(\ell_1 \circledast \ell_2)) \\
= \; & ((\Lambda_{M \otimes_{\mathsf{R}} N})(u \otimes v))((\ell_1 \circledast \ell_2) \circ (\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)) \\
= \; & (\ell_1 \circledast \ell_2)((\Lambda_M \otimes_{\mathsf{R}} \Lambda_N)(u \otimes v)) \\
= \; & (\ell_1 \circledast \ell_2)(\Lambda_M(u) \otimes \Lambda_N(v)) \\
= \; & (\Lambda_M(u)(\ell_1))(\Lambda_N(v)(\ell_2)) \\
= \; & \ell_1(u)\ell_2(v).
\end{aligned} \tag{100}$$

One also has

$$\begin{array}{rl} & ((\Lambda_{(M^*)'\otimes_{\mathsf{R}}(N^*)'} \circ (\Lambda_M \otimes_{\mathsf{R}} \Lambda_N))(u \otimes v))(\ell_1 \circledast \ell_2) \\ = & (\Lambda_{(M^*)'\otimes_R(N^*)'}(\Lambda_M(u) \otimes \Lambda_N(v)))(\ell_1 \circledast \ell_2) \\ = & (\ell_1 \circledast \ell_2)(\Lambda_M(u) \otimes \Lambda_N(v)) \\ = & \ell_1(u)\ell_2(v). \end{array} \tag{101}$$

2. Concerning the second above diagram, one has $(\Lambda_R(1_{\mathsf{R}}))(\ell) = \ell(1_{\mathsf{R}})$ and $((\phi^{-1})'(\psi(1_{\mathsf{R}})))(\ell) = (\psi(1_{\mathsf{R}}) \circ \phi^{-1})(\ell) = id_R(\ell(1_{\mathsf{R}})) = \ell(1_{\mathsf{R}})$.

C.2.4 $\Gamma\colon \mathbb{id} \Rightarrow \mathbb{Alg}_{(\mathsf{R},\tau)} \circ (\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}}$ is a monoidal isomorphism

By Lemma 77 (p. 35),

$$\Gamma := \Gamma^{(\mathsf{R},\tau)}\colon id \Rightarrow Alg_{(\mathsf{R},\tau)} \circ Top^{\mathsf{op}}_{(\mathsf{R},\tau)}\colon \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbf{TopFreeMod}_{(\mathsf{R},\tau)}$$

is natural isomorphism.

Since $(\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}} = (Top^{\mathsf{op}}_{(\mathsf{R},\tau)}, (\Psi^{-1})^{\mathsf{op}}, (\psi^{-1})^{\mathsf{op}})$ is a (strong) monoidal functor from $\mathbb{TopFreeMod}_{(\mathsf{R},\tau)}$ to $\mathbb{FreeMod}^{\mathbb{op}}_{\mathsf{R}}$, it suffices to check that Γ is a monoidal transformation from $\mathbb{id}$ to $\mathbb{Alg}_{(\mathsf{R},\tau)} \circ (\mathbb{Top}^{\mathbb{d}}_{(\mathsf{R},\tau)})^{\mathbb{op}}$.

This turns out to be equivalent to the commutativity of the following diagrams, for each topologically-free (R,τ)-modules (M,σ) and (N,γ).

$$\begin{tikzcd} (M,\sigma) \circledast (N,\gamma) \arrow[r, "\Gamma_{(M,\sigma)} \circledast \Gamma_{(N,\gamma)}"] \arrow[dr, "\Gamma_{(M,\sigma)\circledast(N,\gamma)}"'] & ((M,\sigma)')^* \circledast ((N,\gamma)')^* \arrow[dr, "(\Lambda_{(M,\sigma)'\otimes(N,\gamma)'})^*"] & \\ & & ((M,\sigma)' \otimes (N,\gamma)')^* \arrow[dl, "(\Lambda^{-1}_{(M,\sigma)'\otimes(N,\gamma)'})^*"'] \\ & (((M,\sigma) \circledast (N,\gamma))')^* & \end{tikzcd} \tag{102}$$

and

$$\begin{tikzcd} (R,\tau) \arrow[r, "id"] \arrow[d, "\phi"'] & (R,\tau) \arrow[d, "\Gamma_{(R,\tau)}"] \\ R^* \arrow[r, "(\psi^{-1})^*"'] & ((R,\tau)')^* \end{tikzcd} \tag{103}$$

1. Commutativity of the first diagram is equivalent to the equality

$$\begin{array}{rl} & \Lambda^*_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \circ \Gamma_{(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)} \\ = & (\Lambda_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'})^* \circ (\Gamma_{(M,\sigma)} \circledast_{(\mathsf{R},\tau)} \Gamma_{(N,\gamma)}). \end{array} \tag{104}$$

Let B, D be respective topological bases of (M,σ) and (N,γ). Let $b, b_1 \in B$ and let $d, d_1 \in D$. One has

$$\begin{array}{rl} & ((\Lambda^*_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'} \circ \Gamma_{(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)})(b \circledast d))(b'_1 \otimes d'_1) \\ = & ((\Gamma_{(M,\sigma)\circledast_{(\mathsf{R},\tau)}(N,\gamma)})(b \circledast d))(\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(b'_1 \otimes d'_1)) \\ = & (\Lambda_{(M,\sigma)'\otimes_{\mathsf{R}}(N,\gamma)'}(b'_1 \otimes d'_1))(b \circledast d) \\ = & (b \circledast d)(b'_1 \otimes d'_1) \\ = & b'_1(b)d'_1(d). \end{array} \tag{105}$$

One also has

$$\begin{array}{rl} & ((\Lambda_{(M,\sigma)'} \otimes_{\mathsf{R}} \Lambda_{(N,\gamma)'})^*((\Gamma_{(M,\sigma)} \circledast_{(\mathsf{R},\tau)} \Gamma_{(N,\gamma)})(b \circledast d)))(b'_1 \otimes d'_1) \\ = & ((\Gamma_{(M,\sigma)} \circledast_{(\mathsf{R},\tau)} \Gamma_{(N,\gamma)})(b \circledast d))(\Lambda_{(M,\sigma)'}(b'_1) \otimes \Lambda_{(N,\gamma)'}(d'_1)) \\ = & (\Gamma_{(M,\sigma)}(b) \circledast \Gamma_{(N,\gamma)}(d))(\Lambda_{(M,\sigma)'}(b'_1) \otimes \Lambda_{(N,\gamma)'}(d'_1)) \\ = & (\Lambda_{(M,\sigma)'}(b'_1))(\Gamma_{(M,\sigma)}(b))(\Lambda_{(N,\gamma)'}(d_1)')(\Gamma_{(N,\gamma)}(d)) \\ = & (\Gamma_{(M,\sigma)}(b))(b'_1)(\Gamma_{(N,\gamma)}(d))(d'_1) \\ = & b'_1(b)d'_1(d). \end{array} \tag{106}$$

2. Concerning the second diagram, one has to check that $\Gamma_{(R,\tau)} = (\psi^{-1})^* \circ \phi$. One first observes that $1'_{\mathsf{R}} = id_R$. (Indeed, $1'_{\mathsf{R}}(1_{\mathsf{R}}) = 1_{\mathsf{R}}$.) One has $(\Gamma_{(R,\tau)}(1_{\mathsf{R}}))(1'_{\mathsf{R}}) = 1'_{\mathsf{R}}(1_{\mathsf{R}}) = 1_{\mathsf{R}}$, and

$$\begin{array}{rcl} ((\psi^{-1})^*(\phi(1_{\mathsf{R}})))(1'_{\mathsf{R}}) & = & (\phi(1_{\mathsf{R}}))(\psi^{-1}(1'_{\mathsf{R}})) \\ & = & id_R(\psi^{-1}(id_R)) \\ & = & 1_{\mathsf{R}}. \end{array} \tag{107}$$

C.3 Proof of Lemma 107, p. 51

One first observes that $id_{\mathsf{R}}\colon R \simeq \|(R,\tau)\| = R$.

Secondly, let us check that Ξ is a natural transformation

$$\|-\|\otimes_{\mathsf{R}}\|-\| \Rightarrow \|-\circledast_{(\mathsf{R},\tau)}-\|\colon \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \times \mathbf{TopFreeMod}_{(\mathsf{R},\tau)} \to \mathbf{Mod}_{\mathsf{R}}.$$

Let $(M_i,\sigma_i) \xrightarrow{f_i} (N_i,\gamma_i)$ be a continuous (R,τ)-linear map, $i = 1,2$. Let $u \in M_1$ and $v \in M_2$. Then, $(f_1 \circledast_{(\mathsf{R},\tau)} f_2)(\Xi_{M_1,M_2}(u \otimes v)) = (f_1 \circledast_{(\mathsf{R},\tau)} f_2)(u \circledast v) = f_1(u) \circledast f_2(v)$ (by Eq. (32), p. 46). Likewise $\Xi_{N_1,N_2}((f_1 \otimes_{\mathsf{R}} f_2)(u \otimes v)) = \Xi_{N_1,N_2}(f_1(u) \otimes f_2(v)) = f_1(u) \circledast f_2(v)$. Therefore naturality is obtained.

It remains to check compatibility with the coherence constraints. Let $(M,\sigma), (N,\gamma), (P,\mu)$ be topologically-free (R,τ)-modules. One has to check that the four diagrams below commute.

$$
\begin{tikzcd}
(M \otimes N) \otimes P \arrow[r, "\alpha_{M,N,P}"] \arrow[d, "\Xi_{(M,\sigma),(N,\gamma)} \otimes id_P"'] & M \otimes (N \otimes P) \arrow[d, "id_M \otimes \Xi_{(N,\gamma),(P,\mu)}"] \\
\|(M,\sigma) \circledast (N,\gamma)\| \otimes \|(P,\mu)\| \arrow[d, "\Xi_{(M,\sigma)\circledast(N,\gamma),(P,\mu)}"'] & \|(M,\sigma)\| \otimes \|(N,\gamma) \circledast (P,\mu)\| \arrow[d, "\Xi_{(M,\sigma),(N,\gamma)\circledast(P,\mu)}"] \\
\|((M,\sigma) \circledast (N,\gamma)) \circledast (P,\mu)\| \arrow[r, "\|\alpha_{(M,\sigma),(N,\gamma),(P,\mu)}\|"'] & \|(M,\sigma) \circledast ((N,\gamma) \circledast (P,\mu))\|
\end{tikzcd}
\tag{108}
$$

$$
\begin{tikzcd}
\|(M,\sigma)\| \otimes_{\mathsf{R}} R \arrow[r, "id_M \otimes_{\mathsf{R}} id_R"] \arrow[d, "\rho_M"'] & \|(M,\sigma)\| \otimes_{\mathsf{R}} \|(R,\tau)\| \arrow[d, "\Xi_{(M,\sigma),(R,\tau)}"] \\
\|(M,\sigma)\| & \|(M,\sigma) \circledast_{(\mathsf{R},\tau)} (R,\tau)\| \arrow[l, "\|\rho_{(M,\sigma)}\|"]
\end{tikzcd}
\tag{109}
$$

$$
\begin{tikzcd}
R \otimes_{\mathsf{R}} \|(N,\gamma)\| \arrow[r, "id_R \otimes_{\mathsf{R}} id_N"] \arrow[d, "\lambda_N"'] & \|(R,\tau)\| \otimes_{\mathsf{R}} \|(N,\gamma)\| \arrow[d, "\Xi_{(R,\tau),(N,\gamma)}"] \\
\|(N,\gamma)\| & \|(R,\tau) \circledast_{(\mathsf{R},\tau)} (N,\gamma)\| \arrow[l, "\|\lambda_{(N,\gamma)}\|"]
\end{tikzcd}
\tag{110}
$$

and

$$
\begin{tikzcd}
\|(M,\sigma)\| \otimes_{\mathsf{R}} \|(N,\gamma)\| \arrow[r, "\sigma_{M,N}"] \arrow[d, "\Xi_{(M,\sigma),(N,\gamma)}"'] & \|(N,\gamma)\| \otimes_{\mathsf{R}} \|(M,\sigma)\| \arrow[d, "\Xi_{(N,\gamma),(M,\sigma)}"] \\
\|(M,\sigma) \circledast_{(\mathsf{R},\tau)} (N,\gamma)\| \arrow[r, "\|\sigma_{(M,\sigma),(N,\gamma)}\|"'] & \|(N,\gamma) \circledast_{(\mathsf{R},\tau)} (M,\sigma)\|
\end{tikzcd}
\tag{111}
$$

Let $u \in M$, $v \in N$ and $w \in P$.

1. The top-left sequence of arrows of Diag. (108) acts as follows: $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes (v \circledast w) \mapsto u \circledast (v \circledast w)$, while the right-bottom sequence acts as follows: $(u \otimes v) \otimes w \mapsto (u \circledast v) \otimes w \mapsto (u \circledast v) \circledast w \mapsto u \circledast (v \circledast w)$.

2. One has $\rho_M(u \otimes 1_{\mathsf{R}}) = u$ and $\rho_{(M,\sigma)}(\Xi_{(M,\sigma),(R,\tau)}((id_M \otimes_{\mathsf{R}} id_R)(u \otimes 1_{\mathsf{R}}))) = \rho_{(M,\sigma)}(\Xi_{(M,\sigma),(R,\tau)}(u \otimes 1_{\mathsf{R}}) = \rho_{(M,\sigma)}(u \circledast 1_{\mathsf{R}}) = u$.

3. One has $\lambda_N(1_{\mathsf{R}} \otimes v) = v$ and $\lambda_{(N,\gamma)}(\Xi_{(R,\tau),(N,\gamma)}((id_R \otimes_{\mathsf{R}} id_N)(1_{\mathsf{R}} \otimes v))) = \lambda_{(N,\gamma)}(\Xi_{(R,\tau),(N,\gamma)}(1_{\mathsf{R}} \otimes v)) = \lambda_{(N,\gamma)}(1_{\mathsf{R}} \circledast v) = v$.

4. One has $\Xi_{(N,\gamma),(M,\sigma)}(\sigma_{M,N}(u \otimes v)) = \Xi_{(N,\gamma),(M,\sigma)}(u \circledast v) = v \circledast u$, and $\sigma_{(M,\sigma),(N,\gamma)}(\Xi_{(M,\sigma),(N,\gamma)}(u \otimes v)) = \sigma_{(M,\sigma),(N,\gamma)}(u \circledast v) = v \circledast u$.

D Coalgebras

D.1 Definitions

Let R be a ring. By a R-coalgebra (see [1]) is meant a pair $\mathsf{C} := (C, \delta_{\mathsf{C}})$ with C a R-module, and $C \xrightarrow{\delta_{\mathsf{C}}} C \otimes_{\mathsf{R}} C$ a R-linear map, called the *comultiplication*, which is *coassociative*, i.e., which makes commute the following diagram (where α is the associative constraint of $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$; see Appendix C.1.1, p. 98).

$$\begin{tikzcd}
C \otimes_{\mathsf{R}} C \arrow[d, "id_C \otimes_{\mathsf{R}} \delta_{\mathsf{C}}"'] & C \arrow[l, "\delta_{\mathsf{C}}"'] \arrow[r, "\delta_{\mathsf{C}}"] & C \otimes_{\mathsf{R}} C \arrow[d, "\delta_{\mathsf{C}} \otimes_{\mathsf{R}} id_C"] \\
C \otimes_{\mathsf{R}} (C \otimes_{\mathsf{R}} C) \arrow[rr, "\alpha^{-1}_{C,C,C}"'] & & (C \otimes_{\mathsf{R}} C) \otimes_{\mathsf{R}} C
\end{tikzcd} \tag{112}$$

A *counit* for a coalgebra C is a R-linear map $C \xrightarrow{\epsilon_{\mathsf{C}}} R$ such that the following diagram commutes (where λ and ρ are the left and ring unit constraints of $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$). A triple $\mathsf{C} = (C, \delta_{\mathsf{C}}, \epsilon_{\mathsf{C}})$ consisting of a coalgebra with a counit is called a *counital coalgebra*.

$$\begin{tikzcd}
R \otimes_{\mathsf{R}} C & C \otimes_{\mathsf{R}} C \arrow[l, "\epsilon_{\mathsf{C}} \otimes_{\mathsf{R}} id_C"'] \arrow[r, "id_C \otimes_{\mathsf{R}} \epsilon_{\mathsf{C}}"] & C \otimes_{\mathsf{R}} R \\
& C \arrow[ul, "\lambda^{-1}_C"] \arrow[u, "\epsilon_{\mathsf{C}}"] \arrow[ur, "\rho^{-1}_C"'] &
\end{tikzcd} \tag{113}$$

Given coalgebras C and D a *morphism* of coalgebras, also called a *coalgebra map*, $\mathsf{C} \xrightarrow{f} \mathsf{D}$ is a R-linear map $C \xrightarrow{f} D$ such that the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\delta_{\mathsf{C}}"'] & D \arrow[d, "\delta_{\mathsf{D}}"] \\
C \otimes_{\mathsf{R}} C \arrow[r, "f \otimes_{\mathsf{R}} f"'] & D \otimes_{\mathsf{R}} D
\end{tikzcd} \tag{114}$$

$\mathbf{Coalg}_{\mathsf{R}}$ is the category of R-coalgebras with coalgebra maps.

When C, D both are counital, then a coalgebra map $\mathsf{C} \xrightarrow{f} \mathsf{D}$ is furthermore assumed to be *counit-preserving* meaning that the following diagram also commutes.

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "\epsilon_{\mathsf{C}}"'] & & D \arrow[dl, "\epsilon_{\mathsf{D}}"] \\
& R &
\end{tikzcd} \tag{115}$$

${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}$ is the category of counital R-coalgebras and (counit-preseving) coalgebra maps.

A coalgebra (counital or not) C is said to be *cocommutative* when $\sigma_{C,C} \circ \delta_{\mathsf{C}} = \delta_{\mathsf{C}}$, where σ is the symmetry constraint of $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$. One also defines the full subcategories ${}_{coc}\mathbf{Coalg}_{\mathsf{R}}$ of $\mathbf{Coalg}_{\mathsf{R}}$ and ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$ of ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}$ respectively, spanned by the cocommutative coalgebras.

By direct inspection one has (see Appendix B.2, p. 91)

1. ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}} = \mathbf{Comon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})$,
2. ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}} = {}_{coc}\mathbf{Comon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})$.

D.2 Group-like elements

Let C be a counital coalgebra over R. A member $x \in C$ is a *group-like* when $\delta_{\mathsf{C}}(x) = x \otimes x$ and $\epsilon_{\mathsf{C}}(x) = 1_{\mathsf{R}}$. Let $G\mathsf{C}$ be the set of all group-like elements of C.

When R is a field, then the condition $\epsilon_{\mathsf{C}}(x) = 1_{\mathsf{R}}$ for a group-like allows to exclude zero from $G\mathsf{C}$.

186 Lemma *Let us assume that $\Bbbk$ is a field. Let $x \in X$. x is a group-like if, and only if, $\delta_{\mathsf{C}}(x) = x \otimes x$ and $x \neq 0$.*

Proof: The direct implication is immediate.

Let us assume that for a given $x \in C \smallsetminus \{0\}$, one has $\delta(x) = x \otimes x$. Since ϵ_{C} is a counit, $x\epsilon(x) = x$. So $(\epsilon(x) - 1_{\Bbbk})x = 0$, and thus $\epsilon(x) = 1_{\Bbbk}$, because $x \neq 0$. □

187 Fact *Let $\Bbbk$ be a field, and let C be a counital $\Bbbk$-coalgebra. Then, the set $G\mathsf{C}$ of group-likes of C is $\Bbbk$-linearly independent ([1, Theorem 2.1.2, p. 59]).*

Let $\mathsf{C} \xrightarrow{f} \mathsf{D}$ be a coalgebra map between counital R-coalgebras. It is clear that $f(G\mathsf{C}) \subseteq G\mathsf{D}$ since if $x \in G\mathsf{C}$, then $\delta_{\mathsf{D}}(f(x)) = (f \otimes_{\mathsf{R}} f)(\delta_{\mathsf{C}}(x)) = (f \otimes_{\mathsf{R}} f)(x \otimes x) = f(x) \otimes f(x)$, and $\epsilon_{\mathsf{D}}(f(x)) = \epsilon_{\mathsf{C}}(x) = 1_{\mathsf{R}}$.

Whence one has a *group-like functor* ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}} \xrightarrow{G} \mathbf{Set}$.

In the opposite direction, one may define for a set X, a counital cocommutative R-coalgebra $C_{\mathsf{R}}(X) := (R^{(X)}, d_X, e_X)$, called the *group-like coalgebra on X*, with $d_X(p) = \sum_{x \in X} p(x)\delta_x \otimes \delta_x$ and $e_X(p) = \sum_{x \in X} p(x)$, $x \in X$. Given a map $X \xrightarrow{f} Y$, the unique R-linear map $R^{(X)} \xrightarrow{F_{\mathsf{R}}(f)} R^{(Y)}$ given by $F_{\mathsf{R}}(f)(\delta_x) = \delta_{f(x)}$ (cf. Remark 12, p. 14), is a coalgebra map.

Indeed, $d_Y(F_{\mathsf{R}}(f)(\delta_x)) = d_Y(\delta_{f(x)}) = \delta_{f(x)} \otimes \delta_{f(x)} = (F_{\mathsf{R}}(f) \otimes_{\mathsf{R}} F_{\mathsf{R}}(f))(\delta_x \otimes \delta_x) = (F_{\mathsf{R}}(f) \otimes_{\mathsf{R}} F_{\mathsf{R}}(f))(d_X(\delta_x))$, and $e_Y(F_{\mathsf{R}}(f)(\delta_x)) = e_Y(\delta_{f(x)}) = 1_{\mathsf{R}} = e_X(\delta_x)$, $x \in X$.

Whence one has a functor $\mathbf{Set} \xrightarrow{C_{\mathsf{R}}} {}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}}$, and the following diagram commutes (the unnamed arrow is the evident forgetful functor).

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "C_{\mathsf{R}}"] \arrow[dr, "F_{\mathsf{R}}"'] & {}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}} \arrow[d] \\ & \mathbf{Mod}_{\mathsf{R}} \end{tikzcd} \tag{116}$$

188 Remark *Let $X \xrightarrow{\delta_X} R^{(X)}$ be the map $x \mapsto \delta_x^{\mathsf{R}}$ as already introduced in Remark 10 (p. 14). Then it corestricts as a one-to-one map $X \xrightarrow{\delta_X} GC_{\mathsf{R}}X$ since by definition of C_{R}, $\delta_x \in GC_{\mathsf{R}}X$ for every $x \in X$.*

189 Lemma *C_{R} is a left adjoint of the restriction ${}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}} \xrightarrow{G} \mathbf{Set}$ of ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}} \xrightarrow{G} \mathbf{Set}$.*

Proof: Let X be a set, let $X \xrightarrow{f} G\mathsf{C}$ be a map, where C is a counital cocommutative coalgebra. Let $R^{(X)} \xrightarrow{f^\sharp} C$ be the unique R-linear map such that $f^\sharp(\delta_x) = f(x)$, $x \in X$. Let $p \in R^{(X)}$. Then,

$$\begin{array}{rcl} \delta_{\mathsf{C}}(f^\sharp(p)) & = & \delta_{\mathsf{C}}(\sum_{x \in X} p(x) f(x)) \\ & = & \sum_{x \in X} p(x) \delta_{\mathsf{C}}(f(x)) \\ & = & \sum_{x \in X} p(x) f(x) \otimes f(x) \\ & & \text{(since } f(x) \in G\mathsf{C}) \\ & = & \sum_{x \in X} p(x) f^\sharp(\delta_x) \otimes f^\sharp(\delta_x) \\ & = & (f^\sharp \otimes_{\mathsf{R}} f^\sharp)(\sum_{x \in X} p(x) \delta_x \otimes \delta_x) \\ & = & (f^\sharp \otimes_{\mathsf{R}} f^\sharp)(d_X(p)). \end{array} \tag{117}$$

Moreover, $\epsilon_{\mathsf{C}}(f^\sharp(p)) = \epsilon_{\mathsf{C}}(\sum_{x \in X} p(x) f(x)) = \sum_{x \in X} p(x) = e_X(p)$. Whence $f^\sharp$ is a coalgebra map from $C_{\mathsf{R}}X$ to C.

Let us assume that $C_{\mathsf{R}}X \xrightarrow{h} \mathsf{C}$ is a coalgebra map such that $h(\delta_x) = f(x)$, $x \in X$. Then, as linear maps, $f^\sharp = h$, and so also as coalgebra maps. □

190 Remark *The functor $\mathbf{Set} \xrightarrow{F_{\mathsf{R}}} \mathbf{Mod}_{\mathsf{R}}$ (Remark 12, p. 14) provides a strong symmetric monoidal functor*

$$\mathbb{S}\mathrm{et} = (\mathbf{Set}, \times, 1) \xrightarrow{\mathbb{F}_{\mathsf{R}}} \mathbb{M}\mathrm{od}_{\mathsf{R}} = (\mathbf{Mod}_R, \otimes_{\mathsf{R}}, R)$$

with coherence constraints $F_{\mathsf{R}}(X \times Y) \xrightarrow{\Psi_{X,Y}} F_{\mathsf{R}}(X) \otimes_{\mathsf{R}} F_{\mathsf{R}}(Y)$ *(as in the proof of Lemma 90, p. 42), and* $R \simeq F_{\mathsf{R}}(1)$, $1_{\mathsf{R}} \mapsto (\varnothing \mapsto 1_{\mathsf{R}})$ *(recall that* $1 := \{\varnothing\}$ *is a terminal object of* **Set***). Whence one has an induced functor* ${}_{coc}\mathbf{Comon}(\mathbb{S}\mathbb{e}\mathbb{t}) \xrightarrow{(\mathbb{F}^{\mathsf{d}}_{\mathsf{R}})_m} {}_{\epsilon,coc}\mathbf{Coalg}_{\mathsf{R}} = {}_{coc}\mathbf{Comon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})$ *(cf. Appendix B.3, p. 93). One recovers* C_{R} *with the factorization* $C_{\mathsf{R}} = (\mathbb{F}^{\mathsf{d}}_{\mathsf{R}})_m \circ \Delta$ *(see Remark 174, p. 92, in Appendix B.2 for the definition of* Δ*, and Remark 179, p. 96, in Appendix B.3 for the definition of the functor* $\mathbb{F}_m$ *induced on categories of comonoids).*

D.3 (Pointed) cosemisimple coalgebras

Let $\Bbbk$ be a field.

Let be a C counital $\Bbbk$-coalgebra. By a *subcoalgebra* D of C is meant a subspace D of C such that $\delta_{\mathsf{C}}(D) \subseteq D \otimes_{\Bbbk} D$. Then, under the co-restrictions δ_{D} of δ_{C} and ϵ_{D} of ϵ_{C}, $\mathsf{D} = (D, \delta_{\mathsf{D}}, \epsilon_{\mathsf{D}})$ is a $\Bbbk$-coalgebra on its own right, and the canonical injection $D \hookrightarrow C$ is a coalgebra map $\mathsf{D} \hookrightarrow \mathsf{C}$. One also says that D as defined is a *subcoalgebra* of C.

For each subset E of a coalgebra C there is a least subcoalgebra of C *generated by* E, namely the intersection of all subcoalgebras of C containing E.

191 Theorem *(Fundamental theorem of coalgebras) Let* E *be a finite subset of a counital coalgebra* C*. The subcoalgebra of* C *generated by* E *is finite-dimensional.*

192 Corollary *Each finite-dimensional subspace* V *of a counital coalgebra* C *generates a finite-dimensional subcoalgebra of* C*.*

Proof: See [15, Theorem 1.3.2, p. 21]. □

A subcoalgebra D of C is said to be *simple* if it does not have any subcoalgebras over than (0) and D. The sum of all simple subcoalgebras of C is called the *coradical* of C, and is denoted by $\mathrm{corad}\mathsf{C}$. It is a subcoalgebra of C, and C is *cosemisimple* when $C = \mathrm{corad}\mathsf{C}$. If all simple subcoalgebras of C are one dimensional, then C is called a *pointed* $\Bbbk$-coalgebra.

Of course, any one-dimensional coalgebra is automatically simple.

193 Remark *Let* C *be a counital coalgebra over* $\Bbbk$*. Since* $G\mathsf{C}$ *is* $\Bbbk$*-linearly independent (Fact 187), the subspace* $\langle G\mathsf{C}\rangle$ *spanned by* $G\mathsf{C}$ *is isomorphic to* $\bigoplus_{g \in G\mathsf{C}} \Bbbk g = \Bbbk^{(G\mathsf{C})}$*. It is clearly a cocommutative subcoalgebra of* C *denoted* $\Bbbk G\mathsf{C}$ *(both as a coalgebra and a subcoalgebra, i.e., one denotes in the*

same way the corresponding coalgebra and its underlying subspace). Moreover $G(\Bbbk G\mathsf{C}) = G\mathsf{C}$.

194 Facts *1. There is a one-one correspondence between the group-like elements of* C *and its one-dimensional subcoalgebras ([9, Remark 1.9.8, p. 13]).*

2. Let X *be a set. Then,* $X \simeq GC_{\Bbbk}X$ *under* $X \xrightarrow{\delta_X} \Bbbk^{(X)}$ *([9, Example 1.9.9, p. 13]).*

3. $\Bbbk G\mathsf{C}$ *is cosemisimple (because it is a subcoalgebra of* $\mathrm{corad}(\mathsf{C})$*,* $\mathrm{corad}(\mathsf{C})$ *is the largest cosemisimple subcoalgebra of* C*, and any subcoalgebra of a cosemisimple coalgebra is itself cosemisimple ([7, Exercice 3.1.6, p. 141])).*

195 Lemma $C_{\Bbbk}X$ *is cosemisimple and pointed for each set* X.

Proof: According to the above facts, $GC_{\Bbbk}X \simeq X$, whence $\Bbbk GC_{\Bbbk}X \simeq C_{\Bbbk}X$. Since $\Bbbk GC_{\Bbbk}X$ is cosemisimple, so is $C_{\Bbbk}X$.

Let $S \hookrightarrow C_{\Bbbk}X$ be a simple subcoalgebra. Because $C_{\Bbbk}X = \bigoplus_{x\in X} \Bbbk\delta_x$, and each such one-dimensional subspace $\Bbbk\delta_x$ is simple, it follows that $S \hookrightarrow \Bbbk\delta_x$ for some $x \in X$. By semisimplicity, $S = \Bbbk\delta_x$. □

196 Definition *Let* ${}_{\epsilon,coc}\mathbf{CSCoalg}_{\Bbbk}$ *be the full subcategory of* ${}_{\epsilon,coc}\mathbf{Coalg}_{\Bbbk}$ *spanned by cosemisimple coalgebras.*

Let ${}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$ *be the full subcategory of* ${}_{\epsilon,coc}\mathbf{CSCoalg}_{\Bbbk}$ *spanned by the pointed cosemisimple cocommutative coalgebras.*

197 Remark *The objects of* ${}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$ *are characterized by the relation* $\mathsf{C} \simeq C_{\Bbbk}G\mathsf{C}$ *(as coalgebras) among the objects* C *of* ${}_{\epsilon}\mathbf{Coalg}_{\Bbbk}$.

Equivalently, C *is pointed cosemisimple if, and only if,* $G\mathsf{C}$ *is a basis of* C.

Lemma 195 tells us that $C_{\Bbbk}X$ is an object of ${}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$ for each set X , and thus the functor $\mathbf{Set} \xrightarrow{C_{\Bbbk}} {}_{\epsilon,coc}\mathbf{Coalg}_{\Bbbk}$ corestricts as a functor $\mathbf{Set} \xrightarrow{C_{\Bbbk}} {}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$.

One also observes that when $\Bbbk$ is algebraically closed, then the categories ${}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$ and ${}_{\epsilon,coc}\mathbf{CSCoalg}_{\Bbbk}$ are the same (since over an algebraically closed field, any cocommutative coalgebra is pointed according to [1, Theorem 2.3.3, p. 80]).

198 Lemma $\mathbf{Set} \xrightarrow{C_{\Bbbk}} {}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk}$ *is an equivalence of categories, with equivalence inverse* ${}_{\epsilon,coc}\mathbf{PtCSCoalg}_{\Bbbk} \xrightarrow{G} \mathbf{Set}$.

Proof: This is nearly immediate because $\delta_X : X \simeq GC_{\Bbbk}X$ for each set X, and $C_{\Bbbk}G\mathsf{C} \simeq \mathsf{C}$ under $\delta_g \mapsto g$, for each pointed cosemisimple coalgebra C. □

199 Corollary *Let* $\Bbbk$ *be an algebraically closed field.* $\mathbf{Set} \xrightarrow{C_{\Bbbk}} {}_{\epsilon,coc}\mathbf{CSCoalg}_{\Bbbk}$ *is an equivalence of categories, with equivalence inverse* ${}_{\epsilon,coc}\mathbf{CSCoalg}_{\Bbbk} \xrightarrow{G} \mathbf{Set}$.

D.4 Finite duality

Over a field, there is a nice duality between counital coalgebras and unital algebras that is not briefly reviewed. See [7] for more details.

First of all, recall from Example 173 (p. 92, Appendix B.2) that one has a functorial isomorphism $O : \mathbf{Mon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}) \simeq {}_1\mathbf{Alg}_{\mathsf{R}}$ (concrete over $\mathbf{Mod}_{\mathsf{R}}$). In particular, by an *ideal* of a monoid A in $\mathbf{Mon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})$ is meant an ideal of the corresponding unital algebra $O(\mathsf{A})$.

Secondly, let C be a counital R-coalgebra. Then, $\widetilde{\mathbb{D}_*}(\mathsf{C}) := (C^*, \delta_{\mathsf{C}}^* \circ \Theta_{C,C}, \epsilon_{\mathsf{C}}^* \circ \theta)$ is a monoid in $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$, where Θ is as in Definition 92 (p. 44), and $\theta : \mathsf{R} \to \mathsf{R}^*$ is the isomorphism $\theta(1_{\mathsf{R}}) = id_{\mathsf{R}}$ (and $\theta^{-1}(\ell) = \ell(1_{\mathsf{R}})$). Moreover given a coalgebra map $\mathsf{C} \xrightarrow{f} \mathsf{D}$ one gets a morphism of monoids $\widetilde{\mathbb{D}_*}(\mathsf{D}) \xrightarrow{f^*} \widetilde{\mathbb{D}_*}(\mathsf{C})$.

200 Remark *Let* $(-)^* : \mathbf{Mod}_{\mathsf{R}}^{\mathsf{op}} \to \mathbf{Mod}_{\mathsf{R}}$ *be the usual algebraic dual functor. Then,* $\mathbb{D}_* := ((-)^*, \Theta, \theta)$ *is a lax symmetric monoidal functor from* $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}^{\mathbb{op}}$ *to* $\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}}$. *According to Appendix B.3 (p. 93), it induces a functor* ${}_{\epsilon}\mathbf{Coalg}_{\mathsf{R}}^{\mathsf{op}} = \mathbf{Comon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})^{\mathsf{op}} \xrightarrow{\widetilde{\mathbb{D}_*}} \mathbf{Mon}(\mathbb{M}\mathbb{o}\mathbb{d}_{\mathsf{R}})$.

In the opposite direction, when $\Bbbk$ is a field, there is also a functor $D_{fin} : \mathbf{Mon}(\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{\Bbbk})^{\mathsf{op}} \to {}_{\epsilon}\mathbf{Coalg}_{\Bbbk}$, the *finite dual functor* defined as follows. Let $\mathsf{A} = (A, \mu_{\mathsf{A}}, \eta_{\mathsf{A}})$ be a monoid in $\mathbb{V}\mathbb{e}\mathbb{c}\mathbb{t}_{\Bbbk}$. Let

$$A^0 := \{\, \ell \in A^* : \ker \ell \text{ contains a finite-codimensional ideal of } \mathsf{A} \,\}. \tag{118}$$

Let also $(A \otimes_{\Bbbk} A)^0 := \Theta_{A,A}(A^0 \otimes_{\Bbbk} A^0) \subseteq (A \otimes_{\Bbbk} A)^*$.

The following diagram commutes, and one denotes by $\Theta^0\colon A^0 \otimes_{\Bbbk} A^0 \simeq (A \otimes_{\Bbbk} A)^0$ the corresponding co-restriction of $\Theta_{A,A}$.

$$\begin{tikzcd} A^* \otimes_{\Bbbk} A^* \arrow[r, "\Theta_{A,A}"] & (A \otimes_{\Bbbk} A)^* \\ A^0 \otimes_{\Bbbk} A^0 \arrow[u, hook] \arrow[r, equal, "\sim"] & (A \otimes_{\Bbbk} A)^0 \arrow[u, hook] \end{tikzcd} \tag{119}$$

Moreover, the following diagram also commutes (of course, $incl_{A^0}$ is the canonical inclusion), and one denotes by $A^0 \xrightarrow{\mu^0_{\mathsf{A}}} A^0 \otimes_{\Bbbk} A^0$ the corresponding co-restriction of μ^*_{A}.

$$\begin{tikzcd} A^* \arrow[r, "\mu^*_{\mathsf{A}}"] & (A \otimes_{\Bbbk} A)^* \\ A^0 \arrow[u, hook, "incl_{A^0}"] \arrow[r] & (A \otimes_{\Bbbk} A)^0 \arrow[u, hook] \end{tikzcd} \tag{120}$$

Then, $D_{fin}(\mathsf{A}) \coloneqq (A^0, (\Theta^0)^{-1} \circ \mu^0_{\mathsf{A}}, \theta^{-1} \circ \eta^*_{\mathsf{A}} \circ incl_{A^0})$ is a counital $\Bbbk$-coalgebra ([7, Proposition 1.5.3, p. 35]), called the *finite dual coalgebra* of A. Moreover, given a monoid morphism $\mathsf{A} \xrightarrow{f} \mathsf{B}$, then the following diagram commutes, and we define $B^0 \xrightarrow{f^0} A^0$ as the corresponding co-restriction.

$$\begin{tikzcd} B^* \arrow[r, "f^*"] & A^* \\ B^0 \arrow[u, hook, "incl_{B^0}"] \arrow[r] & A^0 \arrow[u, hook, "incl_{A^0}"'] \end{tikzcd} \tag{121}$$

Then, $D_{fin}(\mathsf{B}) \xrightarrow{f^0} D_{fin}(\mathsf{A})$ is a coalgebra map by [7, Proposition 1.5.4, p. 37]. In conclusion, this provides a functor $D_{fin}\colon \mathbf{Mon}(\mathbb{V}\mathrm{ec}\mathbb{t}_{\Bbbk})^{\mathsf{op}} \to {}_{\epsilon}\mathbf{Coalg}_{\Bbbk}$.

Finally, the aforementioned duality is given by the following result.

201 Theorem *([7, Theorem 1.5.22, p. 44]) D^{op}_{fin} is a left adjoint of $\widetilde{\mathbb{D}_*}$.*

202 Remark *One observes that the above duality does directly provide an equivalence of categories (as is the case of every adjunction; see e.g., [16, Proposition 4.2, p. 16]) when one restricts the adjunction to the full subcategories of* coreflexive coalgebras*, i.e., those coalgebras C such that $\mathsf{C} \simeq D^{\mathsf{op}}_{fin}(\widetilde{\mathbb{D}_*}(\mathsf{C}))$ under the natural inclusion $C \hookrightarrow (C^*)^0$, $u \mapsto (\ell \mapsto \ell(u))$, and of* reflexive algebras*, i.e., those monoids A such that $\mathsf{A} \simeq \widetilde{\mathbb{D}_*}(D^{\mathsf{op}}_{fin}(\mathsf{A}))$ under the linear map $A \to (A^0)^*$, $u \mapsto (\ell \mapsto \ell(u))$.*

203 Corollary *([1, Corollary 2.3.17, pp. 88–89]) One has co-restrictions* ${}_c\mathbf{Mon}(\mathbb{V}\mathbb{ec}\mathbb{t}_{\Bbbk}) \xrightarrow{D_{fin}^{\mathsf{op}}} {}_{\epsilon,coc}\mathbf{Coalg}_{\Bbbk}^{\mathsf{op}}$ *and* ${}_{\epsilon,coc}\mathbf{Coalg}_{\Bbbk}^{\mathsf{op}} \xrightarrow{\widetilde{\mathbb{D}_*}} {}_c\mathbf{Mon}(\mathbb{V}\mathbb{ec}\mathbb{t}_{\Bbbk})$*, and* $D_{fin}^{\mathsf{op}} \dashv \widetilde{\mathbb{D}_*}$ *still is an adjunction.*

E Quotient and product of separately continuous algebras

Let (R,τ) be a topological ring. Let, as in Remark 118, p. 55, ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ (resp. ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$) be the category with objects $\mathsf{A} = ((A,\sigma), m_{\mathsf{A}}, 1_{\mathsf{A}})$ such that (A,σ) is a topological (R,τ)-module, $(A, m_{\mathsf{A}}, 1_{\mathsf{A}})$ is a unital R-algebra (resp. unital continuous R-algebra) and $(A,\sigma)\times(A,\sigma) \xrightarrow{m_{\mathsf{A}}} (A,\sigma)$ is separately continuous together with continuous (unit preserving) algebra maps. One refers to such algebras with a topology as *separately continuous algebras.*

The objective of this appendix is to check that ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ (resp. ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$) is closed under topological products and topological quotients by closed two-sided ideals in the same ways ${}_{1}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ (resp. ${}_{1,c}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$) is (see [23]).

Let I be a two-sided ideal of a separately continuous algebra A. Let us furthermore assume that I is closed in the topology σ. Let $A \xrightarrow{\pi_I} A/I$ be the canonical (R-linear) projection. The *quotient topology* of A/I ([23, Definition 5.1, p. 33], denoted $(A,\sigma)/I$, is the strongest topology on A/I for which π_I is continuous. (According to (the proof of) [23, Theorem 5.7, p. 34], $(A,\sigma)/I$ is a Hausdorff space, because I is closed.) In other words, the quotient topology is the final topology on A/I induced by π_I (see [6]). In particular, U is open in the quotient topology if, and only if, $\pi_I^{-1}(U)$ is open in (A,σ).

It satisfies the following universal property: given a topological space (Y,γ) and a map $A/I \xrightarrow{f} Y$. f is continuous if, and only if, $\pi_I \circ f$ is continuous ([23, Theorem 5.3, p. 33]).

204 Lemma *Let* $\mathsf{A} = ((A,\sigma), m_{\mathsf{A}}, 1_{\mathsf{A}})$ *be an object of* ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *(resp.* ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$*). Let* I *be a closed two-sided ideal of* A*. Then,* $(A,\sigma)/I$ *is an object of* ${}_{1,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$ *(resp.* ${}_{1,c,sep}\mathbf{TopAlg}_{(\mathsf{R},\tau)}$*).*

Proof: According to the proof of [23, Theorem 5.4, p. 33] it suffices to prove that the quotient multiplication m_I of A/I is separately continuous. By definition, $m_I \circ (\pi_I \times \pi_I) = \pi_I \circ m_{\mathsf{A}}$. Let $Q \in A/I$, and $v \in A$ with $\pi_I(v) = Q$.

Let $\rho(v)\colon A \to A$, $u \mapsto m_{\mathsf{A}}(u,v)$ be the right multiplication by v on A, and let $\rho_I(Q)\colon A/I \to A/I$ be the corresponding right multiplication on A/I. By the above, $\rho_I(Q) \circ \pi_I = \pi_I \circ \rho(v)$, which is continuous by assumption on A. Considering the universal property of $(A,\sigma)/I$, it follows that $\rho_I(Q)$ is continuous too. $\square$

Let $(\mathsf{A}_i)_{i\in I}$ be a family of separately continuous (R,τ)-algebras. Let $\prod_{i\in I}(A_i,\sigma_i)$ be the product topological (R,τ)-module. Then, $\prod_{i\in I}(A_i,\sigma_i) \times \prod_{i\in I}(A_i,\sigma_i) \xrightarrow{M} \prod_{i\in I}(A_i,\sigma_i)$ is separately continuous, where by M is meant the multiplication of the product algebra $\prod_{i\in I}(A_i, m_{\mathsf{A}_i}, 1_{\mathsf{A}_i})$. Indeed let $i \in I$. Then, $\pi_i(M(u,v)) = m_{\mathsf{A}_i}(u_i,v_i) = m_{\mathsf{A}_i}((\pi_i(u),\pi_i(v))$. Let $\rho_M(v)$ and $\rho_{m_{\mathsf{A}_i}}(v_i)$ be the corresponding right multiplications. Then, $\pi_i \circ \rho_M(v) = \rho_{m_{\mathsf{A}_i}}(v_i)\circ\pi_i$, so that $\rho_M(v)$ is continuous. Whence $(\prod_{i\in I}(A_i,\sigma_i), M, (1_{\mathsf{A}_i})_{i\in I})$ turns out to be a separately continuous algebra, denoted $\prod_{i\in I}\mathsf{A}_i$.

One may summarize this result as follows.

205 Lemma *A product of a family of separately continuous* (R,τ)*-algebras is a separately continuous* (R,τ)*-algebra.*

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