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Various analytical solutions for the incompressible Euler and Navier-Stokes systems with free surface

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Abstract

In this paper, we propose several time dependent analytical solutions for the incompressible Euler and Navier-Stokes systems with free surface. The given analytical solutions concerns the hydrostatic and non-hydrostatic Euler and Navier-Stokes systems.

1 Introduction

Models arising in fluid dynamics are often based on the Euler or Navier-Stokes equations and are generally difficult to analyze both at the mathematical or numerical level. As a consequence the derivation of simplified models is important, however the design of efficient/validated numerical schemes for such models remains complex. Presence of the free surface complicates the digital study of the models. It is then possible to show only very few properties on the digital plans and often not the convergence of these. A way to circumvent this difficulty is to have access to analytical solutions of the considered problem. Indeed, analytical solutions allow

- to have access to a complete behavior of the variables of a model in a given situation,

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- to find closure relations for unknown quantities during the derivation process of models simpler than the Navier-Stokes equations (e.g. shallow water type models),
- to confront the simulated solutions with reference solutions. And it makes possible the study of the convergence order of a simulated solution towards the reference one.

Many of the analytical solutions which we find in the literature concern two-dimensional shallow water type models, few have solutions distributed on the vertical line. One of the most famous is the paper of Thacker [16]. We spread these works to the three-dimensional case, to the variable density and to several tracers. Our analytical solutions can also treat the case where the wet/dry interface is moving.

This paper is organized as follows. In section 2 we present the notations and the various variants of the Euler and Navier-Stokes system studied in this paper. Then we present our analytical solutions of the Euler and Navier-Stokes systems for non hydrostatic or hydrostatic cases, with a time variable topography or with a variable density. In each case, we give illustrations of the typical situations depicted by the proposed analytical solutions. In the last part, we present an analytical solution for propagative waves in an linearized Euler system and a full Euler system where the free surface is not necessary on the sinusoidal form.

2 The Euler and Navier-Stokes systems

We consider the three-dimensional Navier-Stokes system [11] describing a free surface gravitational flow moving over a bottom topography $z_b(x, y)$.

$$\nabla \cdot \mathbf{U} = 0, \quad (1)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{U}) = \mathbf{g} + \frac{1}{\rho_0} \nabla \cdot \Sigma_T, \quad (2)$$

where $\mathbf{U}(t, x, y, z) = (u, v, w)^T$ is the velocity, p is the fluid pressure, ρ_0 is the density, $\mathbf{g} = (0, 0, -g)^T$ represents the gravity forces and ρ_0 a constant density.

The quantity ∇ denotes $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T$. We assume a Newtonian fluid, μ is the viscosity coefficient. The viscosity stress tensor Σ is given by

$$\Sigma = \mu (\nabla U + (\nabla U)^T).$$

We define the total stress tensor Σ_T by

$$\Sigma_T = -pI_d + \Sigma.$$

We consider a free surface flow (see Fig. 1, therefore we assume

$$z_b(x, y) \leq z \leq \eta(t, x, y) := h(t, x, y) + z_b(x, y)$$

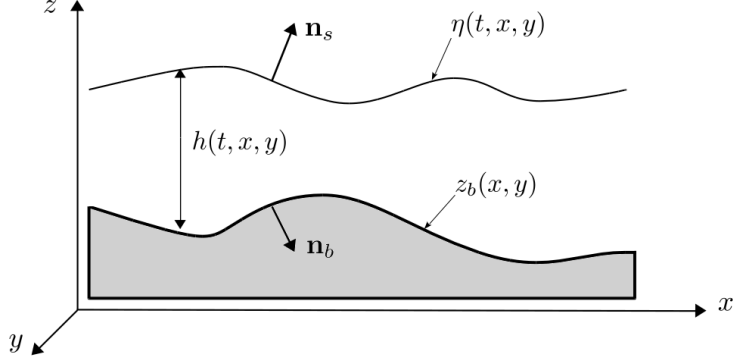


Figure 1: Flow domain with water height $h(t, x, y)$, free surface $\eta(t, x, y)$ and bottom $z_b(x, y)$.

with $z_b(x, y)$ the bottom elevation and $h(t, x, y)$ the water depth.

In Eqs. (1)-(2), the fluid density is supposed constant $\rho(t, x, y, z) = \rho_0 = 1$ and we consider the fluid contains a tracer $\phi(t, x, y, z)$. The tracer ϕ is governed by a transport-diffusion equation

$$\frac{\partial \phi}{\partial t} + \mathbf{U} \cdot \nabla \phi = \mu_\phi \Delta \phi, \quad (3)$$

where μ_ϕ is the tracer diffusivity and Δ is the classical laplacian operator.

2.1 Boundary conditions

At the free surface, the kinematic boundary condition is

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} + v_s \frac{\partial \eta}{\partial y} - w_s = 0, \quad (4)$$

whereas at the bottom we have the non-penetration condition

$$u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y} - w_b = 0, \quad (5)$$

where the subscript s (resp. b) denotes the value of the considered quantity at the free surface (resp. at the bottom).

Let \mathbf{n}_b and \mathbf{n}_s be the unit outward normals at the bottom and at the free surface respectively defined by

$$\mathbf{n}_b = \frac{1}{\sqrt{1 + |\nabla_{x,y} z_b|^2}} \begin{pmatrix} \nabla_{x,y} z_b \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{n}_s = \frac{1}{\sqrt{1 + |\nabla_{x,y} \eta|^2}} \begin{pmatrix} -\nabla_{x,y} \eta \\ 1 \end{pmatrix},$$

where $\nabla_{x,y}$ corresponds to the projection of ∇ on the horizontal plane i.e.

$$\nabla_{x,y} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)^T.$$

At the bottom we prescribe a friction condition given e.g. by a Navier law

$$(\Sigma_T \cdot \mathbf{n}_b) \cdot \mathbf{t}_b^i = -\kappa \mathbf{U}_b \cdot \mathbf{t}_b^i, \quad (6)$$

with $\kappa = \kappa(h, u_b) \geq 0$ a Navier coefficient and \mathbf{t}_b^i , $i = 1, 2$ are two tangential vectors ($\mathbf{n}_b \cdot \mathbf{t}_b^i = 0$).

On the free surface, we impose the condition

$$\Sigma_T \cdot \mathbf{n}_s = -p^a(t, x, y) \mathbf{n}_s + \tau_w(t, x, y) \mathbf{t}_s, \quad (7)$$

where $p^a(t, x, y)$, $\tau_w(t, x, y)$ are two given quantities, p^a (resp. τ_w) mimicks the effects of the atmospheric pressure (resp. the wind blowing at the free surface), \mathbf{t}_s a tangential vector satisfying $\mathbf{n}_s \cdot \mathbf{t}_s = 0$. The boundary conditions associated with Eq. (3) are given by

$$\mu_\phi \nabla \phi|_s \cdot \mathbf{n}_s = F_s^\phi, \quad (8)$$

and

$$\mu_\phi \nabla \phi|_b \cdot \mathbf{n}_b = F_b^\phi. \quad (9)$$

where F_s^ϕ (rest. F_b^ϕ) are the flux of tracer ϕ at the surface (resp. the bottom).

Remark 1 *The system (1)-(9) has to be completed with initial and boundary conditions on lateral boundary (inflow, outflow or wall type) that are not detailed here.*

2.2 The Euler system

The Euler system consists in considering an idealized fluid for which the stress tensor Σ_T reduces to the pressure part. It writes

$$\nabla \cdot \mathbf{U} = 0, \quad (10)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g}, \quad (11)$$

and it is completed with the two kinematic boundary conditions (4),(5). The dynamic boundary condition at the free surface reduces to

$$p_s = p(t, x, y, \eta) = p^a(t, x, y). \quad (12)$$

The tracer ϕ remains governed by Eq. (3) with possibly $\mu_\phi = 0$.

2.3 Hydrostatic models

Hydrostatic models consist in neglecting the vertical acceleration of the fluid i.e.

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \approx 0.$$

These models are very often used for the study of geophysical flows, see [2, 8, 13, 3] for justifications of such models.

The hydrostatic version of the Navier-Stokes system (1)-(2) writes

$$\nabla \cdot \mathbf{U} = 0, \quad (13)$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\tilde{\mathbf{U}} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g} + \frac{1}{\rho_0} \nabla \cdot \tilde{\Sigma}, \quad (14)$$

with $\tilde{\mathbf{U}} = (u, v, 0)$, $\tilde{\Sigma} = (\Sigma_x, \Sigma_y, 0)$ and it is completed with the boundary conditions (4)-(7). The hydrostatic Euler system consists in neglecting the viscous terms in (13)-(14) and is given by

$$\nabla \cdot \mathbf{U} = 0, \quad (15)$$

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\tilde{\mathbf{U}} \otimes \mathbf{U}) + \frac{1}{\rho_0} \nabla p = \mathbf{g}, \quad (16)$$

completed with the boundary conditions (4),(5),(12).

In the hydrostatic Navier-Stokes system (13)-(14) but also in the hydrostatic Euler model (15)-(16), the tracer ϕ remains governed by Eq. (3) with the boundary conditions (8)-(9).

2.4 The hydrostatic Euler system with variable density

When the fluid density is a function of the tracer concentration ϕ i.e. $\rho = \rho(\phi)$, $\rho(\phi)$ being a given function then the hydrostatic Euler system with variable density writes

$$\nabla \cdot \mathbf{U} = 0, \quad (17)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \quad (18)$$

$$\frac{\partial \rho \tilde{\mathbf{U}}}{\partial t} + \nabla \cdot (\rho \tilde{\mathbf{U}} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g}, \quad (19)$$

with $\tilde{\mathbf{U}} = (u, v, 0)$ and the system (17)-(19) is completed with the boundary conditions (4),(5),(12).

2.5 The linearized Euler system

Assuming the velocities u and w are such that $u = u^0 + \mathcal{O}(\varepsilon)$, $v = v^0 + \mathcal{O}(\varepsilon)$, $w = \mathcal{O}(\varepsilon)$ with $\varepsilon \ll 1$ and $u^0 = cst, v^0 = cst$ the approximation in $\mathcal{O}(\varepsilon^2)$ of the Euler system (10)-(11) gives

$$\nabla \cdot \mathbf{U} = 0, \quad (20)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U}^0 \otimes \mathbf{U}) + \frac{\nabla p}{\rho_0} = -\mathbf{g}, \quad (21)$$

with $\mathbf{U}^0 = (u^0, v^0, 0)^T$. And the system (20)-(21) is completed with the boundary conditions (4),(5),(12). It is important to notice that whereas in most cases

the linearized Euler system does not admit any energy balance, when the quantity

$$e_{s,b} = \frac{|\mathbf{U}|_s^2}{2} \frac{\partial \eta}{\partial t} + \frac{|\mathbf{U}|_s^2}{2} \mathbf{U}^0 \cdot \nabla \eta - \frac{|\mathbf{U}|_b^2}{2} \mathbf{U}_b \cdot \nabla z_b,$$

can be written under the conservative form

$$e_{s,b} = \nabla_{x,y} \alpha_{s,b},$$

with $\alpha = \alpha(h, \mathbf{U}^0, \mathbf{U}_s)$ then the linearized Euler system admits an energy balance under the form

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} (E + p^a) dz + \nabla_{x,y} \cdot \left[\int_{z_b}^{\eta} (\mathbf{U}_0 E + \mathbf{U}(p + gz)) dz + \alpha_b^s \right] = h \frac{\partial p^a}{\partial t}, \quad (22)$$

with E defined by

$$E = \frac{u^2 + w^2}{2} + gz.$$

It is worth noticing that whereas the linearized Euler system (20)-(21) completed with (4),(5),(12) contains only terms up to ε , the energy balance (22) necessarily contains terms in $\mathcal{O}(\varepsilon^3)$.

3 Analytical solutions

In this section, we give the analytical solutions for the Navier-Stokes and Euler systems, hydrostatic or not, with constant and variable density.

3.1 Draining of a tank: non-hydrostatic case

Considering the Navier-Stokes system (1)-(3) completed with the boundary conditions (4)-(9) in a tank such that $(x, y) \in [-L/2, L/2]^2$. The following proposition holds.

Proposition 1 *For some $\alpha \in \mathbb{R}_+$, $t_0 \in \mathbb{R}$, $t_1 \in \mathbb{R}_+^*$, $\phi_0 \in \mathbb{R}$, $\theta \in [0, 2\pi] \setminus \{\pi\}$ let us consider the functions h, u, v, w, p, ϕ defined for $t \geq t_0$ by*

$$\begin{aligned} h(t, x, y) &= \alpha f(t), \\ u(t, x, y, z) &= f(t) (x \cos(\theta) + y \sin(\theta)), \\ v(t, x, y, z) &= f(t) \tan\left(\frac{\theta}{2}\right) (x \cos(\theta) + y \sin(\theta)), \\ w(t, x, y, z) &= f(t) (z_b - z), \\ p(t, x, y, z) &= p^{a,0}(t) - 2\mu f(t) + g(h - (z - z_b)) + (h^2 - (z - z_b)^2) f(t)^2, \\ \phi(t, x, y, z) &= \frac{\phi_0}{L} (x \cos(\theta) + y \sin(\theta)) \frac{z - z_b}{h_0}, \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$ and with a flat bottom $z_b(x, y) = z_{b,0} = cst$, $h_0 = h(t_0, x, y) = \alpha/t_1$ and $p^{a,0}(t)$ a given function.

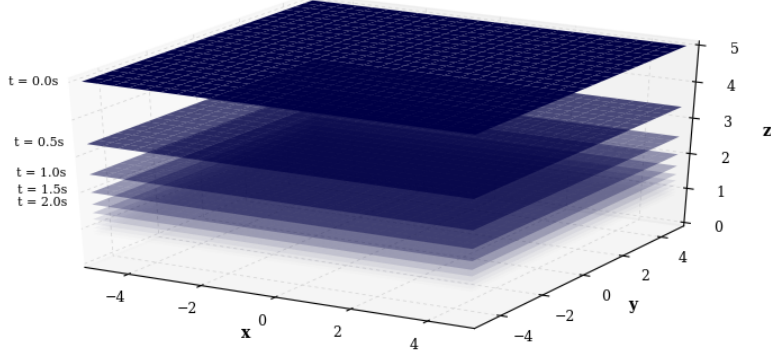


Figure 2: Draining of a tank: free surface every 0.5s from initial time, for parameters set to $\theta = 0$, $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

Then h, u, v, w, p, ϕ as defined previously satisfy the 3d Navier-Stokes system (1)-(3) completed with the boundary conditions (4)-(9) where $\kappa = 0$ in (6), $\tau_w = 0$ in (7) and $F_s^\phi = -F_b^\phi = \mu_\phi \phi_0 (x \cos(\theta) + y \sin(\theta)) / (Lh_0)$ in Eqs. (8)-(9). The appropriate boundary conditions for $x \in \{-L/2, L/2\}$ or $y \in \{-L/2, L/2\}$ are also determined by the expressions of h, u, v, w, p, ϕ given above.

Choosing the viscosity $\mu = 0$, the variables h, u, v, w, p, ϕ become analytical solutions of the 3d Euler system (10)-(11) completed with the boundary conditions (4),(5) and the tracer equation (3) with (8)-(9).

Remark 2 The analytical solution depicted in prop. 1 can be written in the 2d case (x, z) when $\theta = 0$. With obvious notations, the functions h, u, w, p, ϕ defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x) &= \alpha f(t), \\ u(t, x, z) &= x f(t), \\ w(t, x, z) &= (z_b - z) f(t), \\ p(t, x, z) &= p^a(t, x) - 2\mu f(t) + g(h - (z - z_b)) + (h^2 - (z - z_b)^2) f(t)^2, \\ \phi(t, x, z) &= \phi_0 \frac{x}{L} \frac{z - z_b}{h_0}, \end{aligned}$$

with $p^a(t, x) = p^{a,0}(t)$, $h_0 = h(t_0, x) = \alpha/t_1$ and $F_s^\phi = -F_b^\phi = \mu_\phi \phi_0 x / (Lh_0)$ in (4)-(9) are solutions of the 2d incompressible Navier-Stokes system with free surface for any $x \in [-L/2, L/2]$, $\alpha \in \mathbb{R}_+$, $t_1 \in \mathbb{R}_+^*$, $t_0 \in \mathbb{R}$, $\phi_0 \in \mathbb{R}$ and $\kappa = 0$.

proof 3.1 (Proof of prop. 1) The proof of prop. 1 relies on very simple computations since is enough to verify that these solutions are well solutions of the equations of Euler or Navier Stokes given in paragraph 2.3. It is the same thing for all the solutions which will be presented in these articles, we shall not detail the calculations.

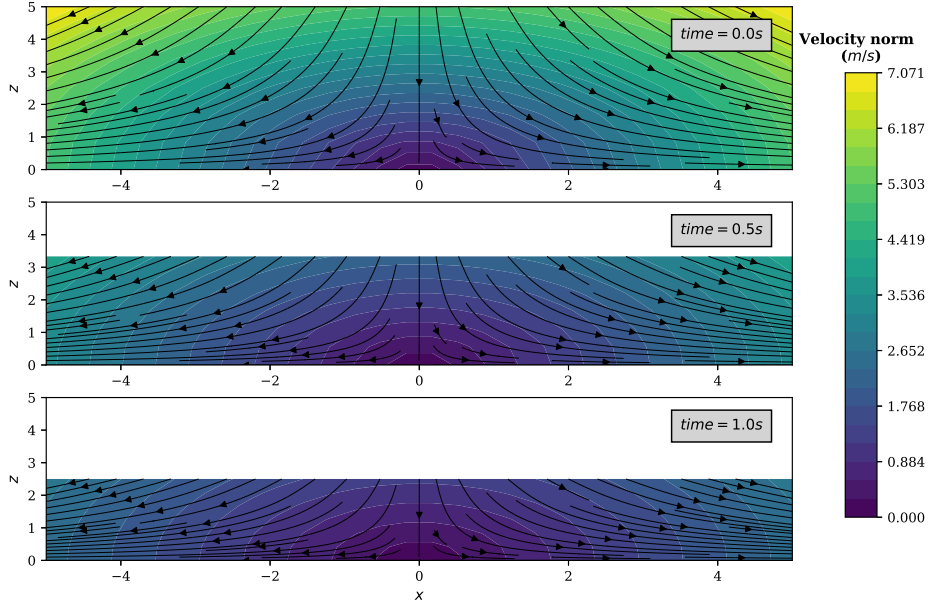


Figure 3: Draining of a tank (non-hydrostatic case): velocity norm and streamlines at $t = 0, 0.5, 1.0$ s, in $(x, y = 0, z)$ slice plane for parameters set to $\theta = 0$, $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

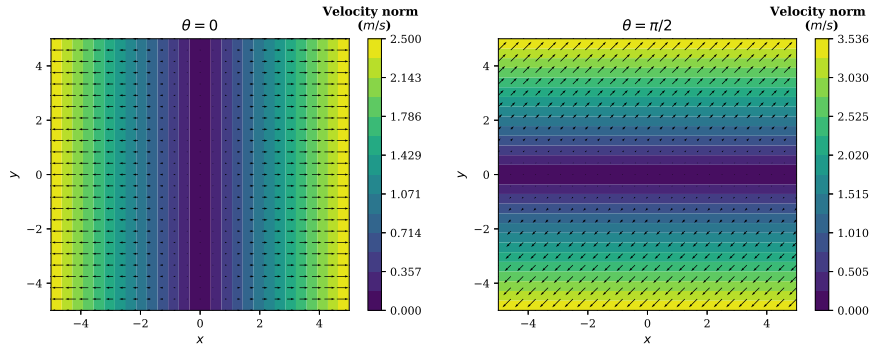


Figure 4: Draining of a tank (non-hydrostatic case): velocity norm and vectors at initial time in $(x, y, z = h_0/2)$ slice plane with $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

3.2 Draining of a tank: hydrostatic case

Considering the hydrostatic Navier-Stokes and Euler systems given in paragraph 2.3 in a tank such that $(x, y) \in [-L/2, L/2]^2$. The two following propositions hold.

Proposition 2 For some $t_0 \in \mathbb{R}$, $t_1 \in \mathbb{R}_+^*$, $\phi_0 \in \mathbb{R}$, $(\alpha, \beta) \in \mathbb{R}_+^2$ such that $\alpha\beta > L$, let us consider the functions h, u, v, w, p, ϕ defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \alpha f(t), \\ u(t, x, y, z) &= \beta \left((z - z_b) - \frac{\alpha}{2} f(t) \right) + f(t)(x \cos^2(\theta) + y \sin^2(\theta)), \\ v(t, x, y, z) &= \beta \left((z - z_b) - \frac{\alpha}{2} f(t) \right) + f(t)(x \cos^2(\theta) + y \sin^2(\theta)), \\ w(t, x, y, z) &= f(t)(z_b - z), \\ p(t, x, y, z) &= p^a(t, x, y) - 2\mu f(t) + g(h - (z - z_b)), \\ \phi(t, x, y, z) &= \frac{\phi_0}{4L} (2x \cos^2(\theta) + 2y \sin^2(\theta) - \beta\alpha)h(t, x, y) + \frac{\beta\phi_0\alpha}{2L}z, \end{aligned}$$

where $f(t) = 1/(t - t_0 + t_1)$ and with a flat bottom $z_b(x, y) = z_{b,0} = cst$ and $p^a(t, x, y) = p^{a,1}(t)$, where $p^{a,1}(t)$ a given function.

Then h, u, v, w, p, ϕ as defined previously satisfy the 3d hydrostatic Navier-Stokes system (13)-(14),(3) completed with the kinematic boundary conditions (4)-(5) and $\kappa = \frac{2\mu\alpha\beta}{h(t,x,y)[\alpha\beta - 2(x\cos^2(\theta) + y\sin^2(\theta))]}$ in (6), $\tau_w = \mu\beta/2$ with $\mathbf{t}_s = \frac{1}{\sqrt{2}}(1, 1)^t$ in (7), $F_s^\phi = F_b^\phi = \frac{\mu\phi}{2L}\phi_0\alpha\beta$ in Eqs. (8)-(9). The appropriate boundary conditions for $x \in \{-L/2, L/2\}$ or $y \in \{-L/2, L/2\}$ are also determined by the expressions of h, v, u, w, ϕ given above.

Choosing the viscosity $\mu = 0$, the variables h, u, v, w, p, ϕ become analytical solutions of the 3d hydrostatic Euler system (15)-(16) completed with the boundary conditions (4),(5) and the tracer equation (3) with (8)-(9).

Remark 3 As in remark 2, the analytical solutions given in prop. 2 are solution of the 2d hydrostatic Navier-Stokes and Euler systems when $\theta = 0$.

Remark 4 Choosing $\beta = 0$ in prop. 2 gives analytical solutions with a velocity field that does not depend on the variable z . It is easy to see that it corresponds to analytical solutions for the classical viscous Saint-Venant system [7].

proof 3.2 (Proof of prop. 3) See proof of prop. 2.

3.3 Radially-symmetrical parabolic bowl

The Thacker' analytical solution [16], corresponds to a periodic oscillation in a parabolic bowl. This analytical solution for the 2d Saint-Venant system is also an analytical solution for the 3d incompressible and hydrostatic Euler system. We propose an extension of the Thacker' radially-symmetrical solution to the

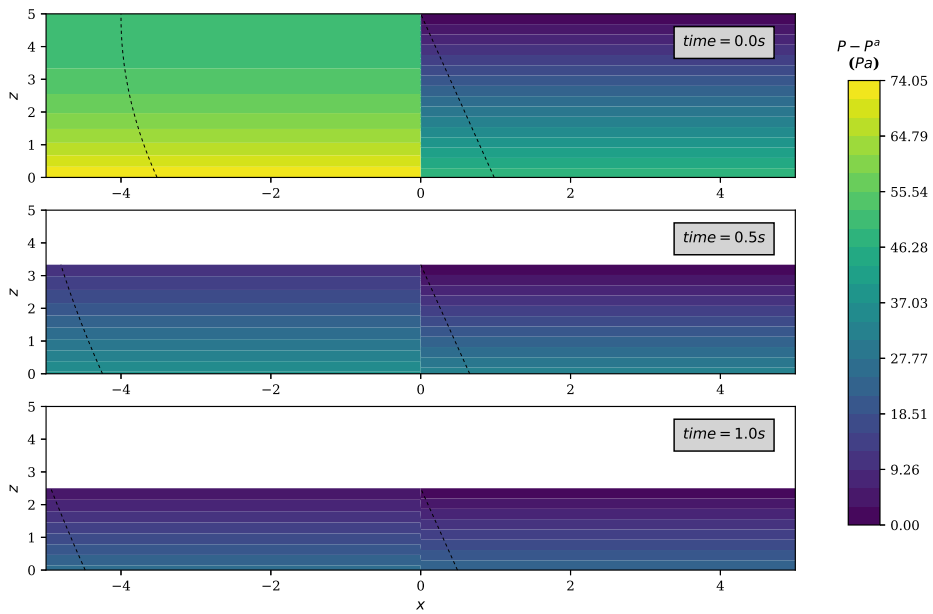


Figure 5: Draining of a tank: pressure in non-hydrostatic case (left, $x < 0$) and hydrostatic case (right $x > 0$), at $t = 0, 0.5, 1.0s$, in $(x, y=0, z)$ plane with parameters set to $\mu = 0$, $\theta = 0$, $\alpha = 5$, $t_1 = 1$, $p^a = 0$, $z_b = 0$ and $L = 10$.

situation where the velocity field depends on the vertical coordinate. This means the proposed solution is analytical for the 3d incompressible hydrostatic Euler system but does not correspond to a shallow water flows.

Proposition 3 For some $t_0 \in \mathbb{R}$, $(\alpha, \beta, \gamma) \in \mathbb{R}_{+*}^3$ such that $\gamma < 1$ let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \max \left\{ 0, \frac{1}{r^2} f \left(\frac{r^2}{\gamma \cos(\omega t) - 1} \right) \right\}, \\ u(t, x, y, z) &= x \left(\beta \left(z - z_b - \frac{h}{2} \right) + \frac{\omega \gamma \sin(\omega t)}{2(1 - \gamma \cos(\omega t))} \right), \\ v(t, x, y, z) &= y \left(\beta \left(z - z_b - \frac{h}{2} \right) + \frac{\omega \gamma \sin(\omega t)}{2(1 - \gamma \cos(\omega t))} \right), \\ w(t, x, y, z) &= -\frac{\partial}{\partial x} \int_{z_b}^z u dz - \frac{\partial}{\partial y} \int_{z_b}^z v dz, \\ p(t, x, y, z) &= g(h + z_b - z), \end{aligned}$$

with $\omega = \sqrt{4\alpha g}$, $r = \sqrt{x^2 + y^2}$ and with a bottom topography defined by

$$z_b(x, y) = \alpha \frac{r^2}{2},$$

and the function f given by

$$f(z) = -\frac{4g}{\beta^2} + \frac{2}{\beta^2} \sqrt{4g^2 + cz + \beta^2 \alpha g (\gamma^2 - 1) z^2},$$

c being a negative constant such that $c \leq 4g^2/(\gamma - 1)$.

Then h, u, v, w, p as defined previously satisfy the 3d hydrostatic Euler system (15)-(16) completed with the boundary conditions (4),(5). The appropriate boundary conditions in lateral boundary are also determined by the expressions of h, u, v, w given above.

Remark 5 If we note R the maximum of $r(x, y)$ in order to have f positive, a simple computation of the zeros of the function f for $t = \pi/\omega$ gives $R = \sqrt{c/(\beta^2 \alpha g (\gamma - 1))}$ with $c \leq 0$

Remark 6 Integrating u and v along z , the vertical velocity component w is obtained

$$w(t, x, y, z) = -\beta(z - z_b)^2 + \left(\beta h - 2\Gamma - \beta \alpha r^2 + \frac{\beta}{2} \left(x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} \right) \right) (z - z_b) - \frac{\alpha r^2}{2} (\beta h - 2\Gamma)$$

where $\partial_x h = 2x\mathcal{F}(t, x, y)$ and $\partial_y h = 2y\mathcal{F}(t, x, y)$ with

$$\mathcal{F}(t, x, y) = -\frac{1}{r^4} f \left(\frac{r^2}{\Lambda} \right) + \frac{1}{r^2 \Lambda} f' \left(\frac{r^2}{\Lambda} \right)$$

and

$$\Lambda = \gamma \cos(\omega t) - 1, \quad \Gamma = -\frac{\omega \gamma \sin(\omega t)}{2\Lambda}$$

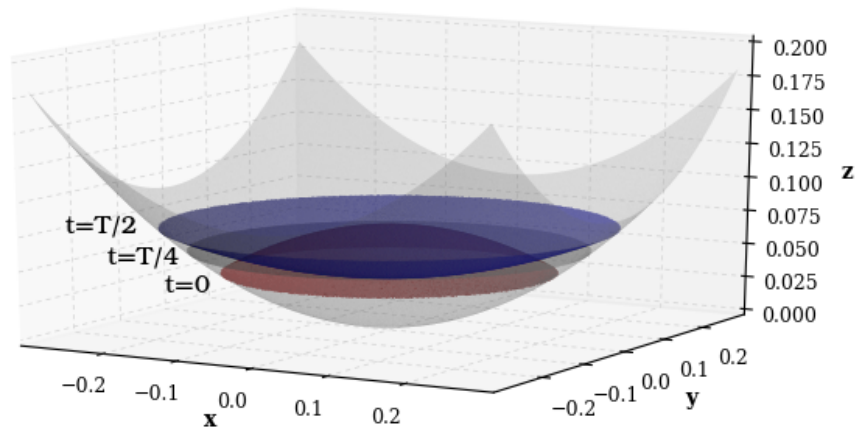


Figure 6: 3D Axisymmetrical parabolic bowl: free surface at $t = 0$ (red), $t = T/4$ (dark grey), $t = T/2$ (blue), with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 1$, $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $c = -1$, $L = 1$.

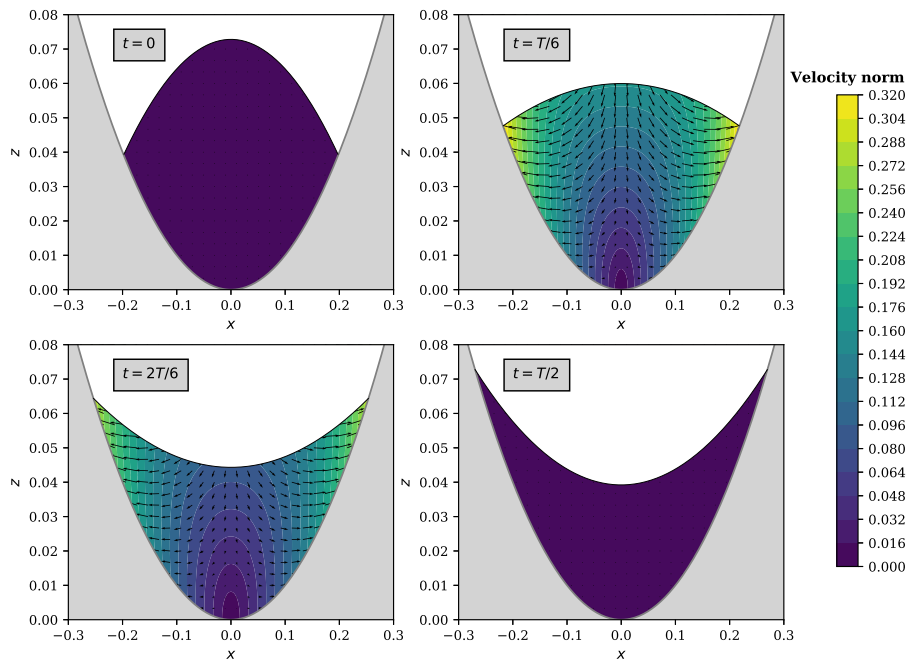


Figure 7: 3D Axisymmetrical parabolic bowl: velocity norm and vectors at $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $h_0 = 1$, $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $c = -1$, $L = 1$.

Remark 7 *The analytical solution depicted in prop. 3 can be written in the 2d case. With obvious notations, we choose $\gamma \leq 2g/(\beta\omega)$ and $c > 0$, and we consider the functions h, u, w, p defined for $t \geq t_0$ by*

$$\begin{aligned} h(t, x) &= \max \left\{ 0, f \left(x - \frac{\gamma}{\omega} \sin(\omega t) \right) \right\}, \\ u(t, x, z) &= \beta \left(z - z_b - \frac{h}{2} \right) + \gamma \cos(\omega t), \\ w(t, x, z) &= -\frac{\partial}{\partial x} \int_{z_b}^z u dz, \\ p(t, x, z) &= g(h + z_b - z), \end{aligned}$$

with a bottom topography defined by $z_b(x) = 2\alpha x^2$, a function f given by

$$f(z) = -\frac{4g}{\beta^2} + \frac{2}{\beta^2} \sqrt{4g^2 - 2c\beta^2 - \beta^2\omega^2 z^2};$$

Then h, u, w, p as defined previously satisfy the 2d hydrostatic Euler system completed with the boundary conditions (4),(5). The appropriate boundary conditions in lateral boundary are also determined by the expressions of h, u, w given above.

Remark 8 *If we note R the maximum of x in order to have f positive, a simple computation of the zeros of the function f for $t = \pi/2\omega$ gives $R = \frac{\gamma + \sqrt{-2c}}{\omega}$ with $c \leq 0$.*

proof 3.3 (Proof of prop. 3) *See proof of prop. 1.*

3.4 Planar surface in parabolic bowl with variable density

In this second Thacker' solution, we consider the case of a planar surface rotating in a parabolic bowl. For the hydrostatic Euler with variable density (17)-(19), the following proposition holds.

Proposition 4 *For any nonnegative function $s \mapsto \rho(s)$ and for some $(a, \alpha, \eta, h_0) \in \mathbb{R}^3 \times \mathbb{R}_+$, let us consider the functions h, u, v, w, p, ϕ defined for $(x, y) \in [-L/2, L/2]^2$, $t \geq t_0$ by*

$$\begin{aligned} h(t, x, y) &= \max \left\{ 0, h_0 - \alpha \frac{(x - \eta \cos(\omega t))^2 + (y - \eta \sin(\omega t))^2}{2} \right\}, \\ u(t, x, y, z) &= -\eta\omega \sin(\omega t), \\ v(t, x, y, z) &= \eta\omega \cos(\omega t), \\ w(t, x, y, z) &= -\alpha\eta\omega (x \sin(\omega t) - y \cos(\omega t)), \\ p(t, x, y, z) &= p^a(t) + \int_z^{h+z_b} \rho(\phi(t, x, y, z_1)) dz_1, \\ \phi(t, x, y, z) &= a(h + z_b - z), \end{aligned}$$

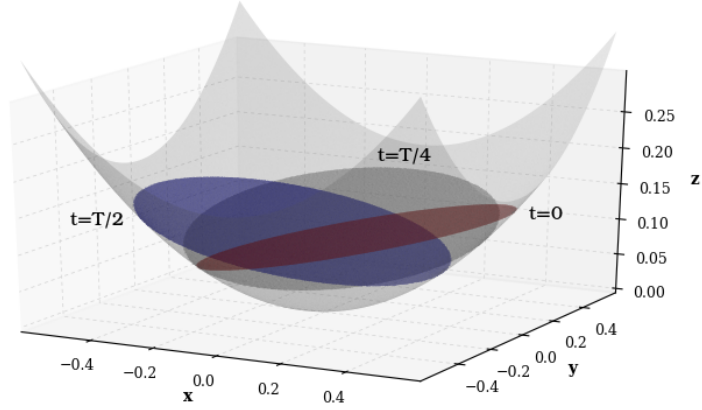


Figure 8: 3D Planar surface in parabolic bowl: free surface at $t = 0$ (red), $t = T/4$ (dark grey), $t = T/2$ (blue), with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1$, $h_0 = 0.1$, $a = 1$, $\alpha = 1$, $L = 4$.

with $\omega = \sqrt{\alpha g}$ and with a bottom topography defined by

$$z_b(x, y) = \alpha \frac{x^2 + y^2}{2},$$

Then h, u, v, w, p, ϕ as defined previously satisfy the 3d hydrostatic Euler system with variable density (17)-(19) completed with the kinematic boundary conditions (4)-(5).

Remark 9 Even if the functions given in the prop. 4 are solutions of the Euler equations with varying density, it is possible that they are unstable. It can be the case in particular when we take an increasing density from the bottom to the surface.

Remark 10 The analytical solution depicted in prop. 4 can be written in the 2d case. With obvious notations, the functions h, u, w, p, ϕ defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x) &= \max \left\{ 0, h_0 - \alpha \frac{(x - \eta \cos(\omega t))^2}{2} \right\}, \\ u(t, x, z) &= -\eta \omega \sin(\omega t), \\ w(t, x, z) &= -\alpha x \eta \omega \sin(\omega t), \\ p(t, x, z) &= p^a(t) + \int_z^{h+z_b} \rho(\phi(t, x, z_1)) dz_1, \\ \phi(t, x, z) &= a(h + z_b - z), \end{aligned}$$

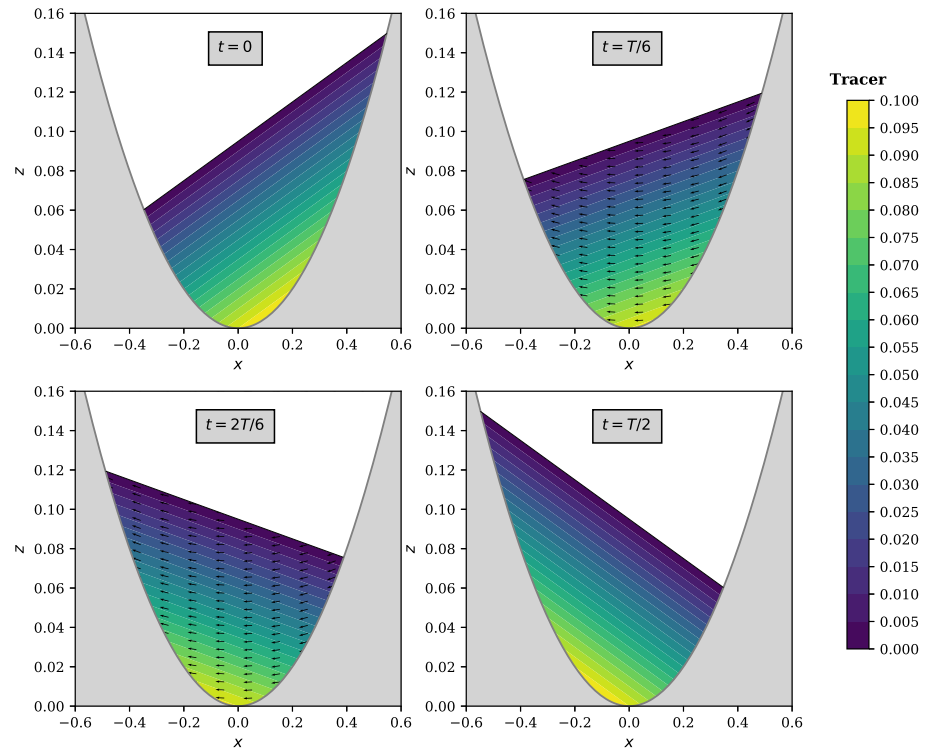


Figure 9: 3D Planar surface in parabolic bowl: free surface, velocity vectors and tracer for $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1, h_0 = 0.1, a = 1, \alpha = 1, L = 4$.

with a bottom topography defined by $z_b(x) = \alpha x^2/2$ and with the kinematic boundary conditions (4)-(5) satisfy the 2d hydrostatic Euler system with variable density for any $(a, \alpha, \eta, h_0) \in \mathbb{R}^3 \times \mathbb{R}_+$.

proof 3.4 (Proof of prop. 4) See proof of prop. 1.

3.5 Planar surface in parabolic bowl with time variable topography

If the topography is a function of time, the non-penetration condition at the bottom (5) must be replaced by

$$\frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y} - w_b = 0. \quad (23)$$

We have then the following proposition

Proposition 5 For some $t_0 \in \mathbb{R}$, $(h_0, \eta, \alpha, \beta) \in \mathbb{R}_+^4$, let us consider the functions h, u, v, w, p defined for $t \geq t_0$ by

$$\begin{aligned} h(t, x, y) &= \max \left\{ 0, h_0 - \frac{\alpha}{4} (\eta \cos(\omega t) - \eta \sin(\omega t) - x + y)^2 \right\}, \\ u(t, x, y, z) &= \beta \left(z - z_b - \frac{h}{2} \right) - \eta \omega \sin(\omega t), \\ v(t, x, y, z) &= \beta \left(z - z_b - \frac{h}{2} \right) + \eta \omega \cos(\omega t), \\ w(t, x, y, z) &= \frac{\partial z_b}{\partial t} - \frac{\partial}{\partial x} \int_{z_b}^z u dz - \frac{\partial}{\partial y} \int_{z_b}^z v dz, \\ p(t, x, y, z) &= g(h + z_b - z), \end{aligned}$$

with $\omega = \sqrt{\alpha g}$ and with a bottom topography defined by

$$z_b(x, y, t) = \alpha \frac{(x - y)^2}{4} + \frac{\alpha \eta}{2} (x + y) (\cos(\omega t) + \sin(\omega t)).$$

Then h, u, v, w, p as defined previously satisfy the 3d hydrostatic Euler system (15)-(16) completed with the boundary conditions (4),(12),(23).

proof 3.5 (Proof of prop. 5) See proof of prop. 1.

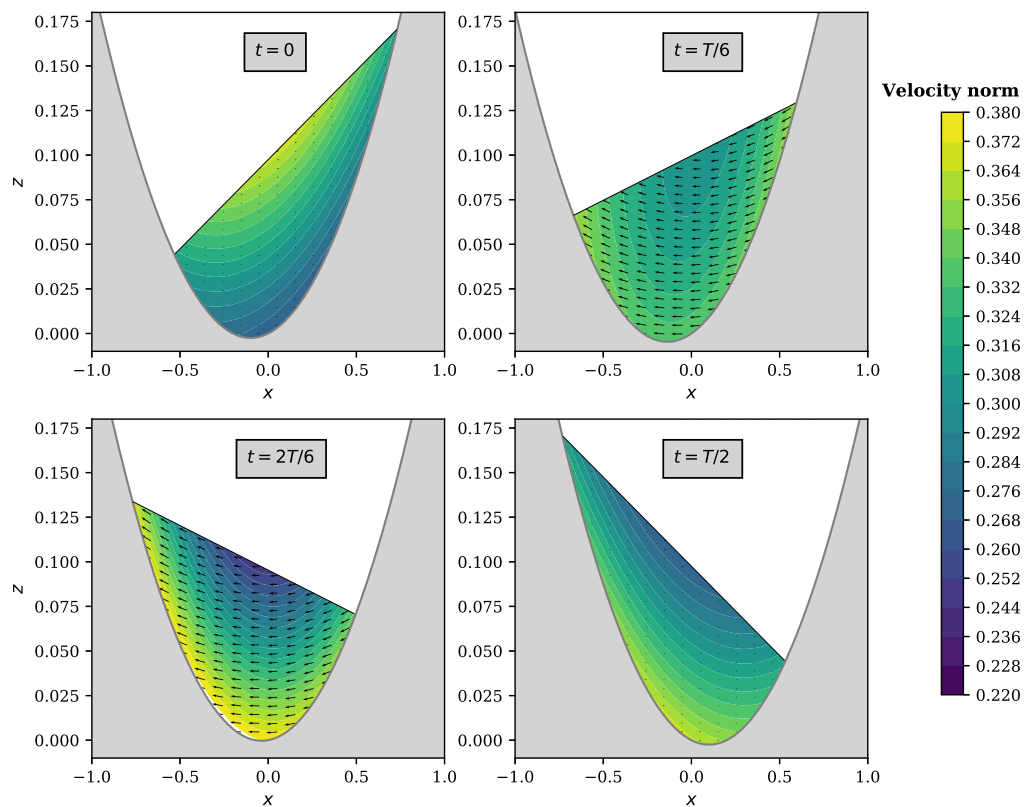
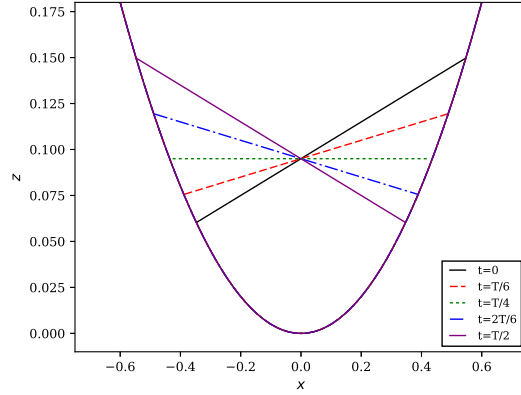
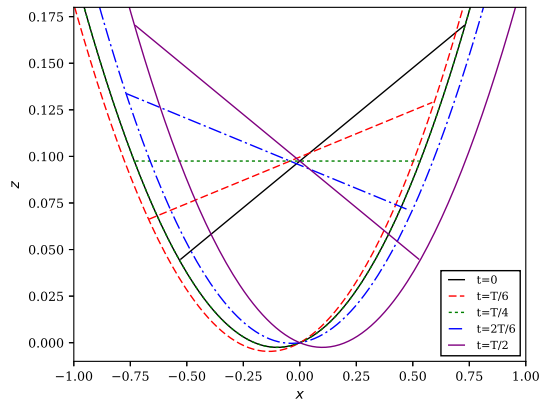


Figure 10: 3D Planar surface in parabolic bowl: free surface, velocity vectors and norm at $t = 0, T/6, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1$, $h_0 = 0.1$, $\alpha = 1$, $\beta = 1$, $L = 4$.



(a)



(b)

Figure 11: Evolution of the topography and free surface for the parabolic bowl with variable density (a) and with variable topography (b) at $t = 0, T/6, T/4, 2T/6, T/2$, in $(x, y = 0, z)$ slice plane with the period T defined by $T = 2\pi/\omega$ and for parameters set to $\eta = 0.1$, $h_0 = 0.1$, $\alpha = 1$, $\beta = 1$, $L = 4$.

3.6 Wave propagation

In this section we look for solutions of the Euler and linearized Euler system. The analytical solutions we propose mainly concerns propagating waves and but some results concerning standing waves are also given. Since we consider planar waves, the analytical solutions are given in 2d in the (x, z) domain i.e. with $v = 0$.

The water wave problem described by the Euler equations with a free surface has been widely studied in the literature see e.g. [6, 14, 12, 9, 10]. The analytical solutions we propose have connections with recent works of W. Strauss, see [4, 15] and references therein.

3.6.1 The LambertW function

In the following paragraphs, we will use the LambertW functions [5]. The LambertW functions have many applications in pure and applied mathematics but also in physics. It is a set of functions, namely the branches of the inverse relation of the function $f(z) = ze^z$ where z is any complex number. In other words, the LambertW function satisfies

$$\text{LambertW}(x)e^{\text{LambertW}(x)} = x.$$

As the equation

$$ze^z = x, \tag{24}$$

has an infinite number of solutions z for each non-zero value of x , LambertW has an infinite number of branches. Exactly one of these branches is analytic at 0. In the following, this branch is referred to as the principal branch of LambertW, and is denoted by $\text{LambertW}(x)$. The other branches all have a branch point at 0, and these branches are often denoted in the litterature by $\text{LambertW}(k, x)$, where k is any non-zero integer. The results presented hereafter are valid for all the branches but for the sake of simplicity, we focus on the principal branch i.e. $\text{LambertW}(0, x) = \text{LambertW}(x)$.

Notice that Eq. (24) admits an unique real solution for $x \geq -1/e$. For $x < -1/e$, all the solutions of Eq. (24) are complex.

3.6.2 Propagating waves for the linearized Euler system

We consider the linearized Euler system (20)-(21) completed by the boundary conditions (4),(12) and (5). Then we have the following result.

Proposition 6 *Let $(a, b, h_0, k, n) \in \mathbb{R}^2 \times \mathbb{R}_+^3$ such that $|a| < 1/e$ and $h_0k/n > \text{LambertW}(-|a|)$. Let f the function defined by*

$$f : x \mapsto -\frac{n}{k} \text{LambertW}\left(a \cos\left(\frac{x}{n} + b\right)\right), \tag{25}$$

Let $p^a(t)$ be any given function.

Up to terms in $\mathcal{O}(e^{-\frac{h_0 k}{n}})$, the functions h , u , w and p defined by

$$h(t, x) = h_0 + f(kx - \omega t), \quad (26)$$

$$u(t, x, z) = -\frac{\omega a}{k} e^{\frac{k}{n}(z-h_0)} \cos\left(\frac{kx - \omega t}{n} + b\right), \quad (27)$$

$$w(t, x, z) = -\frac{n}{k} \frac{\partial u}{\partial x}, \quad (28)$$

$$p(t, x, z) = p^a(t) + g(h_0 - z) - \frac{gn}{k} e^{\frac{k}{n}(z-h_0)} \cos\left(\frac{kx - \omega t}{n} + b\right), \quad (29)$$

are analytical solutions of the linearized Euler system (20)-(21) completed by the boundary conditions (4),(12) and (5) iff the dispersion relation

$$\frac{\omega}{k} = \sqrt{\frac{gn}{k}}, \quad (30)$$

holds.

proof 3.6 (Proof of prop. 6) First, Eq. (28) is obtained using an integration along the vertical axis of the divergence free condition (20) and neglecting the term in $\mathcal{O}(e^{-\frac{h_0 k}{n}})$ corresponding to the vertical velocity at the bottom. Then the proof relies on very simple computations by inserting the expressions (26)-(29) in Eqs. (20)-(21),(4),(12) and (5).

Remark 11 The solutions proposed in prop. 6 for the linearized Euler system are not exactly analytical solutions in the sense that additional terms in $\mathcal{O}(e^{-\frac{h_0 k}{n}})$ appear. But it is important to notice that when $h_0 k \ll 1$, considering e.g. $h_0 = 100 \text{ m}$ $k = 0.2 \text{ m}^{-1}$ and $n = 1$ gives $e^{-\frac{h_0 k}{n}} \approx 10^{-9} \ll 1$.

Remark 12 The solutions proposed in prop. 6 satisfy the energy balance (22) with

$$\alpha_{s,b} = -\frac{g\omega}{4k} \frac{n^2 \text{LambertW}\left(a \cos\left(\frac{x}{n} + b\right)\right)^2}{k^2 \cos\left(\frac{x}{n} + b\right)^2}.$$

3.6.3 Propagating waves for the Euler system

We consider the Euler system (10)-(11) completed by the boundary conditions (4), (5) and (12). Then we have the following result.

Proposition 7 Let $(a, b, h_0, k, n) \in \mathbb{R}^2 \times \mathbb{R}_+^3$ such that $|a| < 1/e$ and $h_0 k/n > \text{LambertW}(-|a|)$. Let f the function defined by

$$f(x) = -\frac{n}{k} \text{LambertW}\left(a \cos\left(\frac{x}{n} + b\right)\right).$$

Up to terms in $\mathcal{O}(e^{-\frac{h_0 k}{n}})$, the functions h , u , w and p defined by

$$h(t, x) = h_0 + f(kx - \omega t),$$

$$u(t, x, z) = \frac{\omega a}{n} e^{\frac{k}{n}(z-h_0)} \cos\left(\frac{kx - \omega t}{n} + b\right),$$

$$w(t, x, z) = -\frac{n}{k} \frac{\partial u}{\partial x},$$

$$p(t, x, z) = g(h_0 - z) - \frac{ga^2 n}{2k} e^{\frac{2k}{n}(z-h_0)} - \frac{gn}{k} e^{\frac{k}{n}(z-h_0)} \cos\left(\frac{kx - \omega t}{n} + b\right) + g \frac{a^2 n}{2k},$$

are analytical solutions of the Euler system (10)-(11) completed by the boundary conditions (4) and (5) iff the dispersion relation (30) holds. At the free surface, the pressure p is such that $p_s = p(t, x, h(t, x)) = p^\alpha(t, x)$ with

$$p^\alpha(t, x) = \frac{ga^2 n}{2k} \left(1 - e^{\frac{2k}{n}(h(t, x) - h_0)}\right). \quad (31)$$

proof 3.7 (Proof of prop. 7) *The proof is similar to the one given in prop. 6 and is not detailed here.*

Remark 13 *When the parameter a is small, surface elevation is close to a sinusoidal function. When $|a|$ is near to $1/e$, the top of the wave is narrower than its bottom. This property is presented in figure 12 (the same structure appear in cnoidal waves). For $|a| = 1/e$, the function f are not differentiable in $(2m + 1)\pi$, $\forall m \in \mathbb{Z}$.*

Remark 14 *In [1], C. J. Amick proved that for any irrotational wave, the angle of inclination with respect to the horizontal must be less than 31.15° . In these analytical solutions, the flow is irrotational and the angle of inclination is less or equal to 45° . But we have here an additional source term given by the pressure at the surface 31 that can justify this inclination.*

3.7 Standing waves

Now we consider the situation of standing waves. Such a situation can occur when two progressive waves of same amplitude travel in opposite direction. Namely, assuming the free surface can be written under the form

$$h(t, x) = a \cos\left(\frac{kx - \omega t}{n}\right) + a \cos\left(\frac{kx + \omega t}{n}\right) + \mathcal{O}(a^2),$$

we have a standing wave given by

$$h(t, x) = 2a \cos\left(\frac{2kx}{n}\right) \cos\left(\frac{2\omega t}{n}\right) + \mathcal{O}(a^2).$$

The results depicted in this paragraph are based on the following remark: for small values of the parameter a , one has the Taylor expansion

$$\text{LambertW}\left(a \cos\left(\frac{kx - \omega t}{n}\right)\right) = a \cos\left(\frac{kx - \omega t}{n}\right) - a^2 \cos^2\left(\frac{kx - \omega t}{n}\right) + \mathcal{O}(a^3).$$

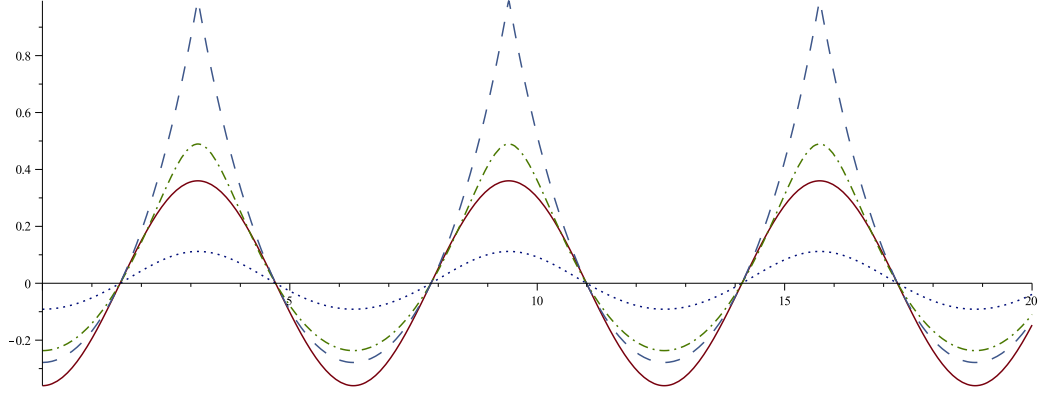


Figure 12: Free surface given by the function $f : x \mapsto -\text{LambertW}(a \cos(x))$ for three value of the parameter a : 0.10 (dot line), 0.30 (dash-dot line), $1/e$ (dash line), and comparison with function $-1/e \cos(x)$ (solid line).

Proposition 8 *Let f the function defined by*

$$f(x) = -\frac{n}{k} \text{LambertW}\left(a \cos\left(\frac{x}{n}\right)\right),$$

where $a, k > 0$ and $n > 0$ are constants.

Up to terms in $\mathcal{O}(e^{-\frac{h_0 k}{n}}, a^2)$, the functions h, u, w and p defined by

$$h(t, x) = h_0 + f(kx - \omega t) + f(kx + \omega t), \quad (32)$$

$$u(t, x, z) = \frac{\omega}{n} e^{\frac{k}{n}(z-h(t,x))} \left(f(kx - \omega t) + f(kx + \omega t) \right), \quad (33)$$

$$w(t, x, z) = -\frac{n}{k} \frac{\partial u}{\partial x}, \quad (34)$$

$$p(t, x, z) = p^a(t) + g(h - z) + \int_z^{h(t,x)} \frac{\partial w}{\partial t} dz, \quad (35)$$

are analytical solutions of the linearized Euler system (20)-(21) completed by the boundary conditions (4), (5) and (12) iff the dispersion relation (30) holds.

Since a Taylor expansion of Eq. (32) gives

$$h(t, x) = h_0 - \frac{2n}{k} \cos\left(\frac{kx}{n}\right) \cos\left(\frac{\omega t}{n}\right) + \mathcal{O}\left(\frac{a^2}{k}\right),$$

the proposed solution corresponds, up to terms in $\mathcal{O}\left(\frac{a^2}{k}\right)$, to a standing wave.

Corollary 1 *Considering a slightly modified version of the solution proposed in*

prop. 8 where (32)-(35) is replaced by

$$\begin{aligned} h(t, x) &= h_0 + f(kx - \omega t) + f(kx + \omega t) + 2\frac{a^2 n}{k} \cos^2\left(\frac{kx}{n}\right), \\ u(t, x, z) &= \frac{\omega}{n} e^{\frac{k}{n}(z-h(t,x))} \left(f(kx - \omega t) + f(kx + \omega t) + 2\frac{a^2 n}{k} \cos^2\left(\frac{kx}{n}\right) \right), \\ w(t, x, z) &= -\frac{n}{k} \frac{\partial u}{\partial x}, \\ p(t, x, z) &= p^a(t) + g(h - z) + \int_z^{h(t,x)} \frac{\partial w}{\partial t} dz, \end{aligned}$$

up to terms in $\mathcal{O}(e^{-\frac{h_0 k}{n}}, a^3)$, the functions h , u , w and p are analytical solutions of the linearized Euler system (20)-(21) completed by the boundary conditions (4), (5) and (12) iff the dispersion relation (30) holds.

proof 3.8 (Proof of prop. 8) The proof relies on very simple computations similar to those performed in the proof of props. 6,7.

Proposition 8 and corollary 1, valid for the linearized Euler system, can be easily extend to the Euler system (10)-(11) completed by the boundary conditions (4), (5) and (12).

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