Bilinear quantum systems on compact graphs: well-posedness and global exact controllability

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Abstract

In the present work, we study the well-posedness and the controllability of the bilinear Schrödinger equation on compact graphs. In particular, we consider the (BSE) $i\partial_t\psi = A\psi + u(t)B\psi$ in $L^2(\mathscr{G},\mathbb{C})$ where $\mathscr{G}$ is a compact graph. The operator A is a self-adjoint Laplacian, B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ with $T>0$. We study interpolation properties of the spaces $D(|A|^{s/2})$ for $s>0$, which lead to the well-posedness of the equation in $D(|A|^{s/2})$ with suitable $s \geq 3$. In such spaces, we attain the global exact controllability of the (BSE) and we provide examples of the main results involving star graphs and tadpole graphs.

1 Introduction

In this paper, we study the evolution of a particle confined in a compact graph type structure $\mathscr{G}$ and subjected to an external field.

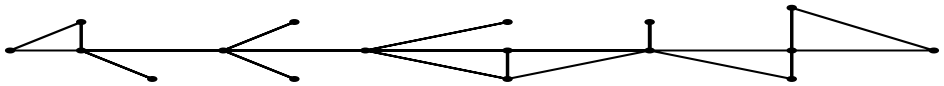


Figure 1: Example of compact graph.

Its dynamics is modeled by the bilinear Schrödinger equation in the Hilbert space $L^2(\mathscr{G},\mathbb{C})$

$$\text{(BSE)} \qquad \begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)B\psi(t), & t\in(0,T), \\ \psi(0)=\psi_0, & T>0. \end{cases}$$

The term $u(t)B$ represents the control field, where the bounded symmetric operator B describes the action of the field and $u \in L^2((0,T),\mathbb{R})$ its intensity. The operator $A=-\Delta$ is a self-adjoint Laplacian equipped with suitable boundary conditions (presented in Section 2). When the (BSE) is well-posed, we call Γ_t^u the unitary propagator generated by $A+u(t)B$.

A natural question of practical implications is whether, given a couple of states, there exists u steering the system from the first state to the second one. In other words, when the (BSE) is *exactly controllable*.

The exact controllability of infinite-dimensional quantum systems is in general a delicate matter. When we consider the linear Schrödinger equation, the controllability and observability properties are reciprocally dual. Different results were developed by addressing the problem directly or by duality (see for instance [Bur91, Leb92, Lio83, LT92]). For results on networks, we refer to [AG18, AJ04, AJK05, MAN17] and to [AN15, DZ06]. Regarding inverse problems, we cite [ALM10, Bel04] for the boundary control approach and [BCV11, IPR12] for uniqueness and stability results via Carleman estimates.

In the current manuscript, we study the *global exact controllability* of the bilinear Schrödinger equation on compact graphs. Before providing further details on the work, we underline that the exact controllability of bilinear quantum systems can not be proved with the classical techniques adopted for

the linear Schrödinger equation. Indeed, the bilinear Schrödinger equation is not exactly controllable in the Hilbert space where it is defined when B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ with $T > 0$; even though it is well-posed in such space. We refer to [BMS82] by Ball, Mardsen and Slemrod where the well-posedness and the non-controllability of the equation are proved.

To overcome this non-controllability result, different works were developed by addressing the controllability of the bilinear Schrödinger equation in suitable sub-spaces of $D(A)$ when $\mathscr{G} = (0,1)$. Let

$$D(-\Delta_D) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})), \qquad -\Delta_D \psi := -\Delta\psi, \quad \forall \psi \in D(-\Delta_D).$$

This idea, introduced by Beauchard in [Bea05], has been mostly popularized by Beauchard and Laurent with [BL10]. In this work, they prove the *well-posedness* and the *local exact controllability* of the equation in $H^s_{(0)} := D(|\Delta_D|^{s/2})$ for $s = 3$ when B is a suitable multiplication operator.
In [Mor14], Morancey proves the *simultaneous local exact controllability* of two or three (BSE) in $H^3_{(0)}$. Such outcome is extended to a global controllability for any finite number (BSE) in $H^4_{(0)}$ by Morancey and Nersesyan in [MN15]. Both the outcomes are provided for suitable multiplicaton operators B.
In [Duc18b], the author ensures the *simultaneous global exact controllability in projection* of infinite (BSE) in $H^3_{(0)}$, while he exhibits the *global exact controllability* of the equation via explicit controls and explicit times in [Duc19]. The result are valid for suitable bounded symmetric operators B.

Even though the global exact controllability of the (BSE) on $\mathscr{G} = (0,1)$ is well-established, the result on generic compact graphs is still an open problem. In the following subsection, we present the main obstacles appearing when we try to adopt the techniques developed in [BL10, Mor14, MN15, Duc18b, Duc19] in the framework of the bilinear Schrödinger equation on compact graphs.

1.1 Novelties of the work

Let $(\lambda_k)_{k\in\mathbb{N}^*}$ be the ordered sequence of eigenvalues of A and $(\phi_k)_{k\in\mathbb{N}^*}$ be a Hilbert basis of $L^2(\mathscr{G},\mathbb{C})$ made by corresponding eigenfunctions. To understand the main difference between studying the controllability of the (BSE) on bounded intervals $\mathscr{G} = (0,1)$ and on generic $\mathscr{G}$, we notice that the spectral gap

$$\inf_{k\in\mathbb{N}^*} |\lambda_{k+1} - \lambda_k| > 0, \tag{1}$$

is only guaranteed when $\mathscr{G} = (0,1)$. This hypothesis is crucial for the techniques developed in the works [BL10, Duc18b, Duc19, Mor14], which can not be directly applied in the current framework. Indeed, the global exact controllability is usually proved by extending a local result, which follows from the solvability of a suitable "moment problem" in ℓ^2. Such result is usually attained by using Ingham's type theorems that are valid when (1) is verified (see for instance [KL05, Theorem 4.3]).

To overcome this problem, we develop a new technique in Appendix B leading to the solvability of this moment problem in suitable sub-spaces of ℓ^2 when the following assumptions are verified. The first condition demands the existence $\mathcal{M} \in \mathbb{N}^*$ and $C > 0$ so that

$$\inf_{k\in\mathbb{N}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| > C. \tag{2}$$

Such hypothesis is always guaranteed when $\mathscr{G}$ is a compact graph (see Remark 2.2 for further details), The second assumption consists in the existence of $C > 0$ and $\tilde{d} \geq 0$ such that

$$|\lambda_{k+1} - \lambda_k| \geq C k^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}^*. \tag{3}$$

The solvability of the moment problem in sub-spaces of ℓ^2 forces us to study the exact controllability in $D(|A|^{\frac{s}{2}})$ with $s \geq 3$ that is not always integer (contrary to [BL10, Duc18b, Duc19, Mor14]). The characterization of these spaces is still unknown when A is a self-adjoint Laplacian on a compact graph $\mathscr{G}$. As a consequence, we provide different *interpolation properties* in order to understand which boundary conditions define the spaces $D(|A|^{\frac{s}{2}})$ when $s \in \mathbb{R}^+ \setminus \mathbb{N}^*$. An example is the following.

Let $\{e_k\}_{k\leq N}$ be the N edges composing $\mathscr{G}$ and $H^s = \prod_{k=1}^N H^s(e_k, \mathbb{C})$ with $s > 0$. Let $D(A)$ be the subspace of those functions in H^2 satisfying Dirichlet or Neumann boundary conditions in the external vertices of $\mathscr{G}$ and Neumann-Kirchhoff boundary conditions (defined in Section 2) in the internal vertices.

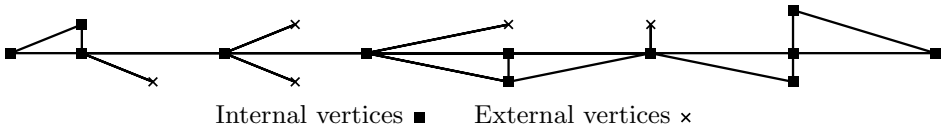


Figure 2: Internal and external vertices in a compact graph.

In this framework, for $H^s_{\mathscr{G}} := D(|A|^{\frac{s}{2}})$ with $s > 0$, we have

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2} \qquad \forall s_1 \in \mathbb{N},\ s_2 \in [0, 1/2).$$

This identity holds under generic assumptions on the problem, but stronger outcomes can be guaranteed by imposing more restrictive conditions. We provide the complete result in Proposition 3.2.

Thanks to the interpolation properties, we attain in Section 3 the *well-posedness* of the bilinear Schrödinger equation in $H^s_{\mathscr{G}}$ with specific $s \geq 3$. In such spaces, we prove that the *global exact controllability* can be ensured for $u \in L^2((0,T),\mathbb{R})$ with $T > 0$ when the identities (2) and (3) are satisfied with suitable parameter $\tilde{d}$. The complete result is provided in Theorem 2.4.

After having provided the abstact controllability result, another difficulty appears when we try to verify if a specific bilinear quantum problem is globally exactly controllable. Indeed, proving the validity of the identity (3) is not an easy task as the spectrum of A is usually not explicit and, more the structure of the graph is complicated, more the spectral behaviour is difficult to characterize. To this purpose, we provide some spectral results in Appendix A based on the Roth's Theorem [Rot56] and outcomes from [DZ06, BK13]. This analysis leads to validity of the identity (3) for the following types of graphs.

Figure 3: Respectively a star graph, a double-ring graph, a tadpole graph and a two-tails tadpole graph.

The spectral gap (3) is valid when the lengths of the edges of the graph $\{L_k\}_{k\leq N}$ are such that the ratios L_k/L_j are *algebraic irrational numbers*. The result is guaranteed independently from the choice of boundary conditions in $D(A)$ in the external vertices, which can be Neumann or Dirichlet type boundary conditions. This outcome leads to the controllability of the following explicit bilinear quantum systems.

Let $\mathscr{G}$ be a *star graph* composed by $N \in \mathbb{N}^*$ edges $\{e_k\}_{k\leq N}$ connected in an internal vertex v. Each e_k is parametrized with a coordinate going from 0 to the length of the edge L_k in the vertex v.

Figure 4: The figure shows the parametrization of a star graph with 4 edges.

Definition 1.1. Let $N \in \mathbb{N}^*$. We denote $\mathcal{AL}(N)$ the set of elements $\{L_j\}_{j\leq N} \in (\mathbb{R}^+)^N$ so that: $\{1, \{L_j\}_{j\leq N}\}$ are linearly independent over $\mathbb{Q}$ and all the ratios L_k/L_j are algebraic irrational numbers.

Theorem 1.2. *Let $\mathscr{G}$ be a four edges star graph. Let $D(A)$ be the set of functions $f \in H^2$ so that:*

- *$f(\tilde{v}) = 0$ for every external vertex $\tilde{v}$ of $\mathscr{G}$ (Dirichlet boundary conditions);*
- *f is continuous at the vertex v and $\sum_{e\ni v} \frac{\partial f}{\partial x_e}(v) = 0$ (Neumann-Kirchhoff boundary conditions).*

Let the control field B be such that, for every $\psi \in L^2(\mathscr{G},\mathbb{C})$,

$$\begin{cases} B\psi(x) = (x - L_1)^4\psi(x), & x \in e_1, \\ B\psi(x) = 0, & x \in \mathscr{G} \setminus e_1. \end{cases}$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^4$ countable such that, for every $\{L_j\}_{j\leq 4} \in \mathcal{AL}(4)\setminus\mathcal{C}$, the (BSE) is globally exactly controllable in

$$H_{\mathscr{G}}^{4+\epsilon} \qquad \epsilon > 0.$$

In other words, for every $\psi^1, \psi^2 \in H_{\mathscr{G}}^{4+\epsilon}$ such that $\|\psi^1\|_{L^2} = \|\psi^2\|_{L^2}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\Gamma_T^u \psi^1 = \psi^2$.

In Theorem 1.2, we notice an interesting phenomenon. The controllability holds even if the control field only acts on one edge of the graph. It is due to the choice of the lengths, which are linearly independent over $\mathbb{Q}$ and such that all the ratios L_k/L_j are algebraic irrational numbers.

Another application of Theorem 2.4 is the following. Let $\mathscr{G}$ be a *tadpole graph* composed by two edges $\{e_1, e_2\}$ connected in an internal vertex v. The edge e_1 is self-closing and parametrized in the clockwise direction with a coordinate going from 0 to L_1 (the length e_1). On the "tail" e_2, we consider a coordinate going from 0 in the to L_2 and we associate the 0 to the external vertex $\tilde{v}$.

Figure 5: The parametrization of the tadpole graph.

Theorem 1.3. *Let $\mathscr{G}$ be a tadpole graph. Let $D(A)$ be the set of functions $f \in H^2$ such that:*

- *$f(\tilde{v}) = 0$ (Dirichlet boundary conditions);*
- *f is continuous at v and $\sum_{e\ni v} \frac{\partial f}{\partial x_e}(v) = 0$ (Neumann-Kirchhoff boundary conditions).*

Let $\mu_1(x) := \sin\left(\frac{2\pi}{L_1}x\right) + x(x - L_1)$, and $\mu_2(x) := x^2 - (2L_1 + 2L_2)x + L_2^2 + 2L_1L_2$. Let B be such that

$$\begin{cases} B\psi(x) = \mu_1(x)\psi(x), & x \in e_1, \\ B\psi(x) = \mu_2(x), & x \in e_2, \end{cases}$$

for every $\psi \in L^2(\mathscr{G},\mathbb{C})$. There exists $\mathcal{C} \subset (\mathbb{R}^+)^2$ countable so that, for each $\{L_1, L_2\} \in \mathcal{AL}(2) \setminus \mathcal{C}$, the (BSE) is globally exactly controllable in $H_{\mathscr{G}}^{4+\epsilon}$ with $\epsilon > 0$.

Another application of Theorem 2.4 is provided by Corollary 2.7 which considers $\mathscr{G} = \{I_j\}_{j\leq N}$ a set of $N \in \mathbb{N}^*$ unconnected intervals. We show that when $\{L_j\}_{j\leq N} \in \mathcal{AL}(N)$, the controllability of the (BSE) can be ensured in the space

$$\prod_{j\leq N} H_{I_j}^{3+\epsilon}.$$

In other words, we ensure the controllability of vectors of functions $\{\psi_j\}_{j\leq N}$ such that $\psi_j \in H_{I_j}^{3+\epsilon}$ for every $j \leq N$. The result differs from the *simultaneous controllability* provided by [MN15] (also by [Duc18b]) that takes in account vectors of functions belonging to the same space.

1.2 Scheme of the work

In Section 2, we present the main results of the work. The *global exact controllability* of the (BSE) is ensured in Theorem 2.4. Theorem 2.5 shows types of graphs satisfying the hypothesis of Theorem 2.4. In Corollary 2.7, we provide an application of Theorem 2.4. In Section 3, Proposition 3.1 attains the *well-posedness* of the (BSE) by using the *interpolation properties* of the spaces $H_{\mathscr{G}}^s$ for $s > 0$ provided by Proposition 3.2. Section 4 exhibits the proof of Theorem 2.4 which follows from the local exact controllability ensured in Proposition 4.1. Theorem 2.5 and Corollary 2.7 are proved in Section 5, while Theorem 1.2 and Theorem 1.3 in Section 6. In Appendix A, we provide some spectral results for the problem, while we study different *moment problems* in Appendix B. In Appendix C, we adapt the perturbation theory techniques developed in [Duc18b, *Appendix B*].

2 Main results

A compact graph $\mathscr{G}$ is a structure composed by $N \in \mathbb{N}^*$ edges $\{e_j\}_{j\leq N}$ of finite lengths $\{L_j\}_{j\leq N}$ connecting $M \in \mathbb{N}^*$ vertices $\{v_j\}_{j\leq M}$. For each $j \leq M$, we denote

$$N(v_j) := \big\{l \in \{1, ..., N\} \mid v_j \in e_l\big\}, \qquad n(v_j) := |N(v_j)|. \tag{4}$$

An edge connected in both sides with a unique vertex is called self-closing or loop. Given a couple of vertices v and $\widetilde{v}$ of $\mathscr{G}$, it is admitted having two or more edges connecting v with $\widetilde{v}$. We respectively call V_e and V_i the external and the internal vertices of $\mathscr{G}$ (see also Figure 2), *i.e.*

$$V_e := \big\{v \in \{v_j\}_{j\leq M} \mid \exists! e \in \{e_j\}_{j\leq N} : v \in e\big\}, \qquad V_i := \{v_j\}_{j\leq M} \setminus V_e. \tag{5}$$

We study graphs equipped with a metric, which parametrizes each edge e_j with a coordinate going from 0 to its length L_j. A graph is compact when it is composed by a finite number of vertices and edges of finite length. We consider functions $f := (f^1, ..., f^N) : \mathscr{G} \to \mathbb{C}$ so that $f^j : e_j \to \mathbb{C}$ for every $j \leq N$ and

$$\mathscr{H} = L^2(\mathscr{G}, \mathbb{C}) = \prod_{j\leq N} L^2(e_j, \mathbb{C}).$$

The Hilbert space $\mathscr{H}$ is equipped with the norm $\|\cdot\|_{L^2}$ and the scalar product

$$\langle \psi, \varphi\rangle_{L^2} := \sum_{j\leq N} \langle \psi^j, \varphi^j\rangle_{L^2(e_j,\mathbb{C})} = \sum_{j\leq N} \int_{e_j} \overline{\psi^j}(x)\varphi^j(x)dx, \qquad \forall \psi, \varphi \in \mathscr{H}.$$

By referring to [BK13], we denote a graph $\mathscr{G}$ as quantum graph when a self adjoint Laplacian A is defined on it. When we introduce a quantum graph $\mathscr{G}$, we are not only introducing the graph $\mathscr{G}$, but also a self-adjoint Laplacian A with domain $D(A)$ characterized by the following boundary conditions.

Boundary conditions. Let $\mathscr{G}$ be a quantum compact graph.

- $(\mathcal{NK})$ A vertex $v \in V_i$ is equipped with Neumann-Kirchhoff boundary conditions when every $f \in D(A)$ is continuous at v and $\sum_{e\ni v} \frac{\partial f}{\partial x_e}(v) = 0$ (the derivatives have ingoing directions in v).
- $(\mathcal{D})$ A vertex $v \in V_e$ is equipped with Dirichlet boundary conditions when $f(v) = 0$ for every $f \in D(A)$.
- $(\mathcal{N})$ A vertex $v \in V_e$ is equipped with Neumann boundary conditions when $\partial_x f(v) = 0$ for every $f \in D(A)$.

Notations. Let $\mathscr{G}$ be a quantum compact graph.

- The graph $\mathscr{G}$ is said to be equipped with $(\mathcal{D})$ (or $(\mathcal{N})$) when every $v \in V_e$ is equipped with $(\mathcal{D})$ (or $(\mathcal{N})$) and every $v \in V_i$ with $(\mathcal{NK})$.
- The graph $\mathscr{G}$ is said to be equipped with $(\mathcal{D}/\mathcal{N})$ when every $v \in V_e$ is equipped with $(\mathcal{D})$ or $(\mathcal{N})$, while every $v \in V_i$ with $(\mathcal{NK})$.

In our framework, the Laplacian A admits purely discrete spectrum (see [Kuc04, *Theorem* 18]). Let

$$(\lambda_k)_{k\in\mathbb{N}^*}, \qquad \Phi := (\phi_k)_{k\in\mathbb{N}^*} \tag{6}$$

respectively be the ordered sequence of eigenvalues of A and a Hilbert basis of $\mathscr{H}$ made by corresponding eigenfunctions. Let $\phi_j(t) = e^{-i\lambda_j t}\phi_j$ and $[r]$ the entire part of $r \in \mathbb{R}$. For $s > 0$, we define the spaces

$$H^s_{\mathcal{NK}} = \Big\{\psi \in H^s \mid \partial_x^n \psi \in C^0(\mathscr{G}, \mathbb{C}),\ \sum_{e\in N(v)} \partial^m_{x_e} f(v) = 0,\ \forall n \in 2\mathbb{N},\ m \in 2\mathbb{N}+1\ n, m < [s+1/2],\ v \in V_i\Big\},$$

$$H^s = \prod_{k=1}^{N} H(e_k, \mathbb{C}), \qquad H^s_{\mathscr{G}} := D(A^{s/2}), \qquad h^s(\mathbb{C}) := \Big\{(a_k)_{k\in\mathbb{N}^*} \subset \mathbb{C} \mid \sum_{k\in\mathbb{N}^*} |k^s a_k|^2 < \infty\Big\}.$$

We respectively equip the spaces $H^s_{\mathscr{G}}$ and $h^s(\mathbb{C})$ with the norms

$$\|\cdot\|_{(s)} := \Big(\sum_{k\in\mathbb{N}^*} |k^s \langle \cdot, \phi_k\rangle_{L^2}|^2\Big)^{\frac{1}{2}}, \qquad \|\mathbf{x}\|_{(s)} := \Big(\sum_{k\in\mathbb{N}^*} |k^s x_k|^2\Big)^{\frac{1}{2}} \qquad \forall \mathbf{x} = (x_k)_{k\in\mathbb{N}^*}.$$

Remark. *Let a vertex v be either connected with one side of the two edges e and $\widetilde{e}$, or with one edge in both sides. The Neumann-Kirchhoff boundary conditions valid for $f \in H^2_{\mathscr{G}}$ (or $H^2_{\mathcal{NK}}$) do not only imply the continuity of f at v but also the continuity of its derivative. For this reason, e and $\widetilde{e}$ can be considered as a unique edge long $|e| + |\widetilde{e}|$, when we consider the spaces $H^s_{\mathscr{G}}$ and $H^s_{\mathcal{NK}}$ with any $s > 0$. This is not true for the spaces H^s, where the continuity of the functions at the vertices is not guaranteed.*

Remark 2.1. *If $0 \notin \sigma(A)$ (the spectrum of A), then $\|\cdot\|_{(s)} \asymp \||A|^{\frac{s}{2}} \cdot\|_{L^2}$, i.e.*

$$\exists\ C_1, C_2 > 0 \quad : \qquad C_1 \|\cdot\|^2_{(s)} \leq \||A|^{\frac{s}{2}} \cdot\|^2_{L^2} = \sum_{k\in\mathbb{N}^*} |\lambda_k^{\frac{s}{2}} \langle \cdot, \phi_k\rangle_{L^2}|^2 \leq C_2 \|\cdot\|^2_{(s)}.$$

Indeed, from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10], there exist $C_3, C_4 > 0$ such that $C_3 k^2 \leq \lambda_k \leq C_4 k^2$ for every $k \geq 2$ and for $k = 1$ if $\lambda_1 \neq 0$ (see Remark A.4 for further details). If $0 \in \sigma(A)$, then $\lambda_1 = 0$ and there exists $c \in \mathbb{R}$ such that $0 \notin \sigma(A + c)$ and $\|\cdot\|_{(s)} \asymp \||A + c|^{\frac{s}{2}} \cdot\|_{L^2}$.

Remark 2.2. *The relation (2) follows from [DZ06, relation (6.6)], which leads to the existence of $\mathcal{M} \in \mathbb{N}^*$ and $\delta' > 0$ such that $\inf_{k\in\mathbb{N}^*} |\sqrt{\lambda}_{k+\mathcal{M}} - \sqrt{\lambda}_k| > \delta' \mathcal{M}$ and*

$$\inf_{k\in\mathbb{N}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \sqrt{\lambda_{\mathcal{M}+1}} \inf_{k\in\mathbb{N}^*} |\sqrt{\lambda}_{k+\mathcal{M}} - \sqrt{\lambda}_k| > \sqrt{\lambda_{\mathcal{M}+1}}\delta'\mathcal{M}.$$

We define the following assumptions on (A, B). Let $\eta > 0$, $a \geq 0$ and $I := \{(j, k) \in (\mathbb{N}^*)^2 : j \neq k\}$.

Assumptions I (η)**.** The operator B satisfies the following conditions.

1. There exists $C > 0$ such that $|\langle \phi_j, B\phi_1\rangle_{L^2}| \geq \frac{C}{j^{2+\eta}}$ for every $j \in \mathbb{N}^*$.
2. For $(j, k), (l, m) \in I$ such that $(j, k) \neq (l, m)$ and such that $\lambda_j - \lambda_k = \lambda_l - \lambda_m$, we have

$$\langle \phi_j, B\phi_j\rangle_{L^2} - \langle \phi_k, B\phi_k\rangle_{L^2} \neq \langle \phi_l, B\phi_l\rangle_{L^2} - \langle \phi_m, B\phi_m\rangle_{L^2}.$$

The first point of Assumptions I quantifies how much the control operator B "mixes" eigenstates of A. The second is necessary to decouple eigenvalue resonances appearing in the proof of the global approximate controllability, which is an important part of the proof of the global exact controllability.

Assumptions II (η, a)**.** Let $Ran(B|_{H^2_{\mathscr{G}}}) \subseteq H^2_{\mathscr{G}}$ and one of the following assumptions be satisfied.

1. When $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$ and $a + \eta \in (0, 3/2)$, there exists $d \in [\max\{a + \eta, 1\}, 3/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}}$.
2. When $\mathscr{G}$ is equipped with $(\mathcal{N})$ and $a + \eta \in (0, 7/2)$, there exist $d \in [\max\{a + \eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ and $Ran(B|_{H^{d_1}_{\mathcal{NK}}}) \subseteq H^{d_1}_{\mathcal{NK}}$.
3. When $\mathscr{G}$ is equipped with $(\mathcal{D})$ and $a + \eta \in (0, 5/2)$, there exists $d \in [\max\{a + \eta, 1\}, 5/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$. If $a + \eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that $Ran(B|_{H^{d_1}}) \subseteq H^{d_1}$.

The validity of Assumptions II not only tells us that B stabilizes $H^2_{\mathscr{G}}$, but also the following fact. The action of B on suitable spaces $H^s_{\mathscr{G}}$ with $s > 2$ preserves the regularity of the functions, even though some boundary conditions are lost. The choice of the parameter s is done according to the boundary conditions defined on the graph and to the values of the inputs η and a. From now on, we omit η and a from the notations of Assumptions I and Assumptions II when these parameters are not relevant.

Definition 2.3. The (BSE) is said to be globally exactly controllable in $H^s_{\mathscr{G}}$ with $s \geq 3$ when, for every $\psi^1, \psi^2 \in H^s_{\mathscr{G}}$ such that $\|\psi^1\|_{L^2} = \|\psi^2\|_{L^2}$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that $\Gamma^u_T \psi^1 = \psi^2$.

Theorem 2.4. *Let $\mathscr{G}$ be a compact quantum graph. Let the spectral gaps (2) and (3) be guaranteed for $\tilde{d} > 0$ and $\mathcal{M} \in \mathbb{N}^*$. If the couple (A, B) satisfies Assumptions I(η) and Assumptions II$(\eta, \tilde{d})$ for some $\eta > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II.*

Proof. See Section 4. □

In the next theorem, we provide the validity of the spectral hypothesis of Theorem 2.4 when $\mathscr{G}$ is one of the graphs introduced in Figure 3. The provided result leads to Theorem 1.2 and Theorem 1.3.

Theorem 2.5. *Let $\{L_j\}_{j\leq N} \in \mathcal{AL}(N)$. Let $\mathscr{G}$ be either a tadpole, a two-tails tadpole, a double-rings graph or a star graph with $N \leq 4$ edges. Let $\mathscr{G}$ be equipped with $(\mathcal{D}/\mathcal{N})$. If the couple (A, B) satisfies Assumptions I(η) and Assumptions II(η, ϵ) for some $\eta, \epsilon > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II.*

Proof. See Section 5. □

Remark 2.6. *Let $\{L_j\}_{j\leq 2} \in \mathcal{AL}(2)$. As explained in Remark 5.1, Theorem 2.5 is also valid when $\mathscr{G}$ is a two-tails tadpole or a star graph with 3 or 4 edges. For the tadpole graphs the claim is valid when the tails are long L_2, while the head L_1. The property is valid in the cases of the star graphs with $N = 3$ (resp. $N = 4$) when two edges are long L_1 and the remaining one (resp. ones) L_2.*

In the following corollary, we provide another result based on Theorem 2.4. We refer to Remark 6.1 for an explicit control field B ensuring the controllability.

Corollary 2.7. *Let $\mathscr{G} = \{I_j\}_{j\leq N}$ be a set of bounded unconnected intervals. Let the couple (A, B) satisfy Assumptions I(η) and Assumptions II(η, ϵ) for some $\eta, \epsilon > 0$. If $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$, then the (BSE) is globally exactly controllable in $\prod_{j\leq N} H^s_{I_j}$ with $s = d + 2$ and d from Assumptions II.*

Proof. See Section 5. □

Remark. *The size of the time in Theorem 2.4, Theorem 2.5 and Corollary 2.7 depends on the initial and the final states of the dynamics. This is due to the global approximate controllability result adopted in the proof of Theorem 2.4. Nevertheless, the local exact controllability (preseted in Proposition 4.1), is valid for any $T > 0$ when the hypotheses of Theorem 2.5 or Corollary 2.7 are satisfied (see Remark 5.2).*

3 Well-posedness and interpolation properties of the spaces $H^s_{\mathscr{G}}$

In the current section, we provide the *well-posedness* of the (BSE).

Proposition 3.1. *Let $\mathscr{G}$ a compact quantum graph. Let (A, B) satisfy Assumptions II($\eta, \tilde{d}$) with $\eta > 0$ and $\tilde{d} \geq 0$. Let $\psi^0 \in H^{2+d}_{\mathscr{G}}$ with d introduced in Assumptions II and $u \in L^2((0, T), \mathbb{R})$. There exists a unique mild solution of (BSE) in $H^{2+d}_{\mathscr{G}}$, i.e. $\psi \in C_0([0, T], H^{2+d}_{\mathscr{G}})$ such that for every $t \in [0, T]$,*

$$\psi(t, x) = e^{-iAt}\psi^0(x) - i\int_0^t e^{-iA(t-s)}u(s)B\psi(s, x)ds. \tag{7}$$

Moreover, there exists $C = C(T, B, u) > 0$ so that $\|\psi\|_{C^0([0,T],H^{2+d}_{\mathscr{G}})} \leq C\|\psi^0\|_{H^{2+d}_{\mathscr{G}}}$, while $\|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2}$ for every $t \in [0, T]$ and $\psi_0 \in H^{2+d}_{\mathscr{G}}$.

Now, we present some *interpolation properties* for the spaces $H^s_{\mathscr{G}}$ with $s > 0$. The proof of Proposition 3.1 is provided in the end of the section.

Proposition 3.2.
1) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$, then*

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2} \quad \text{for} \quad s_1 \in \mathbb{N},\ s_2 \in [0, 1/2).$$

2) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{N})$, then*

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2}_{\mathcal{NK}} \quad \text{for} \quad s_1 \in 2\mathbb{N},\ s_2 \in [0, 3/2).$$

3) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D})$, then*

$$H^{s_1+s_2+1}_{\mathscr{G}} = H^{s_1+1}_{\mathscr{G}} \cap H^{s_1+s_2+1}_{\mathcal{NK}} \quad \text{for} \quad s_1 \in 2\mathbb{N},\ s_2 \in [0, 3/2).$$

Proof. **1) (a) Bounded intervals.** Let $\mathscr{G} = I^{\mathcal{N}}$ be an interval equipped with $(\mathcal{N})$ on the external vertices and $\mathscr{G} = I^{\mathcal{D}}$ be an interval equipped with $(\mathcal{D})$ on the external vertices. From [Gru16, $Definition$ 2.1], for every $s_1 \in 2\mathbb{N}$, $s_2 \in [0, 3/2)$ and $s_3 \in [0, 1/2)$, we have

$$H^{s_1+s_2}_{I^{\mathcal{N}}} = H^{s_1}_{I^{\mathcal{N}}} \cap H^{s_1+s_2}(I^{\mathcal{N}}, \mathbb{C}), \quad H^{s_1+s_2+1}_{I^{\mathcal{D}}} = H^{s_1+1}_{I^{\mathcal{D}}} \cap H^{s_1+s_2+1}(I^{\mathcal{D}}, \mathbb{C}), \quad H^{s_3}_{I^{\mathcal{D}}} = H^{s_3}(I^{\mathcal{D}}, \mathbb{C}). \tag{8}$$

Let $\mathscr{G} = I^{\mathcal{M}}$ be an interval equipped with $(\mathcal{D})$ on one external vertex and $(\mathcal{N})$ on the other. We prove

$$H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1}_{I^{\mathcal{M}}} \cap H^{s_1+s_2}(I^{\mathcal{M}}, \mathbb{C}), \qquad \forall s_1 \in \mathbb{N},\ s_2 \in [0, 1/2). \tag{9}$$

Let $\widetilde{I}^{\mathcal{D}}$ and $\widetilde{I}^{\mathcal{N}}$ respectively be two sub-intervals of $I^{\mathcal{M}}$ of length $\frac{3}{4}|I^{\mathcal{M}}|$. The interval $\widetilde{I}^{\mathcal{D}}$ contains one external vertex of $I^{\mathcal{M}}$, while $\widetilde{I}^{\mathcal{N}}$ contains the other. We consider both the edges as quantum graphs: $\widetilde{I}^{\mathcal{D}}$ is equipped in both the external vertices with $(\mathcal{D})$ and $\widetilde{I}^{\mathcal{N}}$ is equipped with $(\mathcal{N})$. Let χ be the partition of the unity so that $\chi(x) = 1$ in $\widetilde{I}$, $\chi(x) = 0$ in $I^{\mathcal{M}} \setminus I^{\mathcal{D}}$ and $\chi(x) \in (0, 1)$ in $I^{\mathcal{D}} \setminus \widetilde{I}$. There holds

$$\psi(x) = \psi^1(x) + \psi^2(x), \quad \text{with} \quad \psi^1 := \chi\psi \in H^2_{I^{\mathcal{D}}}, \quad \psi^2 := (1-\chi)\psi \in H^2_{I^{\mathcal{N}}}$$

and then $H^2_{I^{\mathcal{M}}} = H^2_{\widetilde{I}^{\mathcal{D}}} \times H^2_{\widetilde{I}^{\mathcal{N}}}$. The same is valid for $L^2(I^{\mathcal{M}}, \mathbb{C})$ and $H^s(I^{\mathcal{M}}, \mathbb{C})$. Thus, for $s \in (0, 2]$,

$$H^s(I^{\mathcal{M}}, \mathbb{C}) = H^s(\widetilde{I}^{\mathcal{D}}, \mathbb{C}) \times H^s(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), \qquad L^2(I^{\mathcal{M}}, \mathbb{C}) = L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}) \times L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}).$$

Let $[\cdot, \cdot]_\theta$ be the complex interpolation of spaces for $0 < \theta < 1$ defined in [Tri95, $Definition$, $Chapter$ 1.9.2]. From [Tri95, $Chapter$ 1.15.1, $Chapter$ 1.15.3], for $s_1 \in \mathbb{N}$ and $s_2 \in [0, 1/2)$, we have

$$\left[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\right]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}}, \qquad \left[L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\right]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

Thanks to [Tri95, $relation$ (12), $Chapter$ 1.18.1], we have $[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}) \times L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}} \times H^2_{\widetilde{I}^{\mathcal{D}}}]_{s_2/2} = [L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}]_{s_2/2} \times [L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}]_{s_2/2}$, which implies

$$H^{s_2}_{I^{\mathcal{M}}} = \left[L^2(I^{\mathcal{M}}, \mathbb{C}), H^2_{I^{\mathcal{M}}}\right]_{s_2/2} = \left[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\right]_{s_2/2} \times \left[L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\right]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}} \times H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

Equivalently, $H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1+s_2}_{\widetilde{I}^{\mathcal{N}}} \times H^{s_1+s_2}_{\widetilde{I}^{\mathcal{D}}}$ that proves to the identity (9) thanks to the validity of (8).

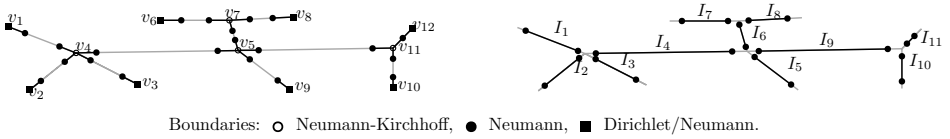
1) (b) Star graphs with equal edges. Let $I^{\mathcal{N}}$ and $I^{\mathcal{M}}$ be two quantum graphs defined on an interval I of length L. We suppose that $I^{\mathcal{N}}$ is equipped with $(\mathcal{N})$, while $I^{\mathcal{M}}$ is equipped with $(\mathcal{D})$ in the external vertex parametrized with 0 and with $(\mathcal{N})$ in the other. We respectively call $A_{\mathcal{N}}$ and $A_{\mathcal{M}}$ the two self-adjoint Laplacians defining $I^{\mathcal{N}}$ and $I^{\mathcal{M}}$. Let $(f^1_j)_{j\in\mathbb{N}^*}$ be a Hilbert basis of $L^2(I, \mathbb{C})$ made by eigenfunctions of $A_{\mathcal{N}}$ and $(f^2_j)_{j\in\mathbb{N}^*}$ a Hilbert basis of $L^2(I, \mathbb{C})$ composed by eigenfunctions of $A_{\mathcal{M}}$
Let $\mathscr{S}$ be a star graph of N edges long L and equipped with $(\mathcal{N})$. The $(\mathcal{N})$ conditions on V_e imply that $\phi_k = (a^1_k \cos(x\sqrt{\lambda_k}), ..., a^N_j \cos(x\sqrt{\lambda_k}))$ with $k \in \mathbb{N}^*$, λ_k the corresponding eigenvalue and $\{a^l_k\}_{l\le N} \subset \mathbb{C}$. The $(\mathcal{NK})$ condition in V_i ensures that $\sin(\sqrt{\lambda_k}L)\sum_{l\le N} a^l_k = 0$ and $a^1_k \cos(\sqrt{\lambda_k}L) = ... = a^N_k \cos\sqrt{\lambda_k}L)$ for every $k \in \mathbb{N}^*$. Each eigenvalue is either of the form $\frac{(n-1)^2\pi^2}{L^2}$, or $\frac{(2n-1)^2\pi^2}{4L^2}$ when $\sum_{l\le N} a^l_k = 0$ with $n \in \mathbb{N}^*$. Hence, for every $k \in \mathbb{N}^*$, there exists $j(k) \in \mathbb{N}^*$ such that

$$\begin{aligned} &\text{either} \quad \phi^l_k = c^l_k f^1_{j(k)} \quad \text{for} \quad c^l_k \in \mathbb{C}, \ |c^l_k| \le 1, \quad \forall l \in \{1, ..., N\}, \\ &\text{or} \quad \phi^l_k = c^l_k f^2_{j(k)} \quad \text{for} \quad c^l_k \in \mathbb{C}, \ |c^l_k| \le 1, \quad \forall l \in \{1, ..., N\}. \end{aligned} \tag{10}$$

In addition, for each $k \in \mathbb{N}^*$ and $m \in \{1, 2\}$, there exist $\widetilde{j}(k) \in \mathbb{N}^*$ and $l \le N$ such that $f^m_k = c^l_{\widetilde{j}(k)}\phi^l_{\widetilde{j}(k)}$ with $c^l_{\widetilde{j}(k)} \in \mathbb{C}$ uniformly bounded in $k \in^*$ and $l \le N$. Thanks to the last identity and to (10),

$$\psi = (\psi^1, ..., \psi^N) \in H^s_{\mathscr{S}} \quad \Longleftrightarrow \quad \psi^l \in H^s_{I^{\mathcal{N}}} \cap H^s_{I^{\mathcal{M}}}, \qquad \forall l \le N. \tag{11}$$

1) (c) Generic graphs. Let $\mathscr{G}$ be equipped with $(\mathcal{D}/\mathcal{N})$ and $\widetilde{L} < \min\{L_k/2 : k \in \{1, ..., N\}\}$. Let $n(v)$ be defined in (4) for every $v \in V_e \cup V_i$. We define the graphs $\widetilde{\mathscr{G}}(v)$ for every $v \in V_i \cup V_e$ and the intervals $\{I_j\}_{j\leq N}$ as follows (see Figure 6 for an explicit example). If $v \in V_i$, then $\widetilde{\mathscr{G}}(v)$ is a star sub-graph of $\mathscr{G}$ equipped with $(\mathcal{N})$ and composed by $n(v)$ edges long $\widetilde{L}$ and connected to the internal vertex v. If $v \in V_e$, then $\widetilde{\mathscr{G}}(v)$ is an interval long $\widetilde{L}$ such that the external vertex v is equipped with the same boundary conditions that v has in $\mathscr{G}$. We impose $(\mathcal{N})$ on the other vertex. For each $v, \hat{v} \in V_e \cup V_i$, the graphs $\widetilde{\mathscr{G}}(v)$ and $\widetilde{\mathscr{G}}(\hat{v})$ have respectively two external vertices w_1 and w_2 lying on the same edge e and such that $w_1 \notin \widetilde{\mathscr{G}}(\hat{v})$. We construct an interval strictly containing w_1 and w_2, strictly contained in e and equipped with $(\mathcal{N})$. We collect those intervals in $\{I_j\}_{j\leq N}$.



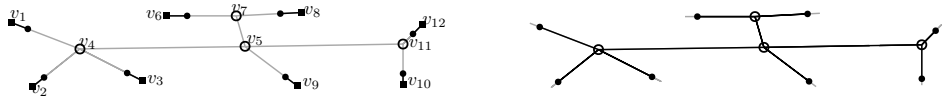
Boundaries: ○ Neumann-Kirchhoff, ● Neumann, ■ Dirichlet/Neumann.

Figure 6: The left and the right figures respectively represent the graphs $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_i\cup V_e}$ and the intervals $\{I_j\}_{j\leq N}$ for a given graph $\mathscr{G}$.

From **1) (a)** and **1) (b)**, for every $v \in V_i \cup V_e$, $j \leq N$, $s_1 \in \mathbb{N}$ and $s_2 \in [0, 1/2)$, we have the validity of the identities $H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)} = H^{s_1}_{\widetilde{\mathscr{G}}(v)} \cap H^{s_1+s_2}(\widetilde{\mathscr{G}}(v), \mathbb{C})$ and $H^{s_1+s_2}_{I_j} = H^{s_1}_{I_j} \cap H^{s_1+s_2}(I_j, \mathbb{C})$. We notice that $G := \{\widetilde{\mathscr{G}}(v_j)\}_{j\leq M} \cup \{I_j\}_{j\leq N}$ covers $\mathscr{G}$. As in **1) (a)**, we see each function of domain $\mathscr{G}$ as a vector of functions of domain G_j with $j \leq M + N$. We use [Tri95, *relation* (12), *Chapter* 1.18.1] as in **1) (a)** and

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2} \quad \text{for} \quad s_1 \in \mathbb{N},\ s_2 \in [0, 1/2).$$

2) Let $\mathscr{G}$ be equipped with $(\mathcal{N})$ and $N_e = |V_e|$. We consider $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ introduced in **1) (c)** and we define $\widetilde{\mathscr{G}}$ from $\mathscr{G}$ as follows (see Figure 7). For every $v \in V_e$, we remove from the edge including v, a section of length $\widetilde{L}/2$ containing v. We equip the new external vertex with $(\mathcal{N})$.



Boundaries: ○ Neumann-Kirchhoff, ● Neumann, ■ Dirichlet/Neumann.

Figure 7: The left and the right figures respectively represent the graphs $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ and $\widetilde{\mathscr{G}}$ for a given graph $\mathscr{G}$.

We call $G' := \{G'_j\}_{j\leq N_e+1} := \{\widetilde{\mathscr{G}}(v)\}_{v\in V_e} \cup \{\widetilde{\mathscr{G}}\}$ which covers $\mathscr{G}$. For every $s_1 \in 2\mathbb{N}$, $s_2 \in [0, 3/2)$, we have $H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)} = H^{s_1}_{\widetilde{\mathscr{G}}(v)} \cap H^{s_1+s_2}$ from (8). The arguments of **1) (a)**, lead to the proof since

$$H^{s_1+s_2}_{\mathcal{NK}} = H^{s_1+s_2}_{\widetilde{\mathscr{G}}} \times \prod_{v\in V_e} H^{s_1+s_2}(\widetilde{\mathscr{G}}(v), \mathbb{C}).$$

3) As in **2)**, the claim follows by considering $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ as intervals equipped with $(\mathcal{D})$ and $\widetilde{\mathscr{G}}$ equipped with $(\mathcal{D})$ in its external vertices. □

Proof of Proposition 3.1. Part of the statement is proved by generalizing the proofs of [BL10, *Lemma* 1] and [BL10, *Proposition* 2], which are designed for the interval $\mathscr{G} = (0, 1)$ equipped with Dirichlet boundary condition. The remaining part consists in exploiting the interpolation properties stated in Proposition 3.2 in order to ensure the well-posedness in higher regularity spaces than $H^3_{\mathscr{G}}$.

1) Preliminaries. Let $T>0$ and the function f be such that $f(s)\in H^{2+d}\cap H^{1+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$ and $t\in(0,T)$. We introduce

$$G(\cdot):=\int_0^{(\cdot)} e^{iA\tau}f(\tau)d\tau.$$

In the first part of the proof, we prove that $G\in C^0([0,T],H^{2+d}_{\mathscr{G}})$ by ensuring the existence of $C(T)>0$ uniformly bounded for T lying on bounded intervals such that

$$\|G\|_{L^\infty((0,T),H^{2+d}_{\mathscr{G}})}\leq C(T)\|f\|_{L^2((0,T),H^{2+d})}.$$

1) (a) Assumptions II.1 . Let $f(s)\in H^3\cap H^2_{\mathscr{G}}$ for almost every $s\in(0,t)$, $t\in(0,T)$ and $f(s)=(f^1(s),...,f^N(s))$. We prove that $G\in C^0([0,T],H^3_{\mathscr{G}})$. The definition of $G(t)$ implies

$$G(t)=\sum_{k=1}^{\infty}\phi_k\int_0^t e^{i\lambda_k s}\langle\phi_k,f(s)\rangle_{L^2}ds,\qquad \|G(t)\|_{(3)}=\Big(\sum_{k\in\mathbb{N}^*}\Big|k^3\int_0^t e^{i\lambda_k s}\langle\phi_k,f(s)\rangle_{L^2}ds\Big|^2\Big)^{\frac12}. \tag{12}$$

We estimate $\langle\phi_k,f(s,\cdot)\rangle_{L^2}$ for each $k\in\mathbb{N}^*$ and $s\in(0,t)$. We suppose that $\lambda_1\neq0$. Let $\partial_x f(s)=(\partial_x f^1(s),...,\partial_x f^N(s)$ be the derivative of $f(s)$ and $P(\phi_k)=(P(\phi_k^1),...,P(\phi_k^N))$ be the primitive of ϕ_k such that $P(\phi_k)=-\frac{1}{\lambda_k}\partial_x\phi_k$. We call ∂e the two points composing the boundaries of an edge e. For every $v\in V_e$, $\tilde v\in V_i$ and $j\in N(\tilde v)$, there exist $a(v),a^j(\tilde v)\in\{-1,+1\}$ such that

$$\langle\phi_k,f(s)\rangle_{L^2}=\frac{1}{\lambda_k^2}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dy+\frac{1}{\lambda_k^2}\sum_{v\in V_i\cup V_e}\sum_{j\in N(v)}a^j(v)\partial_x\phi_k^j(v)\partial_x^2 f^j(s,v). \tag{13}$$

From Remark *A*.4, there exist $C_1>0$ such that $\lambda_k^{-2}\leq C_1k^{-4}$ for every $k\in\mathbb{N}^*$ and

$$\begin{aligned}\Big|k^3\int_0^t e^{i\lambda_k s}\langle\phi_k,f(s)\rangle_{L^2}ds\Big|\leq\frac{C_1}{k}\Bigg(&\sum_{v\in V_i\cup V_e}\sum_{j\in N(v)}\Big|\partial_x\phi_k^j(v)\int_0^t e^{i\lambda_k s}\partial_x^2 f^j(s,v)ds\Big|\\&+\Big|\int_0^t e^{i\lambda_k s}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dyds\Big|\Bigg).\end{aligned} \tag{14}$$

Remark 3.3. *We point out that* $A'\lambda_k^{-1/2}\partial_x\phi_k=\lambda_k\lambda_k^{-1/2}\partial_x\phi_k$ *for every* $k\in\mathbb{N}^*$, *where* $A'=-\Delta$ *is a self-adjoint Laplacian with compact resolvent. Thus,* $\|\lambda_k^{-1/2}\partial_x\phi_k\|^2_{L^2}=\langle\lambda_k^{-1/2}\partial_x\phi_k,\lambda_k^{-1/2}\partial_x\phi_k\rangle_{L^2}=\langle\phi_k,\lambda_k^{-1}A\phi_k\rangle_{L^2}=1$ *and then* $\big(\lambda_k^{-1/2}\partial_x\phi_k\big)_{k\in\mathbb{N}^*}$ *is a Hilbert basis of* $\mathscr{H}$.

Let $\mathbf{a^l}=(a_k^l)_{k\in\mathbb{N}^*},\mathbf{b^l}=(b_k^l)_{k\in\mathbb{N}^*}\subset\mathbb{C}$ for $l\leq N$ be so that $\phi_k^l(x)=a_k^l\cos(\sqrt{\lambda_k}x)+b_k^l\sin(\sqrt{\lambda_k}x)$ and $-a_k^l\sin(\sqrt{\lambda_k}x)+b_k^l\cos(\sqrt{\lambda_k}x)=\lambda_k^{-1/2}\partial_x\phi_k^l(x)$. Now, we have $\mathbf{a^l},\mathbf{b^l}\in\ell^\infty(\mathbb{C})$ since

$$2\geq\|\lambda_k^{-1/2}\partial_x\phi_k^l\|^2_{L^2(e^l)}+\|\phi_k^l\|^2_{L^2(e^l)}=(|a_k^l|^2+|b_k^l|^2)|e_l|,\qquad\forall k\in\mathbb{N}^*,\ l\leq N.$$

Thus, there exists $C_2>0$ so that, for every $k\in\mathbb{N}^*$ and $v\in V_e\cup V_i$, we have $|\lambda_k^{-1/2}\partial_x\phi_k(v)|\leq C_2$. From the validity of the relations (12) and (14), it follows

$$\|G(t)\|_{(3)}\leq C_1C_2\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\Big\|\int_0^t\partial_x^2 f^j(s,v)e^{i\lambda_{(\cdot)}s}ds\Big\|_{\ell^2}+C_1\Big\|\int_0^t\big\langle\lambda_{(\cdot)}^{-1/2}\partial_x\phi_{(\cdot)}(s),\partial_x^3 f(s)\big\rangle_{L^2}e^{i\lambda_{(\cdot)}s}ds\Big\|_{\ell^2}.$$

The last relation and Proposition *B*.6 ensure the existence of $C_3(t),C_4(t)>0$ uniformly bounded for t in bounded intervals such that

$$\|G\|_{H^3_{\mathscr{G}}}\leq C_3(t)\sum_{v\in V_e\cup V_i}\sum_{j\in N(v)}\|\partial_x^2 f^j(\cdot,v)\|_{L^2((0,t),\mathbb{C})}+\sqrt{t}\|f\|_{L^2((0,t),H^3)}\leq C_4(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^3)}. \tag{15}$$

We underline that the identity is also valid when $\lambda_1 = 0$, which is proved by isolating the term with $k = 1$ and by repeating the steps above. For every $t \in [0,T]$, the inequality (15) shows that $G(t) \in H^3_{\mathscr{G}}$. The provided upper bounds are uniform and the Dominated Convergence Theorem leads to

$$G \in C^0([0,T], H^3_{\mathscr{G}}).$$

When $f(s) \in H^5 \cap H^4_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, the techniques just adopted leads to

$$G \in C^0([0,T], H^5_{\mathscr{G}})$$

Let $F(f)(t) := \int_0^t e^{iA\tau} f(\tau) d\tau$ for $f \in \mathscr{H}$ and $t \in (0,T)$. For B a Banach space, let $X(B)$ be the space of functions f so that $f(s) \in B$ for almost every $s \in (0,t)$ and $t \in (0,T)$. The first part of the proof implies

$$F : X(H^3 \cap H^2_{\mathscr{G}}) \longrightarrow C^0([0,T], H^3_{\mathscr{G}}), \qquad F : X(H^5 \cap H^4_{\mathscr{G}}) \longrightarrow C^0([0,T], H^5_{\mathscr{G}}).$$

From a classical interpolation result (see [BL76, *Theorem* 4.4.1] with $n = 1$), we have $F : X(H^{2+d} \cap H^{1+d}_{\mathscr{G}}) \longrightarrow C^0([0,T], H^{2+d}_{\mathscr{G}})$ with $d \in [1,3]$. Thanks to Proposition 3.2, if $d \in [1, 3/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} = H^{2+d} \cap H^{1+d}_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then

$$G \in C^0([0,T], H^{2+d}_{\mathscr{G}}).$$

1) (b) Assumptions II.3 . If $\mathscr{G}$ is equipped with $(\mathcal{D})$, then $H^2_{\mathscr{G}} = H^2_{\mathcal{NK}} \cap H^1_{\mathscr{G}}$ and $H^4_{\mathscr{G}} = H^4_{\mathcal{NK}} \cap H^3_{\mathscr{G}}$ from Proposition 3.2. As above, if $f(s) \in H^3 \cap H^2_{\mathcal{NK}} \cap H^1_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^3_{\mathscr{G}})$, while if $f(s) \in H^5 \cap H^4_{\mathcal{NK}} \cap H^3_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^5_{\mathscr{G}})$. From the interpolation techniques, if $d \in [1, 5/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^d_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ and the proof is attained.

1) (c) Assumptions II.2 . Let $f(s) \in H^4 \cap H^3_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$ and $\mathscr{G}$ be equipped with $(\mathcal{N})$. In this framework, the last term in right-hand side (13) is zero. Indeed, $\partial_x^2 f(s) \in C^0$ as $f(s) \in H^3_{\mathcal{NK}}$ and, for $v \in V_e$, we have $\partial_x \phi_k(v) = 0$ thanks to the $(\mathcal{N})$ boundary conditions (the terms $a^j(v)$ have different signs according to the orientation of the edges connected in v). For every $v \in V_i$, thanks to the $(\mathcal{NK})$ in $v \in V_i$, we have $\sum_{j \in N(v)} a^j(v) \partial_x \phi^j_k(v) = 0$. From (13), we obtain

$$\langle \phi_k, f(s) \rangle_{L^2} = -\frac{1}{\lambda_k^2} \sum_{v \in V_i \cup V_e} \sum_{j \in N(v)} a^j(v) \phi^j_k(v) \partial_x^3 f^j(s,v) + \frac{1}{\lambda_k^2} \int_{\mathscr{G}} \phi_k(y) \partial_x^4 f(s,y) dy.$$

Now, $(\phi_k)_{k \in \mathbb{N}^*}$ is a Hilbert basis of $\mathscr{H}$ and we proceed as in (14) and (15). From Proposition $B.6$, there exists $C_6(t) > 0$ uniformly bounded for t lying in bounded intervals such that $\|G\|_{H^4_{\mathscr{G}}} \leq C_1(t) \|f(\cdot,\cdot)\|_{L^2((0,t),H^4)}$ and $G \in C^0([0,T], H^4_{\mathscr{G}})$. Equivalently, when $f(s) \in H^6 \cap H^5_{\mathcal{NK}} \cap H^4_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, we have $G \in C^0([0,T], H^6_{\mathscr{G}})$. As above, Proposition 3.2 implies that when $d \in [2, 7/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^{2+d}_{\mathscr{G}})$.

2) Conclusion. As $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} H^2_{\mathscr{G}} \subseteq H^{2+d}$, we have $B \in L(H^{2+d}_{\mathscr{G}}, H^{2+d})$ thanks to the arguments of [Duc18b, *Remark* 2.1]. Let $\psi_0 \in H^{2+d}_{\mathscr{G}}$. We consider the map $F : \psi \in C^0([0,T], H^{2+d}_{\mathscr{G}}) \mapsto \phi \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ with

$$\phi(t) = F(\psi)(t) = e^{i\Delta t} \psi_0 - \int_0^t e^{i\Delta(t-s)} u(s) B \psi(s) ds, \qquad \forall t \in [0,T].$$

For every $\psi^1, \psi^2 \in C^0([0,T], H^{2+d}_{\mathscr{G}})$, we have $F(\psi^1)(t) - F(\psi^2)(t) = \int_0^t e^{-iA(t-s)} u(s) B(\psi^1(s) - \psi^2(s)) ds$. From **1)**, there exists $C(t) > 0$ uniformly bounded for t lying on bounded intervals such that

$$\|F(\psi^1) - F(\psi^2)\|_{L^\infty((0,T),H^{2+d}_{\mathscr{G}})} \leq C(T) \|u\|_{L^2((0,T),\mathbb{R})} \| B \|_{L(H^{2+d}_{\mathscr{G}}, H^{2+d})} \|\psi^1 - \psi^2\|_{L^\infty((0,T),H^{2+d}_{\mathscr{G}})}.$$

If $\|u\|_{L^2((0,T),\mathbb{R})}$ is small enough, then F is a contraction and Banach Fixed Point Theorem implies that there exists $\psi \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ such that $F(\psi) = \psi$. When $\|u\|_{L^2((0,T),\mathbb{R})}$ is not sufficiently small, one considers $\{t_j\}_{0 \leq j \leq n}$ a partition of $[0,T]$ with $n \in \mathbb{N}^*$. We choose a partition such that each $\|u\|_{L^2([t_{j-1},t_j],\mathbb{R})}$ is so small that the map F, defined on the interval $[t_{j-1}, t_j]$, is a contraction. Thanks to the Banach Fixed Point Theorem, the existence and the uniqueness of the mild solution is provided. In conclusion, the solution ψ of the (BSE) when $u \in C^0((0,T),\mathbb{R})$ is $C^1((0,T),\mathscr{H})$ and $\partial_t \|\psi(t)\|^2 = 0$, which implies $\|\psi(t)\| = \|\psi(0)\|$ for every $t \in [0,T]$. The generalization for $u \in L^2((0,T),\mathbb{R})$ follows from classical density arguments. □

4 Proof of Theorem 2.4

The result is achieved as in the proof of [Duc18b, *Proposition* 3.4] (also done in [MN15]). In particular, it is obtained by gathering the local exact controllability and the global approximate controllability (both provided below) thanks to the time reversibility of the (BSE).

4.1 Local exact controllability in $H^s_{\mathscr{G}}$

The aim of the section is to prove the local exact controllability in $H^s_{\mathscr{G}}$ when the hypotheses of Theorem 2.4 are satisfied. Let $O^s_{\epsilon,T} := \left\{\psi \in H^s_{\mathscr{G}} \middle|\ \|\psi\|_{L^2} = 1,\ \|\psi - \phi_1(T)\|_{(s)} < \epsilon\right\}$ with $s, T, \epsilon > 0$.

Proposition 4.1. *Let the hypotheses of Theorem 2.4 be satisfied. Let $s = 2 + d$ with d defined in Assumptions II. There exist $T > 0$ and $\epsilon > 0$ such that, for every $\psi \in O^s_{\epsilon,T}$, there exists a control function $u \in L^2((0,T),\mathbb{R})$ such that $\psi = \Gamma^u_T \phi_1$.*

Proof. The result can be proved by ensuring to the surjectivity, for $T > 0$ sufficiently large, of the map

$$\Gamma^{(\cdot)}_T \phi_1 : u \in U \subseteq L^2((0,T),\mathbb{R}) \longmapsto \psi \in O^s_{\epsilon,T} \subset H^s_{\mathscr{G}}, \qquad \Gamma^{(\cdot)}_t \phi_1 = \sum_{k\in\mathbb{N}^*} \phi_k(t)\langle \phi_k(t), \Gamma^{(\cdot)}_t \phi_1\rangle_{L^2}.$$

Let the map α be the sequence with elements $\alpha_k(u) = \langle \phi_k(T), \Gamma^u_T \phi_1\rangle_{L^2}$ for $k \in \mathbb{N}^*$, so that

$$\alpha : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := (x_k)_{k\in\mathbb{N}^*} \in h^s(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

The local controllability can be guaranteed by proving the local surjectivity of the map α in a neighborhood of $\alpha(0) = \delta = (\delta_{k,1})_{k\in\mathbb{N}^*}$ with respect to the h^4 norm. To this end, we use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of $\gamma(v) := (d_u\alpha(0))\cdot\ v$ the Fréchet derivative of α. Let $B_{j,k} := \langle \phi_j, B\phi_k\rangle_{L^2}$ with $j,k \in \mathbb{N}^*$. The map $\gamma : L^2((0,T),\mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := (x_k)_{k\in\mathbb{N}^*} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}$ is the sequence of elements $\gamma_k(v) := -i\int_0^T v(\tau)e^{i(\lambda_k-\lambda_1)s}d\tau B_{k,1}$ with $k \in \mathbb{N}^*$. Let the moment problem

$$x_k/B_{k,1} = -i\int_0^T u(\tau)e^{i(\lambda_k-\lambda_1)\tau}d\tau, \qquad \forall (x_k)_{k\in\mathbb{N}^*} \in T_\delta Q \subset h^s. \tag{16}$$

Proving surjectivity of γ corresponds to ensure the solvability of (16). In other words, we prove that there exists $T > 0$ large enough such that, for every $(x_k)_{k\in\mathbb{N}^*} \in T_\delta Q$, there exists $u \in L^2((0,T),\mathbb{R})$ such that $(x_k)_{k\in\mathbb{N}^*} = (\gamma_k(u))_{k\in\mathbb{N}^*}$. Even though the strategy of the proof is common for this kind of works (see [BL10, Mor14, MN15, Duc18b, Duc19]), proving the solvability of (16) can not be approached with the classical techniques as we can not ensure the validity of the spectral gap $\inf_{k\in\mathbb{N}^*}|\lambda_{k+1} - \lambda_k| > 0$ (as presented in Section 1.1). To this purpose, we refer to the theory developed in Appendix B and, in particular, to Proposition B.5. We notice that $B_{1,1} \in \mathbb{R}$ as B is symmetric, $ix_1/B_{1,1} \in \mathbb{R}$ and $\left(x_k/B_{k,l}\right)_{k\in\mathbb{N}^*} \in h^{d-\eta} \subseteq h^{\tilde{d}}$ thanks to the first point of Assumptions I. Thanks to (2) and (3), the hypotheses of Proposition $B.5$ are satisfied and the solvability of (16) is guaranteed in $h^{\tilde{d}}$. In conclusion, the map γ is surjective and α is locally surjective, which implies the local exact controllability. □

4.2 Global approximate controllability in $H^s_{\mathscr{G}}$

Definition 4.2. The (BSE) is said to be globally approximately controllable in $H^s_{\mathscr{G}}$ with $s > 0$ when, for every $\psi \in H^s_{\mathscr{G}}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi \in H^s_{\mathscr{G}}$ and $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\widehat{\Gamma}\psi - \Gamma^u_T\psi\|_{(s)} < \epsilon$.

Proposition 4.3. *Let (A,B) satisfy Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta > 0$ and $\widetilde{d} \geq 0$. The (BSE) is globally approximately controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ with d from Assumptions II.*

Proof. In the point **1)** of the proof, we suppose that (A,B) admits a non-degenerate chain of connectedness (see [BdCC13, Definition 3]). We treat the general case in the point **2)** .

1) (a) Preliminaries. Let π_m be the orthogonal projector $\pi_m : \mathscr{H} \to \mathscr{H}_m := span\overline{\{\phi_j \ : \ j \leq m\}}^{L^2}$ for every $m \in \mathbb{N}^*$. Up to reordering of $(\phi_k)_{k\in\mathbb{N}^*}$, the couples $(\pi_m A\pi_m, \pi_m B\pi_m)$ for $m \in \mathbb{N}^*$ admit non-degenerate chains of connectedness in $\mathscr{H}_m$. Let $\|\cdot\|_{BV(T)} = \|\cdot\|_{BV((0,T),\mathbb{R})}$ and $|||\cdot|||_{(s)} := |||\cdot|||_{L(H^s_{\mathscr{G}},H^s_{\mathscr{G}})}$ for $s > 0$.

Claim. $\forall\ \widehat{\Gamma} \in U(\mathscr{H}),\ \forall \epsilon > 0,\ \exists N_1 \in \mathbb{N}^*,\ \widetilde{\Gamma}_{N_1} \in U(\mathscr{H})\ :\ \pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} \in SU(\mathscr{H}_{N_1}),$

$$\|\widetilde{\Gamma}_{N_1}\phi_1 - \widehat{\Gamma}\phi_1\|_{L^2} < \epsilon. \tag{17}$$

Let $N_1 \in \mathbb{N}^*$ and $\widetilde{\phi}_1 := \|\pi_{N_1}\widehat{\Gamma}\phi_1\|_{L^2}^{-1}\pi_{N_1}\widehat{\Gamma}\phi_1$. We define $(\widetilde{\phi}_j)_{2\leq j\leq N_1}$ such that $(\widetilde{\phi}_j)_{j\leq N_1}$ is an orthonormal basis of $\mathscr{H}_{N_1}$. The operator $\widetilde{\Gamma}_{N_1}$ is the unitary map such that $\widetilde{\Gamma}_{N_1}\phi_j = \widetilde{\phi}_j$ for every $j \leq N_1$. The provided definition implies $\lim_{N_1\to\infty} \|\widetilde{\Gamma}_{N_1}\phi_1 - \widehat{\Gamma}\phi_1\|_{L^2} = 0$. Thus, for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}^*$ large enough satisfying the claim.

1) (b) Finite dimensional controllability. Let T_{ad} be the set of $(j,k) \in \{1,...,N_1\}^2$ such that $B_{j,k} := \langle\phi_j, B\phi_k\rangle_{L^2} \neq 0$ and $|\lambda_j - \lambda_k| = |\lambda_m - \lambda_l|$ with $m, l \in \mathbb{N}^*$ implies $\{j,k\} = \{m,l\}$ for $B_{m,l} = 0$. For every $(j,k) \in \{1,...,N_1\}^2$ and $\theta \in [0,2\pi)$, we define $E^\theta_{j,k}$ the $N_1 \times N_1$ matrix with elements $(E^\theta_{j,k})_{l,m} = 0$, $(E^\theta_{j,k})_{j,k} = e^{i\theta}$ and $(E^\theta_{j,k})_{k,j} = -e^{-i\theta}$ for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(j,k),(k,j)\}$. Let $E_{ad} = \left\{E^\theta_{j,k} \ : \ (j,k) \in T_{ad},\ \theta \in [0,2\pi)\right\}$ and $Lie(E_{ad})$. Fixed v a piecewise constant control taking value in E_{ad} and $\tau > 0$, we introduce the control system on $SU(\mathscr{H}_{N_1})$

$$\begin{cases} \dot{x}(t) = x(t)v(t), & t \in (0,\tau), \\ x(0) = Id_{SU(\mathscr{H}_{N_1})}. \end{cases} \tag{18}$$

Claim. (18) is controllable, *i.e.* for $R \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1,...,M_p \in E_{ad}$, $\alpha_1,...,\alpha_p \in \mathbb{R}^+$ such that $R = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}$.

For every $(j,k) \in \{1,...,N_1\}^2$, we define the $N_1 \times N_1$ matrices $R_{j,k}$, $C_{j,k}$ and D_j as follow. For $(l,m) \in \{1,...,N_1\}^2 \setminus \{(j,k),(k,j)\}$, we have $(R_{j,k})_{l,m} = 0$ and $(R_{j,k})_{j,k} = -(R_{j,k})_{k,j} = 1$, while $(C_{j,k})_{l,m} = 0$ and $(C_{j,k})_{j,k} = (C_{j,k})_{k,j} = i$. Moreover, for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(1,1),(j,j)\}$, $(D_j)_{l,m} = 0$ and $(D_j)_{1,1} = -(D_j)_{j,j} = i$. We consider the basis of $su(\mathscr{H}_{N_1})$

$$\mathbf{e} := \{R_{j,k}\}_{j,k\leq N_1} \cup \{C_{j,k}\}_{j,k\leq N_1} \cup \{D_j\}_{j\leq N_1}.$$

Thanks to [Sac00, Theorem 6.1], the controllability of (18) is equivalent to prove that $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$ for $su(\mathscr{H}_{N_1})$ the Lie algebra of $SU(\mathscr{H}_{N_1})$. The claim si valid as it is possible to obtain the matrices $R_{j,k}$, $C_{j,k}$ and D_j for every $j,k \leq N_1$ by iterated Lie brackets of elements in E_{ad}.

1) (c) Finite dimensional estimates. Let $\widehat{\Gamma} \in U(\mathscr{H})$ and $\widetilde{\Gamma}_{N_1} \in U(\mathscr{H})$ be defined in **1) (a)**. Thanks to the previous claim and to the fact that $\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1,...,M_p \in E_{ad}$ and $\alpha_1,...,\alpha_p \in \mathbb{R}^+$ such that

$$\pi_{N_1}\widetilde{\Gamma}_{N_1}\pi_{N_1} = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}. \tag{19}$$

Claim. For every $l \leq p$ and $e^{\alpha_l M_l}$ from (19), there exist $\{T^l_n\}_{l\in\mathbb{N}^*} \subset \mathbb{R}^+$ and $\{u^l_n\}_{n\in\mathbb{N}^*}$ such that $u^l_n \in L^2((0,T^l_n),\mathbb{R})$ for every $n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} \|\Gamma^{u^l_n}_{T^l_n}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0, \qquad \forall k \leq N_1, \tag{20}$$

$$\begin{gathered} \sup_{n\in\mathbb{N}^*} \|u^l_n\|_{BV(T^l_n)} < \infty, \qquad \sup_{n\in\mathbb{N}^*} \|u^l_n\|_{L^\infty((0,T^l_n),\mathbb{R})} < \infty, \\ \sup_{n\in\mathbb{N}^*} T^l_n\|u^l_n\|_{L^\infty((0,T^l_n),\mathbb{R})} < \infty. \end{gathered} \tag{21}$$

We consider the results developed in [Cha12, *Section* 3.1 & *Section* 3.2] by Chambrion and leading to [Cha12, *Proposition* 6] since (A,B) admits a non-degenerate chain of connectedness ([BdCC13, Definition 3]). Each $e^{\alpha_l M_l}$ is a rotation in a two dimensional space for every $l \in \{1,...,p\}$ and this work allows

to explicit $\{T_n^l\}_{n\in\mathbb{N}^*}\subset\mathbb{R}^+$ and $\{u_n^l\}_{n\in\mathbb{N}^*}$ satisfying (21) such that $u_n^l\in L^2((0,T_n^l),\mathbb{R})$ for every $n\in\mathbb{N}^*$ and

$$\lim_{n\to\infty}\|\pi_{N_1}\Gamma_{T_n^l}^{u_n^l}\phi_k-e^{\alpha_l M_l}\phi_k\|_{L^2}=0,\qquad\forall k\leq N_1. \tag{22}$$

As $e^{\alpha_l M_l}\in SU(\mathscr{H}_{N_1})$, we have $\lim_{n\to\infty}\|\Gamma_{T_n^l}^{u_n^l}\phi_k-e^{\alpha_l M_l}\phi_k\|_{L^2}=0$ for $k\leq N_1$.

1) (d) Infinite dimensional estimates.

Claim. Let $\widehat{\Gamma}\in U(\mathscr{H})$. There exist $K_1,K_2,K_3>0$ such that for every $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\phi_1-\widehat{\Gamma}\phi_1\|_{L^2}\leq\epsilon$ and

$$\|u\|_{BV(T)}\leq K_1,\qquad\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_2,\qquad T\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_3. \tag{23}$$

Let **1) (c)** be valid with $p=2$. Although, the following result is valid for any $p\in\mathbb{N}^*$. There exists $2\leq l\leq N_1$ such that $e^{\alpha_1 M_1}\phi_1=\phi_l$. Thanks to (20), there exists $n\in\mathbb{N}^*$ large enough such that,

$$\|\Gamma_{T_n^2}^{u_n^2}\Gamma_{T_n^1}^{u_n^1}\phi_1-e^{\alpha_2 M_2}e^{\alpha_1 M_1}\phi_1\|_{L^2}\leq|\!|\!|\,\Gamma_{T_n^2}^{u_n^2}\,|\!|\!|\,\|\Gamma_{T_n^1}^{u_n^1}\phi_1-e^{\alpha_1 M_1}\phi_1\|_{L^2}+\|\Gamma_{T_n^2}^{u_n^2}\phi_l-e^{\alpha_2 M_2}\phi_l\|_{L^2}\leq\epsilon.$$

The identity (19) leads to the existence of $K_1,K_2,K_3>0$ such that for every $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\phi_1-\widetilde{\Gamma}_{N_1}\phi_1\|_{L^2}<\epsilon$ and

$$\|u\|_{BV(T)}\leq K_1,\qquad\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_2,\qquad T\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_3. \tag{24}$$

The relation (17) and the triangular inequality achieve the claim.

1) (e) Global approximate controllability with respect to the L^2-norm. Let $\psi\in\mathscr{H}$ and $\widehat{\Gamma}\in U(\mathscr{H})$.

Claim. There exist $K_1,K_2,K_3>0$ such that for every $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\psi-\widehat{\Gamma}\psi_k\|_{L^2}\leq\epsilon$ and

$$\|u\|_{BV(T)}\leq K_1,\qquad\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_2,\qquad T\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_3. \tag{25}$$

We assume that $\|\psi\|_{L^2}=1$, but the same proof is also valid for the generic case. From the point **1) (d)**, there exist two controls respectively steering ϕ_1 close to ψ and ϕ_1 close to $\widehat{\Gamma}\psi$. Vice versa, thanks to the time reversibility, there exists a control steering ψ close to ϕ_1. In other words, there exist $T_1,T_2>0$, $u_1\in L^2((0,T_1),\mathbb{R})$ and $u_2\in L^2((0,T_2),\mathbb{R})$ such that

$$\|\Gamma_{T_1}^{u_1}\psi-\phi_1\|_{L^2}\leq\epsilon,\qquad\|\Gamma_{T_2}^{u_2}\phi_1-\widehat{\Gamma}\psi\|_{L^2}\leq\epsilon.$$

The chosen controls u_1 and u_2 satisfy (25). The claim is proven as

$$\|\Gamma_{T_2}^{u_2}\Gamma_{T_1}^{u_1}\psi-\widehat{\Gamma}\psi\|_{L^2}\leq\|\Gamma_{T_2}^{u_2}\Gamma_{T_1}^{u_1}\psi-\Gamma_{T_2}^{u_2}\phi_1\|_{L^2}+\|\Gamma_{T_2}^{u_2}\phi_1-\widehat{\Gamma}\psi\|_{L^2}\leq 2\epsilon.$$

1) (f) Global approximate controllability in higher regularity norm. Let $\psi\in H_{\mathscr{G}}^s$ with $s\in[s_1,s_1+2)$ and $s_1\in\mathbb{N}^*$. Let $\widehat{\Gamma}\in U(\mathscr{H})$ be such that $\widehat{\Gamma}\psi\in H_{\mathscr{G}}^s$ and $B:H_{\mathscr{G}}^{s_1}\longrightarrow H_{\mathscr{G}}^{s_1}$.

Claim. There exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\psi-\widehat{\Gamma}\psi\|_{(s)}\leq\epsilon$.

We consider the propagation of regularity developed by Kato in [Kat53]. We notice that $i(A+u(t)B-ic)$ is maximal dissipative in $H_{\mathscr{G}}^{s_1}$ for suitable $c:=\|u\|_{L^\infty((0,T),\mathbb{R})}|\!|\!|\,B\,|\!|\!|_{(s_1)}$. Let $\lambda>c$ and $\widehat{H}_{\mathscr{G}}^{s_1+2}:=D(A^{\frac{s_1}{2}}(i\lambda-A))\equiv H_{\mathscr{G}}^{s_1+2}$. We know that $B:\widehat{H}_{\mathscr{G}}^{s_1+2}\subset H_{\mathscr{G}}^{s_1}\to H_{\mathscr{G}}^{s_1}$ and the arguments of [Duc18b, Remark 2.1] imply that $B\in L(\widehat{H}_{\mathscr{G}}^{s_1+2},H_{\mathscr{G}}^{s_1})$. For $T>0$ and $u\in BV((0,T),\mathbb{R})$, we have

$$M:=\sup_{t\in[0,T]}|\!|\!|\,(i\lambda-A-u(t)B)^{-1}\,|\!|\!|_{L(H_{\mathscr{G}}^{s_1},\widehat{H}_{\mathscr{G}}^{s_1+2})}<+\infty.$$

We know $\|k+f(\cdot)\|_{BV((0,T),\mathbb{R})}=\|f\|_{BV((0,T),\mathbb{R})}$ for $f\in BV((0,T),\mathbb{R})$ and $k\in\mathbb{R}$. Equivalently,

$$N:=\|\!|\, i\lambda-A-u(\cdot)B\,|\!\|_{BV\left([0,T],L(\widehat{H}^{s_1+2}_{\mathscr{G}},H^{s_1}_{\mathscr{G}})\right)}=\|u\|_{BV(T)}\,\|\!|\,B\,|\!\|_{L(\widehat{H}^{s_1+2}_{\mathscr{G}},H^{s_1}_{\mathscr{G}})}<+\infty.$$

We call $C_1:=\|\!|\,A(A+u(T)B-i\lambda)^{-1}\,|\!\|_{(s_1)}<\infty$ and U^u_t the propagator generated by $A+uB-ic$ such that $U^u_t\psi=e^{-ct}\Gamma^u_t\psi$. Thanks to [Kat53, *Section* 3.10], for every $\psi\in H^{s_1+2}_{\mathscr{G}}$, it follows

$$\|(A+u(T)B-i\lambda)U^u_t\psi\|_{(s_1)}\leq Me^{MN}\|(A-i\lambda)\psi\|_{(s_1)}\quad\Longrightarrow\quad\|\Gamma^u_T\psi\|_{(s_1+2)}\leq C_1Me^{MN+cT}\|\psi\|_{(s_1+2)}.$$

For every $T>0$, $u\in BV((0,T),\mathbb{R})$ and $\psi\in H^{s_1+2}_{\mathscr{G}}$, there exists $C=C(K)>0$ depending on $K=\left(\|u\|_{BV(T)},\|u\|_{L^\infty((0,T),\mathbb{R})},T\|u\|_{L^\infty((0,T),\mathbb{R})}\right)$ such that

$$\|\Gamma^u_T\psi\|_{(s_1+2)}\leq C\|\psi\|_{(s_1+2)}. \tag{26}$$

Now, we notice that, for every $\psi\in H^6_{\mathscr{G}}$, from the Cauchy-Schwarz inequality, we have $\|A\psi\|^2_{L^2}\leq\|\psi\|_{L^2}\|A^2\psi\|_{L^2}$ and there exists $C_2>0$ such that $\|A^2\psi\|^4_{L^2}\leq\|A\psi\|^2_{L^2}\|A^3\psi\|^2_{L^2}\leq C_2\|\psi\|_{L^2}\|A^3\psi\|^3_{L^2}$. By following the same idea, for every $\psi\in H^{s_1+2}_{\mathscr{G}}$, there exist $m_1,m_2\in\mathbb{N}^*$ and $C_3,C_4>0$ such that

$$\|A^{\frac{s}{2}}\psi\|^{m_1+m_2}_{L^2}\leq C_3\|\psi\|^{m_1}_{L^2}\|A^{\frac{s_1+2}{2}}\psi\|^{m_2}_{L^2}\quad\Longrightarrow\quad\|\psi\|^{m_1+m_2}_{(s)}\leq C_4\|\psi\|^{m_1}_{L^2}\|\psi\|^{m_2}_{(s_1+2)}. \tag{27}$$

In conclusion, the point **1) (e)**, the relation (26) and the relation (27) ensure the claim.

1) (g) Conclusion. Let d be the parameter introduced by the validity of Assumptions II. If $d<2$, then $B:H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ and the global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<4$. If $d\in[2,5/2)$, then $B:H^{d_1}\to H^{d_1}$ with $d_1\in(d,5/2)$ from Assumptions II. Now, $H^{d_1}_{\mathscr{G}}=H^{d_1}\cap H^2_{\mathscr{G}}$, thanks to Proposition 3.2, and $B:H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ implies $B:H^{d_1}_{\mathscr{G}}\to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<d_1+2$. If $d\in[5/2,7/2)$, then $B:H^{d_1}_{\mathcal{NK}}\to H^{d_1}_{\mathcal{NK}}$ for $d_1\in(d,7/2)$ and $H^{d_1}_{\mathscr{G}}=H^{d_1}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}$ from Proposition 3.2. Now, $B:H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ that implies $B:H^{d_1}_{\mathscr{G}}\to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<d_1+2$.

2) Generalization. Let (A,B) do not admit a non-degenerate chain of connectedness. We decompose

$$A+u(\cdot)B=(A+u_0B)+u_1(\cdot)B,\qquad\qquad u_0\in\mathbb{R},\quad u_1\in L^2((0,T),\mathbb{R}).$$

We notice that, if (A,B) satisfies Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta>0$ and $\tilde{d}\geq0$, then Lemma C.2 and Lemma C.3 are valid. We consider u_0 in the neighborhoods provided by the two lemmas and we denote $(\phi^{u_0}_k)_{k\in\mathbb{N}}$ a Hilbert basis of $\mathscr{H}$ made by eigenfunctions of $A+u_0B$. The point **1)** can be repeated by considering the sequence $(\phi^{u_0}_k)_{k\in\mathbb{N}}$ instead of $(\phi_k)_{k\in\mathbb{N}}$ and the spaces $D(|A+u_0B|^{\frac{s}{2}})$ in substitution of $H^s_{\mathscr{G}}$ with $s>0$. The claim is equivalently proved since $(A+u_0B,B)$ admits a non-degenerate chain of connectedness thanks to Lemma C.2 and $\left\||A+u_0B|^{\frac{s}{2}}\cdot\right\|_{L^2}\asymp\|\cdot\|_{(s)}$ with $s=2+d$ and d from Assumptions II($\eta,\tilde{d}$) thanks to Lemma C.3. □

5 Proofs of Theorem 2.5 and Corollary 2.7

Let $\left(\lambda^{\widetilde{\mathscr{G}}}_k\right)_{k\in\mathbb{N}^*}$ denote the ordered sequence of eigenvalues of A on a compact quantum graph $\widetilde{\mathscr{G}}$.

Proof of Theorem 2.5. Let $\mathscr{G}$ be a tadpole graph equipped with $(\mathcal{D})$ (see Figure 5). Let $\mathscr{G}^{\mathcal{D}}$ be obtained from $\mathscr{G}$ by imposing $(\mathcal{D})$ on v. Let $\mathscr{G}^{\mathcal{N}}$ be the graph obtained by disconnecting e_1 on one side and by imposing $(\mathcal{N})$ on the new external vertex of e_1 (see the first line of Figure 8).

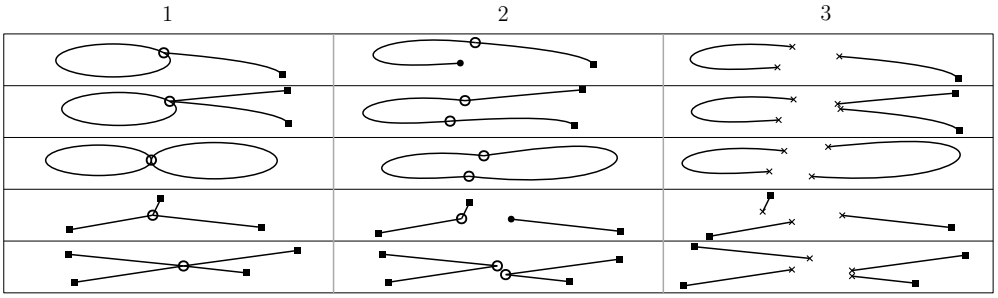


Figure 8: The figure represents the graphs described in the proof of Theorem 2.5. *The column* **1** *shows the graphs* $\mathscr{G}$ *considered: tadpole, two-tails tadpole, double-rings graph, star graph with* $N = 3$ *and star graph with* $N = 4$. *The columns* **2** *and* **3** *respectively provide the corresponding graphs* $\mathscr{G}^N$ *and* $\mathscr{G}^D$.

We notice that $\left(\lambda_k^{\mathscr{G}^{\mathcal{D}}}\right)_{k\in\mathbb{N}^*}$ and $\left(\lambda_k^{\mathscr{G}^{\mathcal{N}}}\right)_{k\in\mathbb{N}^*}$ are the ordered sequences of eigenvalues respectively obtained by reordering $\left\{\frac{k^2\pi^2}{L_j^2}\right\}_{\substack{k\in\mathbb{N}^*\\ j\in\{1,2\}}}$ and $\left\{\frac{(2k-1)^2\pi^2}{4(L_1+L_2)^2}\right\}_{k\in\mathbb{N}^*}$. From Proposition A.3, we have

$$\lambda_k^{\mathscr{G}} \leq \lambda_k^{\mathscr{G}^{\mathcal{D}}} \leq \lambda_{k+1}^{\mathscr{G}}, \qquad \lambda_k^{\mathscr{G}} \leq \lambda_{k+1}^{\mathscr{G}^{\mathcal{N}}} \leq \lambda_{k+1}^{\mathscr{G}}, \qquad \forall k \in \mathbb{N}^*. \tag{28}$$

If $\{L_1, L_2\} \in \mathcal{AL}$, then $\{L_1, L_2, L_1 + L_2\} \in \mathcal{AL}$. The techniques of the proof of Proposition A.2 lead to the existence of $C > 0$ such that, for every $\epsilon > 0$, there holds

$$|\lambda_{k+1}^{\mathscr{G}} - \lambda_k^{\mathscr{G}}| \geq |\lambda_{k+1}^{\mathscr{G}^{\mathcal{N}}} - \lambda_k^{\mathscr{G}^{\mathcal{D}}}| \geq Ck^{-\epsilon}, \qquad \forall k \in \mathbb{N}^*.$$

The relation (3) is verified and the claim is guaranteed by Theorem 2.4. The techniques introduced lead to the claim when $\mathscr{G}$ is a tadpole graph equipped with $(\mathcal{N})$, but also when $\mathscr{G}$ is a two-tails tadpole graph, a double-rings graph or a star graph with $N \leq 4$ edges. In every framework, we impose that $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$. In Figure 8, we represent how to define $\mathscr{G}^{\mathcal{N}}$ and $\mathscr{G}^{\mathcal{D}}$ from the corresponding $\mathscr{G}$. □

Remark 5.1. *The techniques leading to Theorem* 2.5 *can be adopted in order to prove Remark* 2.6. *The peculiarity of the proof is that when* $\mathscr{G}$ *is a star graphs, we construct* $\mathscr{G}^{\mathcal{N}}$ *so that the edges of equal length do not belong to the same connected component composing* $\mathscr{G}^{\mathcal{N}}$.

Proof of Corollary 2.7. As $(\lambda_j)_{j\in\mathbb{N}^*} \subset \left\{\frac{(k-1)^2\pi^2}{4L_j^2}\right\}_{\substack{k,j\in\mathbb{N}^*\\ j\leq N}}$, the claim follows from Proposition A.1. In fact, thanks to the arguments adopted in the proof of Lemma A.2, for every $\epsilon > 0$, there exists $C_1 > 0$ such that $|\lambda_{k+1} - \lambda_k| \geq C_1 k^{-\epsilon}$ for every $k \in \mathbb{N}^*$. In conclusion, Theorem 2.4 attains the proof. □

Remark 5.2. *When the hypotheses of Theorem 2.5 or Corollary 2.7 are satisfied we know that, for every* $\epsilon > 0$, *there exists* $C_1 > 0$ *such that* $|\lambda_{k+1} - \lambda_k| \geq C_1 k^{-\epsilon}$ *for every* $k \in \mathbb{N}^*$. *Now,* $\lambda_k \sim k^2$ *from Remark A.4. The arguments of Remark 2.2 ensure that the validity of* (42) *is guaranteed for* $\delta > 0$ *large as much as desired when* $\mathcal{M} \in \mathbb{N}^*$ *is also sufficiently large. Under these assumptions, the local exact controllability (proposed in Proposition 4.1) is valid for any positive time* $T > 0$ *as the moment problem* (16) *is solvable for any positive time thanks to Proposition B.5 (which is valid for* $T > \frac{2\pi}{\delta}$*).*

6 Proofs of the theorems 1.2 and 1.3

Proof of Theorem 1.2. Let $\mathscr{G}$ be a star graph with 4 edges of lengths $\{L_j\}_{j\leq 4}$ equipped $(\mathcal{D})$. The $(\mathcal{D})$ conditions on the external vertices imply that each eigenfunction ϕ_j with $j \in \mathbb{N}^*$ satisfies $\phi_j^l(0) = 0$ for every $l \leq 4$. Then, $\phi_j(x) = (a_j^1 \sin(x\sqrt{\lambda_j}), a_j^2 \sin(x\sqrt{\lambda_j}), a_j^3 \sin(x\sqrt{\lambda_j}), a_j^4 \sin(x\sqrt{\lambda_j}))$ with $\{a_j^l\}_{l\leq 4} \subset \mathbb{C}$ such that $(\phi_j)_{j\in\mathbb{N}^*}$ forms a Hilbert basis of $\mathscr{H}$, *i.e.* $\sum_{l\leq 4} \int_0^{L_l} |a_j^l|^2 \sin^2(x\sqrt{\lambda_j})dx = 1$, which leads to

$= \sum_{l\leq 4} |a_j^l|^2 \Big(\frac{L_l}{2} + \frac{\cos(L_l\sqrt{\lambda_j})\sin(L_l\sqrt{\lambda_j})}{2\sqrt{\lambda_j}} \Big)$. For every $j \in \mathbb{N}^*$, the $(\mathcal{NK})$ condition in V_i yields that

$$
\begin{aligned}
& a_j^1 \sin(\sqrt{\lambda_j} L_1) = ... = a_j^4 \sin(\sqrt{\lambda_j} L_N), \qquad \sum_{l\leq 4} a_j^l \cos(\sqrt{\lambda_j} L_l) = 0, \\
& \sum_{l\leq 4} \cot(\sqrt{\lambda_j} L_l) = 0, \qquad \sum_{l\leq N} |a_j^l|^2 \sin(L_l\sqrt{\lambda_j})\cos(L_l\sqrt{\lambda_j}) = 0.
\end{aligned} \tag{29}
$$

Now, $1 = \sum_{l=1}^4 |a_j^l|^2 L_l/2$ and the continuity implies $a_j^l = a_j^1 \frac{\sin(\sqrt{\lambda_j} L_1)}{\sin(\sqrt{\lambda_j} L_l)}$ for $l \neq 1$ and $j \in \mathbb{N}^*$, which ensures $|a_j^1|^2 \big(L_1 + \sum_{l=2}^4 L_l \frac{\sin^2(\sqrt{\lambda_j} L_1)}{\sin^2(\sqrt{\lambda_j} L_l)} \big) = 2$. Thus,

$$
|a_j^1|^2 = 2 \prod_{m\neq 1} \sin^2(\sqrt{\lambda_j} L_m) \Bigg(\sum_{k=1}^4 L_k \prod_{m\neq k} \sin^2(\sqrt{\lambda_j} L_m) \Bigg)^{-1}, \quad \forall j \in \mathbb{N}^*. \tag{30}
$$

From (29) and (30), we have $\sum_{l=1}^4 \cos(\sqrt{\lambda_k} L_l) \prod_{m\neq l} \sin(\sqrt{\lambda_k} L_m) = 0$. The validity of [DZ06, *Proposition A.11*] and Remark $A.4$ ensure that, for every $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that, for every $j \in \mathbb{N}^*$,

$$
|a_j^1| = \sqrt{\frac{2}{\sum_{l=1}^4 L_l \sin^{-2}(\sqrt{\lambda_j} L_l)}} \geq \sqrt{\frac{2}{\sum_{l=1}^4 L_l C_1^{-2} \lambda_j^{1+\epsilon}}} \geq \frac{C_2}{j^{1+\epsilon}}. \tag{31}
$$

1) Assumptions I.1 . We notice that $\langle \phi_k^l, B\phi_j^l \rangle_{L^2(e_j,\mathbb{C})} = 0$ for every $2 \leq l \leq 4$ and $k, j \in \mathbb{N}^*$. Let

$$
\begin{aligned}
a_j(x) &:= \frac{2\prod_{m\neq 1} \sin^2(\sqrt{\lambda_j} L_m)}{\sum_{k=2}^4 L_k \sin^2(\sqrt{\lambda_j} x) \prod_{m\neq k,1} \sin^2(\sqrt{\lambda_j} L_m) + x \prod_{m\neq 1} \sin^2(\sqrt{\lambda_j} L_m)}, \\
B_1(x) &:= \frac{-30\sqrt{\lambda_1} x + 20\sqrt{\lambda_1}^3 x^3 + 4\sqrt{\lambda_1}^5 x^5 + 15\sin(2\sqrt{\lambda_1} x)}{40\sqrt{\lambda_1}^5}, \\
B_j(x) &:= 2\frac{-6(\sqrt{\lambda_1} - \sqrt{\lambda_j})x + (\sqrt{\lambda_1} - \sqrt{\lambda_j})^3 x^3 + 6\sin((\sqrt{\lambda_1} - \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} - \sqrt{\lambda_j})^5} \\
&\quad - 2\frac{-6(\sqrt{\lambda_1} + \sqrt{\lambda_j})x + (\sqrt{\lambda_1} + \sqrt{\lambda_j})^3 x^3 + 6\sin((\sqrt{\lambda_1} + \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} + \sqrt{\lambda_j})^5}
\end{aligned}
$$

with $j \in \mathbb{N}^*$. Each function $\widetilde{B}_j(\cdot) := \sqrt{a_1(\cdot)}\sqrt{a_j(\cdot)} B_j(\cdot)$ is non-constant and analytic in $\mathbb{R}^+$, while we notice that $B_{1,j} = \langle \phi_1, B\phi_j \rangle_{L^2} = \widetilde{B}_j(L_1)$ by calculation. The set of positive zeros $\tilde{V}_j$ of each $\widetilde{B}_j$ is a discrete subset of $\mathbb{R}^+$ and $\tilde{V} = \bigcup_{j\in\mathbb{N}^*} \tilde{V}_j$ is countable. For every $\{L_l\}_{l\leq 4} \in \mathcal{AL}(4)$ such that $L_1 \notin \tilde{V}$, we have $|B_{1,j}| \neq 0$ for every $j \in \mathbb{N}^*$. Now, there holds $|B_{1,j}| \sim |a_j| L_1 \sqrt{\lambda_1}\sqrt{\lambda_j} (\lambda_j - \lambda_1)^{-2}$ for every $j \in \mathbb{N}^* \setminus \{1\}$. From Remark $A.4$ and the identity (31), the first point of Assumptions I$(2+\epsilon)$ is verified since, for each $\epsilon > 0$, there exists $C_3 > 0$ such that $|B_{1,j}| \geq \frac{C_3}{j^{4+\epsilon}}$ for every $j \in \mathbb{N}^*$.

2) Assumptions I.2 . Let $(k,j), (m,n) \in I$, $(k,j) \neq (m,n)$ for $I := \{(j,k) \in (\mathbb{N}^*)^2 : j \neq k\}$ and

$$
F_j(x) := a_j(x) \frac{-30\sqrt{\lambda_k} x + 20\sqrt{\lambda_k}^3 x^3 + 4\sqrt{\lambda_k}^5 x^5 + 15\sin(2\sqrt{\lambda_k} x)}{40\sqrt{\lambda_k}^5}.
$$

By calculation, we notice that $B_{j,j} = \langle \phi_j, B\phi_j \rangle_{L^2} = F_j(L_1)$. Moreover, for $F_{j,k,l,m}(x) = F_j(x) - F_k(x) - F_l(x) + F_m(x)$, it follows $F_{j,k,l,m}(L_1) = B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m}$ and $F_{j,k,l,m}(x)$ is a non-constant analytic function for $x > 0$. Furthermore $V_{j,k,l,m}$, the set of the positive zeros of $F_{j,k,l,m}(x)$, is discrete and $V := \bigcup_{\substack{j,k,l,m\in\mathbb{N}^* \\ j\neq k\neq l\neq m}} V_{j,k,l,m}$ is a countable subset of $\mathbb{R}^+$. For each $\{L_l\}_{l\leq 4} \in \mathcal{AL}(4)$ such that $L_1 \notin V \cup \tilde{V}$, Assumptions I$(2+\epsilon)$ are verified.

3) Assumptions II.3 and conclusion. The third point of Assumptions II$(2+\epsilon_1, \epsilon_2)$ is valid for each $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 \in (0, 1/2)$ since B stabilizes $H^2_{\mathscr{G}}$, H^m and $H^m_{\mathcal{NK}}$ for $m \in (0, 9/2)$. Indeed, for every $n \in \mathbb{N}^*$ such that $n < 5$, we have $\partial_x^{n-1}(B\psi)^1(L_1) = ... = \partial_x^{n-1}(B\psi)^4(L_4) = 0$ for every $\psi \in H^n_{\mathcal{NK}}$, which implies $B\psi \in H^n_{\mathcal{NK}}$. From Theorem 2.5, the controllability holds in $H^{4+\epsilon}_{\mathscr{G}}$ with $\epsilon > 0$. $\square$

Proof of Theorem 1.3. Let $\mathscr{G}$ be a tadpole graph equipped with $(\mathcal{D})$ (see Figure 5). Let r be the axis passing along e_2 and crossing e_1 in its middle.

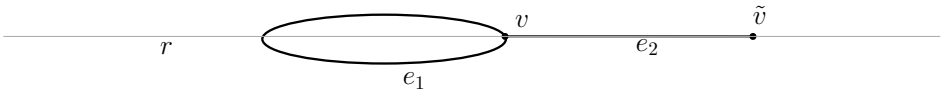


Figure 9: The figure represents the symmetry axis r of the tadpole graph.

The graph $\mathscr{G}$ is symmetric with respect to r and we construct the eigenfunctions $(\phi_k)_{k\in\mathbb{N}^*}$ as a sequence of symmetric or skew-symmetric functions with respect to r. If $\phi_k = (\phi_k^1, \phi_k^2)$ is skew-symmetric, then

$$\phi_k^2 \equiv 0, \quad \phi_k^1(0) = \phi_k^1(L_1/2) = \phi_k^1(L_1) = 0, \quad \partial_x\phi_k^1(0) = \partial_x\phi_k^1(L_1).$$

We denote $(f_k)_{k\in\mathbb{N}^*}$ the skew-symmetric eigenfunctions belonging to the Hilbert basis $(\phi_k)_{k\in\mathbb{N}^*}$ and $(\nu_k)_{k\in\mathbb{N}^*}$ the ordered sequence of corresponding eigenvalues. We set

$$(f_k)_{k\in\mathbb{N}^*} = \left(\left(\sqrt{\frac{2}{L_1}}\sin\left(x\frac{2k\pi}{L_1}\right),0\right)\right)_{k\in\mathbb{N}^*}, \qquad (\nu_k)_{k\in\mathbb{N}^*} := \left(\frac{4k^2\pi^2}{L_1^2}\right)_{k\in\mathbb{N}^*}.$$

If $\phi_k = (\phi_k^1, \phi_k^2)$ is symmetric, then we have $\partial_x\phi_k^1(L_1/2) = 0$ and $\phi_k^1(\cdot) = \phi_k^1(L_1 - \cdot)$. The $(\mathcal{D})$ conditions on $\widetilde{v}$ implies that $(g_k)_{k\in\mathbb{N}^*} := \left(\left(a_k^1\cos\left(\sqrt{\mu_k}(x - \frac{L_1}{2})\right), a_k^2\sin(\sqrt{\mu_k}x)\right)\right)_{k\in\mathbb{N}^*}$ for $\{(a_k^1, a_k^2)\}_{k\in\mathbb{N}^*} \subset \mathbb{C}^2$ are the symmetric eigenfunctions of A corresponding to the eigenvalues $(\mu_k)_{k\in\mathbb{N}^*}$. We characterize $(\mu_k)_{k\in\mathbb{N}^*}$. The $(\mathcal{NK})$ conditions in v ensure that $a_k^1\cos(\sqrt{\mu_k}(L_1/2)) = a_k^2\sin(\sqrt{\mu_k}L_2))$ and

$$2a_k^1\sin(\sqrt{\mu_k}(L_1/2)) + a_k^2\cos(\sqrt{\mu_k}L_2)) = 0 \quad \Longrightarrow \quad 2\tan(\sqrt{\mu_k}(L_1/2)) + \cot(\sqrt{\mu_k}L_2)) = 0. \tag{32}$$

We choose $\{(a_k^1, a_k^2)\}_{k\in\mathbb{N}^*} \subset \mathbb{C}^2$ such that $(\phi_k)_{k\in\mathbb{N}^*}$ obtained by reordering $(f_k)_{k\in\mathbb{N}^*} \cup (g_k)_{k\in\mathbb{N}^*}$ forms an Hilbert basis of $\mathscr{H}$. In particular, the techniques leading to relation (30) in Theorem 1.2 attain

$$|a_k^1|^2 = \frac{2\sin^2(\sqrt{\mu_k}L_2)}{a_k}, \quad |a_k^2|^2 = \frac{2\cos^2(\sqrt{\mu_k}(L_1/2))}{a_k}$$

with $a_k := L_1\cos^2(\sqrt{\mu_k}(L_1/2)) + L_2\sin^2(\sqrt{\mu_k}L_2)$ and $k \in \mathbb{N}^*$. From (32), there holds

$$2\sin(\sqrt{\mu_k}L_2)\sin\left(\sqrt{\mu_k}\frac{L_1}{2}\right) + \cos\left(\sqrt{\mu_k}\frac{L_1}{2}\right)\cos(\sqrt{\mu_k}L_2) = 0.$$

1) Assumptions I.1 . If $\{L_1, L_2\} \in \mathcal{AL}(2)$, then $\{L_1/2, L_2\} \in \mathcal{AL}(2)$. The validity of the two points of Remark $A.6$ is guaranteed for each $l \in \{1, 2\}$ and with $\{L_1/2, L_2\} \in \mathcal{AL}(2)$. The arguments leading to (31) in Theorem 1.2, applied with the identities (38) and (39), imply that

$$\forall\epsilon > 0, \ \ \exists C > 0 \ \ : \ \ |a_k^l| \geq Ck^{-1-\epsilon}, \qquad \forall k \in \mathbb{N}^*, \ \forall l \in \{1, 2\}. \tag{33}$$

Let $B_1 : (\psi^1, \psi^2) \mapsto (h\psi^1, 0)$ and $B_2 : (\psi^1, \psi^2) \mapsto (h_1\psi^1, h_2\psi^2)$ with $h(x) := \sin\left(\frac{2\pi}{L_1}x\right)$, $h_1(x) := x(x - L_1)$ and $h_2(x) := x^2 - (2L_1 + 2L_2)x + L_2^2 + 2L_1L_2$. As h is skew-symmetric with respect to r and h_1 is symmetric, we have

$$\langle f_k, B_1f_k\rangle_{L^2} = \langle g_k, B_1g_k\rangle_{L^2} = 0, \qquad \langle f_k, B_2g_k\rangle_{L^2} = \langle g_k, B_2f_k\rangle_{L^2} = 0.$$

We fix $j \in \mathbb{N}^*$ and by calculation $|\langle f_j^1, Bf_k^1\rangle_{L^2}| = |\langle f_j^1, B_2f_k^1\rangle_{L^2}| \sim k^{-3}$. Now, $\mu_k \sim k^2$ from Remark $A.4$. Thanks to (33) and (39), for every $\epsilon > 0$, there exists $C_1 > 0$ such that, for $k \in \mathbb{N}^*$ large enough,

$$|\langle f_j, Bg_k\rangle_{L^2}| = |\langle f_j, B_1g_k\rangle_{L^2}| \sim \frac{|a_k|}{\mu_k}\left|\sin\left(\left(\sqrt{\mu_k} - \frac{2j\pi}{L_1}\right)\frac{L_1}{2}\right) + \sin\left(\left(\sqrt{\mu_k} + \frac{2j\pi}{L_1}\right)\frac{L_1}{2}\right)\right| \geq \frac{C}{k^{4+\epsilon}}.$$

Moreover, $|\langle g_j, Bg_k\rangle_{L^2}| \sim |a_k^2|^{-1}k^{-2} \sim k^{-3-\epsilon}$ thanks to (33). As in Theorem 1.2, there exists $\tilde{V} \subset \mathbb{R}^+$ countable such that, for every $\{L_1, L_2\} \in \mathcal{AL}(2)$ such that $L_1 \notin \tilde{V}$, we have $|B_{1,k}| \neq 0$ for every $k \in \mathbb{N}^*$. The first point of Assumptions I$(2 + \epsilon)$ is attained and, for every $\epsilon > 0$, there exists $C_2 > 0$ such that

$$|B_{1,k}| \geq C_2k^{-4-\epsilon}, \qquad \forall k \in \mathbb{N}^*.$$

2) Assumptions I.2 . The second point of Assumptions I$(2+\epsilon)$ is verified as in Theorem 1.2 since there exists $V \subset \mathbb{R}^+$ countable such that, for each $\{L_1, L_2\} \in \mathcal{AL}(2)$ such that $L_1 \notin V \cup \tilde{V}$, Assumptions I$(2+\epsilon)$ are verified.

3) Assumptions II.3 and conclusion. The third point of Assumptions II$(2+\epsilon_1, \epsilon_2)$ is valid for $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 \in (0, 1/2)$ since B stabilizes $H^2_{\mathscr{G}}$, H^m and $H^m_{\mathcal{NK}}$ for $m \in \mathbb{N}^*$ similarly to Theorem 1.2. From Theorem 2.5, the controllability holds in $H^{4+\epsilon}_{\mathscr{G}}$ with $\epsilon > 0$. □

Remark 6.1. *The techniques developed in the proofs of theorems* 1.2 *and* 1.3 *can be adopted to ensure the validity of Corollary* 2.7 *when B is the control field*

$$B : \psi = (\psi^1, ...\psi^N) \longmapsto (\varphi^1, ...\varphi^N), \qquad \varphi^l := \sum_{j \leq N} \frac{L_j^{\frac{1}{2}} x^2}{L_l^{\frac{1}{2}}} \psi^j\Big(\frac{L_j}{L_l} x\Big), \quad \forall l \leq N.$$

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A Appendix: Spectral properties

In the current appendix, we characterize $(\lambda_k)_{k\in\mathbb{N}^*}$, the eigenvalues of the Laplacian A in the (BSE), according to the structure of $\mathscr{G}$ and to the definition of $D(A)$.

Proposition A.1. *(Roth's Theorem; [Rot56]) If z is an algebraic irrational number, then for every $\epsilon > 0$ the inequality $\big|z - \frac{n}{m}\big| \leq \frac{1}{m^{2+\epsilon}}$ is satisfied for at most a finite number of $n, m \in \mathbb{Z}$.*

Lemma A.2. *Let $\{L_l\}_{l\leq N_1} \subset \mathbb{R}$ and $\{\tilde{L}_i\}_{i\leq N_2} \subset \mathbb{R}$ with $N_1, N_2 \in \mathbb{N}^*$. Let $\big(\lambda^1_k\big)_{k\in\mathbb{N}^*}$ and $\big(\lambda^2_k\big)_{k\in\mathbb{N}^*}$ be obtained by reordering $\Big\{\frac{k^2\pi^2}{L_l^2}\Big\}_{\substack{k,l\in\mathbb{N}^* \\ l\leq N_1}}$ and $\Big\{\frac{k^2\pi^2}{\tilde{L}_i^2}\Big\}_{\substack{k,i\in\mathbb{N}^* \\ i\leq N_2}}$ respectively. If all the ratios $\tilde{L}_i/L_l$ are algebraic irrational numbers, then for every $\epsilon > 0$ there exists $C > 0$ such that $|\lambda^1_{k+1} - \lambda^2_k| \geq \frac{C}{k^\epsilon}$ for every $k \in \mathbb{N}^*$.*

Proof. For every $k \in \mathbb{N}^*$, there exist $m, n \in \mathbb{N}^*$, $i \leq N_1$ and $l \leq N_2$ such that $\lambda^1_{k+1} = \frac{m^2\pi^2}{L_l^2}$ and $\lambda^2_k = \frac{n^2\pi^2}{\tilde{L}_i^2}$. We suppose $L_l < \tilde{L}_i$. Let z be an algebraic irrational number. From Proposition A.1, we have that, for every $\epsilon > 0$, there exists $C > 0$ such that $|z - n/m| \geq Cm^{-2-\epsilon}$ for every $m, n \in \mathbb{N}^*$. Thus, when $m < n$, for each $\epsilon > 0$, there exists $C_1 > 0$ such that

$$\Big|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\Big| = \Big|\Big(\frac{m\pi}{L_l} + \frac{n\pi}{\tilde{L}_i}\Big)\Big(\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\Big)\Big| \geq \frac{2m\pi}{\tilde{L}_i}\Big|\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\Big| \geq \frac{2C_1\pi^2}{m^\epsilon \tilde{L}_i^2}.$$

If $m \geq n$, then $\big|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\big| \geq \pi^2(L_l^{-2} - \tilde{L}_i^{-2})$, which conclude the proof. □

The following proposition rephrases the results of [BK13, *Theorem* 3.1.8] and [BK13, *Theorem* 3.1.10]. Let $\big(\lambda^{\widehat{\mathscr{G}}}_k\big)_{k\in\mathbb{N}^*}$ be the ordered spectrum of A on a generic compact quantum graph $\widehat{\mathscr{G}}$.

Proposition A.3. [*BK13, Theorem* 3.1.8] *&* [*BK13, Theorem* 3.1.10] *Let $\mathscr{G}$ be a compact graph containing the vertices w and v. Let $\mathscr{G}^{\mathcal{D}}$ be the graph obtained by imposing $(\mathcal{D})$ on w. We have*

$$\lambda^{\mathscr{G}}_k \leq \lambda^{\mathscr{G}^{\mathcal{D}}}_k \leq \lambda^{\mathscr{G}}_{k+1}, \qquad k \in \mathbb{N}^*.$$

Let w and v be equipped with $(\mathcal{NK})$ or $(\mathcal{N})$. If $\mathscr{G}'$ is the graph obtained by merging in $\mathscr{G}$ the vertices w and v in one unique vertex equipped with $(\mathcal{NK})$, then $\lambda^{\mathscr{G}}_k \leq \lambda^{\mathscr{G}'}_k \leq \lambda^{\mathscr{G}}_{k+1}$ for every $k \in \mathbb{N}^*$.

Remark A.4. *Let $\mathscr{G}$ a be compact graphs made by edges of lengths $\{L_l\}_{l\leq N}$. From Proposition A.3,*

$$\exists C_1, C_2 > 0 \quad : \quad C_1 k^2 \leq \lambda^{\mathscr{G}}_k \leq C_2 k^2, \qquad \forall k \geq 2. \tag{34}$$

Indeed, we define $\mathscr{G}^{\mathcal{D}}$ from $\mathscr{G}$ by imposing $(\mathcal{D})$ in each vertex. We denote $\mathscr{G}^{\mathcal{N}}$ from $\mathscr{G}$ by disconnecting each edge and by imposing $(\mathcal{N})$ in each vertex. From Proposition A.3, we have $\lambda_{k-2N}^{\mathscr{G}^{\mathcal{N}}} \leq \lambda_k^{\mathscr{G}} \leq \lambda_{k+M}^{\mathscr{G}^{\mathcal{D}}}$ for $k > 2N$. The sequences $\lambda_k^{\mathscr{G}^{\mathcal{N}}}$ and $\lambda_k^{\mathscr{G}^{\mathcal{D}}}$ are respectively obtained by reordering $\big\{\frac{k^2\pi^2}{L_l^2}\big\}_{\substack{k\in\mathbb{N}^*\\ l\leq N}}$ and $\big\{\frac{(k-1)^2\pi^2}{L_i^2}\big\}_{\substack{k\in\mathbb{N}^*\\ i\leq N}}$. For $l > 2N+1$, $\tilde{m} = \max_{j\leq N} L_j^2$ and $\hat{m} = \min_{j\leq N} L_j^2$, we have

$$\lambda_{l-2N}^{\mathscr{G}^{\mathcal{N}}} \geq \frac{(l-2N-1)^2\pi^2}{N^2\tilde{m}} \geq \frac{l^2\pi^2}{2^{2(2N+1)}N^2\tilde{m}}, \qquad \lambda_{l+M}^{\mathscr{G}^{\mathcal{D}}} \leq \frac{(l+M)^2\pi^2}{\hat{m}} \leq \frac{l^2 2^{2M}\pi^2}{\hat{m}}.$$

The techniques developed in [DZ06, *Appendix A*] in order to prove [DZ06, *Proposition A.11*] lead to following proposition. For $x \in \mathbb{R}$, we denote $E(x)$ the closest integer number to x and

$$|\!|\!| x |\!|\!| = \min_{z\in\mathbb{Z}} |x-z|, \quad F(x) = x - E(x).$$

We notice $|F(x)| = |\!|\!| x |\!|\!|$ and $-\frac{1}{2} \leq F(z) \leq \frac{1}{2}$. Let $\{L_j\}_{j\leq N} \in (\mathbb{R}^+)^N$ and $i \leq N$. We also define

$$n(x) := E\Big(x - \frac{1}{2}\Big), \quad r(x) := F\Big(x - \frac{1}{2}\Big), \quad d(x) := |\!|\!| x - \frac{1}{2} |\!|\!|, \quad \widetilde{m}^i(x) := n\Big(\frac{L_i}{\pi}x\Big).$$

Proposition A.5. *Let $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$ with $N \in \mathbb{N}^*$. Let $(\omega_n)_{n\in\mathbb{N}^*}$ be the unbounded ordered sequence of positive solutions of the equation*

$$\sum_{l\leq N} \sin(xL_l) \prod_{m\neq l} \cos(xL_m) = 0, \qquad x \in \mathbb{R}. \tag{35}$$

For every $\epsilon > 0$, there exists $C_\epsilon > 0$ so that $|\cos(\omega_n L_l)| \geq \frac{C_\epsilon}{\omega_n^{1+\epsilon}}$ for every $l \leq N$ and $n \in \mathbb{N}^*$.

Proof. From [DZ06, *relation (A.3)*], for every $x \in \mathbb{R}$, we obtain the identities

$$2d(x) \leq |\cos(\pi x)| \leq \pi d(x), \qquad 2d\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq \Big|\cos\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\pi\Big)\Big|. \tag{36}$$

As $\cos(\alpha_1 - \alpha_2) = \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)$ for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\widetilde{m}^i(x) + \frac{1}{2} = \frac{L_i}{\pi}x - r\big(\frac{L_i}{\pi}x\big)$ for every $x \in \mathbb{R}$, we have

$$2d\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq |\cos(L_j x)| + \Big|\sin\Big(\pi\frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big|\Big)\Big|. \tag{37}$$

From [DZ06, *relation (A.3)*] and (36), we have the following inequalities $|\sin(\pi|r(\cdot)|)| \leq \pi\,|\!|\!|\,|r(\cdot)|\,|\!|\!| \leq \pi|r(\cdot)| = \pi d(\cdot) \leq \frac{\pi}{2}|\cos(\pi(\cdot))|$, which imply $\big|\sin\big(\pi\frac{L_j}{L_i}\big|r\big(\frac{L_i}{\pi}x\big)\big|\big)\big| \leq \pi\frac{L_j}{L_i}\big|r\big(\frac{L_i}{\pi}x\big)\big| \leq \frac{\pi L_j}{2L_i}|\cos(L_i x)|$ for every $x \in \mathbb{R}$. From (37), there exists $C_1 > 0$ such that, for every $i \leq N$,

$$\prod_{j\neq i} d\Big(\Big(\widetilde{m}^i(x) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq \frac{1}{2^{N-1}} \prod_{j\neq i} |\cos(L_j x)| + C_1|\cos(L_i x)| \quad \forall x \in \mathbb{R}.$$

If there exists $(\omega_{n_k})_{k\in\mathbb{N}^*} \subseteq (\omega_n)_{n\in\mathbb{N}^*}$ such that $|\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$, then $\prod_{j\neq i}|\cos(L_i\omega_{n_k})| \xrightarrow{k\to\infty} 0$ thanks to (35). Equivalently to [DZ06, *relation (A.10)*] (proof of [DZ06, *Proposition A.11*]), there exists a constant $C_2 > 0$ such that, for every $i \in \{0, ..., N\}$, we have

$$C_2|\cos(L_i\omega_n)| \geq \prod_{j\neq i} d\Big(\Big(\widetilde{m}^i(\omega_n) + \frac{1}{2}\Big)\frac{L_j}{L_i}\Big) = \prod_{j\neq i} |\!|\!| \frac{1}{2}\Big(\Big(\widetilde{m}^i(\omega_n) + \frac{1}{2}\Big)\frac{2L_j}{L_i} - 1\Big) |\!|\!|.$$

Now, we have $|\!|\!| \frac{1}{2}(\cdot) |\!|\!| \geq \frac{1}{2}|\!|\!| \cdot |\!|\!|$ and $|\!|\!| (\cdot) - 1 |\!|\!| = |\!|\!| \cdot |\!|\!|$. We consider the Schmidt's Theorem [DZ06, *Theorem A.7*] since $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$. For every $\epsilon > 0$, there exist $C_3, C_4 > 0$ such that, for every $n \in \mathbb{N}^*$, we have $\prod_{j\neq i} \frac{1}{2}|\!|\!| \Big(\widetilde{m}^i(\omega_n) + \frac{1}{2}\Big)\frac{2L_j}{L_i} |\!|\!| \geq \frac{C_3}{(2\widetilde{m}^i(\omega_n)+1)^{1+\epsilon}} \geq \frac{C_4}{\omega_n^{1+\epsilon}}$. □

Remark A.6. *The techniques proving [DZ06, Proposition A.11] and Proposition A.5 lead to the following results. Let $(\omega_n)_{n\in\mathbb{N}^*} \subset \mathbb{R}^+$ be an unbounded sequence and $(\omega_{n_k})_{k\in\mathbb{N}^*}$ any subsequence of $(\omega_n)_{n\in\mathbb{N}^*}$. Let $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$ with $N \in \mathbb{N}^*$ and $l \leq N$.*

1) *If $|\cos(L_l\omega_{n_k})| \xrightarrow{k\to\infty} 0$ implies $\prod_{j\neq l} |\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$ or $\prod_{j\neq l} |\sin(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$, then*

$$\forall \epsilon > 0, \quad \exists C > 0 \quad : \quad |\cos(\omega_n L_l)| \geq C\omega_n^{-1-\epsilon}, \qquad \forall l \leq N,\ n \in \mathbb{N}^*. \tag{38}$$

2) *If $|\sin(L_l\omega_{n_k})| \xrightarrow{k\to\infty} 0$ implies $\prod_{j\neq l} |\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$ or $\prod_{j\neq l} |\sin(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$, then*

$$\forall \epsilon > 0, \quad \exists C > 0 \quad : \quad |\sin(\omega_n L_l)| \geq C\omega_n^{-1-\epsilon}, \qquad \forall l \leq N,\ n \in \mathbb{N}^*. \tag{39}$$

B Appendix: Moment problem

Let $\mathscr{H} = L^2((0,T),\mathbb{R})$ with $T > 0$ and $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$. Let $\mathbf{\Lambda} = (\lambda_k)_{k\in\mathbb{Z}^*}$ be pairwise distinct ordered real numbers such that

$$\exists \mathcal{M} \in \mathbb{N}^*,\ \exists \delta > 0 \quad : \quad \inf_{\{k\in\mathbb{Z}^* \ :\ k+\mathcal{M}\neq 0\}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}. \tag{40}$$

From (40), we notice that there does not exist $\mathcal{M}$ consecutive $k \in \mathbb{Z}^*$ such that $|\lambda_{k+1} - \lambda_k| < \delta$. This leads to a partition of $\mathbb{Z}^*$ in subsets that we call E_m with $m \in \mathbb{Z}^*$. By definition, for every $m \in \mathbb{Z}^*$, if $k, n \in E_m$, then $|\lambda_k - \lambda_n| < \delta(\mathcal{M}-1)$, while if $k \in E_m$ and $n \notin E_m$, then $|\lambda_k - \lambda_n| \geq \delta$. The partition also defines an equivalence relation in $\mathbb{Z}^*$ such that $k, n \in \mathbb{Z}^*$ are equivalent if and only if there exists $m \in \mathbb{Z}^*$ such that $k, n \in E_m$. The sets

$$\{E_m\}_{m\in\mathbb{Z}^*}$$

are the corresponding equivalence classes and $i(m) := |E_m| \leq \mathcal{M} - 1$. For every sequence $\mathbf{x} := (x_l)_{l\in\mathbb{Z}^*}$, we define the vectors

$$\mathbf{x}^m := \{x_l\}_{l\in E_m}$$

for $m \in \mathbb{Z}^*$. Let $\widehat{\mathbf{h}} = (h_j)_{j\leq i(m)} \in \mathbb{C}^{i(m)}$ with $m \in \mathbb{Z}^*$. For every $m \in \mathbb{Z}^*$, we denote $F_m(\widehat{\mathbf{h}}) : \mathbb{C}^{i(m)} \to \mathbb{C}^{i(m)}$ the matrix with elements, for every $j, k \leq i(m)$,

$$F_{m;j,k}(\widehat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l\neq j \\ l\leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

For each $k \in \mathbb{Z}^*$, there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. Let $F(\mathbf{\Lambda})$ be the linear operator on $\ell^2(\mathbb{C})$ such that, for every $\mathbf{x} = (x_l)_{l\in\mathbb{Z}^*} \in D(F(\mathbf{\Lambda}))$,

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \left(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)}\right)_k, \qquad H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \left\{\mathbf{x} := (x_k)_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C}) \ :\ F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{C})\right\}.$$

Proposition B.1. *Let $\mathbf{\Lambda} := (\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of real numbers satisfying (40). Sufficient condition to have $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}(\mathbb{C})$ is the existence of $\tilde{d} \geq 0$ and $C > 0$ such that*

$$|\lambda_{k+1} - \lambda_k| \geq C|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}} \qquad \forall k \in \mathbb{Z}^*. \tag{41}$$

Proof. Thanks to (41), we have $|\lambda_j - \lambda_k| \geq C \min_{l\in E_m} |l|^{-\frac{\tilde{d}}{\mathcal{M}-1}}$ for every $m \in \mathbb{Z}^*$ and $j, k \in E_m$. There exists $C_1 > 0$ such that, for $1 < j, k \leq i(m)$,

$$|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_1\big(\max_{l\in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{k-1} \leq C_1\big(\max_{l\in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{\mathcal{M}-1} \leq C_1 2^{\mathcal{M}\tilde{d}} \min_{l\in E_m} |l|^{\tilde{d}}$$

and $|F_{m;1,1}(\mathbf{\Lambda}^m)| = 1$. Then, there exist $C_2, C_3 > 0$ such that, for $j \leq i(m)$,

$$\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)_{j,j} \leq C_2 \min_{l\in E_m} |l|^{2\tilde{d}}, \quad Tr\Big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\Big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}$$

with $F_m(\mathbf{\Lambda}^m)^*$ the transposed matrix of $F_m(\mathbf{\Lambda}^m)$. Let $\rho(M)$ be the spectral radius of a matrix M and we denote $|||\, M \,||| = \sqrt{\rho(M^*M)}$ its euclidean norm. As $\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)$ is positive-definite, there holds

$$|||\, F_m(\mathbf{\Lambda}^m) \,|||^{\,2} = \rho\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}, \quad \forall m \in \mathbb{Z}^*.$$

In conclusion, $\|F(\mathbf{\Lambda})\mathbf{x}\|^2_{\ell^2} \leq C_3 \|\mathbf{x}\|^2_{h^{\tilde{d}}} < +\infty$ for $\mathbf{x} = (x_k)_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$ as

$$\|F(\mathbf{\Lambda})\mathbf{x}\|^2_{\ell^2} \leq \sum_{m\in\mathbb{Z}^*} |||\, F_m(\mathbf{\Lambda}^m) \,|||^{\,2} \sum_{l\in E_m} |x_l|^2 \leq C_3 \sum_{m\in\mathbb{Z}^*} \min_{l\in E_m} |l|^{2\tilde{d}} \sum_{l\in E_m} |x_l|^2. \qquad \square$$

Corollary B.2. *If $\mathbf{\Lambda} := (\lambda_k)_{k\in\mathbb{Z}^*}$ is an ordered sequence of pairwise distinct real numbers satisfying (40), then $F(\mathbf{\Lambda}) : H(\mathbf{\Lambda}) \to Ran(F(\mathbf{\Lambda}))$ is invertible.*

Proof. As in [DZ06, p. 48], we define $F_m(\mathbf{\Lambda}^m)^{-1}$ the inverse matrix of $F_m(\mathbf{\Lambda}^m)$ for every $m \in \mathbb{Z}^*$. We call $F(\mathbf{\Lambda})^{-1}$ the operator such that $(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k = \big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)}\big)_k$, for every $\mathbf{x} \in Ran(F(\mathbf{\Lambda}))$ and $k \in \mathbb{Z}^*$, which implies $F(\mathbf{\Lambda})^{-1}F(\mathbf{\Lambda}) = Id_{H(\mathbf{\Lambda})}$ and $F(\mathbf{\Lambda})F(\mathbf{\Lambda})^{-1} = Id_{Ran(F(\mathbf{\Lambda}))}$. $\square$

For every $k \in \mathbb{Z}^*$, we have the existence of $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. We define $F(\mathbf{\Lambda})^*$ the infinite matrix such that $(F(\mathbf{\Lambda})^*\mathbf{x})_k = \big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)}\big)_k$ for every $\mathbf{x} = (x_k)_{k\in\mathbb{Z}^*}$ and $k \in \mathbb{Z}^*$, where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed matrix of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$. For $T > 0$, we introduce

$$\mathbf{e} := (e^{i\lambda_j(\cdot)})_{j\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C}), \quad \mathbf{\Xi} := (\xi_k(\cdot))_{k\in\mathbb{Z}^*} = F(\mathbf{\Lambda})^*\mathbf{e} \subset L^2((0,T),\mathbb{C}).$$

Remark B.3. *Thanks to Proposition B.1, when $(\lambda_k)_{k\in\mathbb{Z}^*}$ satisfies (40) and (41), the space $H(\mathbf{\Lambda})$ is dense in $\ell^2(\mathbb{C})$ as $h^{\tilde{d}}$ is dense in ℓ^2. Now, we can consider the infinite matrix $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$ with domain $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*) \subseteq \ell^2(\mathbb{C})$. By transposing each $F_m(\mathbf{\Lambda}^m)$ for $m \in \mathbb{Z}^*$, the arguments of the proof of Corollary B.2 lead to the invertibility of $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \to Ran(F(\mathbf{\Lambda})^*)$ and $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$. Moreover, $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}$ as in Proposition B.1.*

In the following theorem, we rephrase a result of Avdonin and Moran [AM01], which is also proved by Baiocchi, Komornik and Loreti in [BKL02].

Theorem B.4 (Theorem 3.29; [DZ06])**.** *Let $(\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying (40). If $T > 2\pi/\delta$, then $(\xi_k)_{k\in\mathbb{Z}^*}$ forms a Riesz Basis in the space $X := \overline{span\{\xi_k|\ k \in \mathbb{Z}^*\}}^{L^2}$.*

Proposition B.5. *Let $(\omega_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}^+ \cup \{0\}$ be an ordered sequence of real numbers with $\omega_1 = 0$ such that there exist $\tilde{d} \geq 0$, $\delta, C > 0$ and $\mathcal{M} \in \mathbb{N}^*$ with*

$$\inf_{k\in\mathbb{N}^*} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta\mathcal{M}, \qquad |\omega_{k+1} - \omega_k| \geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}^*. \tag{42}$$

Then, for $T > 2\pi/\delta$ and for every $(x_k)_{k\in\mathbb{N}^*} \in h^{\tilde{d}}(\mathbb{C})$ with $x_1 \in \mathbb{R}$,

$$\exists u \in L^2((0,T),\mathbb{R}) \quad : \quad x_k = \int_0^T u(\tau)e^{i\omega_k\tau}d\tau \qquad \forall k \in \mathbb{N}^*. \tag{43}$$

Proof. Let $\mathbf{\Lambda} := (\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of real numbers satisfying (40) and (41). From the definition of Reisz basis ([BL10, *Appendix B.1; Definition 2*]) and [BL10, *Appendix B.1; Proposition 19; 2)*], the map $M : g \in X \mapsto (\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})})_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C})$ is invertible and, for every $k \in \mathbb{Z}^*$, we have

$$\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})} = (F(\mathbf{\Lambda})^*\langle \mathbf{e}, g\rangle_{L^2((0,T),\mathbb{C})})_k.$$

Let $\tilde{X} := M^{-1} \circ F(\mathbf{\Lambda})^*(h^{\tilde{d}}(\mathbb{C}))$. From Remark B.3, we have $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{C})$. The following maps are invertible $(F(\mathbf{\Lambda})^*)^{-1} : Ran(F(\mathbf{\Lambda})^*) \to H(\mathbf{\Lambda})^*$ and

$$(F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto (\langle e^{i\omega_k(\cdot)}, g\rangle_{L^2((0,T),\mathbb{C})})_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C}).$$

For every $(\tilde{x}_k)_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$, there exists $u \in L^2((0,T),\mathbb{C})$ such that $\tilde{x_k} = \int_0^T u(\tau)e^{i\lambda_k\tau}d\tau$ for every $k \in \mathbb{Z}^*$. When $k > 0$, we call $\lambda_k = \omega_k$, while $\lambda_k = -\omega_{-k}$ for $k < 0$ such that $k \neq -1$. The sequence

$(\lambda_k)_{k\in\mathbb{Z}^*\setminus\{-1\}}$ satisfies (40) and (41) with respect to the indices $\mathbb{Z}^*\setminus\{-1\}$. Given $(x_k)_{k\in\mathbb{N}^*}\in h^{\tilde{d}}(\mathbb{C})$, we introduce $(\tilde{x}_k)_{k\in\mathbb{Z}^*\setminus\{-1\}}\in h^{\tilde{d}}(\mathbb{C})$ such that $\tilde{x}_k=x_k$ for $k>0$, while $\tilde{x}_k=\overline{x}_{-k}$ for $k<0$ and $k\neq -1$. As above, there exists $u\in L^2((0,T),\mathbb{C})$ such that $x_1=\int_0^T u(s)ds$ and

$$\tilde{x}_k=\int_0^T u(s)e^{-i\lambda_k s}ds,\ \forall k\in\mathbb{Z}^*\setminus\{-1\}\quad\Longrightarrow\quad\int_0^T u(s)e^{i\omega_k s}ds=x_k=\int_0^T\overline{u}(s)e^{i\omega_k s}ds,\qquad k\in\mathbb{N}^*\setminus\{1\}.$$

If $x_1\in\mathbb{R}$, then u is real and (43) is solvable for $u\in L^2((0,T),\mathbb{R})$. □

Proposition B.6. *Let $(\lambda_k)_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying* (40)*. For every $T>0$, there exists $C(T)>0$ uniformly bounded for T lying on bounded intervals such that*

$$\forall g\in L^2((0,T),\mathbb{C}),\qquad\left\|\int_0^T e^{i\lambda_{(\cdot)}s}g(s)ds\right\|_{\ell^2}\leq C(T)\|g\|_{L^2((0,T),\mathbb{C})}.$$

Proof. **1) Uniformly separated numbers.** Let $(\omega_k)_{k\in\mathbb{N}^*}\subset\mathbb{R}$ be such that $\gamma:=\inf_{k\neq j}|\omega_k-\omega_j|>0$. In the current proof, we adopt the notation $L^2:=L^2((0,T),\mathbb{C})$. Thanks to the Ingham's Theorem [KL05, *Theorem* 4.3], the sequence $\{e^{i\omega_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz Basis in

$$X=\overline{span\{e^{i\omega_k(\cdot)}:\ k\in\mathbb{N}^*\}}^{\,L^2}\subset L^2((0,T),\mathbb{C})\quad\text{when}\quad T>2\pi/\gamma.$$

Now, there exists $C_1(T)>0$ such that $\sum_{k\in\mathbb{N}^*}|\langle e^{i\omega_k(\cdot)},u\rangle_{L^2}|^2\leq C_1(T)^2\|u\|_{L^2}^2$ for every $u\in X$ thanks to [Duc19, *relation* (30)]. Let $P:L^2\longrightarrow X$ be the orthogonal projector. For $g\in L^2$, we have

$$\left\|(\langle e^{i\omega_k(\cdot)},g\rangle_{L^2})_{k\in\mathbb{N}^*}\right\|_{\ell^2}=\left\|(\langle e^{i\omega_k(\cdot)},Pg\rangle_{L^2})_{k\in\mathbb{N}^*}\right\|_{\ell^2}\leq C_1(T)\|Pg\|_{L^2}\leq C_1(T)\|g\|_{L^2}.$$

2) Pairwise distinct numbers. Let $(\lambda_k)_{k\in\mathbb{Z}^*}$ be as in the hypotheses. We decompose $(\lambda_k)_{k\in\mathbb{N}^*}$ in $\mathcal{M}$ sequences $(\lambda_k^j)_{k\in\mathbb{N}^*}$ with $j\leq\mathcal{M}$ such that $\inf_{k\neq l}|\lambda_k^j-\lambda_l^j|>\delta\mathcal{M}$ for every $j\leq\mathcal{M}$. Now, for every $j\leq\mathcal{M}$, we apply the point **1)** with $(\omega_k)_{k\in\mathbb{N}^*}=(\lambda_k^j)_{k\in\mathbb{N}^*}$. For every $T>2\pi/\delta\mathcal{M}$ and $g\in L^2$, there exists $C(T)>0$ uniformly bounded for T in bounded intervals such that

$$\left\|(\langle e^{i\lambda_k(\cdot)},g\rangle_{L^2})_{k\in\mathbb{N}^*}\right\|_{\ell^2}\leq\sum_{j=1}^{\mathcal{M}}\left\|(\langle e^{i\lambda_k^j(\cdot)},g\rangle_{L^2})_{k\in\mathbb{N}^*}\right\|_{\ell^2}\leq\mathcal{M}C(T)\|g\|_{L^2},.$$

3) Conclusion. We know $\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2}\leq\mathcal{M}C(T)\|g\|_{L^2}$ for every $g\in L^2$ and, for $T>2\pi/\delta\mathcal{M}$, we choose the smallest value possible for $C(T)$. When $T\leq 2\pi/\delta\mathcal{M}$, for $g\in L^2$, we define $\tilde{g}\in L^2((0,2\pi/\delta\mathcal{M}+1),\mathbb{C})$ such that $\tilde{g}=g$ on $(0,T)$ and $\tilde{g}=0$ in $(T,2\pi/\delta\mathcal{M}+1)$. Then

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2}=\left\|\int_0^{2\pi/\delta\mathcal{M}+1}e^{i\lambda_{(\cdot)}\tau}\tilde{g}(\tau)dt\right\|_{\ell^2}\leq\mathcal{M}C(2\pi/\delta\mathcal{M}+1)\|g\|_{L^2}.$$

Let $0<T_1<T_2<+\infty$, $g\in L^2(0,T_1)$ and $\tilde{g}\in L^2(0,T_2)$ be defined as $\tilde{g}=g$ on $(0,T_1)$ and $\tilde{g}=0$ on (T_1,T_2). We apply the last inequality to $\tilde{g}$ that leads to $C(T_1)\leq C(T_2)$. □

C Analytic perturbation

The aim of the appendix is to adapt the perturbation theory techniques provided in [Duc18b, *Appendix B*], where the (BSE) is considered on $\mathscr{G}=(0,1)$ and A is the Dirichlet Laplacian. As in the mentioned appendix, we decompose $u(t)=u_0+u_1(t)$, for u_0 and $u_1(t)$ real. Let $A+u(t)B=A+u_0B+u_1(t)B$. We consider u_0B as a perturbative term of A. Let $(\lambda_j^{u_0})_{j\in\mathbb{N}^*}$ be the ordered spectrum of $A+u_0B$ corresponding to some eigenfunctions $(\phi_j^{u_0})_{j\in\mathbb{N}^*}$. We refer to the definition of $\{E_m\}_{m\in\mathbb{Z}^*}$ provided in the first part of Appendix B. We denote $n:\mathbb{N}^*\to\mathbb{N}^*$, $s:\mathbb{N}^*\to\mathbb{N}^*$ and $p:\mathbb{N}^*\to\mathbb{N}^*$ those applications respectively mapping $j\in\mathbb{N}^*$ in $n(j),s(j),p(j)\in\mathbb{N}^*$ such that

$$j\in E_{n(j)},\qquad\lambda_{s(j)}=\inf\{\lambda_k>\lambda_j\mid k\notin E_{n(j)}\},\qquad\lambda_{p(j)}=\sup\{k\in E_{n(j)}\}.$$

The proofs of [Duc18b, *Lemma B.2* & *Lemma B.3*] lead to next lemma. Let $j\in\mathbb{N}^*$ and $P_j^{\perp}$ be the projector onto $\overline{span\{\phi_m\ :\ m\notin E_{n(j)}\}}^{\,L^2}$.

Lemma C.1. *Let (A,B) satisfy Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta>0$ and $\widetilde{d}\geq 0$. There exists a neighborhood $U(0)$ of $u=0$ in $\mathbb{R}$ such that there exists $c>0$ so that*

$$\|\!|\!| \left(A+u_0B-\nu_k\right)^{-1} |\!|\!| \leq c, \qquad \nu_k := (\lambda_{s(k)}-\lambda_{p(k)})/2, \qquad \forall u_0\in U(0),\ \forall k\in\mathbb{N}^*.$$

Moreover, for $u_0\in U(0)$, the operator $(A+u_0P_k^{\perp}B-\lambda_k^{u_0})$ is invertible with bounded inverse from $D(A)\cap Ran(P_k^{\perp})$ to $Ran(P_k^{\perp})$ for every $k\in\mathbb{N}^*$.

Lemma C.2. *Let (A,B) satisfy Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta>0$ and $\widetilde{d}\geq 0$. There exists a neighborhood $U(0)$ of $u=0$ in $\mathbb{R}$ such that, up to a countable subset Q and for every $(k,j),(m,n)\in I:=\{(j,k)\in(\mathbb{N}^*)^2 : j\neq k\}$, $(k,j)\neq(m,n)$, we have*

$$\lambda_k^{u_0}-\lambda_j^{u_0}-\lambda_m^{u_0}+\lambda_n^{u_0}\neq 0, \qquad \langle\phi_k^{u_0},B\phi_j^{u_0}\rangle_{L^2}\neq 0, \qquad \forall u_0\in U(0)\setminus Q.$$

Proof. For $k\in\mathbb{N}^*$, we decompose $\phi_k^{u_0}=a_k\phi_k+\sum_{j\in E^*_{n(k)}}\beta_j^k\phi_j+\eta_k$, where $a_k\in\mathbb{C}$, $\{\beta_j^k\}_{j\in\mathbb{N}^*}\subset\mathbb{C}$ and η_k is orthogonal to ϕ_l for every $l\in E_{n(k)}$. Moreover, $\lim_{|u_0|\to 0}|a_k|=1$ and $\lim_{|u_0|\to 0}|\beta_j^k|=0$ for every $j,k\in\mathbb{N}^*$. We denote $E^*_{n(k)}:=E_{n(k)}\setminus\{k\}$ for every $k\in\mathbb{N}^*$ and

$$\lambda_k^{u_0}\phi_k^{u_0}=(A+u_0B)\Big(a_k\phi_k+\sum_{j\in E^*_{n(k)}}\beta_j^k\phi_j+\eta_k\Big)=a_k(A+u_0B)\phi_k+\sum_{j\in E^*_{n(k)}}\beta_j^k(A+u_0B)\phi_j+(A+u_0B)\eta_k.$$

Now, Lemma $C.1$ leads to the existence of $C_1>0$ such that, for every $k\in\mathbb{N}^*$,

$$\eta_k=-\left(\left(A+u_0P_k^{\perp}B-\lambda_k^{u_0}\right)P_k^{\perp}\right)^{-1}u_0\Big(a_kP_k^{\perp}B\phi_k+\sum_{j\in E^*_{n(k)}}\beta_j^kP_k^{\perp}B\phi_j\Big) \tag{44}$$

and $\|\eta_k\|_{L^2}\leq C_1|u_0|$. We compute $\lambda_k^{u_0}=\langle\phi_k^{u_0},(A+u_0B)\phi_k^{u_0}\rangle_{L^2}$ for every $k\in\mathbb{N}^*$ and

$$\lambda_k^{u_0}=\Big(\lambda_k|a_k|^2+\sum_{j\in E^*_{n(k)}}\lambda_j|\beta_j^k|^2\Big)+\langle\eta_k,(A+u_0B)\eta_k\rangle_{L^2}+u_0\sum_{j,l\in E^*_{n(k)}}\overline{\beta_j^k}\beta_l^kB_{j,l}$$
$$+u_0|a_k|^2B_{k,k}+2u_0\Re\Big(\sum_{j\in E^*_{n(k)}}\beta_j^k\langle\eta_k,B\phi_j\rangle_{L^2}+\overline{a_k}\sum_{j\in E^*_{n(k)}}\beta_j^kB_{k,j}+\overline{a_k}\langle\phi_k,B\eta_k\rangle_{L^2}\Big).$$

Thanks to (44), it follows $\langle\eta_k,(A+u_0B)\eta_k\rangle_{L^2}=O(u_0^2)$ for every $k\in\mathbb{N}^*$. Let

$$\widehat{a}_k:=\frac{|a_k|^2+\sum_{j\in E^*_{n(k)}}|\beta_j^k|^2}{1-\|\eta_k\|_{L^2}^2}, \qquad \widetilde{a}_k:=\frac{|a_k|^2+\sum_{j\in E^*_{n(k)}}\lambda_j/\lambda_k|\beta_j^k|^2}{1-\|\eta_k\|_{L^2}^2}.$$

As $\|\eta_k\|_{L^2}\leq C_1|u_0|$ for every $k\in\mathbb{N}^*$, it follows $\lim_{|u_0|\to 0}|\widehat{a}_k|=1$ uniformly in k. Thanks to

$$\lim_{k\to+\infty}\inf_{j\in E^*_{n(k)}}\lambda_j{\lambda_k}^{-1}=\lim_{k\to+\infty}\sup_{j\in E^*_{n(k)}}\lambda_j{\lambda_k}^{-1}=1,$$

we have $\lim_{|u_0|\to 0}|\widetilde{a}_k|=1$ uniformly in k. Now, there exists f_k such that $\lambda_k^{u_0}=\widetilde{a}_k\lambda_k+u_0\widehat{a}_kB_{k,k}+u_0f_k$ where $\lim_{|u_0|\to 0}f_k=0$ uniformly in k (the relation is also valid when $\lambda_k=0$). For each $(k,j),(m,n)\in I$ such that $(k,j)\neq(m,n)$, there exists $f_{k,j,m,n}$ such that $\lim_{|u_0|\to 0}f_{k,j,m,n}=0$ uniformly in k,j,m,n and

$$\lambda_k^{u_0}-\lambda_j^{u_0}-\lambda_m^{u_0}+\lambda_n^{u_0}=\widetilde{a}_k\lambda_k-\widetilde{a}_j\lambda_j-\widetilde{a}_m\lambda_m+\widetilde{a}_n\lambda_n+u_0f_{k,j,m,n}+u_0(\widehat{a}_kB_{k,k}-\widehat{a}_jB_{j,j}-\widehat{a}_mB_{m,m}$$
$$+\widehat{a}_nB_{n,n})=\widetilde{a}_k\lambda_k-\widetilde{a}_j\lambda_j-\widetilde{a}_m\lambda_m+\widetilde{a}_n\lambda_n+u_0(\widehat{a}_kB_{k,k}-\widehat{a}_jB_{j,j}-\widehat{a}_mB_{m,m}+\widehat{a}_nB_{n,n})+O(u_0^2).$$

Thanks to the second point of Assumptions I, there exists $U(0)$ a neighborhood of $u=0$ in $\mathbb{R}$ small enough such that, for each $u\in U(0)$, we have that every function $\lambda_k^{u_0}-\lambda_j^{u_0}-\lambda_m^{u_0}+\lambda_n^{u_0}$ is not constant and analytic. Now, $V_{(k,j,m,n)}=\{u\in D\big|\ \lambda_k^u-\lambda_j^u-\lambda_m^u+\lambda_n^u=0\}$ is a discrete subset of D and

$$V=\{u\in D\big|\ \exists((k,j),(m,n))\in I^2:\lambda_k^u-\lambda_j^u-\lambda_m^u+\lambda_n^u=0\}$$

is a countable subset of D, which achieves the proof of the first claim. The second relation is proved with the same technique. For $j,k\in\mathbb{N}^*$, the analytic function $u_0\to\langle\phi_j^{u_0},B\phi_k^{u_0}\rangle_{L^2}$ is not constantly zero since $\langle\phi_j,B\phi_k\rangle_{L^2}\neq 0$ and $W=\{u\in D\big|\ \exists(k,j)\in I:\langle\phi_j^{u_0},B\phi_k^{u_0}\rangle_{L^2}=0\}$ is a countable subset of D. $\square$

Lemma C.3. *Let (A,B) satisfy Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta>0$ and $\widetilde{d}\geq 0$. Let $T>0$ and $s=d+2$ for d introduced in Assumptions II. Let $c\in\mathbb{R}$ such that $0\notin\sigma(A+u_0B+c)$ (the spectrum of $A+u_0B+c$) and such that $A+u_0B+c$ is a positive operator. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that,*

$$\forall u_0\in U(0),\qquad \left\||A+u_0B+c|^{\frac{s}{2}}\cdot\right\|_{L^2}\asymp\|\cdot\|_{(s)}. \tag{45}$$

Proof. Let D be the neighborhood provided by Lemma C.2. The proof follows the one of [Duc18b, *Lemma B.6*]. We suppose that $0\notin\sigma(A+u_0B)$ and $A+u_0B$ is positive such that we can assume $c=0$. When $c\neq 0$, the proof follows from the same arguments. Thanks to Remark 2.1, we have $\|\cdot\|_{(s)}\asymp\||A|^{\frac{s}{2}}\cdot\|_{L^2}$. We prove the existence of $C_1,C_2,C_3>0$ such that, for every $\psi\in D(|A+u_0B|^{\frac{s}{2}})=D(|A|^{\frac{s}{2}})$,

$$\|(A+u_0B)^{\frac{s}{2}}\psi\|_{L^2}\leq C_1\|A^{\frac{s}{2}}\psi\|_{L^2}+C_2\|\psi\|_{L^2}\leq C_3\|A^{\frac{s}{2}}\psi\|_{L^2}. \tag{46}$$

Let $s/2=k\in\mathbb{N}^*$. The relation (46) is proved by iterative argument. First, it is true for $k=1$ when $B\in L(D(A))$ as there exists $C>0$ such that $\|AB\psi\|_{L^2}\leq C\,|||\,B\,|||_{L(D(A))}\|A\psi\|_{L^2}$ for $\psi\in D(A)$. When $k=2$ if $B\in L(\mathscr{H})$ and $B\in L(D(A^{k_1}))$ for $1\leq k_1\leq 2$, then there exist $C_4,C_5>0$ such that

$$\begin{aligned}\|(A+u_0B)^2\psi\|_{L^2}&\leq\|A^2\psi\|_{L^2}+|u_0|^2\|B^2\psi\|_{L^2}+|u_0|\|AB\psi\|_{L^2}+|u_0|\|BA\psi\|_{L^2}\\&\leq\|A^2\psi\|_{L^2}+|u_0|^2\,|||\,B^2\,|||\,\|\psi\|_{L^2}+C_4|u_0|\,|||\,B\,|||_{L(D(A^{k_1}))}\|\psi\|_{(k_1)}+|u_0|\,|||\,B\,|||\,\|\psi\|_{(2)}\end{aligned}$$

and $\|(A+u_0B)^2\psi\|_{L^2}\leq C_5\|A^2\psi\|_{L^2}$ for every $\psi\in D(A^2)$. Second, we assume (46) being valid for $k\in\mathbb{N}^*$ when $B\in L(D(A^{k_j}))$ for $k-j-1\leq k_j\leq k-j$ and for every $j\in\{0,...,k-1\}$. We prove (46) for $k+1$ when $B\in L(D(A^{k_j}))$ for $k-j\leq k_j\leq k-j+1$ and for every $j\in\{0,...,k\}$. Now, there exists $C>0$ such that $\|A^kB\psi\|_{L^2}\leq C\,|||\,B\,|||_{D(A^{k_0})}\|A^{k_0}\psi\|_{L^2}$ for every $\psi\in D(A^{k+1})$. Thus, as $\|(A+u_0B)^{k+1}\psi\|_{L^2}=\|(A+u_0B)^k(A+u_0B)\psi\|_{L^2}$, there exist $C_6,C_7>0$ such that

$$\|(A+u_0B)^{k+1}\psi\|_{L^2}\leq C_6(\|A^{k+1}\psi\|_{L^2}+|u_0|\|A^kB\psi\|_{L^2}+\|A\psi\|_{L^2}+|u_0|\|B\psi\|_{L^2})\leq C_7\|A^{k+1}\psi\|_{L^2}$$

for every $\psi\in D(A^{k+1})$. As in the proof of [Duc18b, *Lemma B.6*], the relation (46) is valid for any $s\leq k$ when $B\in L(D(A^{k_0}))$ for $k-1\leq k_0\leq s$ and $B\in L(D(A^{k_j}))$ for $k-j-1\leq k_j\leq k-j$ and for every $j\in\{1,...,k-1\}$. The opposite inequality follows by decomposing $A=A+u_0B-u_0B$.

In our framework, Assumptions II ensure that the parameter s is $2+d$. Indeed, if the second point of Assumptions II is verified for $s\in[4,11/2)$, then B preserves $H^{d_1}_{\mathcal{NK}}$ and $H^2_{\mathscr{G}}$ for d_1 introduced in Assumptions II. Proposition 3.2 claims that $B:H^{d_1}_{\mathscr{G}}\to H^{d_1}_{\mathscr{G}}$ and the argument of [Duc18b, *Remark 2.1*] implies $B\in L(H^{d_1}_{\mathscr{G}})$. Thus, the identity (45) is valid because $B\in L(\mathscr{H})$, $B\in L(H^2_{\mathscr{G}})$ and $B\in L(H^{d_1}_{\mathscr{G}})$ with $d_1>s-2$. If the third point of Assumptions II is verified for $s\in[4,9/2)$, then $B\in L(\mathscr{H})$, $B\in L(H^2_{\mathscr{G}})$ and $B\in L(H^{d_1}_{\mathscr{G}})$ for $d_1\in[d,9,2)$. The claim follows thanks to Proposition 3.2 since B stabilizes H^{d_1} and $H^2_{\mathscr{G}}$ for d_1 introduced in Assumptions II. If $s<4$ instead, then the conditions $B\in L(\mathscr{H})$ and $B\in L(H^2_{\mathscr{G}})$ are sufficient to guarantee (45). □

Remark C.4. *The techniques developed in the proof of Lemma C.3 imply the following claim. Let (A,B) satisfy Assumptions I(η) and Assumptions II($\eta,\tilde{d}$) for $\eta>0$ and $\widetilde{d}\geq 0$. Let $0<s_1<d+2$ for d introduced in Assumptions II. Let $c\in\mathbb{R}$ such that $0\notin\sigma(A+u_0B+c)$ and such that $A+u_0B+c$ is a positive operator. We have There exists a neighborhood $U(0)\subset\mathbb{R}$ of 0 so that, for any $u_0\in U(0)$, we have $\||A+u_0B+c|^{\frac{s_1}{2}}\cdot\|_{L^2}\asymp\|\cdot\|_{(s_1)}$.*

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