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Some analytical solutions for validation of free surface flow computational codes

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We present several time dependent analytical solutions for the incompressible Euler system with free surface. These analytical solutions give quantitative descriptions of some physical phenomena and can be used as reference solutions when validating numerical simulation codes. They concern fluid flows governed by Euler equations with or without hydrostatic hypothesis including wet/dry interface, variable density, wide variety of boundary conditions and possibly non-uniform bottom topographies. Some of these analytical solutions concern shallow water type models or free surface Navier-Stokes model i.e. with a viscous fluid.

1. Introduction

Models arising in fluid dynamics are often based on the Euler equations and are generally difficult to analyze both at the mathematical and numerical level. As a consequence the derivation of simplified models is important, despite this the design of efficient/validated numerical schemes for such very models remains complex. The presence of the free surface coupled with the non-linearities complicates the numerical analysis of such models. Even if some discrete stability properties can be proved (consistency, invariant domains. . .), some of them e.g. discrete entropy inequalities are hardly accessible and in most of the cases the proof of the convergence of the numerical scheme is out of reach.

This paper proposes a list of analytical test cases that allow to validate the efficiency of the numerical tools. These analytical solutions can be adapted to two-dimensional shallow water type models or to models where the velocity is distributed along the vertical direction. Some of the proposed solutions include

- wet/dry interfaces,
- the hydrostatic assumption,
- a variable density (the density depends on a tracer concentration).

To validate the boundary conditions, we have analytical solutions in a cubic tank with open boundary and in a semi-open domain with hyperbolic topography and open boundary. In such cases we have access to the value of all variables and if necessary all derivatives of these variables at the boundary.

Some of these analytical solutions are also solutions of the free surface Navier-Stokes system for a viscous fluid meaning they satisfy a minimum energy principle.

It is important to notice that these analytical solutions correspond to possible physical configurations, even if they are expressed over simple geometrical domains. To compare

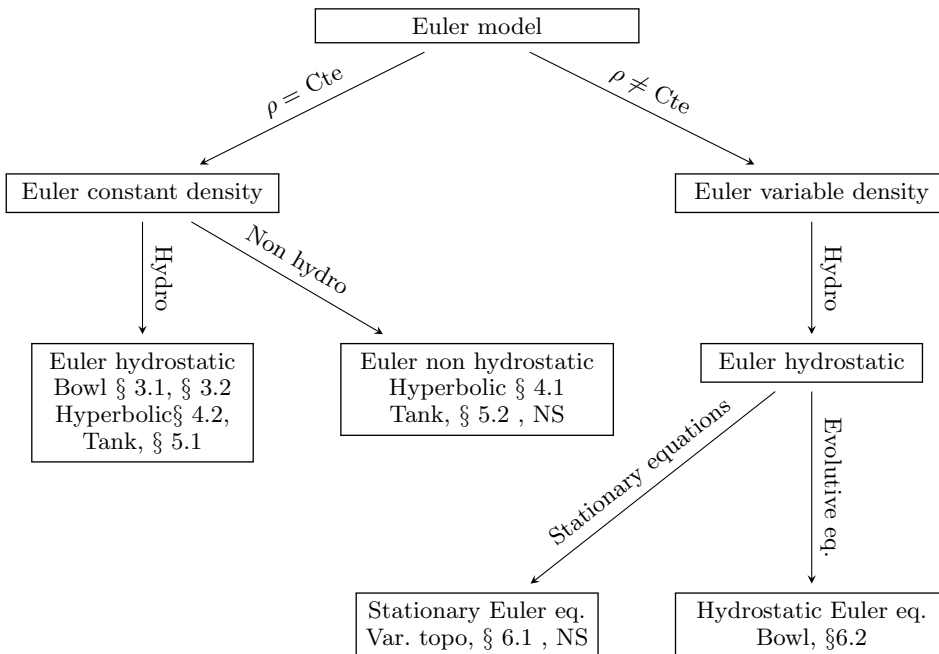


FIGURE 1. Links between the analytical solutions proposed and the models considered. With indication of the shape of the domain (Bowl, Tank, Hyperbolic or Various shape), the fact that they are also solutions of Navier-Stokes (NS) and the paragraph where they are presented.

them to a laboratory experience or a numerical simulation, it is necessary to be able to impose the suitable boundary conditions, what is practicable without difficulties in a numerical simulation but can become more complex in a laboratory experiment.

All the graphics presented in this document represent the exact analytical solutions, but we are able to obtain a good approximation of them by using the Freshkiss3D simulation code (Freshkiss3d 2017) by imposing only the geometry of the domain and the corresponding initial and boundary conditions. With these analytical solutions, it is possible to test simulation codes and to obtain convergence curves when increasing the numerical scheme order and the mesh resolution, see for example (Allgeyer *et al.* 2019). Figure 1 presents the simplified distribution of the proposed analytical solutions and the corresponding model. Indication *Bowl*, *Hyperbolic*, *Tank* or *Var. topo.* (for various topography) refers to the shape of the basin; *NS* refers to the fact that the solutions are also valid for Navier-Stokes system and § precedes the section number to be referred to.

This paper is organized as follows. In Section 2 we present the notations and the various forms of the Euler and the Navier-Stokes systems used in this paper. In Section 3 we propose some extensions of the solution proposed by Thacker (Thacker 1981) in a parabolic bowl in order to have a velocity distributed along the vertical direction, a moving bottom or a variable density. In Section 4 we consider a domain with a hyperbolic topography for which we can exhibit some solutions in which the horizontal components of the velocity do not depend on the vertical axis. In Section 5 we present a cubic tank domain that permits to test the boundary conditions. Finally, in Section 6 we present some analytical solutions for the hydrostatic system with variable density.

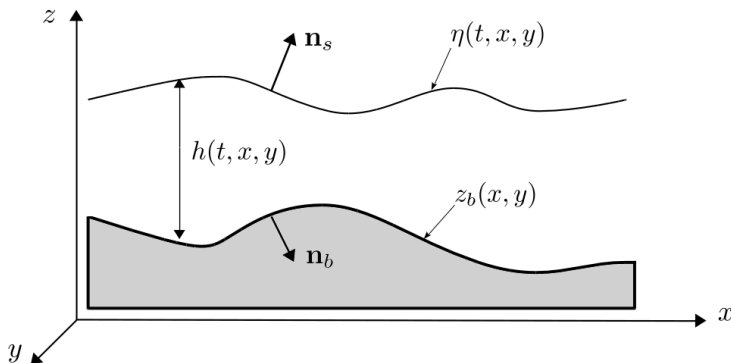


FIGURE 2. Flow domain with water height $h(t, x, y)$, free surface $\eta(t, x, y)$ and bottom $z_b(x, y)$.

2. The Euler and Navier-Stokes systems

2.1. Free surface Navier-Stokes models

Let us first describe below the mass and momentum equations with associated boundary conditions for which we will propose analytical solutions. We assume in this part that the density is constant and that the bottom topography does not depend on the time. We consider the three-dimensional Navier-Stokes system describing a free surface gravitational flow moving over a bottom topography $z_b(x, y)$ (unit: m) with constant density.

$$\nabla \cdot \mathbf{U} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{U}) = \mathbf{g} + \frac{1}{\rho_0} \nabla \cdot \Sigma_{\mathbf{T}}, \quad (2.2)$$

where $\mathbf{U}(t, x, y, z) = (u, v, w)^T$ is the velocity (unit: m s^{-1}), p is the fluid pressure (unit: Pa), ρ_0 is the density (unit: kg m^{-3}) assumed to be constant and $\mathbf{g} = (0, 0, -g)^T$ represents the gravity forces (unit: m s^{-2}). The quantity nabla denotes $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T$. We assume a Newtonian fluid, μ is the viscosity coefficient (unit: Pa s). The viscosity stress tensor Σ (unit: Pa) is given by

$$\Sigma = \mu (\nabla \mathbf{U} + (\nabla \mathbf{U})^T), \quad (2.3)$$

and we denote the total stress tensor by $\Sigma_T = -pI_d + \Sigma$. We consider a free surface flow (see Fig. 2), therefore we assume

$$z_b(x, y) \leq z \leq \eta(t, x, y) := h(t, x, y) + z_b(x, y),$$

with $h(t, x, y)$ the water depth (unit: m).

2.2. Boundary conditions

At the free surface, the kinematic boundary condition is

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} + v_s \frac{\partial \eta}{\partial y} - w_s = 0, \quad (2.4)$$

whereas at the bottom we have the non-penetration condition

$$u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y} - w_b = 0, \quad (2.5)$$

where $U_s = (u_s, v_s, w_s)^T = (u(t, x, y, \eta(t, x, y)), v(t, x, y, \eta(t, x, y)), w(t, x, y, \eta(t, x, y)))^T$, $U_b = (u_b, v_b, w_b)^T = (u(t, x, y, z_b(x, y)), v(t, x, y, z_b(x, y)), w(t, x, y, z_b(x, y)))^T$.

We also prescribe dynamic boundary conditions as follows. Let \mathbf{n}_b and \mathbf{n}_s be the unit outward normals at the bottom and at the free surface respectively defined by

$$\mathbf{n}_b = \frac{1}{\sqrt{1 + |\nabla_{x,y} z_b|^2}} \begin{pmatrix} \nabla_{x,y} z_b \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{n}_s = \frac{1}{\sqrt{1 + |\nabla_{x,y} \eta|^2}} \begin{pmatrix} -\nabla_{x,y} \eta \\ 1 \end{pmatrix},$$

where $\nabla_{x,y}$ corresponds to the projection of ∇ on the horizontal plane i.e. $\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T$.

At the bottom we prescribe a friction condition given e.g. by a Navier law

$$(\Sigma_T \cdot \mathbf{n}_b) \cdot \mathbf{t}_b^i = -\kappa \mathbf{U}_b \cdot \mathbf{t}_b^i, \quad (2.6)$$

with $\kappa = \kappa(h, U_b) \geq 0$ a Navier coefficient (unit: $\text{kg m}^{-2} \text{s}^{-1}$) and \mathbf{t}_b^i , $i = 1, 2$ are two tangential vectors ($\mathbf{n}_b \cdot \mathbf{t}_b^i = 0$).

At the free surface, we impose the condition

$$\Sigma_T \cdot \mathbf{n}_s = -p^a(t, x, y) \mathbf{n}_s + \tau_w(t, x, y) \mathbf{t}_s, \quad (2.7)$$

where $p^a(t, x, y)$ and $\tau_w(t, x, y)$ are two given quantities, p^a (resp. τ_w) mimicks the effects of the atmospheric pressure (resp. the wind blowing at the free surface), \mathbf{t}_s being a tangential vector satisfying $\mathbf{n}_s \cdot \mathbf{t}_s = 0$.

The system (2.1)-(2.7) has to be completed with initial and boundary conditions at the lateral boundaries (inflow, outflow or wall type) that are not detailed here but will be defined by each analytical solution.

2.3. The Euler system

The Euler system consists in considering an idealized fluid for which the total stress tensor Σ_T reduces to the pressure part. It writes

$$\nabla \cdot \mathbf{U} = 0, \quad (2.8)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g}, \quad (2.9)$$

and it is completed with the two kinematic boundary conditions (2.4), (2.5). The dynamic boundary condition at the free surface reduces to

$$p_s = p(t, x, y, \eta) = p^a(t, x, y). \quad (2.10)$$

We can also consider that the fluid contains a passive tracer $\phi(t, x, y, z)$ governed by a transport equation

$$\frac{\partial \phi}{\partial t} + \mathbf{U} \cdot \nabla \phi = 0. \quad (2.11)$$

This tracer is assumed to have no impact on the flow. This differs from the case of fluid with variable density depicted in Section 6.

2.4. Hydrostatic models

Hydrostatic models consist in neglecting the vertical acceleration of the fluid i.e.

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \approx 0. \quad (2.12)$$

Hence, the hydrostatic Euler system writes

$$\nabla \cdot \mathbf{U} = 0, \quad (2.13)$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\tilde{\mathbf{U}} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g}, \quad (2.14)$$

with $\tilde{\mathbf{U}} = (u, v, 0)$, completed with the boundary conditions (2.4), (2.5) and (2.10). This means that

$$p(t, x, y, z) = p^a(t, x, y) + \rho_0 g(h(t, x, y) + z_b(x, y) - z). \quad (2.15)$$

The hydrostatic version of the Navier-Stokes system (2.1)-(2.2) is obtained in the same way by neglecting in addition viscous component in the third equations of (2.2). This gives

$$\nabla \cdot \mathbf{U} = 0, \quad (2.16)$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\tilde{\mathbf{U}} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g} + \frac{1}{\rho_0} \nabla \cdot \tilde{\Sigma}, \quad (2.17)$$

with $\tilde{\Sigma} = (\Sigma_x, \Sigma_y, 0)$ and it is completed with the boundary conditions (2.4)-(2.7).

These hydrostatic models are very often used for the study of geophysical flows, see Brenier (1999); Grenier (1999); Bresch *et al.* (2004); Masmoudi & Wong (2012) for justifications of such models.

3. Parabolic bowl

This first set of analytical solutions are extensions of the solutions proposed by Thacker (Thacker 1981). The solution initially proposed by Thacker corresponds to the solution of shallow water equations i.e. an hydrostatic flow where the horizontal velocity (u, v) does not depend on the vertical coordinate z . Thacker's approach requires a parabolic shape of the basin and its solution implies to treat wet/dry interfaces. We propose here two extensions of Thacker's solutions, the first one having a velocity distributed along the vertical axis, and the second one valid with a time dependant topography.

3.1. Hydrostatic Euler model and parabolic topography

We propose here an extension of the Thacker's solutions with curved surface Thacker (1981) where components of the velocity are functions of the z variable. The following

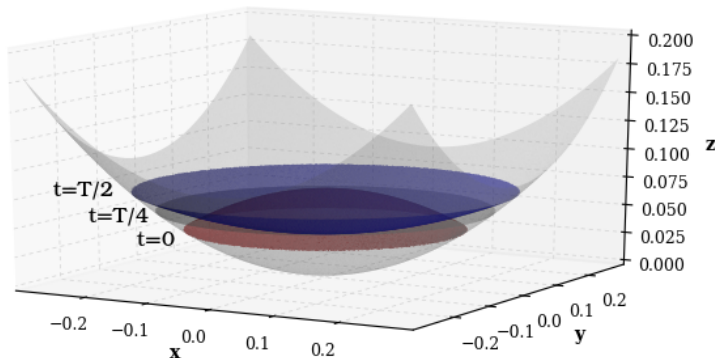


FIGURE 3. 3D Axisymmetrical parabolic bowl: free surface at $t = 0$ (red), $t = T/4$ (dark grey), $t = T/2$ (blue), with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 1$, $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $c = -1$, $L = 1$.

proposition gives an analytical solution for the hydrostatic Euler model presented in Section 2.4.

PROPOSITION 1. *For some $t_0 \in \mathbb{R}$, $(\alpha, \beta, \gamma) \in \mathbb{R}_{+*}^3$, such that $\gamma < 1$ let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by*

$$\begin{aligned}
 h(t, x, y) &= \max \left\{ 0, \frac{1}{r^2} f \left(\frac{r^2}{\gamma \cos(\omega t) - 1} \right) \right\}, \\
 u(t, x, y, z) &= x \left(\beta \left(z - z_b - \frac{h}{2} \right) + \frac{\omega \gamma \sin(\omega t)}{2(1 - \gamma \cos(\omega t))} \right), \\
 v(t, x, y, z) &= y \left(\beta \left(z - z_b - \frac{h}{2} \right) + \frac{\omega \gamma \sin(\omega t)}{2(1 - \gamma \cos(\omega t))} \right), \\
 w(t, x, y, z) &= -\beta(z - z_b)^2 + \left(\beta h - 2\Gamma - \beta \alpha r^2 + \frac{\beta}{2} \left(x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} \right) \right) (z - z_b) \\
 &\quad - \frac{\alpha r^2}{2} (\beta h - 2\Gamma) \\
 p(t, x, y, z) &= g(h + z_b - z),
 \end{aligned}$$

with $\omega = \sqrt{4\alpha g}$, $r = \sqrt{x^2 + y^2}$, $\Gamma = -\frac{\omega \gamma \sin(\omega t)}{2\Lambda}$, a bottom topography defined by

$$z_b(x, y) = \alpha \frac{r^2}{2},$$

and the function f given by

$$f(z) = -\frac{4g}{\beta^2} + \frac{2}{\beta^2} \sqrt{4g^2 + cz + \beta^2 \alpha g (\gamma^2 - 1) z^2},$$

c being a nonpositive constant such that $c \leq 4g^2/(\gamma - 1)$.

Then h, u, v, w and p as defined previously satisfy the 3d hydrostatic Euler system (2.13)-(2.14) completed with the boundary conditions (2.4), (2.5) and (2.10) with $p^\alpha = \text{cst}$.

Proof. The proof of Prop. 1 relies on simple (but tedious) computations since it is enough to verify that the proposed expressions are exact solutions of the Euler equations

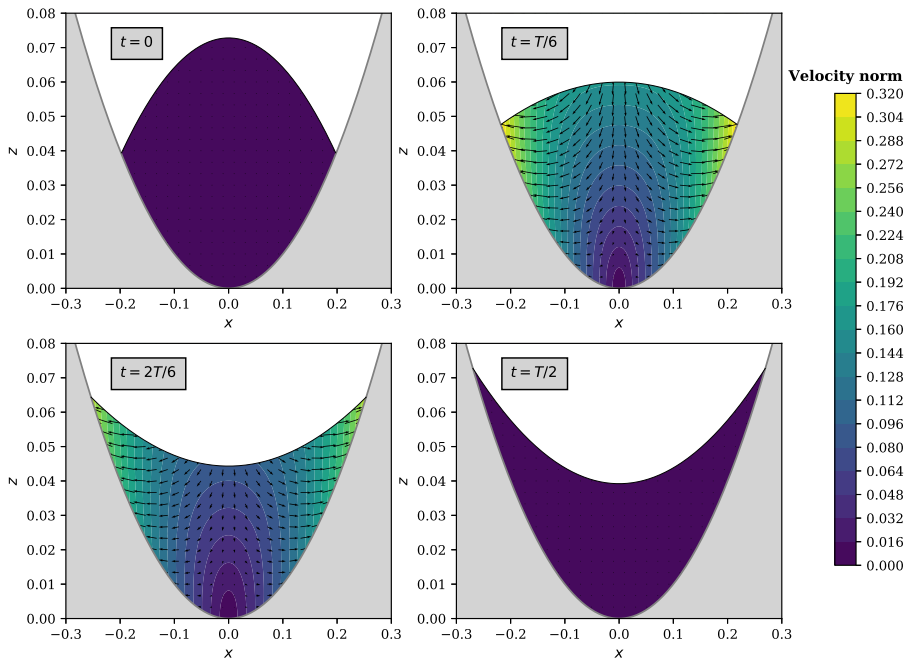


FIGURE 4. 3D Axisymmetrical parabolic bowl: velocity field and norm at $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 1, \alpha = 2, \beta = 1, \gamma = 0.3, c = -1, L = 1$.

given in paragraph 2.4. Taking the limit when $\beta \rightarrow 0$, the solutions proposed here are similar to those proposed by Thacker. \square

In Figure 3, the shape of the free surface of the analytical solution is plotted at different times. In (Allgeyer *et al.* 2019), convergence curves towards the analytical solution are obtained with the numerical code (Freshkiss3d 2017). In Figures 4, we represented an axial section of the analytical solution at four moments ($t = 0, T/6, T/3, T/2$) for a parameters set $h_0 = 1, \alpha = 2, \beta = 1, \gamma = 0.3, c = -1, L = 1$. The arrows represent the velocity field and the color shading the velocity norm.

The analytical solution proposed in prop. 1 can be also expressed in a two dimensional (x, z) -domain and the following corollary holds.

COROLLARY 1. *The analytical solution depicted in prop. 1 can be written in the 2d case. With obvious notations and $\beta > 0$, we choose $\gamma \leq 2g/(\beta\omega)$, $c > 0$, and we consider the functions h, u, w, p defined for $t \geq t_0$ by*

$$\begin{aligned} h(t, x) &= \max \left\{ 0, f \left(x - \frac{\gamma}{\omega} \sin(\omega t) \right) \right\}, \\ u(t, x, z) &= \beta \left(z - z_b - \frac{h}{2} \right) + \gamma \cos(\omega t), \\ w(t, x, z) &= 4\alpha x (\beta z + \gamma \cos(\omega t)), \\ p(t, x, z) &= g(h + z_b - z), \end{aligned}$$

with a bottom topography defined by $z_b(x) = 2\alpha x^2$, a function f given by

$$f(z) = -\frac{4g}{\beta^2} + \frac{2}{\beta^2} \sqrt{4g^2 - 2c\beta^2 - \beta^2\omega^2 z^2}.$$

Then h, u, w, p as defined previously satisfy the 2d hydrostatic Euler system completed with the boundary conditions (2.4),(2.5) and 2.10 with $p^\alpha = cst$.

Proof. It is enough to verify that the expressions of u, w , and h are solutions of the following equations

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_{z_b}^{z_b+h} u(x, z) dz &= 0, \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + g \frac{\partial h + z_b}{\partial x} &= 0, \end{aligned}$$

where w is then given thanks to the incompressibility condition by

$$w(t, x, z) = -\frac{\partial}{\partial x} \int_{z_b}^z u dz.$$

□

3.2. Hydrostatic Euler model and time dependant topography

In many situations e.g. earthquake generated tsunamis, the topography of the domain varies with respect to time t . In such a context, the non-penetration condition at the bottom (2.5) must be replaced by

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y} - w_b = 0. \quad (3.1)$$

In the two dimensional case (x, z) , the following proposition gives analytical solutions for the 2d hydrostatic Euler system.

PROPOSITION 2. *For some $t_0 \in \mathbb{R}$, $(h_0, \alpha, \beta, \gamma) \in \mathbb{R}_+^4$, let us consider the functions h, u, w, p defined for $t \geq t_0$ by*

$$\begin{aligned} h(t, x) &= \max \left\{ 0, h_0 - \frac{\alpha}{4} (\gamma \cos(\omega t) - \gamma \sin(\omega t) - x)^2 \right\}, \\ u(t, x, z) &= \beta \left(z - z_b - \frac{h}{2} \right) - \gamma \omega \sin(\omega t), \\ w(t, x, z) &= \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x} \int_{z_b}^z u dz, \\ p(t, x, z) &= g(h + z_b - z), \end{aligned}$$

with $\omega = \sqrt{\alpha g}$ and with a bottom topography defined by

$$z_b(x, t) = \alpha \frac{x^2}{4} + \frac{\alpha \gamma}{2} x (\cos(\omega t) + \sin(\omega t)).$$

Then h, u, w, p as defined previously satisfy the 3d hydrostatic Euler system (2.13)-(2.14) completed with the boundary conditions (2.4), (2.10), (3.1) (with $p^\alpha = cst$) and restricted to the (x, z) -plane (i.e. assuming $y = 0$ and $v = 0$).

Proof. The proof relies on simple computations. We only have to verify that Eq. (2.14) is satisfied since Eq. (2.13) gives the expression of $w(t, x, z)$. Inserting the expressions of u, h and z_b in Eq. (2.14), we observe that this equation is satisfied under the hypothesis $\omega = \sqrt{\alpha g}$. □

Figure 5 represents of the evolution of the analytical solution in the moving domain

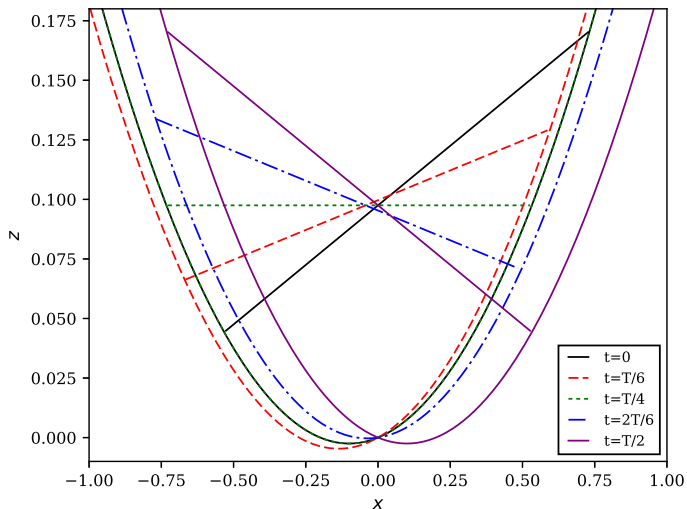


FIGURE 5. Analytical solution of prop. 2, evolution of the topography and free surface for the parabolic bowl with topography at $t = 0, T/6, T/4, 2T/6, T/2$, in (x, z) plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 0.1$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.1$, $L = 4$.

given in prop. 2. We can observe that in the computation of $\eta = h + z_b$, quadratic terms disappear, thus the free surface remain planar.

Proposition 2 can be extended in the three dimensional case leading to the following proposition.

PROPOSITION 3. *For some $t_0 \in \mathbb{R}$, $(h_0, \alpha, \beta, \gamma) \in \mathbb{R}_+^4$, let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by*

$$h(t, x, y) = \max \left\{ 0, h_0 - \frac{\alpha}{4} (\gamma \cos(\omega t) - \gamma \sin(\omega t) - x + y)^2 \right\},$$

$$u(t, x, y, z) = \beta \left(z - z_b - \frac{h}{2} \right) - \gamma \omega \sin(\omega t),$$

$$v(t, x, y, z) = \beta \left(z - z_b - \frac{h}{2} \right) + \gamma \omega \cos(\omega t),$$

$$w(t, x, y, z) = \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x} \int_{z_b}^z u dz - \frac{\partial}{\partial y} \int_{z_b}^z v dz,$$

$$p(t, x, y, z) = g(h + z_b - z),$$

with $\omega = \sqrt{\alpha g}$ and with a bottom topography defined by

$$z_b(x, y, t) = \alpha \frac{(x - y)^2}{4} + \frac{\alpha \gamma}{2} (x + y) (\cos(\omega t) + \sin(\omega t)).$$

Then h, u, v, w, p as defined previously satisfy the 3d hydrostatic Euler system (2.13)-(2.14) completed with the boundary conditions (2.4), (2.10) and (3.1).

However, this solution presents some numerical difficulties since the moving domain is a kind of oscillating gutter, indeed there are open boundaries in the direction of $y = x$ and it is necessary to impose on these boundaries incoming or outgoing conditions that

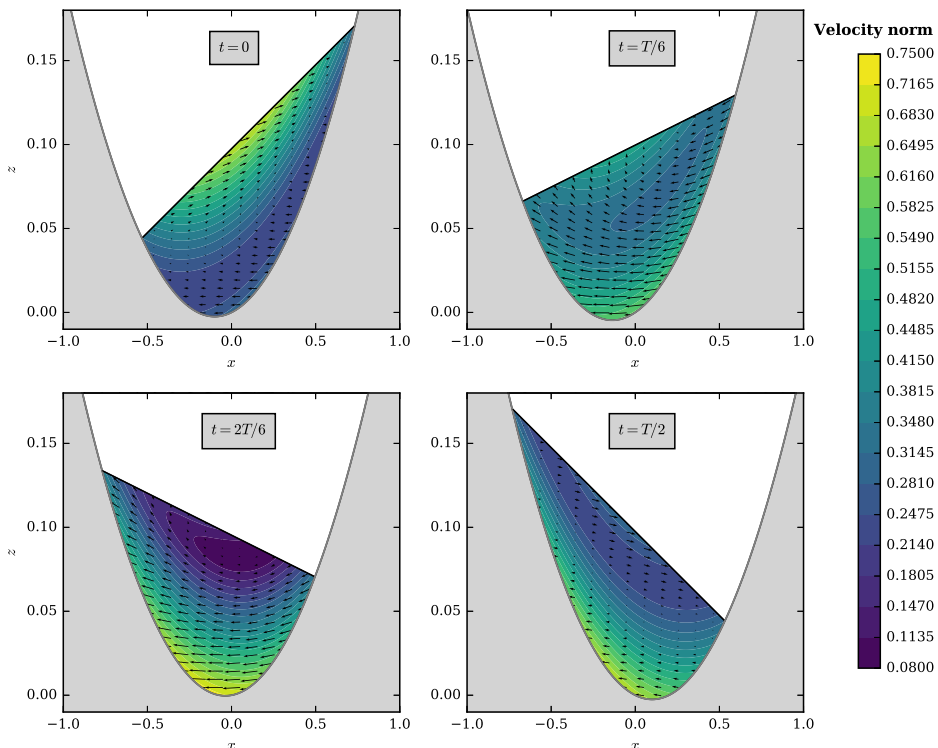


FIGURE 6. Analytical solution of prop. 3, planar surface in the parabolic bowl: free surface, velocity field and norm at $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 0.1$, $\alpha = 1$, $\beta = 6$, $\gamma = 0.1$, $L = 4$.

are given by the solution indicated above. For example, we can set a positive value for a parameter L and impose the flow on boundaries $(x - y = -L$ and $x - y = L)$.

Figure 6 represents a cross section ($y = 0$) of the velocity field associated with prop. 3 (the velocity norm is associated with the three dimensional components of the velocity).

4. Hyperbolic topography

In this section, we are interested in characterizing the shallow water analytical solutions of the Euler system. More precisely, we show that under a reasonable hypothesis, we can find all the solutions of Euler free surface equations whose horizontal velocity does not depend on z . More precisely, we exhibit the analytical solutions of the system (2.8)-(2.9) completed with the boundary conditions (2.4), (2.5) and (2.10) having a velocity field of the form $u = \bar{u}(t, x, y)$ and $v = \bar{v}(t, x, y)$. Moreover, it is possible to exhibit the shallow water analytical solutions of the hydrostatic Euler equations (2.13)-(2.14) and also for Navier-Stokes equations (2.1)-(2.2). At the numerical level, these solutions permits to test free surface Euler equations with open boundary and a wet/dry interface.

4.1. Non hydrostatic Euler equations

PROPOSITION 4. For some $(\alpha, \beta, c_0) \in \mathbb{R}^{+3}$, $(b_0, t_0) \in \mathbb{R}^2$, $t_1 \in \mathbb{R}_+^*$, $\theta \in [0, 2\pi]$, let us consider the function h, u, v, w and p defined for $t \geq t_0$, $(x, y) \in \mathbb{R}^{+2}$ and $z \in [z_b(x, y), \eta(x, y, t)]$ by

$$\begin{aligned} h(t, x, y) &= \max \{0, \alpha f(t) - b_0 - z_b(x, y)\}, \\ u(t, x, y, z) &= f(t)(x \cos \theta + y \sin \theta + \beta) \cos \theta, \\ v(t, x, y, z) &= f(t)(x \cos \theta + y \sin \theta + \beta) \sin \theta, \\ w(t, x, y, z) &= -f(t)(z + b_0), \\ p(t, x, y, z) &= p^{a,0}(t) + f^2(t)(\eta(x, y, t) - z^2) + (2b_0 f^2(t) + g)(\eta - z), \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$, $z_b(x, y) = \frac{c_0}{x \cos \theta + y \sin \theta + \beta} - b_0$, and $p^{a,0}(t)$ a given function.

Then h, u, v, p as defined previously satisfy the Euler system (2.8)-(2.9) completed with the boundary conditions (2.4), (2.5) and (2.10).

Proof. It is not very difficult to ensure that the above expressions are a solution of the Euler system (2.8)-(2.9) completed with the boundary conditions (2.4), (2.5) and (2.10). We will detail here how they are obtained.

For the sake of simplicity, the computations are carried out in 2d, that corresponds to take $\theta = 0$ and to consider only the (x, z) direction.

Assuming $u = \bar{u}(t, x)$, the divergence free condition (2.8) coupled with (2.5) gives

$$w = -z \frac{\partial \bar{u}}{\partial x} + \frac{\partial(z_b \bar{u})}{\partial x}, \quad (4.1)$$

and the two components of (2.9) give

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{\partial p}{\partial x} &= 0, \quad (4.2) \\ -z \left(\frac{\partial^2 \bar{u}}{\partial x \partial t} + \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \left(\frac{\partial \bar{u}}{\partial x} \right)^2 \right) + \frac{\partial^2(z_b \bar{u})}{\partial x \partial t} + \bar{u} \frac{\partial^2(z_b \bar{u})}{\partial x^2} - \frac{\partial \bar{u}}{\partial x} \frac{\partial(z_b \bar{u})}{\partial x} + \frac{\partial p}{\partial z} &= -g. \quad (4.3) \end{aligned}$$

From Eqs. (4.2), (4.3) and (2.10), it comes that the pressure p satisfies

$$p(t, x, \eta) = 0, \quad \frac{\partial^2 p}{\partial x \partial z} = \frac{\partial^3 p}{\partial z^3} = 0,$$

therefore the pressure p has necessarily the form

$$\bar{p} = \frac{a(t)}{2} (\eta^2 - z^2) + b(t) (\eta - z), \quad (4.4)$$

where $a = a(t)$ and $b = b(t)$ are two functions to be determined.

Hence the shallow water solutions of the incompressible Euler system with free surface

are characterized by

$$\frac{\partial h}{\partial t} + \frac{\partial(h\bar{u})}{\partial x} = 0, \quad (4.5)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \left(a(t)\eta + b(t) \right) \frac{\partial \eta}{\partial x} = 0, \quad (4.6)$$

$$\frac{\partial^2 \bar{u}}{\partial x \partial t} + \bar{u} \frac{\partial^2 \bar{u}}{\partial x^2} - \left(\frac{\partial \bar{u}}{\partial x} \right)^2 = -a(t), \quad (4.7)$$

$$\frac{\partial^2(z_b \bar{u})}{\partial x \partial t} + \bar{u} \frac{\partial^2(z_b \bar{u})}{\partial x^2} - \frac{\partial \bar{u}}{\partial x} \frac{\partial(z_b \bar{u})}{\partial x} = b(t) - g. \quad (4.8)$$

Now subtracting to Eq. (4.7) the derivative of Eq. (4.6) w.r.t. the variable x gives

$$2 \left(\frac{\partial \bar{u}}{\partial x} \right)^2 = a(t) - \frac{\partial}{\partial x} \left(\left(a(t)\eta + b(t) \right) \frac{\partial \eta}{\partial x} \right). \quad (4.9)$$

Likewise, subtracting Eq. (4.7) multiplied by z_b to Eq. (4.8) gives

$$\frac{\partial \bar{u}}{\partial t} \frac{\partial z_b}{\partial x} + \bar{u}^2 \frac{\partial^2 z_b}{\partial x^2} + \bar{u} \frac{\partial \bar{u}}{\partial x} \frac{\partial z_b}{\partial x} + a(t)z_b = b(t) - g.$$

The solution $\eta(t, x)$ of Eq. (4.6) can be obtained explicitly if we assume $a(t) > 0$ and then $h(t, x)$ is given by

$$h(t, x) = -z_b(x) - \frac{b(t)}{a(t)} + \frac{1}{a(t)} \sqrt{a(t) \left(F_1(t) - \bar{u}^2 - 2 \int^x \partial_t \bar{u} ds \right)}, \quad (4.10)$$

where $t \mapsto F_1(t)$ is any function such that $F_1(t) \geq \bar{u}^2 + 2 \int^x \partial_t \bar{u} ds$.

A solution of the derivative of equation (4.7) w.r.t. the variable x is given by

$$\bar{u}(t, x) = \frac{u_1(x)}{t - t_0 + t_1}. \quad (4.11)$$

Let us assume that (4.11) holds true. Inserting (4.11) into Eq. (4.5), we can write this equation formally under the form

$$\frac{\partial h u_1}{\partial x} + \frac{t - t_0 + t_1}{u_1} \frac{\partial h u_1}{\partial t} = 0, \quad (4.12)$$

then we apply the characteristic method following $t'(x) = \frac{t - t_0 + t_1}{u_1}$, that gives $t(x) = (t - t_0 + t_1) \int^x \frac{1}{u_1(s)} ds$ and we can exhibit a formal form of $h(x, t)$:

$$h(t, x) = \frac{F_0 \left((t - t_0 + t_1) e^{-\int^x \frac{ds}{u_1(s)}} \right)}{u_1(x)}, \quad (4.13)$$

where $\xi \mapsto F_0(\xi)$ is any function.

Likewise, inserting the expression of \bar{u} given by (4.11) into Eq. (4.10) gives another expression for $h(t, x)$ under the form

$$h(t, x) = -z_b(x) - \frac{b(t)}{a(t)} + \frac{1}{a(t)(t - t_0 + t_1)} \sqrt{a(t) \left((t - t_0 + t_1)^2 F_1(t) - u_1^2(x) + 2 \int^x u_1(s) ds \right)}. \quad (4.14)$$

Now inserting the expression (4.11) into Eq. (4.7) implies that necessarily

$$a(t) = \frac{a_0}{(t - t_0 + t_1)^2},$$

with $a_0 \in \mathbb{R}$. Similarly, inserting the expression (4.11) into Eq. (4.8) allows to obtain the expression $b(t)$ under the form

$$b(t) = g + \frac{2b_0}{(t - t_0 + t_1)^2},$$

with $b_0 \in \mathbb{R}$.

Thus, equation (4.14) gives

$$h(t, x) = -z_b(x) - \frac{b_0}{a_0} - \frac{g}{a_0}(t - t_0 + t_1)^2 + \frac{1}{\sqrt{a_0}} \sqrt{(t - t_0 + t_1)^2 F_1(t) - u_1^2(x) + 2 \int^x u_1(s) ds}. \quad (4.15)$$

The two expressions for $h(t, x)$, namely (4.13) and (4.15) are compatible only if the primitive function associated to $\frac{1}{u_1(x)}$ is a logarithmic function of x , leading to

$$u_1(x) = \gamma x + \beta, \quad (4.16)$$

with $(\beta, \gamma) \in \mathbb{R}^2$. Inserting (4.16) into (4.7) gives that

$$\gamma^2 + \gamma = a. \quad (4.17)$$

Now from Eq. (4.14) we can set $-u_1^2(x) + 2 \int^x u_1(s) ds = C$ in order to have $h(t, x) + z_b(x)$ only depending on time t i.e.

$$\frac{\partial \eta}{\partial x} = 0. \quad (4.18)$$

This property inserted into (4.6) gives $\gamma = 1$, and using equation (4.17), $a_0 = 2$.

Finally we have obtained that

$$u_1(x) = x + \beta, \quad (4.19)$$

and $h = z_b(x) + F_\eta(t)$. These two expressions inserted in Eq. (4.5) give

$$\begin{cases} z_b(x) = \frac{c_0}{x+\beta} - b_0 \\ h(t, x) = \frac{\alpha}{t-t_0+t_1} - \frac{c_0}{x+\beta} \end{cases} \quad (4.20)$$

These expressions are valid only if $h(t, x) \geq 0$. Then (4.19), (4.20), (4.11) give a proof of proposition 4 in the two-dimensional (x, z) case (when $\theta = 0$). \square

COROLLARY 2. *If we assume that all the solutions of equation (4.7) can be written under the form (4.11), proposition 4 gives all the solutions of the free surface Euler equations in which u and v do not depend of z .*

Figure 7 shows the velocity fields at time $t = 0, 0.05, 0.1$ and 0.15 second for a given set of the parameters value ($\alpha = 0.5$, $\beta = 0.1$, $b_0 = 0.4$, $t_0 - t_1 = 2.$, $c_0 = 1.2$). The solution proposed in proposition 4 is not a solution of the hydrostatic formulation of the equation, but we can easily deduce a solution with the hydrostatic hypothesis by reversing the direction of the velocity. Unfortunately, a space-time-dependent pressure is necessary at the surface, this result is described in the following paragraph.

4.2. Hydrostatic Euler equations

If we consider the hydrostatic Euler system (2.13), (2.14), the following result holds.

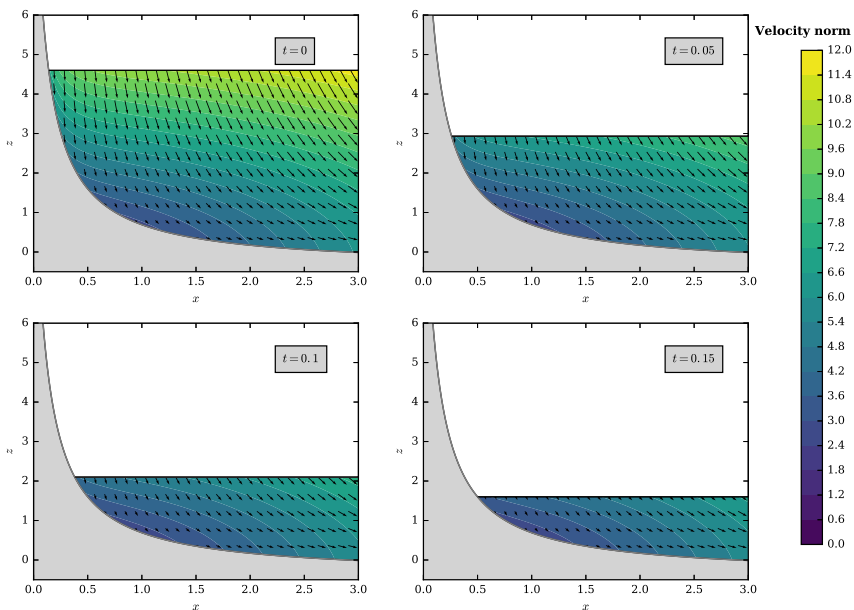


FIGURE 7. Decreasing bathymetry in $1/x$: velocity norm and vectors at $t = 0, 0.05, 0.1, 0.15$ in $(x, y = 0, z)$ for parameters set to $\alpha = 0.5, \beta = 0.1, b_0 = 0.4, t_0 - t_1 = 2., c_0 = 1.2$.

PROPOSITION 5. For some $(\alpha, \beta, c_0, t_0) \in \mathbb{R}^{+3}, (b_0, t_0) \in \mathbb{R}^2, t_1 \in \mathbb{R}_+^*, \theta \in [0, 2\pi]$, let us consider the function h, u, v, w and p defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \max\{0, \alpha f(t) - b_0 - zb(x, y)\}, \\ u(t, x, y, z) &= -(x \cos \theta + y \sin \theta + \beta) \cos \theta / f(t), \\ v(t, x, y, z) &= -(x \cos \theta + y \sin \theta + \beta) \sin \theta / f(t), \\ w(t, x, y, z) &= (b_0 + z) / f(t), \\ p(t, x, y, z) &= p^{a,0}(t) + p^{a,1}(x, t) + g(\eta - z), \end{aligned}$$

where $f(t) = t - t_0 + t_1$ and with bottom $z_b(x, y) = \frac{c_0}{x \cos \theta + y \sin \theta + \beta} - b_0, p^{a,0}(t)$ a given function and

$$p^{a,1}(x, y, t) = -((x \cos \theta + y \sin \theta)^2 + 2u_{1,1}(x \cos \theta + y \sin \theta)) / f(t)^2.$$

Then h, u, v, p as defined previously satisfy the hydrostatic Euler system (2.13) (2.14) completed with the boundary conditions (2.4), (2.5) and (2.10).

Proof. The main idea of the proof is to observe that the analytical solution proposed in Prop. 4 satisfies

$$\frac{\partial w}{\partial t} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}. \quad (4.21)$$

It is sufficient to reverse the direction of the components of the velocity to cancel the non hydrostatic part of the third momentum equation. Compared to Prop. 4, with opposite values of u and v , the water depth increases in time. We have then to find the pressure term allowing to verify exactly the two other momentum equations. \square

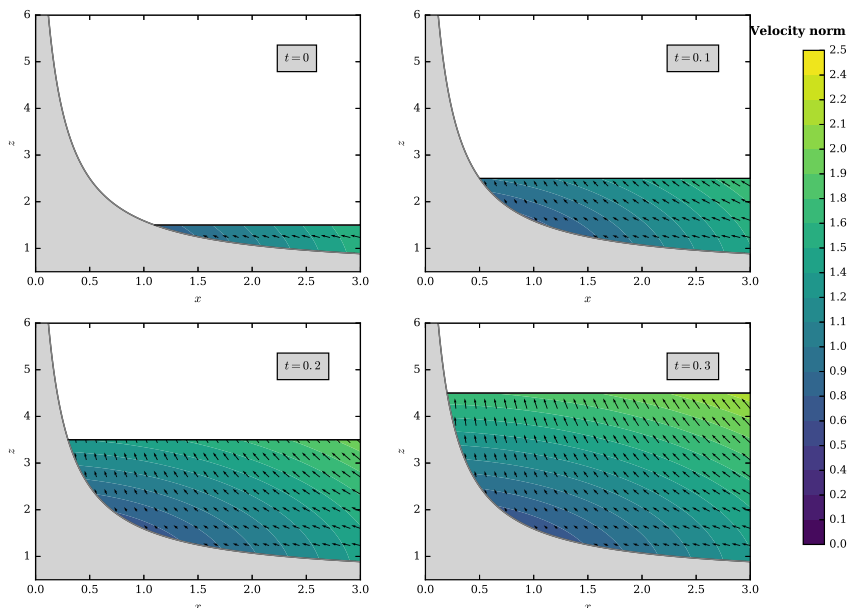


FIGURE 8. Decreasing bathymetry in $1/x$: velocity norm and vectors at $t = 0, 0.1, 0.2, 0.3$ in $(x, y = 0, z)$ for parameters set to $\alpha = 1, \beta = 0.1, b_0 = -1, t_0 = 2., c_0 = 1.2$.

Figure 8 shows the velocity fields at times $t = 0, 0.1, 0.2$ and 0.3 second for a given set of the parameters value ($\alpha = 0.5, \beta = 0.1, b_0 = 0.4, t_0 - t_1 = 2., c_0 = 1.2$).

5. Tank

We consider in this section that the domain is a cubic tank with lateral artificial boundary conditions. The free surface remains horizontal and decreases linearly with time. With this hypothesis, we can exhibit some exact solution of the Euler and Navier-Stokes equations, either under the hydrostatic assumption or in the non hydrostatic case.

These analytical solutions permit to test the good implementation of the boundary conditions on artificial boundaries in a simulation code. Indeed, the solutions are exact in all the domain, including the boundaries and we can then compute all the derivatives of these solutions at the boundaries. For example, if we want to test a condition such that $U \cdot n = \phi$ on a boundary, ϕ being a given function, we can impose $U \cdot n = U_a \cdot n$ where U_a is the analytical solution which is given here.

5.1. *Euler and Navier-Stokes equations with hydrostatic hypothesis*

Considering the hydrostatic Euler system given in paragraph 2.4 in a tank such that $(x, y) \in [-L/2, L/2]^2$. The two following propositions hold.

PROPOSITION 6. *For some $t_0 \in \mathbb{R}$, $t_1 \in \mathbb{R}_+^*$, $\phi_0 \in \mathbb{R}$, $(\alpha, \beta) \in \mathbb{R}_+^2$ such that $\alpha\beta > L$, let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by*

$$\begin{aligned} h(t, x, y) &= \alpha f(t), \\ u(t, x, y, z) &= \beta \left((z - z_b) - \frac{\alpha}{2} f(t) \right) + f(t)(x \cos^2 \theta + y \sin^2 \theta), \\ v(t, x, y, z) &= \beta \left((z - z_b) - \frac{\alpha}{2} f(t) \right) + f(t)(x \cos^2 \theta + y \sin^2 \theta), \\ w(t, x, y, z) &= f(t)(z_b - z), \\ p(t, x, y, z) &= p^a(t, x, y) + g(h - (z - z_b)), \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$ and with a flat bottom $z_b(x, y) = z_{b,0} = cst$ and $p^a(t, x, y) = p^{a,1}(t)$, with $p^{a,1}(t)$ a given function.

Then h, u, v, w, p as defined previously satisfy the 3d hydrostatic Euler system (2.13)-(2.14) completed with the boundary conditions (2.4),(2.5) and (2.10). The appropriate boundary conditions for $x \in \{-L/2, L/2\}$ or $y \in \{-L/2, L/2\}$ are also determined by the expressions of h, u, v, w given above.

The vertical acceleration (2.12) in this analytical solution is equal to $2z/(t - t_0 + t_1)^2 < 2h^3/\alpha^2$. The hydrostatic hypothesis is justified only for values of h small enough that corresponds to a shallow water regime.

Proof. The proof relies on very simple computations since it is enough to verify that these solutions are solutions of the equations of hydrostatic Euler given in paragraph 2.4. \square

REMARK 1. *Notice that The solution proposed in prop. 6 can be written in the 2d (x, z) case by taking $\theta = 0$.*

REMARK 2. *Choosing $\beta = 0$ in prop. 6 gives analytical solutions with a velocity field that does not depend on the variable z . It is easy to see that it corresponds to analytical solutions for the classical viscous Saint-Venant system Gerbeau & Perthame (2001).*

Figure 9 represents the free surface elevation at different times ($t = 0.0s, 0.5s, 1s, 1.5s, 2.0s$) for $\alpha = 5$, $L = 10$, $t_0 = 0$ and $t_1 = 1$. This elevation is the same in all the tank solutions proposed in this paper.

5.2. *Euler and Navier-Stokes equations without hydrostatic hypothesis*

Considering the Euler system (2.8)-(2.9) in a tank such that $(x, y) \in [-L/2, L/2]^2$. The following proposition holds.

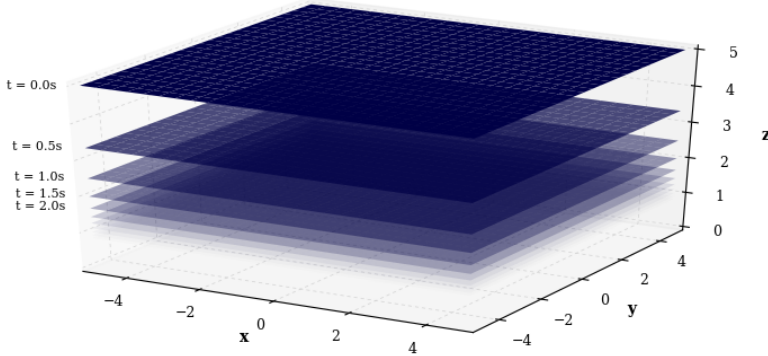


FIGURE 9. Draining of a tank: free surface every 0.5s from initial time, for parameters set to $\alpha = 5$, $t_1 = 0$, $t_1 = 1$ and $L = 10$.

PROPOSITION 7. For some $\alpha \in \mathbb{R}_+$, $(t_0, \gamma, \delta) \in \mathbb{R}^3$, $t_1 \in \mathbb{R}_+^*$, $\theta \in]0, 2\pi[\setminus \{\pi\}$ let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \alpha f(t), \\ u(t, x, y, z) &= f(t) (x \cos \theta + y \sin \theta) + \gamma z + \delta, \\ v(t, x, y, z) &= f(t) (x \cos \theta + y \sin \theta) \tan \frac{\theta}{2} + \frac{\cos \theta - 1}{\sin \theta} \gamma z - \frac{\delta}{\tan \theta}, \\ w(t, x, y, z) &= f(t) (z_b - z), \\ p(t, x, y, z) &= p^{a,0}(t) + g(h - (z - z_b)) + (h^2 - (z - z_b)^2) f(t)^2, \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$ and with a flat bottom $z_b(x, y) = z_{b,0} = cst$, $h_0 = h(t_0, x, y) = \alpha/t_1$ and $p^{a,0}(t)$ a given function.

Then h, u, v, w, p as defined previously satisfy the 3d Euler system (2.8)-(2.9) completed with the boundary conditions (2.4), (2.5) and (2.10). The appropriate boundary conditions for $x \in \{-L/2, L/2\}$ or $y \in \{-L/2, L/2\}$ are also determined by the expressions of h, u, v, w, p given above.

Proof. The proof relies on very simple computations since it is enough to verify that these solutions are solutions of the equations of Euler given in paragraph 2.3. \square

We can observe that the velocity has only components of the first degree in x, y or z . The stress tensor (2.3) is then equal to zero and the solution is also the solution of the Navier-Stokes equations. On the computational point of view, it is an interesting test since simulating such a solution of the Navier-Stokes system, all the components of the stress tensor Σ defined by (2.3) have to remain zero.

For $\theta = 0$ the flow is only in the direction of x and we can consider only a two dimensional version of the solution in (x, z) coordinates. Nevertheless, the angle θ does not represent the direction of the flow in the (x, y) plane.

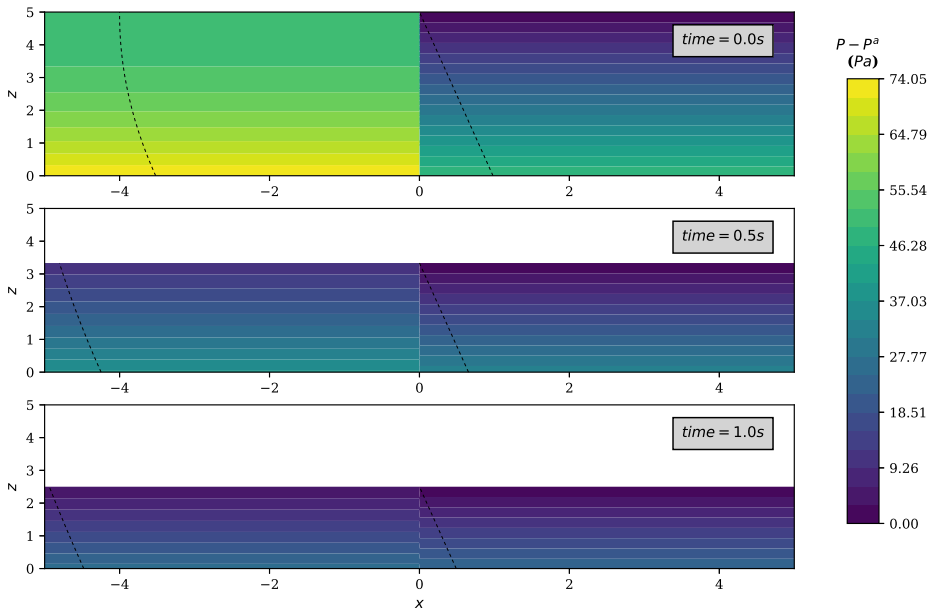


FIGURE 10. Draining of a tank: non-hydrostatic pressure (left, $x < 0$) and hydrostatic pressure (right $x > 0$), at $t = 0, 0.5$ and 1 s, in $(x, y=0, z)$ plane with parameters set to $\mu = 0$, $\theta = 0$, $\alpha = 5$, $t_0 = 1$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$. Dashed line represents pressure profile

In Figure 10 we present a vertical profile of the analytical solution. With the given data, $h(t) = 5/(t + 1)$, we have at initial time, $h(0) = 5$ and at $t = 1$, $h(1) = \frac{5}{2}$. For $x > 0$ (right part of the figure), the pressure is hydrostatic and increases linearly with the depth whereas for $x < 0$ (left part of the figure), the pressure includes a non hydrostatic component (quadratic in $z - z_b$).

5.3. Non hydrostatic Euler equations with passive tracer

Considering the Euler system given in Section 2.1 by equations (2.8)-(2.9) including the tracer equation (2.11) and formulated in a tank such that $(x, y) \in [-L/2, L/2]^2$, the following proposition holds.

PROPOSITION 8. For some $\alpha \in \mathbb{R}_+$, $t_0 \in \mathbb{R}$, $t_1 \in \mathbb{R}_+^*$, $\phi_0 \in \mathbb{R}$, $\theta \in [0, 2\pi]/\{\pi\}$ let us consider the functions h, u, v, w, p, ϕ defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \alpha f(t), \\ u(t, x, y, z) &= f(t) (x \cos \theta + y \sin \theta), \\ v(t, x, y, z) &= f(t) (x \cos \theta + y \sin \theta) \tan \frac{\theta}{2}, \\ w(t, x, y, z) &= f(t) (z_b - z), \\ p(t, x, y, z) &= p^{a,0}(t) + g(h - (z - z_b)) + (h^2 - (z - z_b)^2) f(t)^2, \\ \phi(t, x, y, z) &= \frac{\phi_0}{L} (x \cos \theta + y \sin \theta) \frac{z - z_b}{h_0}, \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$ and with a flat bottom $z_b(x, y) = z_{b,0} = cst$, $h_0 = h(t_0, x, y) = \alpha/t_1$ and $p^{a,0}(t)$ a given function.

Then h, u, v, w, p and ϕ satisfy the 3d Euler system (2.8)-(2.9) completed with the

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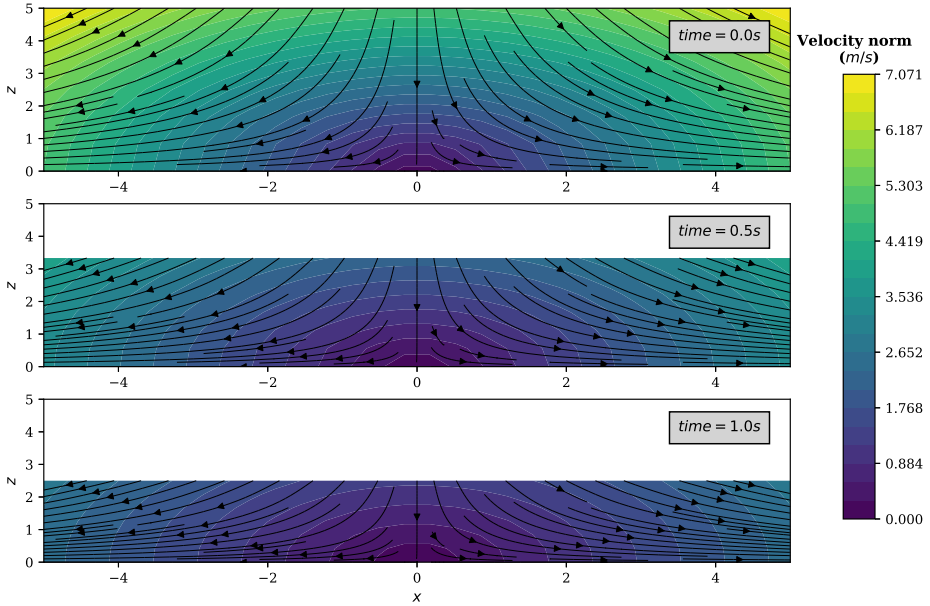


FIGURE 11. Draining of a tank (non-hydrostatic case): velocity norm and streamlines at $t = 0, 0.5, 1.0s$, in $(x, y = 0, z)$ slice plane for parameters set to $\theta = 0$, $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

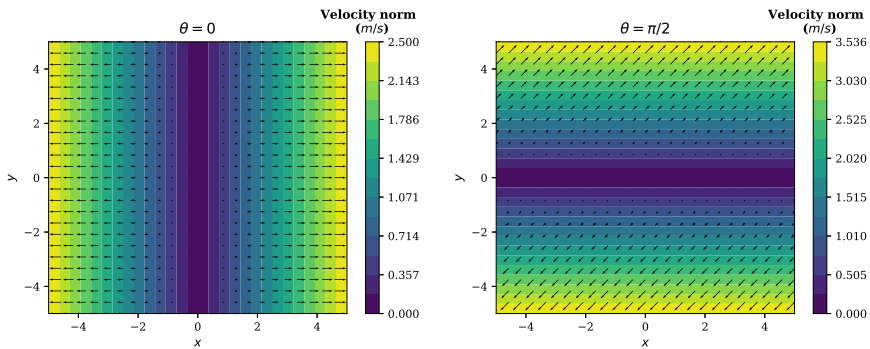


FIGURE 12. Draining of a tank (non-hydrostatic case): velocity norm and vectors at initial time in $(x, y, z = h_0/2)$ slice plane with $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

boundary conditions (2.4),(2.5),(2.10) and the tracer equation (2.11). The appropriate boundary conditions for $x \in \{-L/2, L/2\}$ or $y \in \{-L/2, L/2\}$ are also determined by the expressions of h, u, v, w, p given above.

Proof. The proof relies on very simple computations since it is enough to verify that these solutions are solutions of the equations of Euler given in paragraph 2.3. \square

COROLLARY 3. *The analytical solution depicted in prop. 8 is also solution of the Euler system with a passive tracer – having a concentration ϕ – governed by a transport diffusion*

$$\frac{\partial \phi}{\partial t} + \mathbf{U} \cdot \nabla \phi - \mu_\phi \Delta \phi = 0. \quad (5.1)$$

In this case, we need to consider additional boundary conditions given by

$$\mu_\phi \nabla \phi|_s \cdot \mathbf{n}_s = \mu_\phi \phi_0 (x \cos \theta + y \sin \theta) / (Lh_0), \quad (5.2)$$

and

$$\mu_\phi \nabla \phi|_b \cdot \mathbf{n}_b = -\mu_\phi \phi_0 (x \cos \theta + y \sin \theta) / (Lh_0). \quad (5.3)$$

where subscript *s* (resp. *b*) refers to the free surface (resp. bottom).

6. Hydrostatic Euler system with variable density

In this section we consider a fluid where the density is a function of the tracer concentration ϕ i.e. $\rho = \rho(\phi)$ (typically, ϕ can represent the temperature or the salinity of the fluid). $\rho(\phi)$ being a given function, the hydrostatic Euler system with variable density writes

$$\nabla \cdot \mathbf{U} = 0, \quad (6.1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \quad (6.2)$$

$$\frac{\partial \rho \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\rho \tilde{\mathbf{U}} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g}, \quad (6.3)$$

with $\tilde{\mathbf{U}} = (u, v, 0)$ and the system (6.1)-(6.3) is completed with the boundary conditions (2.4),(2.5),(2.10).

In the general case we consider that the tracer $\phi(t, x, y, z)$ (unit: kg m^3) is governed by a transport equation (2.11).

An intermediate approach is to use the Boussinesq approximation that consists in considering

$$\rho = \rho_0 + f(T),$$

with $\rho_0 = \text{cst}$ and $f(T) \ll \rho_0$ and in neglecting the density variations except in the gravity terms. Then the system (6.1)-(6.3) is replaced by

$$\nabla \cdot \mathbf{U} = 0, \quad (6.4)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \quad (6.5)$$

$$\rho_0 \frac{\partial \tilde{\mathbf{U}}}{\partial t} + \rho_0 \nabla \cdot (\tilde{\mathbf{U}} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g}. \quad (6.6)$$

In many situations the Boussinesq approximation is justified and many numerical codes use it. We do not know how to describe an analytical solution in this case, but we can obtain a large number of analytical solutions for the stationary case. They are good candidates to test the semi-discretization in space of the hydrostatic Euler system. Classically, these solutions correspond to a stable stratification of the fluid if $\frac{\partial \rho}{\partial z} \leq 0$.

6.1. Stationnary solutions of Euler Boussinesq equations

We restrict here to the 2d (x, z) case. The stationary solutions of the system (6.4)-(6.6) are characterized (after simple computations) by

$$\frac{\partial}{\partial x} \int_{z_b}^{\eta} u dz = 0, \quad (6.7)$$

$$w = -\frac{\partial}{\partial x} \int_{z_b}^z u dz, \quad (6.8)$$

$$u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0, \quad (6.9)$$

$$\rho_0 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial x} \int_z^{\eta} \rho g dz = 0. \quad (6.10)$$

$$p(x, z) = p_a + \int_z^{\eta} \rho(x, \xi) d\xi. \quad (6.11)$$

Considering the stationary hydrostatic Euler system with a free surface and a varying density (6.7)-(6.10) completed with the boundary conditions (2.4), (2.5) in an open channel such that $x \in [0, L]$, the following proposition holds.

PROPOSITION 9. *For some $(\rho_0, \alpha, t_0) \in \mathbb{R}_+^3$, $C_0 \in \mathbb{R}$ for any non-negative function $h(x)$, let us consider the functions u, w, ρ defined for $t \geq t_0$ by*

$$\begin{aligned} u(x, z) &= f_1(x) + \frac{g\alpha}{2\rho_0} z^2, \\ w(x, z) &= -f_1'(x)(z - z_b(x)) + f_1(x)z_b'(x) + \frac{\alpha g}{2\rho_0} z_b^2(x)z_b'(x), \\ \rho(x, z) &= \rho_0 - \alpha f_1(x)(z - z_b(x)) - \frac{\alpha^2 g}{6\rho_0} (z^3 - z_b^3(x)), \end{aligned}$$

with the function $f_1(x)$ defined by

$$f_1(x) = \frac{\alpha g}{2\rho_0} (h^2(x) - z_b^2(x)) - \frac{1}{\sqrt{3}\rho_0} \sqrt{(\alpha^2 g^2 h^3(x) - 6\rho_0^2 g)(h(x) + z_b(x)) - 6C_0}$$

and the bottom topography $z_b(x)$ given by

$$\begin{aligned} z_b(x) &= \frac{2}{3} \frac{-\alpha^2 g h^3(x) + 3K_0 \alpha \rho_0 - 6\rho_0^2}{h^2(x) \alpha^2 g} \\ &\pm 2\sqrt{6} \frac{\sqrt{K_0 \alpha^3 g \rho_0 h^3(x) - \alpha^2 g \rho_0^2 h^3(x) - 6K_0 \alpha \rho_0^3 - 3C_0 \alpha^2 h^2(x) + 6\rho_0^4}}{3\alpha^2 g h^2(x)}. \end{aligned} \quad (6.12)$$

Then h, u, w and ρ satisfy the 2d stationary hydrostatic Euler system with free surface, Boussineq hypothesis and varying density (6.7)-(6.11) completed with the boundary conditions (2.4),(2.5). The appropriate boundary conditions for $x \in \{0, L\}$ are also determined by the expressions of h, u, w, ρ given above.

Proof. The main step to obtain these analytical solutions is to find some conditions on the structure of these solutions in order to transform the considered partial differential equations into ordinary differential equations. Here we are making arbitrary choices, and maybe other choices would lead to other analytical solutions. For instance, here we assume that $u(x, z) = f_1(x) + f_2(z)$ where f_1 and f_2 are arbitrary functions.

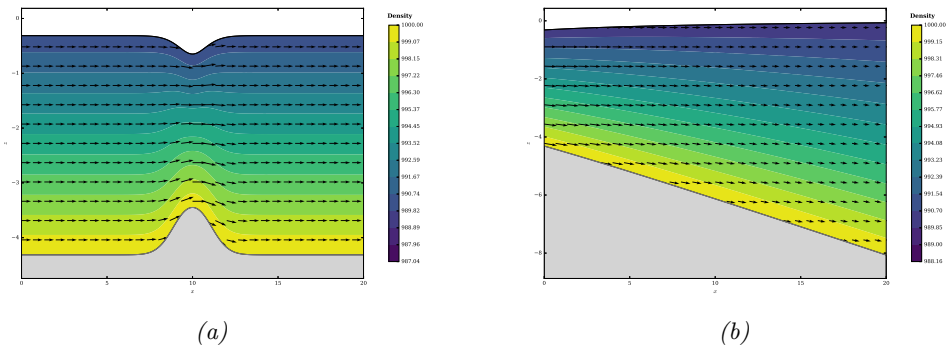


FIGURE 13. Analytical stationary solutions with (a) $\alpha = 10$, $fg = 9.81$, $\rho_0 = 1000$, $K_0 = 10$, $C_0 = 10$ and $h(x) = 4(1 - \frac{1}{5}\exp(-(x - 10)^2/2))$; (a) $\alpha = 1$, $fg = 9.81$, $\rho_0 = 1000$, $K_0 = 10$, $C_0 = 5$ and $h(x) = 4 + x/5$

From (6.8), we obtain

$$w(x, z) = -zf_1'(x) + (z_b(x)f_1(x))' + zb'(x)f_2(z_b(x)). \quad (6.13)$$

Now we assume that the function f_2 is given by $f_2 : z \mapsto az^2$ with $a \in \mathbb{R}$. Then, using equation (6.7), we have for $K_0 \in \mathbb{R}$

$$\int_{z_b}^{\eta} u dz = K_0$$

and using the previous assumption for u and f_2 , we find

$$f_1(x) = -\frac{a}{3}(\eta^2(x) + z_b(x)\eta(x) + z_b^2(x)) + \frac{K_0}{h(x)}. \quad (6.14)$$

Next, we have to find a *good* formulation for $\rho(x, z)$. To simplify equation (6.9), we can assume

$$\rho(x, z) = \alpha \left(\frac{a}{3}(z_b^3(x) - z^3) + f_1(x)(z_b(x) - z) \right) + \rho_0, \quad (6.15)$$

where α is related to a by the relation $a = \frac{\alpha g}{2\rho_0}$. \square

Then, if we insert all the previous results in equation (6.10), we find a quadratic equation in $z_b(x)$ with a solution given by (6.12).

REMARK 3. *Cubic terms in the density expression imply that the solution obtained depends on the reference level of the variable z . In other words, the solution presented cannot simply be translated vertically because it would no longer verify the equations.*

Figure 13 represents two of the possible stationary solutions for the problem (6.7)-(6.11) completed with the boundary conditions (2.4), (2.5). In Figure 13-it (a), $h(x)$ is given by $h(x) = 4(1 - \frac{1}{5}\exp(-(x - 10)^2/2))$ for $x \in [0, 10]$ and the other parameters are $\rho = 1000$, $g = 9.81$, $\alpha = 2$, $K_0 = 10$, $C_0 = 10$. In Figure 13-it (b), $h(x)$ is given by $h(x) = 4 + x/5$ and the other parameters are $\rho = 1000$, $g = 9.81$, $\alpha = 1$, $K_0 = 10$, $C_0 = 5$. These solutions are stable in the sense and we are able to obtain them numerically by imposing the shape of the domain and the boundary conditions.

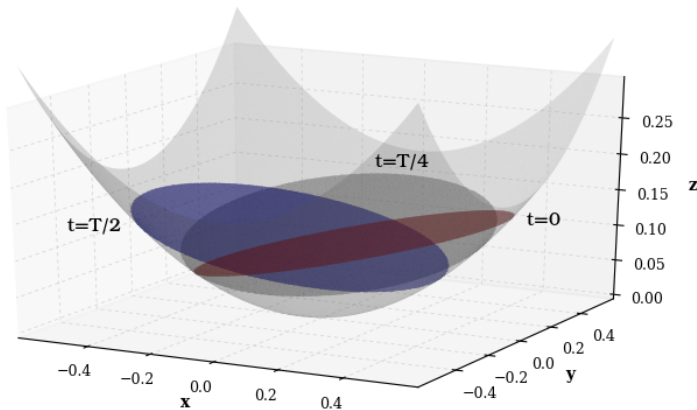


FIGURE 14. Analytical solution of prop. 10, 3D planar surface in parabolic bowl: free surface at $t = 0$ (red), $t = T/4$ (dark grey), $t = T/2$ (blue), with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1$, $h_0 = 0.1$, $a = 1$, $\alpha = 1$, $L = 4$.

6.2. Radially-symmetrical parabolic bowl with variable density

The analytical solution proposed here is based on the analytical solution presented by Thacker (Thacker 1981) when the free surface remains planar. The initial Thacker's solution is valid for the 2d Saint-Venant system and also for the 3d incompressible and hydrostatic Euler system with constant density. Several extensions of these solutions have already been given in Section 3. The extension proposed here gives a solution for the hydrostatic Euler system with variable density (6.1)-(6.3). The following proposition holds.

PROPOSITION 10. *For any nonnegative function $\rho : s \mapsto \rho(s)$ and for some $(\alpha, \eta, a, h_0) \in \mathbb{R}^2 \times \mathbb{R}_+^2$, let us consider the functions h, u, v, w, p, ϕ defined for $(x, y) \in [-L/2, L/2]^2$, $t \geq t_0$ by*

$$h(t, x, y) = \max \left\{ 0, h_0 - \alpha \frac{(x - \eta \cos(\omega t))^2 + (y - \eta \sin(\omega t))^2}{2} \right\},$$

$$u(t, x, y, z) = -\eta \omega \sin(\omega t),$$

$$v(t, x, y, z) = \eta \omega \cos(\omega t),$$

$$w(t, x, y, z) = -\alpha \eta \omega (x \sin(\omega t) - y \cos(\omega t)),$$

$$p(t, x, y, z) = p^a(t) + \int_z^{h+z_b} \rho(\phi(t, x, y, z_1)) dz_1,$$

$$\phi(t, x, y, z) = a(h + z_b - z),$$

with $\omega = \sqrt{\alpha g}$ and with a bottom topography defined by

$$z_b(x, y) = \alpha \frac{x^2 + y^2}{2}.$$

Then h, u, v, w, p, ϕ as defined previously satisfy the 3d hydrostatic Euler system with variable density (6.1)-(6.3) completed with the kinematic boundary conditions (2.4)-(2.5).

In the Figure 15 we can observe that the density isovalue lines remain parallel to the planar surface.

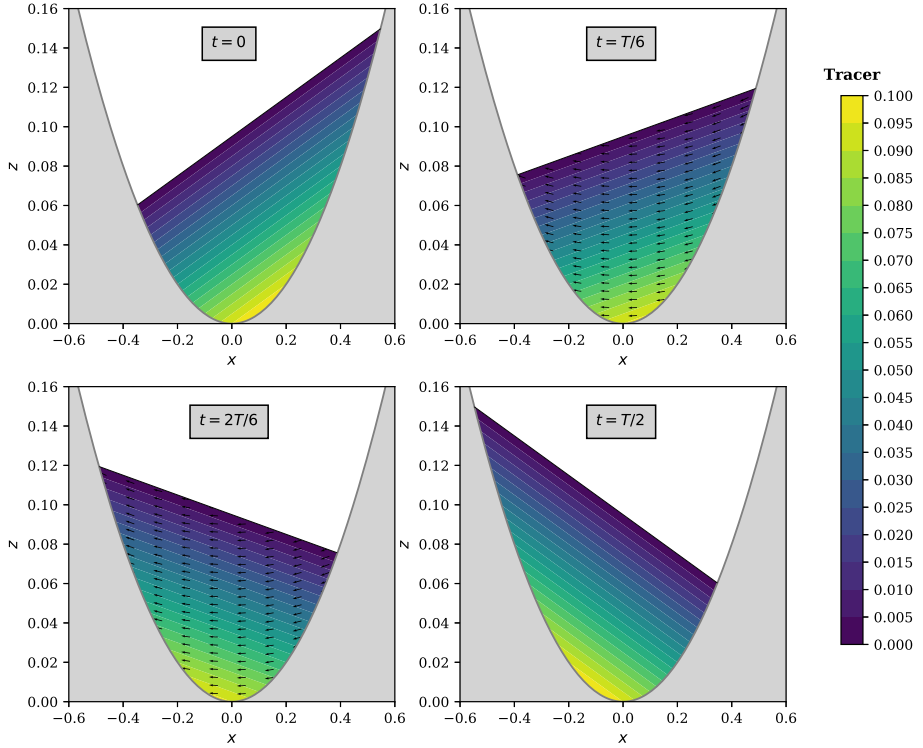


FIGURE 15. Analytical solution of prop. 10, 3D planar surface in parabolic bowl: free surface, velocity vectors and tracer for $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1$, $h_0 = 0.1$, $a = 1$, $\alpha = 1$, $L = 4$.

Situations where $\frac{\partial \rho}{\partial z} \geq 0$ can be encountered in practice (upwellings, Rayleigh-Bénard instabilities, ...). The analytical solutions given in prop. 10 exhibit situations where $\frac{\partial \rho}{\partial z}$ remains nonnegative along the time, of course, such solutions are unstable in the sense that they cannot be reproduced neither by laboratory experiments nor captured at the discrete level.

Proof. Considering the Thacker's solution with constant density, we have only to verify that the chosen density $\rho(t, x, y, z) = a(\eta(t, x, y) - z)$ does not disturb this solution. We have $\partial_z p = -\rho(t, x, y, z)g$ due to the hydrostatic hypothesis. If we note $\eta(t, x, y) = h(t, x, y) + z_b(x, y)$, $\rho(t, x, y, z) = \phi(\eta(t, x, y) - z)$ and $p(\eta(t, x, y)) = 0$, for $\phi \in C^1(\mathbb{R})$ we

have:

$$\begin{aligned}
\nabla p &= g \nabla \int_z^{\eta(t,x,y)} \rho(t,x,y,\xi) d\xi \\
&= g \nabla \int_z^{\eta(t,x,y)} \phi(\eta(t,x,y) - \xi) d\xi \\
&= g \int_z^{\eta(t,x,y)} \nabla \phi(\eta(t,x,y) - \xi) d\xi + g \nabla \eta(t,x,y) \phi(0) \\
&= g \int_z^{\eta(t,x,y)} \phi'(\eta(t,x,y) - \xi) \nabla \eta(t,x,y) d\xi + g \nabla \eta(t,x,y) \phi(0) \\
&= g \nabla \eta(t,x,y) \int_{\eta(t,x,t)-z}^0 \phi'(s) ds + g \nabla \eta(t,x,y) \phi(0) \quad \text{with } s = \eta(t,x,y) - \xi \\
&= g \phi(\eta(t,x,y) - z) \nabla \eta(t,x,y) \\
&= g \rho(t,x,y,z) \nabla \eta(t,x,y).
\end{aligned}$$

If we consider the momentum equation (6.3), using the previous evaluation of the pressure term, we can conclude that u is solution of equation (2.14) that corresponds to the momentum equation with constant density.

But generally, this expression of the density does not verify the conservation equation (6.2)

$$\partial_t \rho + \nabla \cdot (\rho u) = 0.$$

In the proposed analytical solution, since u and w are not dependent of x and z , then we have

$$\begin{aligned}
\partial_t \rho + u \nabla \rho + w \partial_z \rho &= \phi'(\eta - z) \partial_t \eta + \phi'(\eta - z) u \nabla \eta - \phi'(\eta - z) w \\
&= \phi'(\eta - z) (\partial_t \eta + u \nabla \eta - w) = 0
\end{aligned}$$

since at the free surface the kinematic boundary condition gives $\partial_t \eta + u \nabla \eta - w = 0$. \square

The analytical solution depicted in prop. 10 can be written in the 2d case.

COROLLARY 4. *With obvious notations, the functions h, u, w, p, ϕ defined for $t \geq t_0$ by*

$$\begin{aligned}
h(t,x) &= \max \left\{ 0, h_0 - \alpha \frac{(x - \eta \cos(\omega t))^2}{2} \right\}, \\
u(t,x,z) &= -\eta \omega \sin(\omega t), \\
w(t,x,z) &= -\alpha x \eta \omega \sin(\omega t), \\
p(t,x,z) &= p^a(t) + \int_z^{h+z_b} \rho(\phi(t,x,z_1)) dz_1, \\
\phi(t,x,z) &= a(h + z_b - z),
\end{aligned}$$

with a bottom topography defined by $z_b(x) = \alpha x^2/2$ and with the kinematic boundary conditions (2.4)-(2.5) satisfy the 2d hydrostatic Euler system with variable density for any $(a, \alpha, \eta, h_0) \in \mathbb{R}^3 \times \mathbb{R}_+$.

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