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# Regular Switching Components

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## Abstract

We consider a problem of Discrete Tomography which consists in reconstructing a lattice set  $S \subset \mathbb{Z}^2$  with given horizontal and vertical X-rays (in other words, with prescribed number of points in each row and column). Without complementary assumption, the problem can be solved in polynomial time [1, 2]. Many variants add the constraint to find a solution in a chosen class  $\mathcal{C}$ . Among others, the problem is NP-complete for the class  $\mathcal{HV}$  of HV-convex lattice sets [3] and it becomes polynomial for the class  $\mathcal{HV4}$  of the HV-convex polyominoes [4]. Twenty years after these results, its complexity remains unknown for the class  $\mathcal{P}$  of the convex lattice sets (in other words two-dimensional lattice polytopes).

The difficulty of this problem comes from combinatorial structures called *switching components*. Given the border of a solution (its feet), switching components are finite sequences of points  $(p_i)_{1 \leq i \leq 2l}$  with the property that either the points with odd indices, or the points with even indices are in a solution  $S$ . This binary choice is encoded in a boolean variable associated with the switching component. Then the convexity constraints are simply encoded by clauses (2-clauses for HV-convexity and 3-clauses for convexity).

The purpose of the paper is to investigate the properties of the switching components and their relations induced by convexity. We divide the switching components in two classes: *regular* if their turning angle is constant, *irregular* otherwise. We prove that the 4-connected regular switching components have equal boolean variable. It leads to merge them in extended switching components. If all switching components are regular, we prove that the extended switching components are all independent (then the number of

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solutions with the considered feet is  $2^n$  where  $n$  is the number of extended switching components) and that they are geometrically ordered.

*Keywords:* Discrete Tomography, X-rays, HV-convex lattice sets

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## 1. Introduction

### *A word about some friends of Maurice Nivat*

The scientific community involved in the field of Discrete Tomography started his work 25 years ago. Maurice Nivat promoted the field by co-organizing or participating to some weeks of workshop in Dagstuhl (1997), Thionville (1999), Sienna (2000), Oberwolfach (2000). He met new friends and presented the field to younger researchers who became his students and friends. This tribute to Maurice Nivat is an opportunity to honor also the memory of three of his friends who unfortunately died much too early: Alain Daurat, the last PhD student of Maurice (1973-2011), Alberto Del Lungo (1965-2003) [5] and Attila Kuba (1953-2006).

### *1.1. Open problems in Discrete Tomography*

The field of Discrete Tomography [6, 7, 8] started in the mid 1990s years with a report of failure. The classical algorithms of Computerized Tomography fail to reconstruct lattice sets as requested for instance in Electron Microscopy for the investigation of crystals [9, 10]. Due to the technical principle providing the measurements and the complexity of the considered problems, a special attention has been given on dimension 2. The state of the art around this question was already rich of different results of complexity. The most fundamental one is due to D. Gale and R.J Ryser in 1957. They proved independently that if it exists, a lattice set with prescribed number of points in each row and column can be computed in polynomial time [1, 2]. This central problem has been extended in several directions including the following:

- By increasing the dimension of the lattice, the problem becomes NP-hard from dimension 3 [11]. The question can also be related with timetables and multi-commodity flow problems [12, 13].
- By increasing the number of X-rays (let us precise that an *X-ray* is the vector which counts the number of points of a lattice set in the

30 sequence of lines of given direction), from 3 directions, the problem  
31 becomes NP-hard [14].

- 32 • By reconstructing no more one set from its X-rays but simultaneously  
33 several sets (with empty intersections) from their own X-rays [15], the  
34 problem is NP-hard from only two sets [16].
- 35 • By adding geometrical or topological constraints as we are going to  
36 precise in the following.

37 Instead of searching for any lattice set with the prescribed X-rays as in  
38 the initial problem investigated by D. Gale and H.G Ryser [1, 2], we can  
39 search for a solution satisfying complementary constraints. It might change  
40 completely the complexity of the problem. The constraints which are added  
41 can be topological, for instance by searching for 4-connected solutions (4-  
42 connected finite subsets of  $\mathbb{Z}^2$  are called *polyominoes*), or geometrical, as the  
43 research of convex solutions. This area of research has been open by two  
44 seminal papers providing deep results.

- 45 • The first one is due to Richard Gardner and Peter Grizmann[17]. They  
46 considered the reconstruction of convex lattice sets with different num-  
47 ber of directions of X-rays. They characterized the sets of  $n$  directions  
48 for which any set is uniquely determined by its X-rays. For 2 or 3  
49 directions, their exist always ambiguous pairs or triplet of X-rays. For  
50  $n \geq 7$  directions, all lattice sets are uniquely determined by their X-  
51 rays. For  $3 < n < 7$ , the so-called cross-ratios of the directions provide  
52 a characterization of the sets of direction providing either uniqueness,  
53 or ambiguous X-rays [17]. With the directions of X-rays providing  
54 uniqueness, these results have been completed by a polynomial time  
55 algorithm of reconstruction by Sara Brunetti and Alain Daurat [18].
- 56 • The second paper on which we focus our attention is the reconstruction  
57 of HV-convex polyominoes of  $\mathbb{Z}^2$  from their horizontal and vertical X-  
58 rays. Helena Barucci, Alberto Del Lungo, Renzo Pinzani and Maurice  
59 Nivat proved in 1996 that it can be solved in polynomial time [4].

60 The polynomial time algorithm reconstructing 4-connected HV-convex  
61 lattice sets from their horizontal and vertical X-rays is all the more notewor-  
62 thy that almost all the other neighboring problems are NP-hard. Deciding

63 whether there exists a solution in the classes of H-convex, V-convex, HV-  
64 convex, 4-connected, H-convex and 4-connected lattice sets is NP-complete  
65 [3, 4] in these five cases. The main remaining open question with orthogonal  
66 X-rays is the complexity of the reconstruction of lattice polytopes or in other  
67 words convex lattice sets (for a lattice set, *convex* means that it is equal to  
68 the intersection of its real convex hull with the lattice). It's a challenging  
69 problem whose adjacency with the reconstruction of HV-convex polyominoes  
70 has drawn a new attention on the classical algorithm of [4]. Although older  
71 than 20 years, this approach has still a shadow zone. It concerns the last  
72 step of the algorithm. Given the feet of the lattice set, either the algorithm  
73 stops during the filling operations and there is no solution, or the last step is  
74 required to find solutions. Several experts have noticed that if the algorithm  
75 arrives at the fourth step, there are always solutions (unfortunately only oral  
76 communication). Although it can be proved quite easily in the cases b), d)  
77 and f) of Fig.20, this remark remains a conjecture.

78 This unexplained and prevalent property shows that it remains a real  
79 interest in digging the heritage of Maurice Nivat and his co-authors. They let  
80 the combinatorial structures of the switching components unexplored while  
81 their investigation is of major interest for the reconstruction of convex lattice  
82 sets, a question which remains an open question despite the current attention  
83 on this topic [19]).

#### 84 1.2. Results

85 We provide in this paper new results in the framework of the reconstruction  
86 of lattice sets from their horizontal and vertical X-rays with constraints  
87 of convexity or HV-convexity. The original approach for solving these prob-  
88 lems of reconstruction (also used in [20, 18]) works in 4 steps.

- 89 1. Fix the feet of the solutions (the points on the boundary of the region  
90 of interest).
- 91 2. Proceed to a sequence of filling operations.
- 92 3. Partition the remaining undetermined points in switching components  
93 and express the constraint of HV-convexity (or convexity) with 2-  
94 clauses.
- 95 4. Use a 2-SAT solver to provide solutions.

96 The fourth step of the algorithm is however very often useless. In most  
97 cases, the 2-SAT instance is reduced to a set of equalities on boolean vari-  
98 ables which can be solved without any 2-SAT solver. The purpose of the

99 paper is to provide the definitions, classifications and properties which ex-  
100 plain the degeneracy of the set of constraints induced by HV-convexity. We  
101 introduce a partition of the switching components in two classes according to  
102 the variation of their turning angle: Switching components are closed path  
103 which might turn sometimes on the left (anticlockwise) and sometimes on  
104 the right (clockwise). If the path is always turning on the same direction,  
105 the switching component is said *regular*, and *irregular* otherwise. A larger  
106 part of the paper but not all is devoted to the regular case.

107 The main lemma of the paper (Lemma 1) is that the regular switching  
108 components at Euclidean distance 1 are necessarily equal. It follows that in  
109 the case where all the switching components are regular, any pair of switch-  
110 ing component is either equal, or independent (Theorem 1). In other words,  
111 if the HV-convexity makes two regular switching components dependent and  
112 relates them by a 2-clause, then, at some other points of the path, HV-  
113 convexity enforces the converse dependency and provides their equality. It  
114 leads to merge the equal switching components in *extended switching compo-*  
115 *nents*. These objects are all independent. It explains why in the case where  
116 all switching components are regular, the 2-SAT instance is reduced to a set  
117 of equalities (these instances are of course trivial to solve). It follows the  
118 structural property:

119 If all switching components are regular, the number of solutions (with the  
120 chosen feet) is  $2^n$  where  $n$  is the number of extended switching components  
121 (Property 1).

122 This property does not hold in general: with irregular switching compo-  
123 nents, the number of solutions is not necessarily a power of 2 (an example is  
124 drawn in Fig.1).

125 As the condition of regularity of all the switching components can appear  
126 as a bit restrictive, we investigate the different cases where it might occur. It  
127 leads to prove new properties of the switching components and provide some  
128 counter-examples going against the intuition.

- 129 • We prove that each switching component visits necessarily the four  
130 borders (South West, North West, North East and South East) of the  
131 set of undetermined points (Property 2).
- 132 • Among all the possible configurations of the feet, in one case, they are  
133 all regular while in the other, irregular switching components might oc-  
134 cur (Fig.20). In this second case, only one sub-case among five provides

135 both regular and irregular switching components (this counter-intuitive  
 136 case is illustrated Fig.21). For all the others: switching components are  
 137 either all regular, or all irregular.

- 138 • If all switching components are regular, we prove that the extended  
 139 switching components obtained by merging the 4-connected switching  
 140 components are well ordered (Property 3).

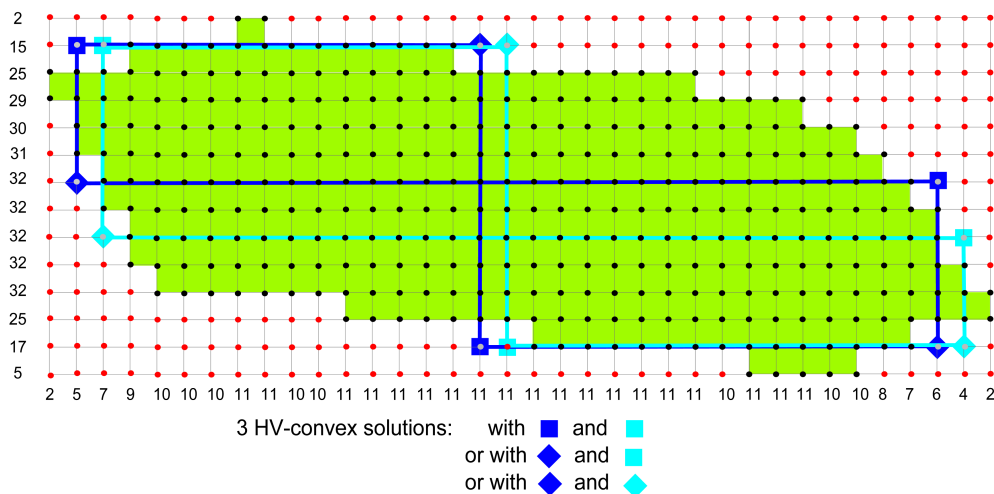


Figure 1: **Counter-example with irregular switching components.** The two switching components (blue and cyan) are irregular since their path does not have a constant turning angle. With such switching components, the number of HV-convex solutions is not necessarily a power of 2 while it is always the case with regular switching components (Property 1).

141 These results do not only provide a better understanding of the classical  
 142 algorithm used for reconstructing HV-convex lattice set. The simplification  
 143 that it introduces in the case of regular switching components might also be  
 144 useful for solving the open question of the reconstruction of convex and no  
 145 more HV-convex lattice sets from their horizontal and vertical X-rays.

146 In Sec.2, we present the definitions from HV-convexity to the notion of  
 147 switching components by passing through the original algorithm of recon-  
 148 struction of [4]. In Sec.3, we provide the main result and its proof about the  
 149 degeneracy of the constraints in the regular case. At last, Sec. 4 is devoted  
 150 to the complementary results (properties 2 and 3).

151 **2. Definitions and notations**

152 *2.1. HV-convexity*

153 There exist several kinds of convexity in  $\mathbb{Z}^d$  or even simply in  $\mathbb{Z}^2$ : direc-  
 154 tional convexity such as HV-convexity,  $Q$ -convexity and just *convexity* some-  
 155 times called *digital convexity* or *strong convexity*. We introduce HV-convex  
 156 and convex lattice sets.

157 **Definition 1.** A lattice set  $S \subset \mathbb{Z}^2$  is *HV-convex* if its intersection with any  
 158 horizontal or vertical line is a set of consecutive points (Fig.2).

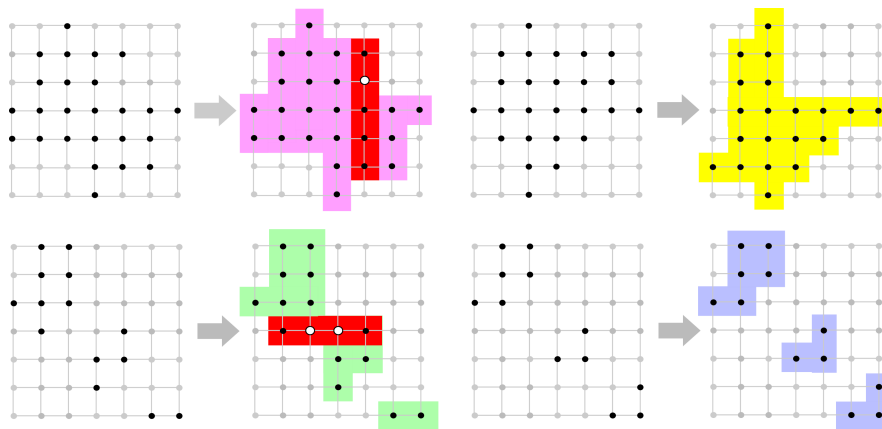


Figure 2: **HV-Convexity.** The left lattice sets are not HV-convex since their intersection with some vertical or horizontal line (in red) has holes. The two right lattice sets are HV-convex.

159 **Definition 2.** A lattice set  $S \subset \mathbb{Z}^2$  is *convex* if it is equal to the intersection  
 160 of its (real) convex hull with the lattice (Fig.3).

161 In other words a finite convex lattice set is equivalent to a lattice poly-  
 162 tope. We introduce notations for the classes of finite lattice sets used in the  
 163 following. The set of the HV-convex 4-connected lattice sets (HV-convex  
 164 polyominoes) is denoted  $\mathcal{HV}4$ . The set of the HV convex lattice sets is deno-  
 165 ted  $\mathcal{HV}$ . The set of the convex lattice sets is denoted  $\mathcal{P}$  ( $\mathcal{P}$  for lattice  
 166 polytopes). Then we have the two inclusions  $\mathcal{P} \subset \mathcal{HV}$  and  $\mathcal{HV}4 \subset \mathcal{HV}$ .  
 167 Notice that both convex and HV-convex lattice sets are not necessarily 4 or  
 168 even 8-connected.



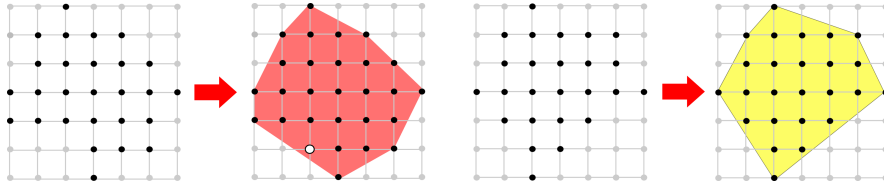


Figure 3: **Convexity.** The left lattice set is not convex since its convex hull (in red) contains a lattice exterior point (in white). The right lattice set is convex.

169 *2.2. Horizontal and vertical X-rays*

170 An *X-ray* is the sequence of the cardinalities of the intersection between  
 171 a given lattice set and the consecutive diophantine lines having a given di-  
 172 rection. In the case of the vertical and horizontal directions, it leads to the  
 173 following definition.

174 **Definition 3.** Given a finite lattice set  $S \subset \mathbb{Z}^2$  included in the rectangle  
 175  $[1, m] \times [1, n]$ , its vertical X-ray is the vector  $V(S) \subset \mathbb{Z}^m$  whose coordinate  
 176  $v_i(S)$  is the number of points of  $S$  in the vertical line  $x = i$  (we have  $V_i(S) =$   
 177  $|\{(x, y) \in S | x = i\}|$ ). The horizontal X-ray of  $S$  is the vector  $H(S) \subset \mathbb{Z}^m$   
 178 counting the number of points of  $S$  in each horizontal line  $y = j$  (we have  
 179  $H_j(S) = |\{(x, y) \in S | y = j\}|$ ).

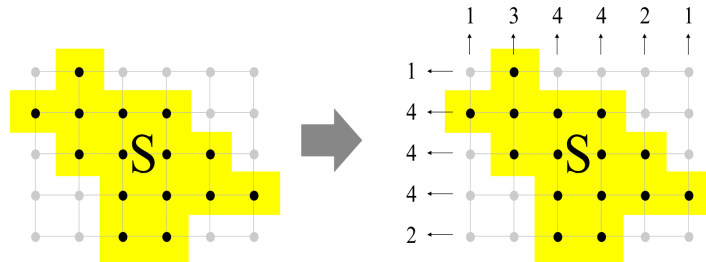


Figure 4: **The horizontal and vertical X-rays** of a given lattice set  $S$ .

180 *2.3. Problem statement*

181 We introduce the problem of decision of the existence of a finite lattice  
 182 set having the double property to belong to a specific class  $\mathcal{C}$  and with the  
 183 given X-rays:

184 **Problem 1** ( $DT_{\mathcal{C}}(h, v)$ ).

185 *Given a class  $\mathcal{C}$  of lattice sets*

186 **Input:** *Two vectors  $v \in \mathbb{Z}^m$ ,  $h \in \mathbb{Z}^n$*

187 **Output:** *does there exist a lattice set  $S \in \mathcal{C}$  with  $V(S) = v$  and  $H(S) = h$  ?*

188 The problems  $DT_{\mathcal{C}}$  are known to be polynomial for the classes  $\mathcal{C}$  of the  
189 whole lattice sets [1, 2] and for the class  $\mathcal{HV}4$  of the HV-convex polyominoes  
190 [4]. Its complexity is unknown for the class of the convex polytopes  $\mathcal{P}$  and is  
191 NP-complete for many other (H-convex, V-convex, HV-convex, 4-connected)  
192 [3, 4].

#### 193 2.4. The feet

194 We introduce the feet of a lattice set  $S$  as the subsets of its points with  
195 minimal and maximal coordinates.

196 **Definition 4.** Given a lattice set  $S \subset \mathbb{Z}^2$ , we consider its South, East, North  
197 and West feet as the sets

- 198 •  $\text{South}(S) = S \cap \{y = y_{\min}(S)\}$  where  $y_{\min}(S)$  is the min ordinate of  
199 the points of  $S$ .
- 200 •  $\text{West}(S) = S \cap \{x = x_{\min}(S)\}$  where  $x_{\min}(S)$  is the min abscissa of the  
201 points of  $S$ .
- 202 •  $\text{North}(S) = S \cap \{y = y_{\max}(S)\}$  where  $y_{\max}(S)$  is the max ordinate of  
203 the points of  $S$ .
- 204 •  $\text{East}(S) = S \cap \{x = x_{\max}(S)\}$  where  $x_{\max}(S)$  is the max abscissa of the  
205 points of  $S$ .

#### 206 2.5. Steps of the original algorithm

207 The original algorithm for solving  $DT_{\mathcal{HV}4}$  is strongly dependent on HV-  
208 convexity. It is only guaranteed to terminate in a polynomial time for the  
209 class  $\mathcal{HV}4$  of the HV-convex polyominoes. Its principle can however be con-  
210 sidered for any instance  $DT_{\mathcal{C}}(h, v)$  with a class  $\mathcal{C}$  of lattice sets which are all  
211 HV-convex ( $\mathcal{C} \subset \mathcal{HV}$ ). It can be for instance  $\mathcal{HV}4$ ,  $\mathcal{P}$  or even  $\mathcal{HV}$  itself. The  
212 algorithm can be decomposed in four steps [4].

- 213 1. Fix the feet (Fig.6).

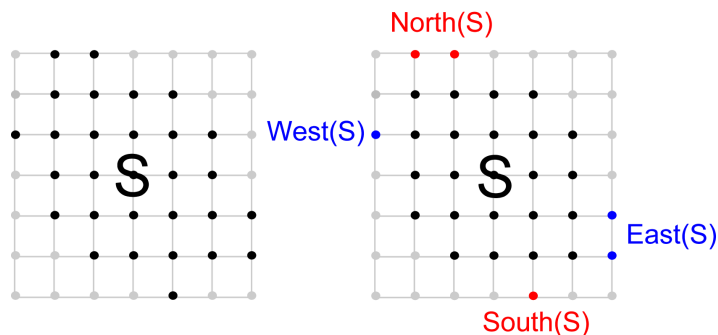


Figure 5: **The feet.** The four feet of a finite lattice set.

- 214 2. Filling operations: Try to satisfy the X-rays and the geometrical con-  
 215 straints until the filling operations do not allow to determine any new  
 216 point.  
 217 3. Decompose the remaining undetermined points in switching compo-  
 218 nents and provide the 2-clauses expressing HV-convexity.  
 219 4. Solve the 2-SAT instance.

220 In many cases, the main part of the computation is done in the second  
 221 step.

222 *2.6. Filling operations*

223 Under the assumption of the chosen feet, the filling operations are working  
 224 with a partition of the lattice in three sets:

- 225 • The set In contains the points which are known to belong to all solu-  
 226 tions.
- 227 • The set Out contains the points which are known to be excluded from  
 228 all solutions.
- 229 • The set Shell is the set of the undetermined points.

230 The task of the filling operations is to take the constraints into account in  
 231 order to include or exclude from the solutions as much points as possible and  
 232 thus decrease the shell. If a point of In has to be added in Out or conversely,  
 233 this contradiction leads to the conclusion that the considered position of the

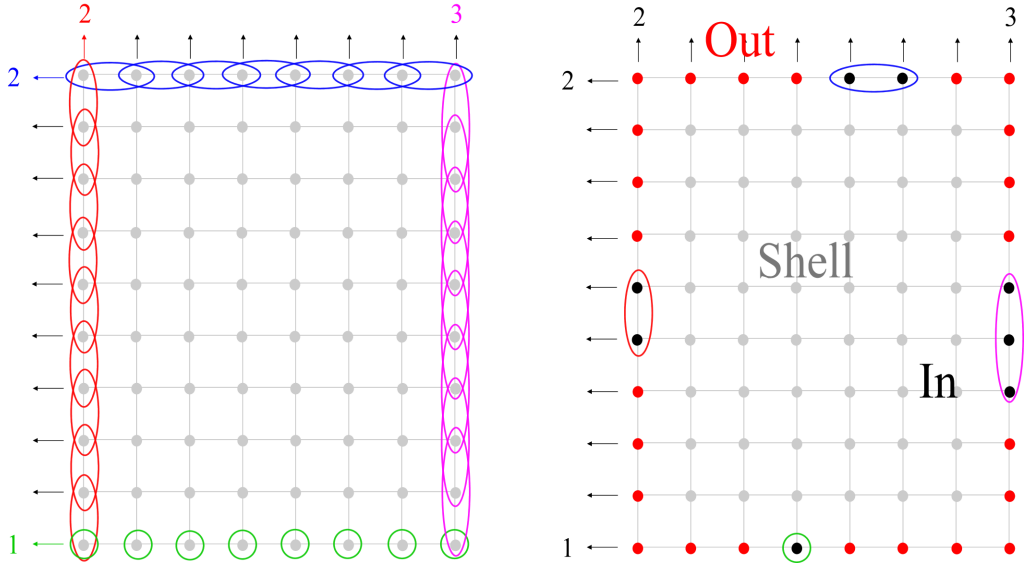


Figure 6: **Step 1.** On the left, the different possible positions of the feet. On the right, a position of the feet is chosen. The points are added in In (in black) while the other extreme points which have not been chosen are added in Out (in red). The shell (in grey) is the set of the points which are not yet determined.

234 feet admits no solution. The next step starts as the filling operations did not  
 235 find contradictions and are no more able to decrease the shell.

236 There exist different kinds of filling operations depending on the chosen  
 237 class  $\mathcal{C}$ . Their common property of HV-convexity and the prescribed X-rays  
 238 provide however the main operations. First operations are uniquely related  
 239 with HV-convexity. The intermediary lattice points between two points of  
 240 In on the same row or on the same column can be added in In. If there  
 241 are two points  $p \in \text{In}$  and  $p' \in \text{Out}$  on the same row or column, the points  
 242 behind  $p'$  can be added in Out. There are also the filling operations which  
 243 take the prescribed X-rays into account with HV-convexity (Fig.7). We refer  
 244 to [20, 18] for a more complete presentation of the operations.

245 Some of the filling operations can be much more complex. It is the case of  
 246 the operation based on the 4-connectivity used for reconstructing HV-convex  
 247 polyominoes [4].

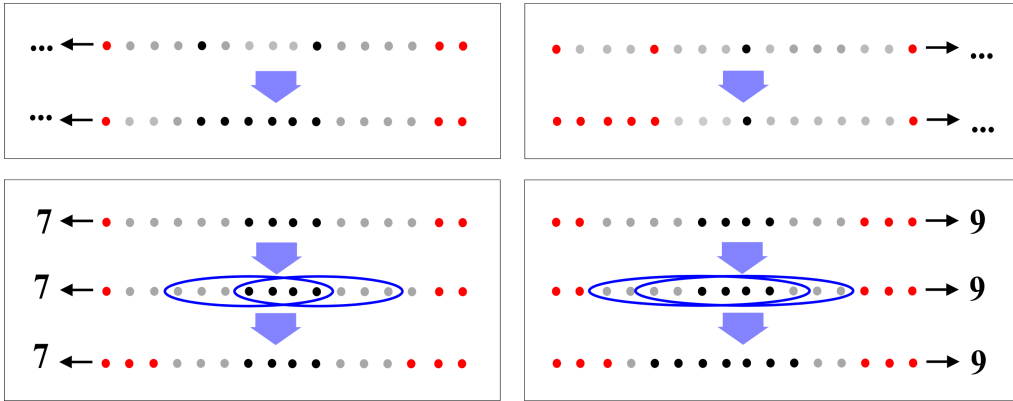


Figure 7: **Four filling operations.**

248 *2.7. 8-connectivity of In and shell partition in SE, NE, NW and SW*

249 We did not restrict the framework of the paper to the reconstruction of a  
 250 specific class  $\mathcal{C}$  of lattice sets, except the property that it has to be HV-convex  
 251 but for going further, we need a first assumption:

252 We assume that the set In obtained at the end of the filling operations  
 253 is 8-connected. At the step 1, the set In is initialized as the union of the  
 254 feet. In most cases, they are not 8-connected. If the filling operations allow  
 255 to connect them, then at the end of the filling operations, we can decompose  
 256 the shell in 4 parts according to their relative position towards the set In.  
 257 Otherwise, there is no intrinsic decomposition.

258 The assumption that the set In is 8-connected allows to decompose the  
 259 shell in four borders. These four borders located between In and Out are  
 260 denoted SE, NE, NW and SW for South-East, North-East, North-West and  
 261 South-West (Fig.8).

262 Once defined, the main property of the four layers is that they are struc-  
 263 tured by pairs, North and South, East and West.

264 *2.8. Corresponding points*

265 We define vertical correspondences between the South and the North  
 266 points and horizontal correspondences between the West and the East points.

267 **Definition 5.** We consider an instance  $DT_{\mathcal{C}}(h, v)$  with X-rays  $h$  and  $v$  and a  
 268 class  $\mathcal{C} \subset \mathcal{HV}$ . We assume that the set In obtained after the filling operations  
 269 is 8-connected so that the shell has been decomposed in four borders.

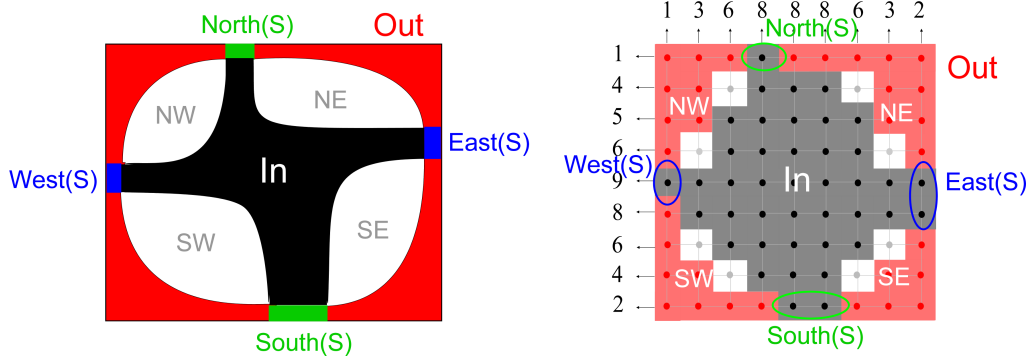


Figure 8: **The 4 borders of the shell.** If the set In is 8-connected, the shell can be decomposed in its South-East, North-East, North-West and South-West borders respectively denoted SE, NE, NW and SW.

270 The vertical correspondent of a point  $\underline{p} = (i, j)$  of the South borders  
 271  $SE \cup SW$  is the point  $\bar{p} = (i, j + v_j)$ .

272 The vertical correspondent of a point  $\bar{p} = (i, j)$  of the North borders  
 273  $NE \cup NW$  is the point  $\underline{p} = (i, j - v_j)$ .

274 The horizontal correspondent of a point  $p| = (i, j)$  of the West borders  
 275  $SW \cup NW$  is the point  $|p = (i + h_j, j)$ .

276 The horizontal correspondent of a point  $|p = (i, j)$  of the East borders  
 277  $SE \cup NE$  is the point  $p| = (i - h_j, j)$ .

278 The vertical and horizontal correspondents have a few properties.

- 279 1. The horizontal and vertical correspondents of an undetermined point  $p$   
 280 are also undetermined (otherwise, the filling operations allow to assign  
 281  $p$  to In or Out).
- 282 2. Correspondences are symmetric relations.
- 283 3. The distance between  $p$  and its correspondents being  $h_j$  and  $v_i$ , if  $p$  is  
 284 in a solution, its correspondents are outside.
- 285 4. A point  $p$  and its correspondent  $p'$  cannot be outside from a solution  
 286 in the meantime because the segment of the solution is necessarily in  
 287 between and there are not enough points.

288 It follows that if  $p$  and  $p'$  are correspondents,  $p$  is in a solution  $S$  if and only  
 289 if  $p'$  is not in  $S$ .

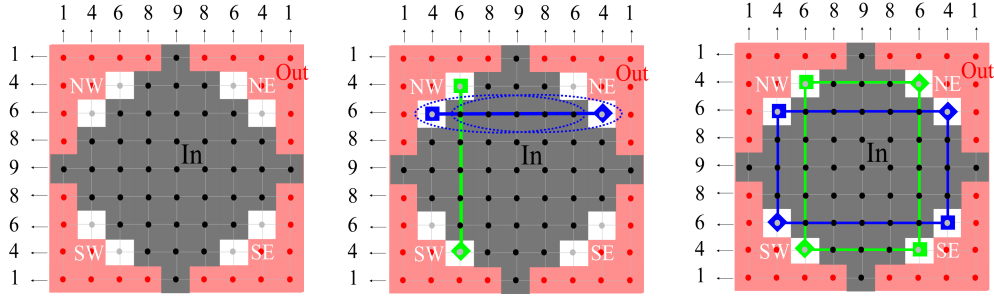


Figure 9: **Corresponding points.** On the left, a configuration with the sets In (black points), Out (red points) and Shell (white cells, grey points) of the undetermined points. In the middle, a pair of vertical correspondents (green) and a pair of horizontal correspondents (blue). We represent the points alternatively with squares or diamonds. Notice that the segment represented by the dotted ellipse has only two possible positions. Due to its length, if it contains the square, it does not contain the diamond and conversely. On the right, the correspondences define closed paths called switching components which provide a partition of the undetermined points. For each switching components, either the squares belong to a solution, or the diamonds.

290 *2.9. Switching components*

291 The correspondences between the points define paths. The undetermined  
 292 point  $p_1$  has an horizontal correspondent  $p_2$  which has a vertical correspon-  
 293 dent  $p_3$  which has an horizontal correspondent  $p_4$  and so on until coming back  
 294 to  $p_1$  as vertical correspondent (since the shell is finite, the path is necessarily  
 295 closed). This path is called *switching component* (Fig.9 and 10).

296 **Definition 6.** We consider an instance  $DT_C(h, v)$  and a position of the feet  
 297 providing a 8-connected set In. The shell obtained at the end of the filling  
 298 operations is decomposed in SE, NE, NW, SW.

299 A switching component  $P$  is a cyclic sequence of corresponding points  $p_r$   
 300 with indices  $r \in \mathbb{Z}/2l\mathbb{Z}$  (the length of the sequence is  $2l$ ) so that  $p_{2k}$  and  $p_{2k+1}$   
 301 are vertical correspondents while  $p_{2k-1}$  and  $p_{2k}$  are horizontal correspondents.  
 302 We denote  $P[0]$  the set of the points of the sequence with even indices and  
 303  $P[1]$  the set of the points with odd indices.

304 By construction, the two sets  $P[0]$  and  $P[1]$  have the same horizontal and  
 305 vertical X-rays, since each point has a unique vertical and horizontal corre-  
 306 spondent in the other set (they are respectively represented with diamonds  
 307 and squares in all following figures). In the same way than the corresponding

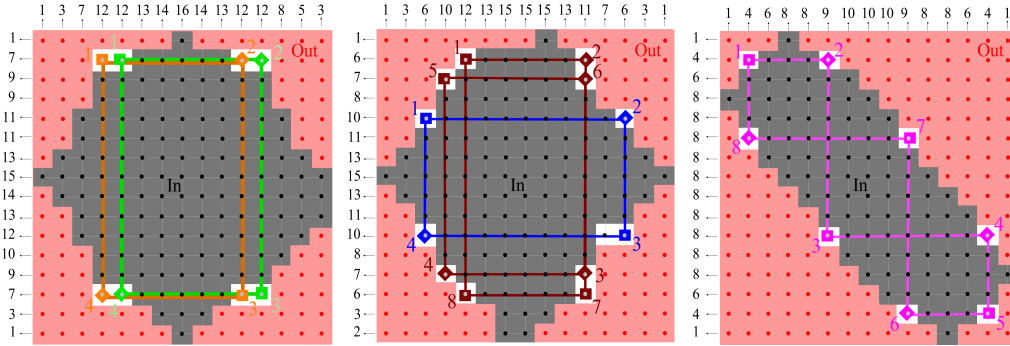


Figure 10: **Switching components.** On the left, two switching components  $P^1$  and  $P^2$ . The squares represent their points with even indices (the set  $P[0]$ ) and the diamonds the points with odd indices (the set  $P[1]$ ). Notice that there are here only 2 convex solutions since the convexity enforces to add  $P^1[0] \cup P^2[0]$  or  $P^1[1] \cup P^2[1]$  in the solution while sets including  $P^1[0] \cup P^2[1]$  or  $P^1[1] \cup P^2[0]$  have also the prescribed X-rays but are not convex. In the middle, regular switching component are not restricted to only one turn (the brown switching component makes 2 turns). On the right, an irregular switching component. It turns clockwise at point 2 and anticlockwise at point 3. On the contrary, the switching components of the two left images are regular (they turn always clockwise).

308 points, any solution  $S$  of  $DT_C(h, v)$  with the considered feet falls in one of  
 309 the two possible cases:

- 310 • either  $P[0] \subset S$  and  $P[1] \cap S = \emptyset$ ,
- 311 • or  $P[1] \subset S$  and  $P[0] \cap S = \emptyset$ .

312 In other words, considering a switching component  $P$ , a solution contains  
 313 either the points of the switching component with odd indices, or the ones  
 314 with even indices.

### 315 2.10. Boolean variables for encoding the state of each switching component

316 According to previous remark, in a solution  $S$  of  $DT_C(h, v)$  with prescribed  
 317 X-rays and fixed feet, a switching component  $P$  can have only two states:

- 318 •  $P(S) = 0$  if  $P[0]$  is in  $S$  and  $P[1]$  is excluded.
- 319 •  $P(S) = 1$  if  $P[1]$  is in  $S$  and  $P[0]$  is excluded.



320 The value  $P(S)$  is a boolean variable. If we consider now the whole sets of  
 321 switching components  $P^q$  for the indices  $q$  going from  $q = 1$  to the number  $Q$   
 322 of switching components, the values of the  $Q$  boolean variables  $P^q(S)$  provide  
 323 a complete characterization of  $S$ . Moreover, the  $2^Q$  possible assignments of  
 324 the  $Q$  boolean variables all provide a set  $S$  with the prescribed horizontal  
 325 and vertical X-rays. The problem is to determine whether there exists an  
 326 assignment which provides a solution  $S$  in the chosen class  $\mathcal{C}$ .

### 327 2.11. Regular switching components

328 Before starting the investigation of the relationships between the switch-  
 329 ing components, we introduce two groups according to the regularity of their  
 330 turning angle.

331 **Definition 7.** A switching component is regular if for all indices, the ori-  
 332 ented angle between vectors  $p_k - p_{k-1}$  and  $p_{k+1} - p_k$  is a constant of  $k$  (in  
 333 other words, the switching component turns always clockwise, or always anti-  
 334 clockwise (Fig.10)).

335 The switching components which have clockwise and anticlockwise turn-  
 336 ing angles are simply said *irregular* (Fig.10).

## 337 3. Main result

338 We prove that the boolean variables associated to regular switching com-  
 339 ponents at distance 1 (they have two 4-neighbors at some points) are neces-  
 340 sarily equal. This main lemma allows to state different properties.

### 341 3.1. Encoding the convexity with clauses

342 The classical approach developed in [4] for reconstructing HV-convex  
 343 polyominoes is to search for an assignment of the boolean variable  $p^q(S)$   
 344 leading to an HV-convex solution  $S$ . The approach passes through the en-  
 345 coding of the HV-convexity constraint in a conjunction of 2-clauses. It leads  
 346 to a 2-SAT instance (Fig.11).

347 The same approach holds for encoding convexity with clauses (Fig.11).  
 348 They are however no more restricted to 2-clauses. 3-clauses are necessary  
 349 with the difficulty that 3-SAT is no more polynomial but NP-complete. It  
 350 explains why the original polynomial time algorithm for the class  $\mathcal{HV4}$  of  
 351 convex polyominoes has not been extended to the class of convex polytopes  
 352  $\mathcal{P}$ . We can however believe that some properties of the switching components  
 353 might be of interest for avoiding the use of 3-clauses.

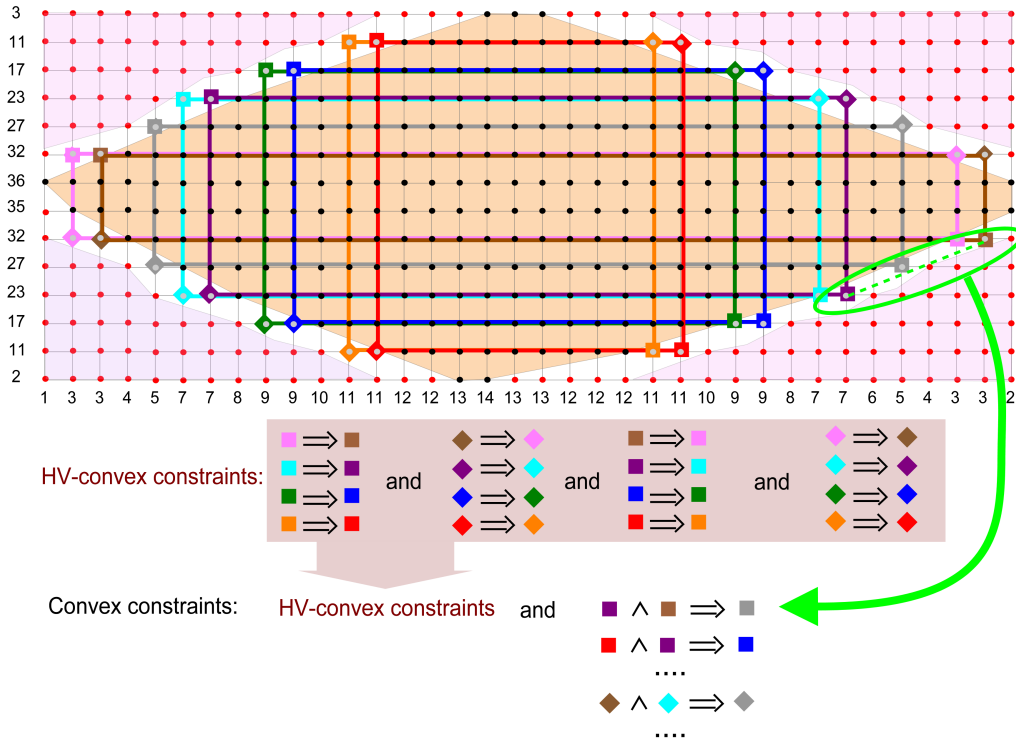


Figure 11: **2 and 3 clauses encoding convexity.** The switching components being encoded in boolean variables (represented here geometrically with colored squares and diamonds), HV-convexity is expressed by a conjunction of 2-clauses. Consequently, the research of an HV-convex solution is reduced to a 2-SAT instance that can be solved in polynomial time. For expressing the convexity (not HV-convexity but the one of Definition 2) in the same manner, we need 3-clauses with the difficulty that 3-SAT is NP-complete.

354 *3.2. Regular switching components at distance 1 are equal*

355 We provide an important lemma whose consequences have been observed  
 356 several times in the framework of the reconstruction of HV-convex polyomi-  
 357 noes. By convention, we choose the initial point  $p_1$  of the regular switching  
 358 components in the North West border NW (Property 2 shows that it is always  
 359 possible).

360 **Lemma 1.** *We consider an instance of  $DT_C(h, v)$  with a class  $\mathcal{C} \subset \mathcal{HV}$ . The*  
 361 *four feet being chosen, we assume that the filling operations have connected*  
 362 *them. It provides a partition of the shell in its NE, NW, SW, SE borders*  
 363 *and thus in switching components (Fig.11). For any pair  $P$  and  $P'$  of regular*

364 *switching components having a pair of points at Euclidean distance 1, the*  
 365 *HV-converity constraint of a solution  $S$  enforces  $P$  and  $P'$  to be equal.*

366 Lemma 1 means that neighboring regular switching components are nec-  
 367 essarily equal.

368 PROOF. We assume without loss of generality that the points  $p_1 = (x_1, y_1)$   
 369 and  $p'_1 = (x'_1, y'_1)$  both belong to the North West border NW and that  $p'_1$  is  
 370 the right 4-neighbor of  $p_1$  (Fig.12).

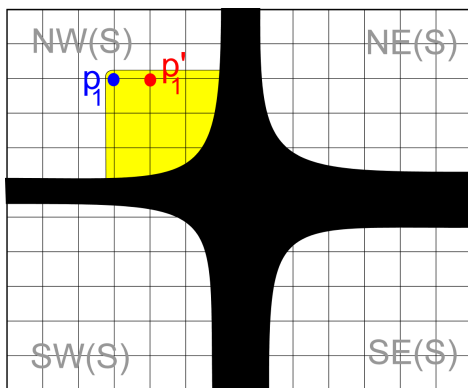


Figure 12: **Assumption of Lemma 1.** We assume that the two considered switching components have two points at Euclidean distance 1. By symmetries, we can assume without loss of generality that  $p'_1$  is the right 4-neighbor of  $p_1$ . It follows that if  $p_1$  is in the solution  $S$  (the odd part  $P[1]$  of the switching component  $P$  is in  $S$  namely  $P(S) = 1$ ), then  $p'_1$  is also in  $S$  (the odd part  $P'[1]$  of the switching component  $P'$  is also in  $S$  namely  $P'(S) = 1$ ). The black zone is a sketch of the set  $\text{In}$ . It underlines the property that with regular switching components horizontal correspondences are only between NW and NE or SW and SE while vertical correspondences are only between NW and SW or NE and SE.

371 We consider the positions of the points  $p_0 = (x_0, y_0)$  (vertical correspon-  
 372 dent of  $p_1$ ),  $p_2$  and  $p_3 = (x_3, y_3)$ . We distinguish three cases according to the  
 373 comparison of  $y_0$  and  $y_3$ .

- 374 • Case 1.  $y_0 < y_3$
- 375 • Case 2.  $y_0 = y_3$
- 376 • Case 3.  $y_0 > y_3$

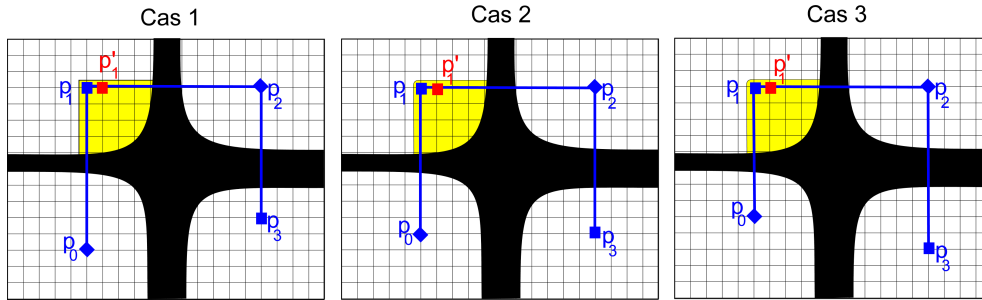


Figure 13: **The three possible configurations used for proving Lemma 1.** We develop the three cases in the following.

377 The three cases are developed in figures 14, 15 and 16. We follow the path  
 378 of the second switching component  $P'$  by starting from  $p'_1$ . This starting point  
 379 is in the region denoted  $A$  and we build the oriented graph of correspondences  
 380 between the regions reached by  $P'$ . This graph provides a description of all  
 381 the possible path and thus of all possible configurations starting from the  
 382 region  $A$ .

383 According to the relative position of  $p'_1$  towards  $p_1$ , the equality  $P(S) = 1$   
 384 implies  $P'(S) = 1$ . Some regions provide the converse relation:  $P'(S) = 1$   
 385 implies  $P(S) = 1$  or equivalently  $P(S) = 0$  implies  $P'(S) = 0$ , and thus  
 386 provide the wanted equality. These regions are colored in grey in all the  
 387 figures. Therefore the task of the proof is to show that any closed path  
 388 starting from  $p'_1$  passes necessarily through a grey region.

389 In the first and second cases (Fig.14 and 15), either the path passes  
 390 through a grey region and provides the equality, or it cannot come back in  
 391 the initial region  $A$  which is excluded. It provides the equality.

392 In the third case, the analysis is a bit more complicated. We proceed in  
 393 two steps. We introduce 4 regions denoted  $A$ ,  $B$ ,  $C$  and  $D$  represented in  
 394 Fig.16. The points  $p'_1$  and  $p'_2$  are respectively in  $A$  and  $B$ . Let us assume  
 395 now that the switching component  $P'$  has two horizontal correspondents  
 396 respectively in  $C$  and  $D$ . These two points and the two points  $p'_1$  and  $p'_2$   
 397 define four (grey) regions as in previous cases and as drawn in Fig.16. If  
 398 the switching component  $P$  passes through one of these regions, then the HV-  
 399 convexity provides the equality between the switching components  $P$  and  
 400  $P'$ . If the switching component does not pass through the four grey regions,  
 401 then the path of the switching component  $P$  after  $p_3$  is restricted to an area

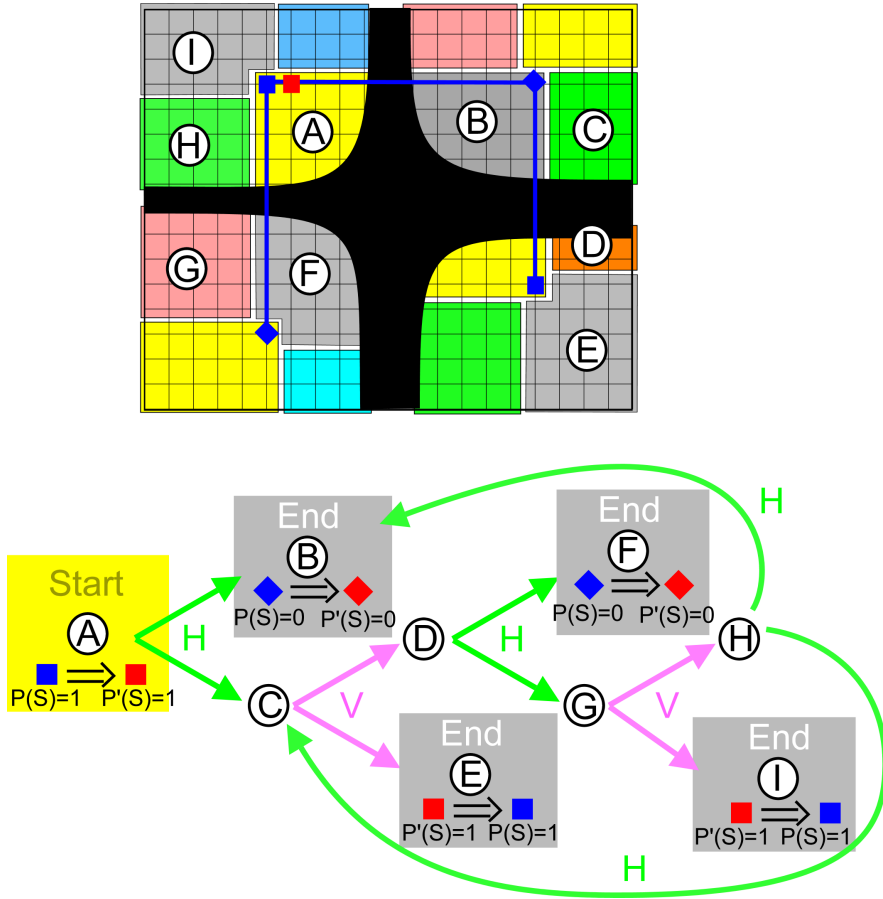


Figure 14: **Case 1**. In the first case, we decompose the four borders in different regions and follow the path of the switching component  $P'$ . It starts with a point  $p'_1$  in the region  $A$ . The path ends also necessarily in  $A$ . Due to the relative position of  $p'_1$  towards  $p_1$ , we have  $P(S) = 1 \implies P'(S) = 1$ . The horizontal correspondents of the points in region  $A$  are either in region  $B$ , or in region  $C$ . The graph of the correspondences between the regions that we can reach from region  $A$  is drawn below. If the switching component  $P'$  passes through one of the grey regions  $B$  or  $F$ , the HV-convexity leads to  $P(S) = 0 \implies P'(S) = 0$ . If it passes through the grey regions  $E$  or  $I$ , then HV-convexity leads to  $P'(S) = 1 \implies P(S) = 1$ . Consequently if the switching component passes through one of the four grey regions, we have the equality  $P(S) = P'(S)$ . We notice at last that the only way to avoid the grey regions is to follow the path  $(CDGH)^*$  but it is excluded because their in no way to end the path in region  $A$ .

402 (in green on the figure) which does not allow to come back to the initial

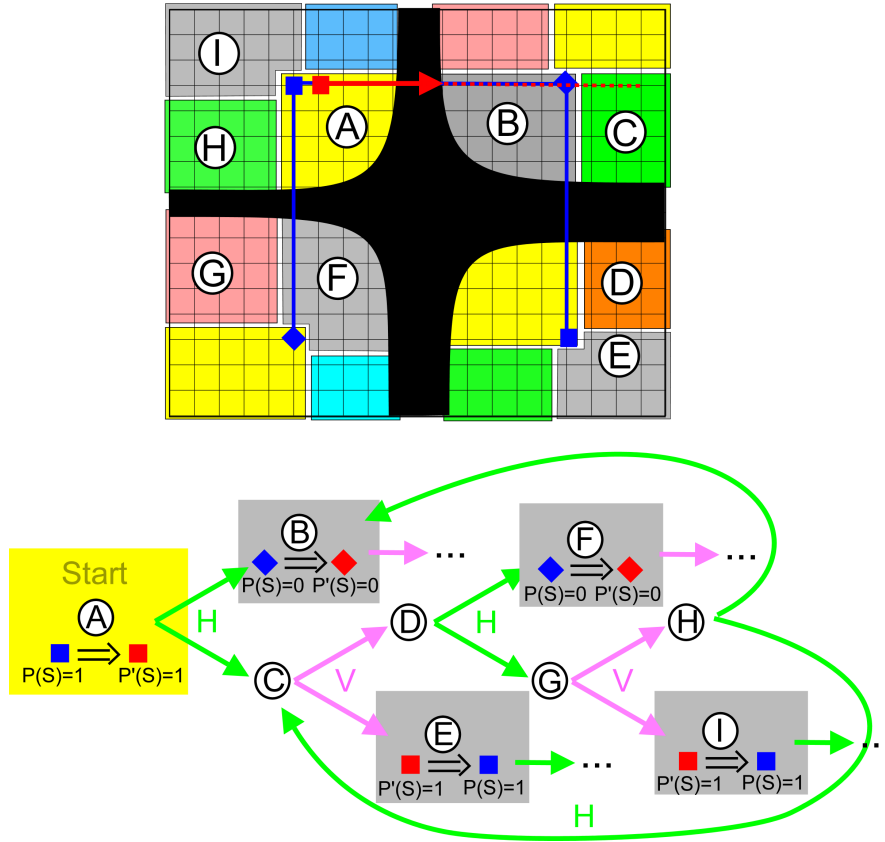


Figure 15: **Case 2.** The second case is similar to the first one. Starting from a point  $p'_1$  in the region  $A$ , the path of the switching component either passes through a grey region which provides the equality  $P(S) = P'(S)$  or has a cycle  $(CDGH)^*$  which does not allow to end the path in region  $A$ . This last possibility is excluded.

403 point. This case can be excluded. We provide a partial conclusion (i): If the  
 404 switching component  $P'$  has two horizontal correspondents in the regions  $C$   
 405 and  $D$ , the equality of  $P$  and  $P'$  is guaranteed.

406 We come back to the path followed by the switching component  $P'$ . We  
 407 partition the undetermined points in regions according to their positions  
 408 relatively to  $p_0$ ,  $p_1$ ,  $p_2$  and  $p_3$  and investigate again the regions crossed by  
 409 the switching component  $P'$ . The grey regions drawn in Fig.17 are the ones  
 410 which, due to HV-convexity, provide the equality of the switching component  
 411  $P(S) = P'(S)$ . By building the diagram of the correspondences between the

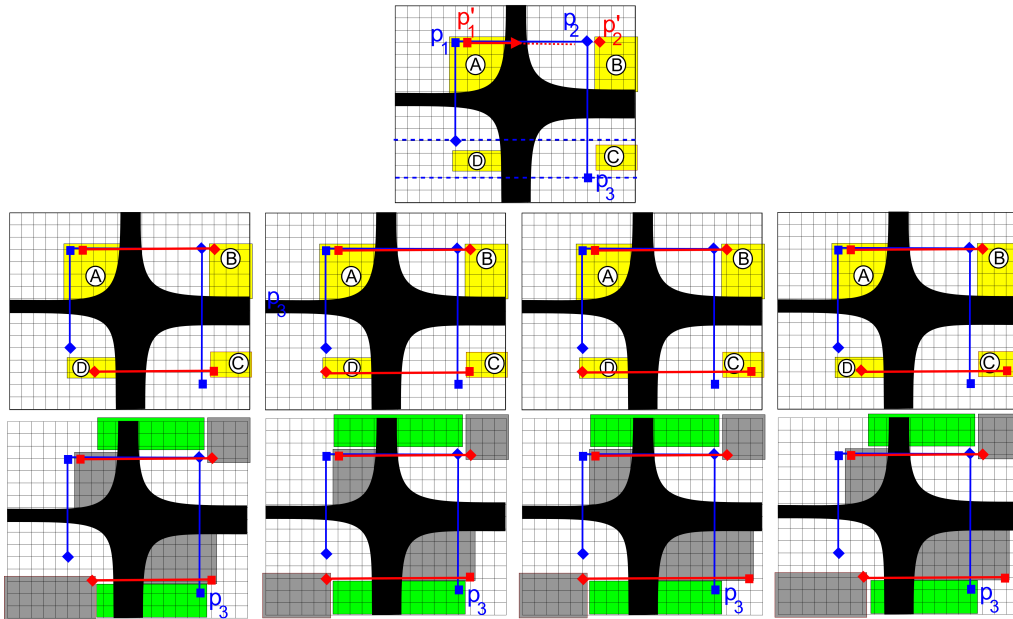


Figure 16: **Case 3. Preliminary analysis.** We introduce four regions  $A$ ,  $B$ ,  $C$  and  $D$ . The points  $p'_1$  and  $p'_2$  (in red) are respectively in  $A$  and  $B$ . Let us assume that there is a pair of horizontal correspondents of  $P'$  in the regions  $C$  and  $D$ . Different sub-cases might occur but the consequences are the same. If we consider now the path of the switching component  $P$  after  $p_3$  (in blue), either it passes through one of the grey regions which provide the equality between  $P$  and  $P'$ , or it is maintained in the green area which does not allow to come back to  $p_1$ . This last case is excluded with the consequence that if  $P'$  has two horizontal correspondents in  $B$  and  $C$ , we have the equality of the switching components (for any solution  $S$ , we have  $P(S) = P'(S)$ ).

412 regions, it appears that  $P'$  passes through a grey region, or passes through the  
 413 edge  $BC$  for which the conclusion (i) provides already the equality  $P = P'$ ,  
 414 or is not closed which is in contradiction with its definition. It follows that in  
 415 all valid cases, we have the equality between the two switching components.  
 416 □

### 417 3.3. Extended switching components are independent

418 The main lemma 1 has a corollary in the case where all the switching  
 419 components are regular.

420 **Theorem 1.** *We consider an instance of  $DT_{\mathcal{C}}(h, v)$  with a class  $\mathcal{C} \subset \mathcal{HV}$ .*  
 421 *The four feet being chosen, we assume that the filling operations have con-*

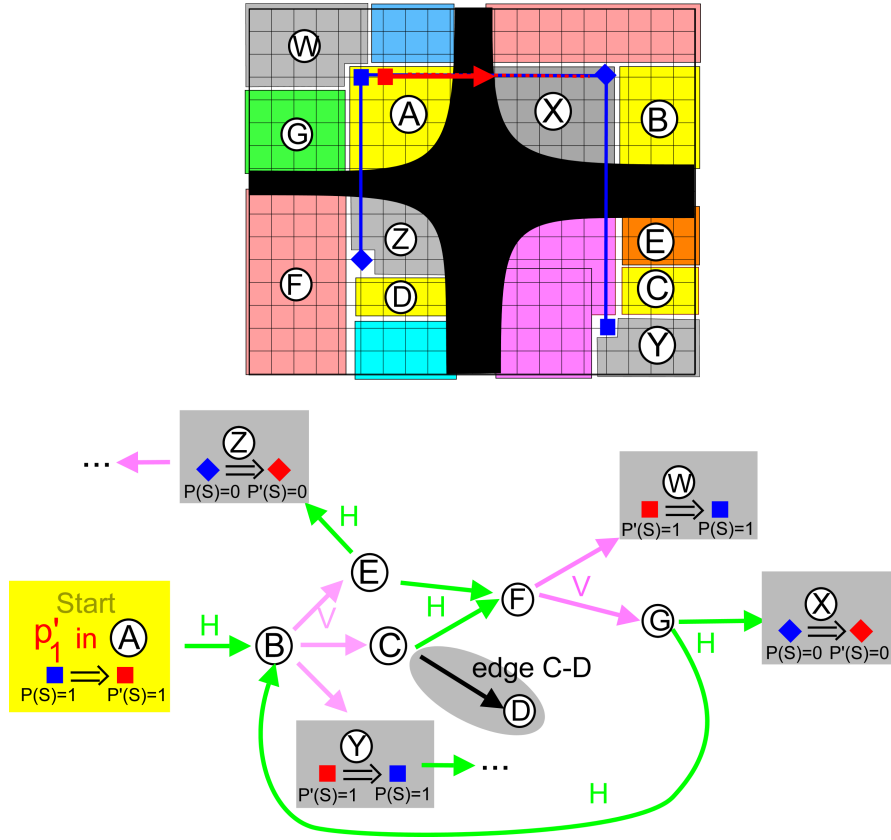


Figure 17: **Case 3. Final analysis.** We partition the borders in regions according to their relative positions towards the points  $p_0, p_1, p_2$  and  $p_3$ . Due to HV-convexity, if the switching component  $P'$  has a point in one of the four grey regions  $W, X, Y$  and  $Z$ , then we have the equality  $P(S) = P'(S)$ . The oriented graph of the horizontal and vertical correspondences between regions is drawn below. It characterizes the possible path issued from  $p'_1$ . Either it passes through a grey region, or it passes through the edge  $CD$  for which we have already proved the equality (i), or it can not be closed and as previously, this case can be excluded. In the three valid cases,  $P(S) = P'(S)$ .

422 nected them. It provides a partition of the shell in its NE, NW, SW, SE  
 423 borders and thus in switching components (Fig.11). We assume that all the  
 424 switching components are regular and keep the convention that the point  $p_1$  of  
 425 any switching component is in NW. Then any pair of switching components  
 426  $P$  and  $P'$  is either equal (for any solution  $S$ ,  $P(S) = P'(S)$ ), or independent.

427 *Proof.* Let us assume that the HV-convexity enforces two switching compo-



428 nents  $P$  and  $P'$  to be dependent. It means that they are in a configuration  
 429 where if a point  $p_k$  of  $P$  is in a solution  $S$ , then the point  $p'_{k'}$  of  $P'$  is also  
 430 in  $S$ . By symmetries, we assume  $p_k$  is in the North West border. It follows  
 431 that  $p'_{k'}$  is its South East quadrant (Fig.18). There is thus a 4-connected  
 432 path of undetermined points from  $p_k$  to  $p'_{k'}$ . By induction, Lemma 1 proves  
 433 that their switching component are all equal and thus provides the equality  
 434 between  $P(S) = P'(S)$ .  $\square$

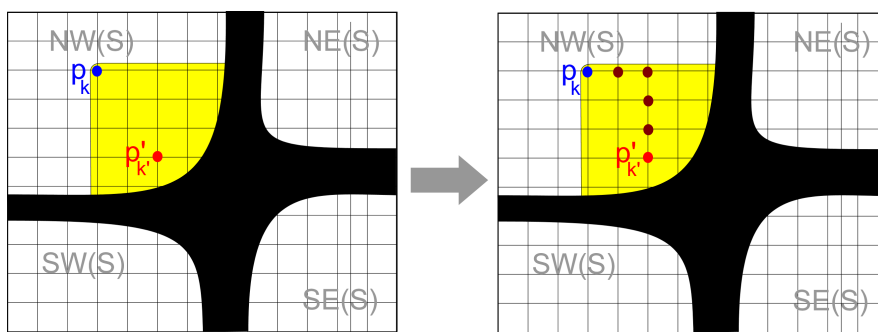


Figure 18: **4-connected path.** HV-convex dependencies between switching components are consequences of a point configurations like the one drawn above. If  $p_k$  is in a solution  $S$ , then by HV-convexity,  $p'_{k'}$  is also necessarily in  $S$ . In any case, we can connect  $p_k$  and  $p'_{k'}$  by a 4-connected path of undetermined points (in brown). As their switching components are all assumed to be regular, according to lemma 1, all the switching components of these brown points are equal. It provides the equality between the switching components of the blue and red points.

435 Theorem 1 states that we have equal switching components (they are  
 436 connected by 4-connected path) and independent switching components. It  
 437 leads to merge the switching components which are equal in *extended switch-*  
 438 *ing components*.

439 **Definition 8.** We consider an instance of  $DT_{\mathcal{C}}(h, v)$  with a class  $\mathcal{C} \subset \mathcal{HV}$   
 440 and positions of the feet providing a partition of the shell in switching com-  
 441 ponents.

442 An extended switching component  $\bar{P}$  is the union of all the switching  
 443 components  $P'$  (considered as the set of their points) whose boolean variables  
 444 are equal to  $P$  due to HV-convexity (Fig.19).

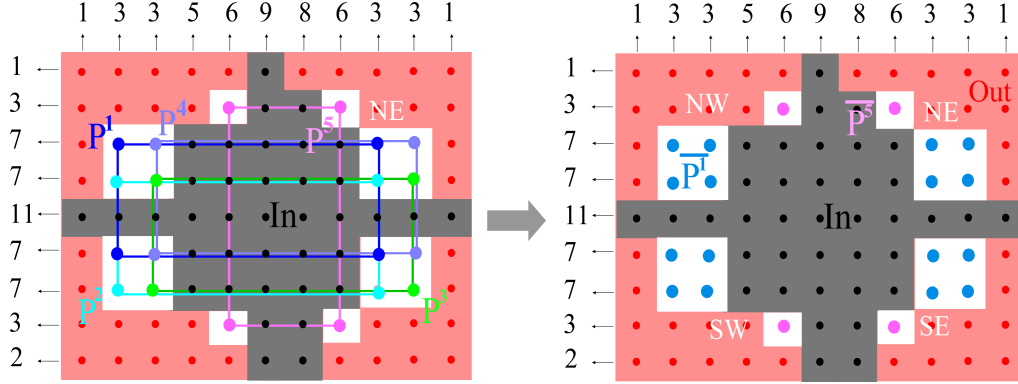


Figure 19: **Extended switching components.** On the left, due to convexity, the switching components  $P^1, P^2, P^3, P^4$  are necessarily equal while  $P^5$  is independent. On the right, we merge  $P^1, P^2, P^3, P^4$  in the extended switching component  $\bar{P}^1$  and  $P^5$  becomes  $\bar{P}^5$ . Extended switching components are less structured, they are just sets of points.

445 Any extended switching component  $\bar{P}$  is a subset of the shell. We denote  
 446  $\bar{P}(\text{NW}), \bar{P}(\text{NE}), \bar{P}(\text{SE})$  and  $\bar{P}(\text{SW})$  its four subsets in the four borders of  
 447 the shell. By definition, for any solution  $S$  of the instance  $DT_C(h, v)$ , either  
 448  $\bar{P}(\text{NW}) \cup \bar{P}(\text{SE})$  is in  $S$  and  $(\bar{P}(\text{SW}) \cup \bar{P}(\text{NE})) \cap S = \emptyset$ , or  $\bar{P}(\text{SW}) \cup \bar{P}(\text{NE})$   
 449 is in  $S$  and  $(\bar{P}(\text{NW}) \cup \bar{P}(\text{SE})) \cap S = \emptyset$ . Theorem 1 means that if all the  
 450 switching components are regular, all extended switching components are  
 451 independent. An obvious corollary of Theorem 1 is therefore the following  
 452 property:

453 **Property 1.** *We consider an instance of  $DT_C(h, v)$  with a class  $\mathcal{C} \subset \mathcal{HV}$ .  
 454 The four feet being chosen, we assume that the filling operations have con-  
 455 nected them. It provides a partition of the shell in its NE, NW, SW, SE  
 456 borders and consequently in switching components (Fig.11).  
 457 If all the switching components are regular, then the number of solutions  
 458 (with the chosen feet) is  $2^n$  where  $n$  is the number of extended switching  
 459 components.*

460 Property 1 does not hold with irregular switching components as drawn  
 461 in Fig.1. We end this section with a last remark about extended switching  
 462 components in the regular case. Under the assumption that all the switch-  
 463 ing components are regular, distinct switching components pass necessarily  
 464 through distinct rows and columns. Otherwise it means that they share a row

465 or column, and they would have been merged according to Lemma 1 under  
 466 the same principle than in Fig.18. This property will be used for ordering  
 467 the extended switching components.

#### 468 4. Complementary results

469 We come back to the general switching components and provide several  
 470 properties. We are in particular interested in the characterization of the  
 471 configurations where all the switching components are regular.

##### 472 4.1. Any switching component visits the 4 borders

473 Due to their constant turning angle, the regular switching components  
 474 have a cyclic property. With an initial point  $p_1$  chosen in NW, we have  
 475  $p_{1+4k} \in NW$ ,  $p_{2+4k} \in NE$ ,  $p_{3+4k} \in SE$ ,  $p_{4k} \in SW$ . It does not hold with irreg-  
 476 ular switching components. We can even ask whether an irregular switching  
 477 component could avoid one of the 4 borders NE, NW, SW, SE. It cannot  
 478 happen:

479 **Property 2.** *Any switching component passes through the four borders NE,*  
 480 *NW, SW, SE.*

481 *Proof.* We assume that there exists a switching component  $P$  which does  
 482 not pass through the border NE. If  $p_1$  is in a South border, then  $p_2$  is a  
 483 North point and thus in NW. Otherwise  $p_1$  is a North point and thus in  
 484 NW. We start from this point  $p_k \in NW$  (with  $k$  equal to 1 or 2) and follow  
 485 the path of the switching component. The horizontal correspondent  $p_{k+1}$  of  
 486  $p_k$  is necessarily in SE. The vertical correspondent  $p_{k+2}$  of  $p_{k+1}$  is therefore  
 487 in NW. By induction, for any positive integer  $n$ , we have  $p_{k+2n} \in NW$  and  
 488  $p_{k+2n} \in SE$ . The turning angles are alternatively  $+\frac{\pi}{2}$  and  $-\frac{\pi}{2}$ . Such a path  
 489 can not come back to the point  $p_k$ . It leads to a contradiction.  $\square$

##### 490 4.2. Structure of the switching components according to the positions of the 491 feet

492 We are interested now in the different possible configurations of the po-  
 493 sitions of the feet. By symmetries, we can reduce them to 6 cases drawn  
 494 Fig.20.

- 495 • In case a), different structures of switching components are possible  
 496 but they are necessarily irregular.

- 497 • In case b), the structure of the switching components is constrained.  
 498 They are all irregular. The point  $p_{k+8}$  is in the same area of the shell  
 499 than  $p_k$ . Moreover, starting from an initial point  $p_1$  in the South West  
 500 border, there is no area containing simultaneously points of switching  
 501 components with odd and even indices. It follows that the constant  
 502 assignment  $P[S] = 0$  for all switching components, or the constant  
 503 assignment  $P[S] = 1$  are two trivial solutions of the instance. In other  
 504 words, there are at least two solutions.
- 505 • In case c), we can have switching components with different structures,  
 506 but necessarily irregular.
- 507 • In case d), the structure of the switching components (all irregular) is  
 508 constrained. The point  $p_{k+6}$  is in the same area of the shell than  $p_k$ .  
 509 As in b) with points  $p_1 \in \text{SW}$ , there are at least two trivial solutions  
 510 obtained with constant assignments of the boolean variables.
- 511 • The case e) is the only one which can provide irregular and regular  
 512 switching components. It is illustrated Fig.20. This example goes  
 513 against intuition.
- 514 • In the case f), all the switching components are regular.

515 It remains a lot of open questions about irregular switching components.  
 516 The case e) is the only one where irregular and regular switching components  
 517 might coexist (Fig.21). In the cases b), d), and f), the structures of the  
 518 switching components guarantee that the two constant assignment of the  
 519 boolean variables are trivial solution. In the configurations, a), c) and e), we  
 520 still don't know if it is possible to have an expression of the HV-convexity  
 521 constraints in 2-clauses which is not feasible.

#### 522 4.3. *Extended regular switching components can be ordered*

523 We introduce an order relation on the points of the shell. For a pair of  
 524 points  $p(x, y)$  and  $p'(x', y')$ , we denote  $p < p'$  if  $x < x'$  or if  $x = x'$  and  $y < y'$   
 525 for the points of the West borders of the shell and  $p < p'$  if  $x > x'$  or  $x = x'$   
 526 and  $y < y'$  for the East borders.

527 An order relation  $<$  defined on an arbitrary set of points  $X$  induces an  
 528 order relation  $A < B$  between a pair of subsets  $A \subset X$  and  $B \subset X$  if for  
 529 any pair  $a \in A$  and  $b \in B$ , we have  $a < b$ . The constraint to order sets is

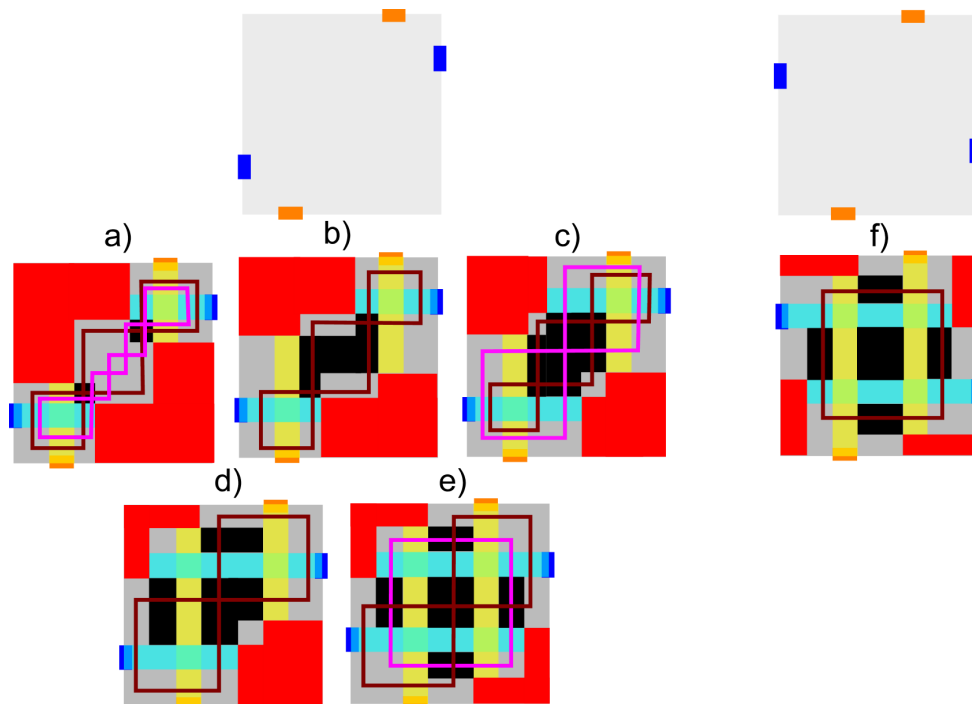


Figure 20: **The 6 possible configurations of the feet.** By symmetries, we can reduce the set of the possible configurations of the feet to the two cases drawn above. Below, we consider the values of the X-rays associated with the feet. They determine points in In and in Out which are colored in yellow for the South and North feet and in blue for the East and West feet. The shell is in grey (except the points issued from the feet, In is black and Out is red). Regular switching components are only possible in the cases e) and f). Irregular switching components are possible in the cases a), b), c), d) and e).

530 that their points are well-ordered. Is it the case with switching components?  
 531 Unfortunately not, the points of switching components are not necessarily  
 532 well ordered, even in the regular case (Fig.22). But it becomes true with  
 533 extended switching components in the case where the switching components  
 534 are all regular.

535 **Property 3.** *In the case where all switching components are regular, the*  
 536 *extended switching components can be ordered.*

537 *Proof.* We assume that switching components have been defined and that  
 538 they are all regular. We use the previous remark that distinct extended  
 539 switching components pass necessarily through distinct rows and columns.

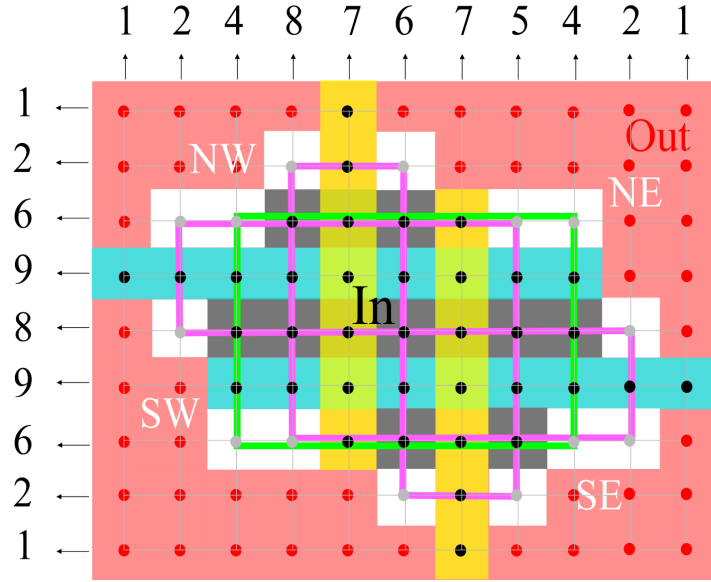


Figure 21: **An instance with regular and irregular switching components.** The only configuration which might provide both regular and irregular switching components is e). In all the other cases, the switching components are either all regular, or all irregular.

540 They cannot share a row or a column. The second remark is that they are  
 541 necessarily 4-connected with the set In. On each border, they follow the  
 542 boundary of In (Fig.23). Then the only case which can prevent them to be  
 543 ordered is if one extended switching component is surrounded by another.  
 544 We assume that such a configuration exist and show that it leads to a con-  
 545 tradiction.

546 We consider 4-connected components of extended switching components  
 547 that we call 4CC. We assume that a 4CC denoted  $\overline{P}_1$  of the switching compo-  
 548 nent  $\overline{P}$  is surrounded by two 4CC  $\overline{P}'_1$  and  $\overline{P}'_2$  of another extended switching  
 549 component  $\overline{P}'$ . It means  $\overline{P}'_1 < \overline{P}_1 < \overline{P}'_2$ . We can assume without loss of  
 550 generality that  $\overline{P}'_2$  is upper than all the 4CC of  $\overline{P}$  (otherwise we invert  $\overline{P}$  and  
 551  $\overline{P}'$ ). By definition, there exists a path of horizontal correspondences connect-  
 552 ing  $\overline{P}'_1$  and  $\overline{P}'_2$ . With the same number of turns around In, the surrounded  
 553 4CC  $\overline{P}_1$  arrives on a 4CC of  $\overline{P}$  that we denote  $\overline{P}_2$ . We notice therefore that  
 554 horizontal and vertical correspondences preserve the order of the 4CCs (since  
 555 they don't share rows or columns). Then by turning around In, it follows  
 556 from the initial assumption  $\overline{P}'_1 < \overline{P}_1$  that we have also  $\overline{P}'_2 < \overline{P}_2$ . It contradicts

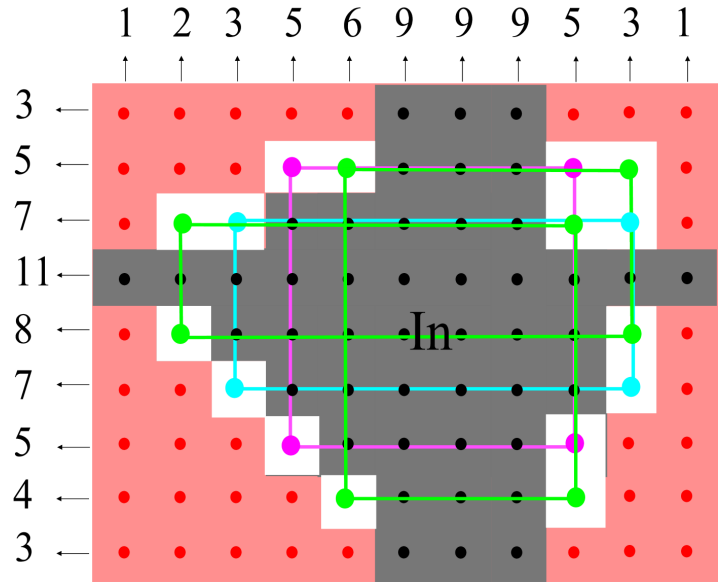


Figure 22: **A instance with regular switching components which can not be ordered directly.** The green switching components surrounds the two others.

557 the initial assumption that  $\overline{P}_2^j$  is upper than any 4CC of  $P$ .

558

□

559 It follows from all these complementary properties that the extended  
 560 switching components have a simple structure along the boundary of the  
 561 set In: they are independent and well ordered. This property is a step for-  
 562 ward in the direction of the reconstruction of convex lattice sets in the case  
 563 where all the switching components are regular.

564 **5. Conclusion**

565 Although they have been widely used for reconstructing HV-convex lat-  
 566 tice sets from their horizontal and vertical X-rays, switching components  
 567 remain quite mysterious combinatorial objects. We started in this paper the  
 568 investigation of the regular case which occurs if the feet are not placed in  
 569 opposite corners of the window (case f of Fig.20). Then the switching com-  
 570 ponents are all regular. The main lemma 1 and Theorem 1 show that their  
 571 boolean variables used at the last step of the HV-convex reconstruction are

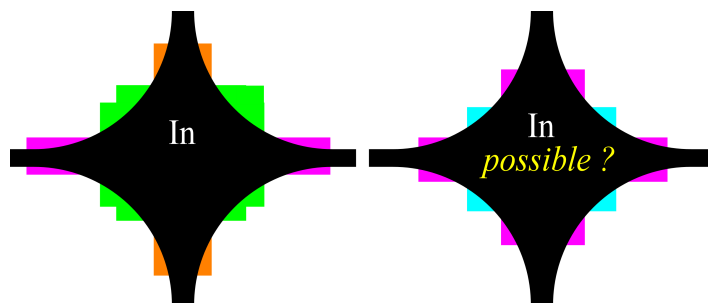


Figure 23: **Ordering the extended switching components.** The extended switching components follow the boundary of the set  $\text{In}$  (in black). They can be ordered if and only if the right case, with an extended switching component surrounding another one can be excluded.

572 either equal, or independent. Switching components are equal as soon as  
 573 they have two 4-neighbors among their points. We merge them in extended  
 574 switching components. While regular switching components might still have  
 575 complex configurations, these new objects can be easily ordered along the  
 576 boundary of the set  $\text{In}$ .

577 The proofs of the results required the development of new approaches  
 578 based on the investigation of oriented graphs of regions. The regular switch-  
 579 ing components have revealed their structure. A similar work would be of  
 580 interest with irregular switching components. A better understanding of their  
 581 relations could be for instance useful to close the remaining question: is it  
 582 true that the constraints expressing the HV-convexity of the solution are al-  
 583 ways feasible? We hope at last that it can help to determine the complexity  
 584 of  $DT_{\mathcal{P}}$  for the class  $\mathcal{P}$  of the convex lattice sets.

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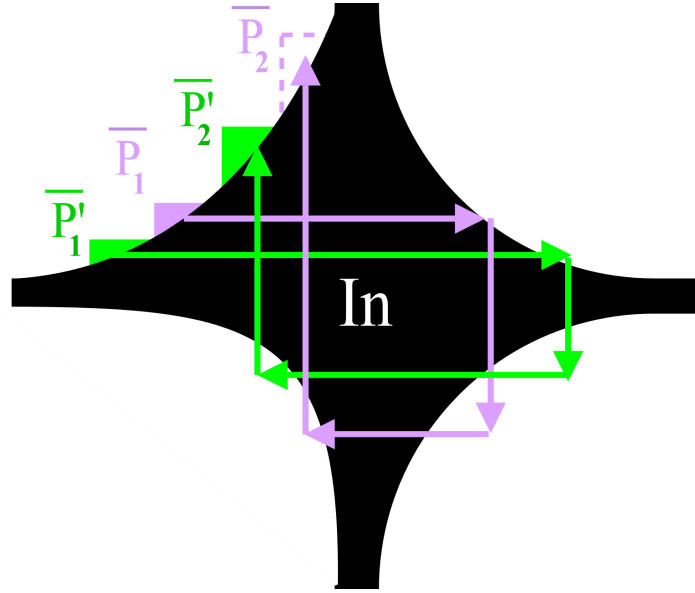


Figure 24: **Sketch of proof.** The initial assumption is  $\overline{P}'_1 < \overline{P}_1 < \overline{P}'_2$ . We can assume w.l.g that there is no 4-connected component of the switching component  $\overline{P}$  above  $\overline{P}'_2$ . This assumption is contradicted by the construction of a 4-connected component  $\overline{P}'_2$  of  $\overline{P}$  above  $\overline{P}'_2$ . Thus extended switching components cannot interlace or surround themselves.

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