Bilinear quantum systems on compact graphs: well-posedness and global exact controllability

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Abstract

In the present work, we study the bilinear Schrödinger equation $i\partial_t\psi = A\psi + u(t)B\psi$ in $L^2(\mathscr{G},\mathbb{C})$ where $\mathscr{G}$ is a compact graph. The operator A is a self-adjoint Laplacian, B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ is the control with $T > 0$. We study interpolation properties of the spaces $D(|A|^{s/2})$ for $s > 0$, which allow to prove the well-posedness of the equation in $D(|A|^{s/2})$ with $s \geq 3$. In such spaces, we attain the global exact controllability of the bilinear Schrödinger equation under suitable assumptions on $\mathscr{G}$. We provide examples of the main results involving star graphs and tadpole graphs.

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Keywords: Bilinear Schrödinger equation, global exact controllability, quantum compact graphs, star graphs, tadpole graphs, moments problems.

1 Introduction

In this paper, we study the evolution of a particle confined in a compact graph type structure $\mathscr{G}$ (*e.g.* Figure 1) and subjected to an external field. Its dynamics is modeled by the bilinear Schrödinger equation in the Hilbert space $\mathscr{H} := L^2(\mathscr{G},\mathbb{C})$

$$\text{(BSE)} \qquad \begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0,T), \\ \psi(0) = \psi_0, & T > 0. \end{cases}$$

The term $u(t)B$ represents the control field, where the symmetric operator B describes the action of the field and $u \in L^2((0,T),\mathbb{R})$ its intensity. The operator $A = -\Delta$ is a self-adjoint Laplacian. When the (BSE) is well-posed, we call Γ_t^u the unitary propagator generated by $A + u(t)B$.

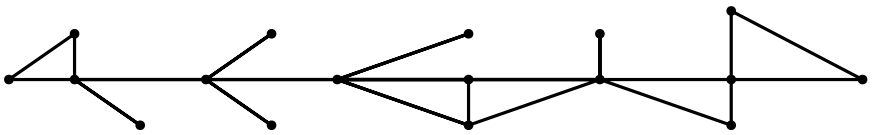

Figure 1: Example of compact graph

A natural question of practical implications is whether, given a couple of states, there exists u steering the system from the first state to the second one. In other words, when the (BSE) is *exactly controllable*. The (BSE) is said to be *approximately controllable* when, for any couple of states, it is possible to drive the system from the first state as close as desired to the second one with a suitable control u and in finite time. Each type of controllability is said to be *simultaneous* when it is simultaneously satisfied between more couples of states with the same control.

The use of graph theory in mathematics and physics is nowadays gaining more and more popularity. In control theory, problems involving graphs have been popularized in the very last decades and many results are still missing. In fact, a complete theory is far from being formulated as the interaction between the components of a graph may generate unexpected phenomena. On this peculiarity, we refer to [DZ06] by Dáger and Zuazua where the boundary controllability is studied for various partial differential equations. Nevertheless, the controllability of the bilinear Schrödinger equation on graphs is still an open problem. For this reason, we study well-posedness and global exact controllability of the (BSE) in suitable subspaces of $D(A)$.

The choice of considering subspaces of $D(A)$ is classical for this type of results and it is due to the seminal work [BMS82] on bilinear systems by Ball, Mardsen and Slemrod. Even though they ensure that the (BSE) admits a unique solution in $\mathscr{H}$, they also prove that, for $u \in L^2_{loc}((0,\infty),\mathbb{R})$, the exact controllability of the bilinear Schrödinger equation can not be achieved in $\mathscr{H}$ and in $D(A)$ when $B : D(A) \to D(A)$ (see [BMS82, *Theorem* 3.6]).

Because of the Ball, Mardsen and Slemrod result, many authors have considered weaker notions of controllability when $\mathscr{G} = (0,1)$. Let

$$D(A_D) = H^2((0,1),\mathbb{C}) \cap H^1_0((0,1),\mathbb{C})), \qquad A_D\psi := -\Delta\psi, \qquad \forall \psi \in D(A_D).$$

In [BL10], Beauchard and Laurent prove the *well-posedness* and the *local exact controllability* of the bilinear Schrödinger equation in $H^s_{(0)} := D(A_D^{s/2})$ for $s = 3$, when B is a multiplication operator for suitable $\mu \in H^3((0,1),\mathbb{R})$.
In [Mor14], Morancey proves the *simultaneous local exact controllability* of two or three (BSE) in $H^3_{(0)}$ for suitable $B = \mu \in H^3((0,1),\mathbb{R})$.
In [MN15], Morancey and Nersesyan extend the previous result. They achieve the *simultaneous global exact controllability* of finitely many bilinear Schrödinger equations in $H^4_{(0)}$ for suitable $B = \mu \in H^4((0,1),\mathbb{R})$.

In [Ducb], the author ensures the *simultaneous global exact controllability in projection* of infinite (BSE) in $H^3_{(0)}$ for suitable bounded symmetric B. Under similar assumptions, the author exhibits the *global exact controllability* of the bilinear Schrödinger equation between eigenstates via explicit controls and explicit times in [Duca].

The *global approximate controllability* of the (BSE) is proved with many different techniques in literature. Some of the existing results are the following. The outcome is achieved with Lyapunov techniques by Mirrahimi in [Mir09] and by Nersesyan in [Ner10]. Adiabatic arguments are considered by Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti in [BCMS12] and [BGRS15]. Lie-Galerking methods are used by Boscain, Boussaïd, Caponigro, Chambrion and Sigalotti in [BdCC13] and [BCS14].

1.1 Preliminaries

Let $\mathscr{G}$ be a compact graph composed by $N \in \mathbb{N}$ edges $\{e_j\}_{j\leq N}$ of lengths $\{L_j\}_{j\leq N}$ and $M \in \mathbb{N}$ vertices $\{v_j\}_{j\leq M}$. We call V_e and V_i the external and the internal vertices of $\mathscr{G}$, *i.e.*

$$V_e := \big\{v \in \{v_j\}_{j\leq M} \mid \exists! e \in \{e_j\}_{j\leq N} : v \in e\big\}, \qquad V_i := \{v_j\}_{j\leq M} \setminus V_e.$$

We study graphs $\mathscr{G}$ equipped with a metric parametrizing each e_k with a coordinate going from 0 to the length of the edge L_k. We recall that a graph is said to be compact when it composed by a finite number of vertices and edges of finite length.

We consider a compact metric graph $\mathscr{G}$ as domain of functions $f := (f^1, ..., f^N) : \mathscr{G} \to \mathbb{C}$ so that $f^j : e_j \to \mathbb{C}$ with $j \leq N$. For $s > 0$, we denote

$$\mathscr{H} = L^2(\mathscr{G}, \mathbb{C}) = \prod_{j=1}^{N} L^2(e_j, \mathbb{C}), \qquad H^s := H^s(\mathscr{G}, \mathbb{C}) = \prod_{j=1}^{N} H^s(e_j, \mathbb{C}).$$

The Hilbert space $\mathscr{H}$ is equipped with the norm $\|\cdot\|$ and the scalar product

$$\langle \psi, \phi \rangle := \langle \psi, \varphi \rangle_{\mathscr{H}} = \sum_{j\leq N} \langle \psi^j, \varphi^j \rangle_{L^2(e_j,\mathbb{C})} = \sum_{j\leq N} \int_{e_j} \overline{\psi^j}(x)\varphi^j(x)dx, \quad \forall \psi, \varphi \in \mathscr{H}.$$

In the (BSE), the operator A is a self-adjoint Laplacian such that the functions in $D(A)$ satisfy the following boundary conditions. Each $v \in V_i$ is equipped with *Neumann-Kirchhoff* boundary conditions when

$$f \text{ is continuous in } v, \qquad \sum_{e\ni v} \frac{\partial f}{\partial x_e}(v) = 0, \qquad \forall f \in D(A).$$

The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions). The external vertices V_e are equipped with *Dirichlet* or *Neumann* type boundary conditions, *i.e.* for every $v \in V_e$,

$$\text{either} \quad f(v) = 0 \ \ (\textit{Dirichlet}), \quad \textit{or} \quad \frac{\partial f}{\partial x}(v) = 0 \ \ (\textit{Neumann}) \quad \forall f \in D(A).$$

For every compact graph, the operator A admits purely discrete spectrum (see [Kuc04, *Theorem* 18]). We call $\{\lambda_j\}_{j\in\mathbb{N}}$ the non-decreasing sequence of eigenvalues of A and $\{\phi_j\}_{j\in\mathbb{N}}$ a Hilbert basis of $\mathscr{H}$ composed by corresponding eigenfunctions.

1.2 Novelties of the work

The main difference between studying the controllability of the bilinear Schrödinger equation on $\mathscr{G} = (0,1)$ and on generic graph $\mathscr{G}$ is that

$$\inf_{k\in\mathbb{N}} |\lambda_{k+1} - \lambda_k| \geq 0 \qquad \text{when} \quad \mathscr{G} = (0,1), \tag{1}$$

which is an important hypothesis in the works [BL10], [Ducb], [Duca] and [Mor14]. Unfortunately, the identity (1) is not guaranteed when $\mathscr{G}$ is a generic compact graphs. Nevertheless, there exist $\mathcal{M} \in \mathbb{N}$ and $\delta > 0$ so that

$$\inf_{k\in\mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| > \delta\mathcal{M} \tag{2}$$

(see Remark 2.2 for further details). To ensure controllability results, we introduce a weaker assumption on the spectral gap and we assume that

$$\exists C > 0,\ \tilde{d} \geq 0 \quad : \quad |\lambda_{k+1} - \lambda_k| \geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}. \tag{3}$$

Proving the validity of the identity (3) is not an easy task as the spectrum of A is usually unexplicit. In addition, the more the structure of the graph is complicated, the more the spectral behaviour is difficult to characterize.

By using Roth's Theorem [Rot56], we prove the validity of the identity (3) for the following types of graphs.

Figure 2: Respectively a star graph, a double-ring graph, a tadpole graph and a two-tails tadpole graph.

The spectral gap is valid when all the ratios L_k/L_j are *algebraic irrational numbers* independently from the choice of boundary conditions of $D(A)$ in the external vertices, which can be both Dirichlet, or Neumann type.

Afterwards, we study the spaces $H^s_{\mathscr{G}}$ with $s > 0$ and we ensure different *interpolation properties*. When $D(A)$ is equipped with Dirichlet or Neumann boundary conditions in V_e and Neumann-Kirchhoff in V_i, we show that

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2} \qquad \forall s_1 \in \mathbb{N}\cup\{0\},\ s_2 \in [0,1/2).$$

This identity holds under generic assumptions on the problem, but stronger outcomes can be guaranteed by imposing more restrictive conditions. We provide the complete result in Proposition 3.2.

The interpolation properties are crucial for *well-posedness* of the bilinear Schrödinger equation in $H^s_{\mathscr{G}}$ with specific $s \geq 3$. In such spaces, we prove the *global exact controllability* when the identities (2) and (3) are satisfied with suitable parameter $\tilde{d}$. The complete result is provided in Theorem 2.3. Two interesting applications of Theorem 2.3 are the following examples that respectively involve a *star graph* and a *tadpole graph*.

Let $\mathscr{G}$ be a *star graph* composed by $N \in \mathbb{N}$ edges $\{e_k\}_{k\leq N}$. Each e_k is parametrized with a coordinate going from 0 to the length of the edge L_k. We set the coordinate 0 in the external vertex belonging to e_k.

Figure 3: The figure shows the parametrization of a star graph with 4 edges.

Definition 1.1. For every $N \in \mathbb{N}$, we define $\mathcal{AL}(N) \subset (\mathbb{R}^+)^N$ as follows. For every $\{L_j\}_{j\leq N} \in \mathcal{AL}(N)$, the numbers $\big\{1, \{L_j\}_{j\leq N}\big\}$ are linearly independent over $\mathbb{Q}$ and all the ratios L_k/L_j are algebraic irrational numbers.

Example 1.2. *Let $\mathscr{G}$ be a star graph with four edges of lengths $\{L_j\}_{j\leq 4}$ and $D(A)$ be equipped with Dirichlet boundary conditions in V_e and Neumann-Kirchhoff in V_i. Let $B : \psi = (\psi^1, \psi^2, \psi^3, \psi^4) \longmapsto \big((x - L_1)^4\psi^1, 0, 0, 0\big)$ for every $\psi \in \mathscr{H}$. There exists $\mathcal{C} \subset (\mathbb{R}^+)^4$ countable such that, for every $\{L_j\}_{j\leq 4} \in \mathcal{AL}(4) \setminus \mathcal{C}$, the (BSE) is globally exactly controllable in*

$$H^{4+\epsilon}_{\mathscr{G}} \qquad \epsilon \in (0, 1/2).$$

In other words, for every $\psi^1, \psi^2 \in H^{4+\epsilon}_{\mathscr{G}}$ such that $\|\psi^1\| = \|\psi^2\|$, there exist $T > 0$ and $u \in L^2((0,T), \mathbb{R})$ such that $\Gamma^u_T\psi^1 = \psi^2$.

Proof. See Section 6. □

In Example 1.2, we notice an interesting phenomenon. The controllability holds even if the control field only acts on one edge of the graph. It is due to the choice of the lengths, which are linearly independent over $\mathbb{Q}$ and such that all the ratios L_k/L_j are algebraic irrational numbers.

Let $\mathscr{G}$ be a *tadpole graph* composed by two edges $\{e_1, e_2\}$. The self-closing edge e_1 is parametrized in the clockwise direction with a coordinate going from 0 to L_1 (the length e_1). On the "tail" e_2, we consider a coordinate going from 0 to L_2 and we associate the 0 to the external vertex.

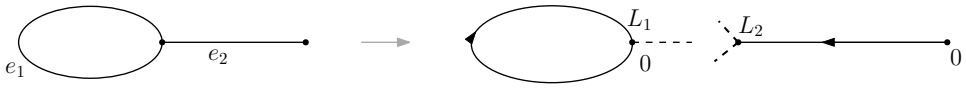

Figure 4: The parametrization of the tadpole graph.

Example 1.3. *Let $\mathscr{G}$ be a tadpole graph. Let $D(A)$ be equipped with Dirichlet boundary conditions in V_e and Neumann-Kirchhoff in V_i. Let*

$$\mu_1(x) := \sin\Big(\frac{2\pi}{L_1}x\Big) + x(x - L_1), \qquad \mu_2(x) := x^2 - (2L_1 + 2L_2)x + L_2^2 + 2L_1L_2$$

and $B : \psi = (\psi^1, \psi^2) \longmapsto (\mu_1\psi^1, \mu_2\psi_2)$ for every $\psi \in \mathscr{H}$. There exists $\mathcal{C} \subset (\mathbb{R}^+)^2$ countable so that, for each $\{L_1, L_2\} \in \mathcal{AL}(2) \setminus \mathcal{C}$, the (BSE) is globally exactly controllable in

$$H^{4+\epsilon}_{\mathscr{G}} \qquad \epsilon \in (0, 1/2).$$

Proof. See Section 6. □

The techniques adopted in Example 1.2 and Example 1.3 are also valid if we consider Neumann boundary condition in the external vertices.

Let $\{I_j\}_{j\leq N}$ be a set of unconnected intervals with $N \in \mathbb{N}$ and $\Gamma^{u,j}_t$ be the propagator generated by $A_j + u(t)B_j$ with

$$B_j := B|_{L^2(I_j,\mathbb{C})}, \qquad A_j := A|_{L^2(I_j,\mathbb{C})}, \qquad H^s_{I_j} := D\big(|A_j|^{\frac{s}{2}}\big), \qquad s > 0.$$

The following result, denoted *contemporaneous controllability*, follows from Theorem 2.3 when we consider $\mathscr{G} = \{I_j\}_{j\leq N}$.

Example 1.4. *Let $\{I_j\}_{j\leq N}$ with $N \in \mathbb{N}$ be a set of unconnected intervals and $D(A)$ be equipped with Dirichlet boundary conditions. Let*

$$B : \psi = (\psi^1, ...\psi^N) \mapsto \left(\sum_{j\leq N} \frac{L_j^{\frac{1}{2}}x^2}{L_1^{\frac{1}{2}}} \psi^j\Big(\frac{L_j}{L_1}x\Big), ..., \sum_{j\leq N} \frac{L_j^{\frac{1}{2}}x^2}{L_N^{\frac{1}{2}}} \psi^j\Big(\frac{L_j}{L_N}x\Big) \right).$$

There exists $\mathcal{C} \subset (\mathbb{R}^+)^N$ countable such that, for every $\{L_j\}_{j\leq N} \in \mathcal{AL}(N)\backslash\mathcal{C}$, the (BSE) is contemporaneously globally exactly controllable in

$$\prod_{j\leq N} H^{3+\epsilon}_{I_j} \qquad \epsilon \in (0, 1/2).$$

In other words, for every $\psi_1, \psi_2 \in \prod_{j\leq N} H^{3+\epsilon}_{I_j}$ such that $\|\psi_1\| = \|\psi_2\|$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that $\Gamma^{u,j}_T \psi^j_1 = \psi^j_2$ for every $j \leq N$.

Proof. See Section 6. □

The *contemporaneous controllability* is deeply different from the *simultaneous controllability* provided by [Mor14], [MN15] and [Ducb] where the authors consider sequences of functions belonging to the same space.

1.3 Scheme of the work

In Section 2, we present the main results of the work. The *global exact controllability* of the (BSE) is ensured in Theorem 2.3. Theorem 2.4 shows types of graphs satisfying the hypothesis of Theorem 2.3. The *contemporaneous controllability* is introduced in Corollary 2.6.
In Section 3, Proposition 3.1 provides the *well-posedness* of the (BSE). We attain *interpolation properties* of the spaces $H^s_{\mathscr{G}}$ for $s > 0$ in Proposition 3.2. Section 4 exhibits the proof of Theorem 2.3, while the proofs of Theorem 2.4 and Corollary 2.6 are provided in Section 5.
In Section 6, we explain Example 1.2, Example 1.3 and Example 1.4.
In Appendix A, we prove some spectral results by using classical theorems on the approximation of real numbers by rational ones.
We treat the solvability of the so-called *moments problems* in Appendix B.
In Appendix C, we adapt the perturbation theory techniques exposed in [Ducb, *Appendix A*].

2 Main results

Let $\mathscr{G}$ be a compact graph composed by N edges $\{e_j\}_{j\leq N}$ of lengths $\{L_j\}_{j\leq N}$ connecting M vertices $\{v_j\}_{j\leq M}$. For each $j \leq M$, we denote

$$N(v_j) := \big\{l \in \{1, ..., N\} \mid v_j \in e_l\big\}, \qquad n(v_j) := |N(v_j)|. \tag{4}$$

We respectively call $(\mathcal{NK})$, $(\mathcal{D})$ and $(\mathcal{N})$ the *Neumann-Kirchhoff*, *Dirichlet* and *Neumann* boundary conditions for the $D(A)$.

When we consider the self-adjoint operator A on $\mathscr{G}$, $\mathscr{G}$ is called *quantum graph*. By denoting $\mathscr{G}$ as a compact quantum graph, we are implicitly introducing a Laplacian A equipped with self-adjoint boundary conditions.

We say that a quantum graph $\mathscr{G}$ is equipped with one of the previous boundary conditions in a vertex v, when each $f \in D(A)$ satisfies it in v. A quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies $(\mathcal{D})$ or $(\mathcal{N})$ in v and, for every $v \in V_i$, the function f verifies $(\mathcal{NK})$ in v. We say that a quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$ (or $(\mathcal{N})$-$(\mathcal{NK})$) when, for every $f \in D(A)$ and $v \in V_e$, the function f satisfies $(\mathcal{D})$ (or $(\mathcal{N})$) in v and verifies $(\mathcal{NK})$ in every $v \in V_i$.

Let $\phi_j(t) = e^{-i\lambda_j t}\phi_j$ and $[r]$ be the entire part of $r \in \mathbb{R}$. For $s > 0$, let

$$H^s_{\mathcal{NK}} := \Big\{\psi \in H^s \mid \partial_x^{2n}\psi \text{ is continuous in } v, \ \ \forall n \in \mathbb{N}\cup\{0\},\ n < \big[(s+1)/2\big]; \\ \sum_{e\in N(v)} \partial_{x_e}^{2n+1} f(v) = 0,\ \forall n \in \mathbb{N}\cup\{0\},\ n < \big[s/2\big],\ \forall v \in V_i\Big\},$$

$$H^s_{\mathscr{G}} = H^s_{\mathscr{G}}(\mathscr{G}, \mathbb{C}) := D(A^{s/2}), \qquad \|\cdot\|_{(s)} := \|\cdot\|_{H^s_{\mathscr{G}}} = \Big(\sum_{k\in\mathbb{N}} |k^s\langle\cdot, \phi_k\rangle|^2\Big)^{\frac{1}{2}},$$

$$h^s(\mathbb{C}) := \Big\{ \{a_k\}_{k\in\mathbb{N}} \subset \mathbb{C} \;\Big|\; \sum_{k\in\mathbb{N}} |k^s a_k|^2 < \infty \Big\}, \qquad \|\cdot\|_{(s)} := \Big(\sum_{k\in\mathbb{N}} |k^s \cdot|^2 \Big)^{\frac{1}{2}}.$$

Remark 2.1. *If $0 \notin \sigma(A)$ (the spectrum of A), then $\|\cdot\|_{(s)} \asymp \||A|^{\frac{s}{2}} \cdot \|$, i.e.*

$$\exists\ C_1, C_2 > 0 \quad : \qquad C_1 \|\cdot\|_{(s)}^2 \leq \||A|^{\frac{s}{2}} \cdot \|^2 = \sum_{k\in\mathbb{N}} |\lambda_k^{\frac{s}{2}} \langle \cdot, \phi_k \rangle|^2 \leq C_2 \|\cdot\|_{(s)}^2.$$

Indeed, from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10], there exist $C_3, C_4 > 0$ such that $C_3 k^2 \leq \lambda_k \leq C_4 k^2$ for every $k \geq 2$ and for $k = 1$ if $\lambda_1 \neq 0$ (see Remark A.4 for further details). If $0 \in \sigma(A)$, then $\lambda_1 = 0$ and there exists $c \in \mathbb{R}$ such that $0 \notin \sigma(A + c)$ and $\|\cdot\|_{(s)} \asymp \||A + c|^{\frac{s}{2}} \cdot \|$.

Remark 2.2. *The relation (2) follows from [DZ06, relation (6.6)], which leads to the existence of $\mathcal{M} \in \mathbb{N}$ and $\delta' > 0$ such that $\inf_{k\in\mathbb{N}} |\sqrt{\lambda}_{k+\mathcal{M}} - \sqrt{\lambda}_k| > \delta' \mathcal{M}$ and*

$$\inf_{k\in\mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \sqrt{\lambda_2} \inf_{k\in\mathbb{N}} |\sqrt{\lambda}_{k+\mathcal{M}} - \sqrt{\lambda}_k| > \sqrt{\lambda_2} \delta' \mathcal{M}.$$

We point out that it is possible to set $\mathcal{M} \geq M + N + 1$ (even though this value is not optimal). This property can be deduced from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10] adopted as in Remark A.4.

Now, we define the following assumptions on the couple (A, B). Let $\eta > 0$, $a \geq 0$ and $I := \{(j,k) \in \mathbb{N}^2 : j \neq k\}$.

Assumptions (I(η))**.** The operator B satisfies the following conditions.

1. There exists $C > 0$ such that $|\langle \phi_j, B\phi_1 \rangle| \geq \frac{C}{j^{2+\eta}}$ for every $j \in \mathbb{N}$.
2. For $(j,k), (l,m) \in I$ such that $(j,k) \neq (l,m)$ and $\lambda_j - \lambda_k = \lambda_l - \lambda_m$,

$$\implies \quad \langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0.$$

Assumptions (II(η, a))**.** Let $Ran(B|_{H^2_{\mathscr{G}}}) \subseteq H^2_{\mathscr{G}}$ and one of the following assumptions be satisfied.

1. When $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ and $a + \eta \in (0, 3/2)$, there exists $d \in [\max\{a + \eta, 1\}, 3/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^2_{\mathscr{G}}$.
2. When $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $a + \eta \in (0, 7/2)$, there exist $d \in [\max\{a + \eta, 2\}, 7/2)$ and $d_1 \in (d, 7/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ and $Ran(B|_{H^{d_1}_{\mathcal{NK}}}) \subseteq H^{d_1}_{\mathcal{NK}}$.
3. When $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$ and $a + \eta \in (0, 5/2)$, there exists $d \in [\max\{a+\eta, 1\}, 5/2)$ such that $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$. If $a + \eta \geq 2$, then there exists $d_1 \in (d, 5/2)$ such that $Ran(B|_{H^{d_1}}) \subseteq H^{d_1}$.

From now on, we omit η and a from the notations of Assumptions I and Assumptions II when these parameters are not relevant.

Theorem 2.3. *Let $\mathscr{G}$ be a compact quantum graph. Let*

$$\exists \tilde{d} \geq 0,\ C > 0 \quad : \quad |\lambda_{k+1} - \lambda_k| \geq C k^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}. \tag{5}$$

If the couple (A, B) satisfies Assumptions I(η) and Assumptions II($\eta, \tilde{d}$) for some $\eta > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II.
In other words, for every $\psi^1, \psi^2 \in H^s_{\mathscr{G}}$ such that $\|\psi^1\| = \|\psi^2\|$, there exist $T > 0$ and $u \in L^2((0,T), \mathbb{R})$ such that $\Gamma^u_T \psi^1 = \psi^2$.

Proof. See Paragraph 4. □

In the next theorem, we provide the validity of the spectral hypothesis of Theorem 2.3 when $\mathscr{G}$ is one of the graphs introduced in Figure 2. The provided result leads to Example 1.2 and Example 1.3.

Theorem 2.4. *Let $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$. Let $\mathscr{G}$ be either a tadpole, a two-tails tadpole, a double-rings graph or a star graph with $N \leq 4$ edges. Let $\mathscr{G}$ be equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$. If the couple (A, B) satisfies Assumptions I(η) and Assumptions II(η, ϵ) for some $\eta, \epsilon > 0$, then the (BSE) is globally exactly controllable in $H^s_{\mathscr{G}}$ for $s = 2 + d$ and d from Assumptions II.*

Proof. See Paragraph 5. □

Remark 2.5. *Let $\{L_j\}_{j \leq 2} \in \mathcal{AL}(2)$. As explained in Remark* 5.1*, Theorem* 2.4 *is also valid when $\mathscr{G}$ is:*

1) *a two-tails tadpole with one edge long L_1 and the others L_2;*

2) *a* 3 *edges star graph with one edge long L_1 and the others L_2;*

3) *a* 4 *edges star graph with two edges long L_1 and the others L_2.*

In the following corollary, we provide the *contemporaneous controllability* introduced by Example 1.4. The result is consequence of Theorem 2.3.

Corollary 2.6. *Let $\mathscr{G} = \{I_j\}_{j \leq N}$ be a compact quantum graph composed by bounded unconnected intervals. Let the couple (A, B) satisfy Assumptions I(η) and Assumptions II(η, ϵ) for some $\eta, \epsilon > 0$. If $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$, then the (BSE) is contemporaneously globally exactly controllable in*

$$\prod_{j \leq N} H^s_{I_j} \qquad \text{with} \quad s = d + 2$$

and d from Assumptions II. In other words, for every $\psi_1, \psi_2 \in \prod_{j \leq N} H^s_{I_j}$ such that $\|\psi_1\| = \|\psi_2\|$, there exist $T > 0$ and $u \in L^2((0,T), \mathbb{R})$ such that

$$\Gamma^{u,j}_T \psi^j_1 = \psi^j_2, \qquad \forall j \leq N.$$

Proof. See Paragraph 5. □

3 Well-posedness and interpolation properties of the spaces $H^s_{\mathscr{G}}$

In the current section, we provide the *well-posedness* of the (BSE).

Proposition 3.1. *Let $\mathscr{G}$ a compact quantum graph. Let the couple (A,B) satisfy Assumptions II$(\eta,\tilde{d})$ with $\eta,\tilde{d}>0$.*

1) *Let $T>0$ and $f\in L^2((0,T),H^{2+d}\cap H^{1+d}_{\mathcal{NK}}\cap H^2_{\mathscr{G}})$ with d from Assumptions II. Let $t\mapsto G(t)=\int_0^t e^{iA\tau}f(\tau)d\tau$. The map $G\in C^0([0,T],H^{2+d}_{\mathscr{G}})$ and there exists $C(T)>0$ uniformly bounded for T lying on bounded intervals so that*

$$\|G\|_{L^\infty((0,T),H^{2+d}_{\mathscr{G}}}\leq C(T)\|f\|_{L^2((0,T),H^{2+d})}.$$

2) *Let $\psi^0\in H^{2+d}_{\mathscr{G}}$ with d introduced in Assumptions II and $u\in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of (BSE) in $H^{2+d}_{\mathscr{G}}$, i.e. a function $\psi\in C_0([0,T],H^{2+d}_{\mathscr{G}})$ such that for every $t\in[0,T]$,*

$$\psi(t,x)=e^{-iAt}\psi^0(x)-i\int_0^t e^{-iA(t-s)}u(s)B\psi(s,x)ds. \tag{6}$$

Moreover, there exists $C=C(T,B,u)>0$ so that

$$\|\psi\|_{C^0([0,T],H^{2+d}_{\mathscr{G}})}\leq C\|\psi^0\|_{H^{2+d}_{\mathscr{G}}},\quad \|\psi(t)\|=\|\psi^0\|,\quad \forall t\in[0,T],\ \psi_0\in H^{2+d}_{\mathscr{G}}.$$

Now, we present some *interpolation properties* for the spaces $H^s_{\mathscr{G}}$ with $s>0$. The proof of Proposition 3.1 is provided in the end of the section.

Proposition 3.2.

1) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$, then*

$$H^{s_1+s_2}_{\mathscr{G}}=H^{s_1}_{\mathscr{G}}\cap H^{s_1+s_2}(\mathscr{G},\mathbb{C})\quad \textit{for}\quad s_1\in\mathbb{N}\cup\{0\},\ s_2\in[0,1/2).$$

2) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{N})$-$(\mathcal{NK})$, then*

$$H^{s_1+s_2}_{\mathscr{G}}=H^{s_1}_{\mathscr{G}}\cap H^{s_1+s_2}_{\mathcal{NK}}\quad \textit{for}\quad s_1\in 2\mathbb{N}\cup\{0\},\ s_2\in[0,3/2).$$

3) *If the compact quantum graph $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$, then*

$$H^{s_1+s_2+1}_{\mathscr{G}}=H^{s_1+1}_{\mathscr{G}}\cap H^{s_1+s_2+1}_{\mathcal{NK}}\quad \textit{for}\quad s_1\in 2\mathbb{N}\cup\{0\},\ s_2\in[0,3/2).$$

Proof. We recall that by defining $\mathscr{G}$ as a quantum graph, we are implicitly introducing a self-adjoint Laplacian A.

1) (a) Bounded intervals. Let $\mathscr{G} = I^{\mathcal{N}}$ be an interval equipped with $(\mathcal{N})$ on the external vertices V_e. From [Gru16, *Definition* 2.1],

$$H^{s_1+s_2}_{I^{\mathcal{N}}} = H^{s_1}_{I^{\mathcal{N}}} \cap H^{s_1+s_2}(I^{\mathcal{N}}, \mathbb{C}), \qquad \forall s_1 \in 2\mathbb{N} \cup \{0\},\ s_2 \in [0, 3/2). \tag{7}$$

Let $\mathscr{G} = I^{\mathcal{D}}$ be an interval equipped with $(\mathcal{D})$ on the external vertices. From [Gru16, *Definition* 2.1], for $s_1 \in 2\mathbb{N} \cup \{0\}$, $s_2 \in [0, 3/2)$ and $s_3 \in [0, 1/2)$,

$$H^{s_1+s_2+1}_{I^{\mathcal{D}}} = H^{s_1+1}_{I^{\mathcal{D}}} \cap H^{s_1+s_2+1}(I^{\mathcal{D}}, \mathbb{C}), \qquad H^{s_3}_{I^{\mathcal{D}}} = H^{s_3}(I^{\mathcal{D}}, \mathbb{C}). \tag{8}$$

Let $\mathscr{G} = I^{\mathcal{M}}$ be an interval equipped with $(\mathcal{D})$ on one external vertex v_1 and $(\mathcal{N})$ on the other v_2. We prove that

$$H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1}_{I^{\mathcal{M}}} \cap H^{s_1+s_2}(I^{\mathcal{M}}, \mathbb{C}), \qquad \forall s_1 \in \mathbb{N} \cup \{0\},\ s_2 \in [0, 1/2). \tag{9}$$

We consider the interval $\widetilde{I}^{\mathcal{D}} \subseteq I^{\mathcal{M}}$ of length $\frac{3}{4}|I^{\mathcal{M}}|$ as a quantum graph containing v_1 and equipped in both the external vertices with $(\mathcal{D})$. We denote $\widetilde{I}^{\mathcal{N}} \subseteq I^{\mathcal{M}}$ an interval of length $\frac{3}{4}|I^{\mathcal{M}}|$, containing v_2 and equipped in both the external vertices with $(\mathcal{N})$. Let χ be the partition of the unity so that $\chi(x) = 1$ in $\widetilde{I}$, $\chi(x) = 0$ in $I^{\mathcal{M}} \setminus I^{\mathcal{D}}$ and $\chi(x) \in (0, 1)$ in $I^{\mathcal{D}} \setminus \widetilde{I}$. There holds $\psi^1 := \chi\psi \in H^2_{I^{\mathcal{D}}}$, $\psi^2 := (1-\chi)\psi \in H^2_{I^{\mathcal{N}}}$ and

$$\psi(x) = \psi^1(x) + \psi^2(x) \quad \Longrightarrow \quad H^2_{I^{\mathcal{M}}} = H^2_{\widetilde{I}^{\mathcal{D}}} \ \times\ H^2_{\widetilde{I}^{\mathcal{N}}}.$$

The same is valid for $L^2(I^{\mathcal{M}}, \mathbb{C})$ and $H^s(I^{\mathcal{M}}, \mathbb{C})$. Thus, for $s \in (0, 2]$,

$$H^s(I^{\mathcal{M}}, \mathbb{C}) = H^s(\widetilde{I}^{\mathcal{D}}, \mathbb{C}) \times H^s(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), \qquad L^2(I^{\mathcal{M}}, \mathbb{C}) = L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}) \times L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}).$$

Let $[\cdot, \cdot]_\theta$ be the complex interpolation of two spaces for $0 < \theta < 1$ defined in [Tri95, *Definition*, *Chapter* 1.9.2]. From [Tri95, *Remark* 1, *Chapter* 1.15.1] and [Tri95, *Theorem*, *Chapter* 1.15.3], for $s_1 \in \mathbb{N} \cup \{0\}$ and $s_2 \in [0, 1/2)$,

$$\big[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}}, \qquad \big[L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

Thanks to [Tri95, *relation* (12), *Chapter* 1.18.1], the interpolation of two products of spaces is the product of the two respective interpolations and

$$H^{s_2}_{I^{\mathcal{M}}} = \Big[L^2(I^{\mathcal{M}}, \mathbb{C}), H^2_{I^{\mathcal{M}}}\Big]_{s_2/2} = \Big[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}) \ \times\ L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}} \ \times\ H^2_{\widetilde{I}^{\mathcal{D}}}\Big]_{s_2/2}$$
$$= \Big[L^2(\widetilde{I}^{\mathcal{N}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{N}}}\Big]_{s_2/2} \ \times\ \Big[L^2(\widetilde{I}^{\mathcal{D}}, \mathbb{C}), H^2_{\widetilde{I}^{\mathcal{D}}}\Big]_{s_2/2} = H^{s_2}_{\widetilde{I}^{\mathcal{N}}} \ \times\ H^{s_2}_{\widetilde{I}^{\mathcal{D}}}.$$

Equally, $H^{s_1+s_2}_{I^{\mathcal{M}}} = H^{s_1+s_2}_{\widetilde{I}^{\mathcal{N}}} \times H^{s_1+s_2}_{\widetilde{I}^{\mathcal{D}}}$ that leads to (9) thanks to (7) and (8).

(b) Star graphs with equal edges. Let $A_{\mathcal{N}}$ be a Laplacian on an interval I of length L and equipped with $(\mathcal{N})$. Let $I^{\mathcal{N}}$ be the relative quantum graph and $\{f_j^1\}_{j\in\mathbb{N}}$ be an Hilbert basis of $L^2(I,\mathbb{C})$ made by eigenfunctions of $A_{\mathcal{N}}$. Let $A_{\mathcal{M}}$ be a Laplacian on I equipped with $(\mathcal{D})$ in the external vertex parametrized with 0 and with $(\mathcal{N})$ in the other. We call $I^{\mathcal{M}}$ the relative quantum graph and $\{f_j^2\}_{j\in\mathbb{N}}$ a Hilbert basis of $L^2(I,\mathbb{C})$ composed by eigenfunctions of $A_{\mathcal{M}}$.

Let $\mathscr{S}$ be a star graph of N edges long L and equipped with $(\mathcal{N})$-$(\mathcal{NK})$. The $(\mathcal{N})$ conditions on V_e imply that each ϕ_k is $(a_k^1\cos(x\sqrt{\lambda_k}),...,a_j^N\cos(x\sqrt{\lambda_k}))$ with λ_k the corresponding eigenvalue and $\{a_j^l\}_{l\leq N}\subset\mathbb{C}$. The $(\mathcal{NK})$ condition in V_i ensures that $\sin(\sqrt{\lambda_k}L)\sum_{l\leq N}a_k^l=0$ and

$$a_k^1\cos(\sqrt{\lambda_k}L)=...=a_k^N\cos\sqrt{\lambda_k}L),\qquad \forall k\in\mathbb{N}.$$

Each eigenvalue is either of the form $\frac{(n-1)^2\pi^2}{L^2}$, or $\frac{(2n-1)^2\pi^2}{4L^2}$ when $\sum_{l\leq N}a_k^l=0$ with $n\in\mathbb{N}$. Hence, for every $k\in\mathbb{N}$, there exists $j(k)\in\mathbb{N}$ such that

$$\begin{array}{ccccccc}\text{either} & \phi_k^l=c_k^l f_{j(k)}^1 & \text{for} & c_k^l\in\mathbb{C}, & |c_k^l|\leq 1, & \forall l\in\{1,...,N\},\\ \text{or} & \phi_k^l=c_k^l f_{j(k)}^2 & \text{for} & c_k^l\in\mathbb{C}, & |c_k^l|\leq 1, & \forall l\in\{1,...,N\}.\end{array}\tag{10}$$

After, for each $k\in\mathbb{N}$ and $m\in\{1,2\}$, there exist $\widetilde{j}\in\mathbb{N}$ and $l\leq N$ such that $f_k^m=c_{\widetilde{j}}^l\phi_{\widetilde{j}}^l$ with $c_{\widetilde{j}}^l\in\mathbb{C}$ and $|c_{\widetilde{j}}^l|\leq 1$. Thanks to the last identity and to (10),

$$\psi=(\psi^1,...,\psi^N)\in H^s_{\mathscr{S}}\quad\Longleftrightarrow\quad\psi^l\in H^s_{I^{\mathcal{N}}}\cap H^s_{I^{\mathcal{M}}},\qquad\forall l\leq N.\tag{11}$$

Now, we consider each edge e_j composing $\mathscr{S}$ as I (introduced above) since every e_j is long L. Let $I^{\mathcal{M}}$ and $I^{\mathcal{N}}$ be defined above and $H^s(\mathscr{S},\mathbb{C})=(H^s(I,\mathbb{C}))^N$. For $s_1\in\mathbb{N}\cup\{0\}$ and $s_2\in[0,1/2)$, from (11), we have

$$\psi\in H^{s_1+s_2}(\mathscr{S},\mathbb{C})\cap H^{s_1}_{\mathscr{S}}\quad\Longleftrightarrow\quad\psi^l\in H^{s_1+s_2}(I,\mathbb{C})\cap H^{s_1}_{I^{\mathcal{N}}}\cap H^{s_1}_{I^{\mathcal{M}}},\quad\forall l\leq N.$$

The relations (7) and (8) imply that $\psi^l\in H^{s_1+s_2}(I,\mathbb{C})\cap H^{s_1}_{I^{\mathcal{N}}}\cap H^{s_1}_{I^{\mathcal{M}}}$ for every $l\leq N$ if and only if $\psi^l\in H^{s_1+s_2}_{I^{\mathcal{N}}}\cap H^{s_1+s_2}_{I^{\mathcal{M}}}$ for every $l\leq N$, which is valid if and only if $\psi\in H^{s_1+s_2}_{\mathscr{S}}$ thanks to (11). In conclusion, we have

$$H^{s_1+s_2}_{\mathscr{S}}=H^{s_1}_{\mathscr{S}}\cap H^{s_1+s_2}(\mathscr{S},\mathbb{C}).$$

(c) Generic graphs. Let $\mathscr{G}$ be equipped with $(\mathcal{D}/\mathcal{N})$-$(\mathcal{NK})$ and $\widetilde{L}<\min\{L_k/2:k\in\{1,...,N\}\}$. Let $n(v)$ be defined in (4) for every $v\in V_e\cup V_i$. We define the graphs $\widetilde{\mathscr{G}}(v)$ for every $v\in V_i\cup V_e$ and the intervals $\{I_j\}_{j\leq N}$ as follows (see Figure 5 for an explicit example).
If $v\in V_i$, then $\widetilde{\mathscr{G}}(v)$ is a star sub-graph of $\mathscr{G}$ equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and composed by $n(v)$ edges long $\widetilde{L}$ and connected to the internal vertex v.

If $v \in V_e$, then $\widetilde{\mathscr{G}}(v)$ is an interval long $\widetilde{L}$ such that the external vertex v is equipped with the same boundary conditions that v has in $\mathscr{G}$. We impose $(\mathcal{N})$ on the other vertex.
For each $v, \hat{v} \in V_e \cup V_i$, the graphs $\widetilde{\mathscr{G}}(v)$ and $\widetilde{\mathscr{G}}(\hat{v})$ have respectively two external vertices w_1 and w_2 lying on the same edge e and such that $w_1 \notin \widetilde{\mathscr{G}}(\hat{v})$. We construct an interval strictly containing w_1 and w_2, strictly contained in e and equipped with $(\mathcal{N})$. We collect those intervals in $\{I_j\}_{j\leq N}$.

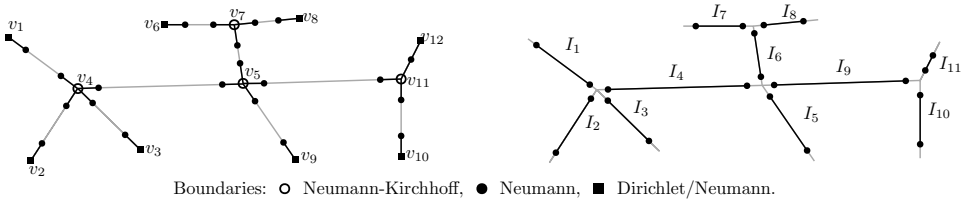

Figure 5: The left and the right figures respectively represent the graphs $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_i\cup V_e}$ and the intervals $\{I_j\}_{j\leq N}$ for a given graph $\mathscr{G}$.

From **1) (a)** and **1) (b)**, for $v \in V_i \cup V_e$, $j \leq N$, $s_1 \in \mathbb{N}\cup\{0\}$ and $s_2 \in [0, 1/2)$,

$$H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)} = H^{s_1}_{\widetilde{\mathscr{G}}(v)} \cap H^{s_1+s_2}(\widetilde{\mathscr{G}}(v), \mathbb{C}), \qquad H^{s_1+s_2}_{I_j} = H^{s_1}_{I_j} \cap H^{s_1+s_2}(I_j, \mathbb{C}).$$

We notice that $G := \{\widetilde{\mathscr{G}}(v_j)\}_{j\leq M} \cup \{I_j\}_{j\leq N}$ covers $\mathscr{G}$. As in **1) (a)**, we see each function of domain $\mathscr{G}$ as a vector of functions of domain G_j with $j \leq M + N$. We use [Tri95, *relation* (12), *Chapter* 1.18.1] as in **1) (a)** and

$$H^{s_1+s_2}_{\mathscr{G}} = H^{s_1}_{\mathscr{G}} \cap H^{s_1+s_2}(\mathscr{G}, \mathbb{C}) \quad \text{for} \quad s_1 \in \mathbb{N} \cup \{0\},\ s_2 \in [0, 1/2).$$

2) Let $\mathscr{G}$ be equipped with $(\mathcal{N})$-$(\mathcal{NK})$ and $N_e = |V_e|$. We consider $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ introduced in **1) (c)** and we define $\widetilde{\mathscr{G}}$ from $\mathscr{G}$ as follows (see Figure 6). For every $v \in V_e$, we remove from the edge including v, a section of length $\widetilde{L}/2$ containing v. We equip the new external vertex with $(\mathcal{N})$.

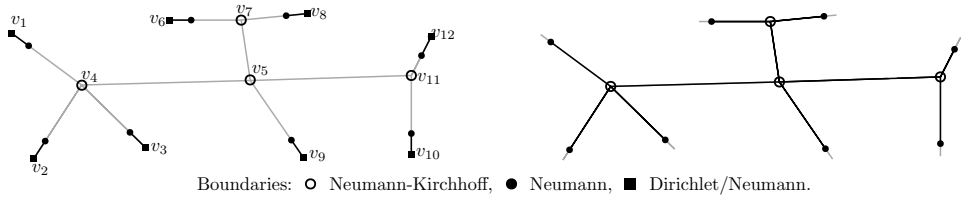

Figure 6: The left and the right figures respectively represent the graphs $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ and $\widetilde{\mathscr{G}}$ for a given graph $\mathscr{G}$.

We call $G' := \{G'_j\}_{j\leq N_e+1} := \{\widetilde{\mathscr{G}}(v)\}_{v\in V_e} \cup \{\widetilde{\mathscr{G}}\}$ which covers $\mathscr{G}$. For every $s_1 \in 2\mathbb{N} \cup \{0\}$, $s_2 \in [0, 3/2)$, we have $H^{s_1+s_2}_{\widetilde{\mathscr{G}}(v)} = H^{s_1}_{\widetilde{\mathscr{G}}(v)} \cap H^{s_1+s_2}$ from (7).

The arguments of **1) (a)**, also adopted in **1) (c)**, lead to the proof since

$$H^{s_1+s_2}_{\mathcal{NK}}(\mathscr{G},\mathbb{C}) = H^{s_1+s_2}_{\widetilde{\mathscr{G}}} \times \prod_{v\in V_e} H^{s_1+s_2}(\widetilde{\mathscr{G}}(v),\mathbb{C}).$$

3) As in **2)**, the claim follows by considering $\{\widetilde{\mathscr{G}}(v)\}_{v\in V_e}$ as intervals equipped with $(\mathcal{D})$ and $\widetilde{\mathscr{G}}$ equipped with $(\mathcal{D})$ in its external vertices. □

Proof of Proposition 3.1.

1) (a) Assumptions II.1 . Let $f(s) \in H^3 \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$, $t \in (0,T)$ and $f(s) = (f^1(s),...,f^N(s))$. We prove that $G \in C^0([0,T], H^3_{\mathscr{G}})$. The definition of $G(t)$ implies $G(t) = \sum_{k=1}^{\infty} \phi_k \int_0^t e^{i\lambda_k s}\langle \phi_k, f(s)\rangle ds$ and

$$\|G(t)\|_{(3)} = \Big(\sum_{k\in\mathbb{N}} \Big| k^3 \int_0^t e^{i\lambda_k s}\langle \phi_k, f(s)\rangle ds\Big|^2\Big)^{\frac{1}{2}}. \tag{12}$$

We estimate $\langle \phi_k, f(s,\cdot)\rangle$ for each $k \in \mathbb{N}$ and $s \in (0,t)$. We suppose that $\lambda_1 \neq 0$. Let $\partial_x f(s) = (\partial_x f^1(s),...,\partial_x f^N(s)$ be the derivative of $f(s)$ and $P(\phi_k) = (P(\phi_k^1),...,P(\phi_k^N))$ be the primitive of ϕ_k such that $P(\phi_k) = -\frac{1}{\lambda_k}\partial_x\phi_k$. We call ∂e the two points composing the boundaries of an edge e. For every $v \in V_e$, $\tilde{v} \in V_i$ and $j \in N(\tilde{v})$, there exist $a(v), a^j(\tilde{v}) \in \{-1,+1\}$ such that

$$\begin{aligned}\langle \phi_k, f(s)\rangle &= \frac{1}{\lambda_k}\langle \phi_k, \partial_x^2 f(s)\rangle = \frac{1}{\lambda_k^2}\int_{\mathscr{G}} \partial_x\phi_k(y)\partial_x^3 f(s,y)dy \\ &+ \frac{1}{\lambda_k^2}\sum_{v\in V_i}\sum_{j\in N(v)} a^j(v)\partial_x\phi_k^j(v)\partial_x^2 f^j(s,v) + \frac{1}{\lambda_k^2}\sum_{v\in V_e} a(v)\partial_x\phi_k(v)\partial_x^2 f(s,v).\end{aligned} \tag{13}$$

From [BK13, *Theorem* 3.1.8] and [BK13, *Theorem* 3.1.10], there exist $C_1 > 0$ such that $\lambda_k^{-2} \leq C_1 k^{-4}$ for every $k \in \mathbb{N}$ (see Remark *A*.4), then

$$\begin{aligned}\Big| k^3 \int_0^t e^{i\lambda_k s}\langle \phi_k, f(s)\rangle ds\Big| &\leq \frac{C_1}{k}\Bigg(\sum_{v\in V_e}\Big|\partial_x\phi_k(v)\int_0^t e^{i\lambda_k s}\partial_x^2 f(s,v)ds\Big| \\ &+ \sum_{v\in V_i}\sum_{j\in N(v)}\Big|\partial_x\phi_k^j(v)\int_0^t e^{i\lambda_k s}\partial_x^2 f^j(s,v)ds\Big| + \Big|\int_0^t e^{i\lambda_k s}\int_{\mathscr{G}}\partial_x\phi_k(y)\partial_x^3 f(s,y)dyds\Big|\Bigg).\end{aligned} \tag{14}$$

Remark 3.3. *We point out that* $A'\lambda_k^{-1/2}\partial_x\phi_k = \lambda_k\lambda_k^{-1/2}\partial_x\phi_k$ *for every* $k \in \mathbb{N}$, *where* $A' = -\Delta$ *is a self-adjoint Laplacian with compact resolvent. Thus,* $\|\lambda_k^{-1/2}\partial_x\phi_k\|^2 = \langle \lambda_k^{-1/2}\partial_x\phi_k, \lambda_k^{-1/2}\partial_x\phi_k\rangle = \langle \phi_k, \lambda_k^{-1}A\phi_k\rangle = 1$ *and then* $\{\lambda_k^{-1/2}\partial_x\phi_k\}_{k\in\mathbb{N}}$ *is a Hilbert basis of* $\mathscr{H}$.

Let $\mathbf{a^l} = \{a_k^l\}, \mathbf{b^l} = \{b_k^l\} \subset \mathbb{C}$ for $l \leq N$ be so that $\phi_k^l(x) = a_k^l \cos(\sqrt{\lambda_k}x) + b_k^l \sin(\sqrt{\lambda_k}x)$ and $-a_k^l \sin(\sqrt{\lambda_k}x) + b_k^l \cos(\sqrt{\lambda_k}x) = \lambda_k^{-1/2}\partial_x\phi_k^l(x)$. Now,

$$2 \geq \|\lambda_k^{-1/2}\partial_x\phi_k^l\|^2_{L^2(e^l)} + \|\phi_k^l\|^2_{L^2(e^l)} = (|a_k^l|^2 + |b_k^l|^2)|e_l|$$

for every $k \in \mathbb{N}$ and $l \in \{1, ..., N\}$. Thus, $\mathbf{a^l}, \mathbf{b^l} \in \ell^\infty(\mathbb{C})$ and there exists $C_2 > 0$ such that, for every $k \in \mathbb{N}$ and $v \in V_e \cup V_i$, we have $|\lambda_k^{-1/2}\partial_x\phi_k(v)| \leq C_2$. Thanks to the identities (12) and (14), it follows

$$\begin{aligned} \|G(t)\|_{(3)} \leq C_1C_2 \sum_{v\in V_e\cup V_i} \sum_{j\in N(v)} \Big\| \int_0^t \partial_x^2 f^j(s,v) e^{i\lambda_{(\cdot)}s} ds \Big\|_{\ell^2} \\ + C_1 \Big\| \int_0^t \big\langle \lambda_{(\cdot)}^{-1/2}\partial_x\phi_{(\cdot)}(s), \partial_x^3 f(s)\big\rangle e^{i\lambda_{(\cdot)}s} ds \Big\|_{\ell^2}. \end{aligned} \tag{15}$$

From Proposition $B.6$ and (15), there exist $C_3(t), C_4(t) > 0$ uniformly bounded for t in bounded intervals such that

$$\|G\|_{H^3_{\mathscr{G}}} \leq C_3(t) \sum_{v\in V_e\cup V_i} \sum_{j\in N(v)} \|\partial_x^2 f^j(\cdot, v)\|_{L^2((0,t),\mathbb{C})} + \sqrt{t}\|f\|_{L^2((0,t),H^3)} \tag{16}$$

and $\|G\|_{H^3_{\mathscr{G}}} \leq C_4(t)\|f(\cdot,\cdot)\|_{L^2((0,t),H^3)}$. We underline that the identity is also valid when $\lambda_1 = 0$, which is proved by isolating the term with $k = 1$ and by repeating the steps above. For every $t \in [0, T]$, the inequality (16) shows that $G(t) \in H^3_{\mathscr{G}}$. The provided upper bounds are uniform and the Dominated Convergence Theorem leads to $G \in C^0([0, T], H^3_{\mathscr{G}})$.
Let $f(s) \in H^5 \cap H^4_{\mathscr{G}}$ for almost every $s \in (0, t)$ and $t \in (0, T)$. The same techniques adopted above shows that $G \in C^0([0, T], H^5_{\mathscr{G}})$.

We denote $F(f)(t) := \int_0^t e^{iA\tau} f(\tau)d\tau$ for $f \in \mathscr{H}$ and $t \in (0, T)$. Let $X(B)$ be the space of functions f so that $f(s)$ belongs to a Banach space B for almost every $s \in (0, t)$ and $t \in (0, T)$. The first part of the proof implies

$$F : X(H^3 \cap H^2_{\mathscr{G}}) \longrightarrow C^0([0, T], H^3_{\mathscr{G}}), \qquad F : X(H^5 \cap H^4_{\mathscr{G}}) \longrightarrow C^0([0, T], H^5_{\mathscr{G}}).$$

From a classical interpolation result (see [BL76, $Theorem$ 4.4.1] with $n = 1$), we have $F : X(H^{2+d} \cap H^{1+d}_{\mathscr{G}}) \longrightarrow C^0([0, T], H^{2+d}_{\mathscr{G}})$ with $d \in [1, 3]$. Thanks to Proposition 3.2, if $d \in [1, 3/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}} = H^{2+d} \cap H^{1+d}_{\mathscr{G}}$ for almost every $s \in (0, t)$ and $t \in (0, T)$, then $G \in C^0([0, T], H^{2+d}_{\mathscr{G}})$. The proof is achieved when the first point Assumptions II is verified.

(b) Assumptions II.3 . If $\mathscr{G}$ is equipped with $(\mathcal{D})$-$(\mathcal{NK})$, then $H^2_{\mathscr{G}} = H^2_{\mathcal{NK}} \cap H^1_{\mathscr{G}}$ and $H^4_{\mathscr{G}} = H^4_{\mathcal{NK}} \cap H^3_{\mathscr{G}}$ from Proposition 3.2. As above, if $f(s) \in H^3 \cap H^2_{\mathcal{NK}} \cap H^1_{\mathscr{G}}$ for almost every $s \in (0, t)$ and $t \in (0, T)$, then $G \in C^0([0, T], H^3_{\mathscr{G}})$, while if $f(s) \in H^5 \cap H^4_{\mathcal{NK}} \cap H^3_{\mathscr{G}}$ for almost every $s \in (0, t)$ and $t \in (0, T)$, then $G \in C^0([0, T], H^5_{\mathscr{G}})$. From the interpolation techniques, if $d \in [1, 5/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^d_{\mathscr{G}}$ for almost every $s \in (0, t)$ and $t \in (0, T)$, then $G \in C^0([0, T], H^{2+d}_{\mathscr{G}})$ and the proof is attained.

(c) Assumptions II.2 . Let $f(s) \in H^4 \cap H^3_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$ and $\mathscr{G}$ be equipped with $(\mathcal{N})$. In this framework, the last line of (13) is zero. Indeed, $\partial_x^2 f(s) \in C^0$ as $f(s) \in H^3_{\mathcal{NK}}$ and, for $v \in V_e$, we have $\partial_x \phi_k(v) = 0$ thanks to the $(\mathcal{N})$ boundary conditions (the terms $a^j(v)$ assume different signs according to the orientation of the edges connected in v). After, for every $v \in V_i$, thanks to the $(\mathcal{NK})$ in $v \in V_i$, we have $\sum_{j \in N(v)} a^j(v) \partial_x \phi_k^j(v) = 0$. From (13), we obtain

$$\langle \phi_k, f(s) \rangle = -\frac{1}{\lambda_k^2} \int_{\mathscr{G}} \partial_x \phi_k(y) \partial_x^3 f(s,y) dy = -\frac{1}{\lambda_k^2} \sum_{v \in V_e} a(v) \phi_k(v) \partial_x^3 f(s,v)$$
$$- \frac{1}{\lambda_k^2} \sum_{v \in V_i} \sum_{j \in N(v)} a^j(v) \phi_k^j(v) \partial_x^3 f^j(s,v) + \frac{1}{\lambda_k^2} \int_{\mathscr{G}} \phi_k(y) \partial_x^4 f(s,y) dy.$$

Now, $\{\phi_k\}_{k \in \mathbb{N}}$ is a Hilbert basis of $\mathscr{H}$ and we proceed as in (14), (15) and (16). From Proposition $B.6$, there exists $C_6(t) > 0$ uniformly bounded for t lying in bounded intervals such that $\|G\|_{H^4_{\mathscr{G}}} \leq C_1(t) \|f(\cdot,\cdot)\|_{L^2((0,t),H^4)}$.

If $f(s) \in H^4 \cap H^3_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^4_{\mathscr{G}})$. Equivalently when $f(s) \in H^6 \cap H^5_{\mathcal{NK}} \cap H^4_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, we have $G \in C^0([0,T], H^6_{\mathscr{G}})$. As above, Proposition 3.2 implies that when $d \in [2, 7/2)$ and $f(s) \in H^{2+d} \cap H^{1+d}_{\mathcal{NK}} \cap H^2_{\mathscr{G}}$ for almost every $s \in (0,t)$ and $t \in (0,T)$, then $G \in C^0([0,T], H^{2+d}_{\mathscr{G}})$.

2) As $Ran(B|_{H^{2+d}_{\mathscr{G}}}) \subseteq H^{2+d} \cap H^{1+d}_{\mathcal{NK}} H^2_{\mathscr{G}} \subseteq H^{2+d}$, we have $B \in L(H^{2+d}_{\mathscr{G}}, H^{2+d})$ thanks to the arguments of [Ducb, *Remark* 1.1]. For every $\psi \in H^{2+d}_{\mathscr{G}}$, let

$$t \mapsto F(\psi)(t) = e^{-iAt} - \int_0^t e^{-iA(t-s)} u(s) B\psi(s) ds \in C^0([0,T], H^{2+d}_{\mathscr{G}}).$$

For every $\psi^1, \psi^2 \in H^{2+d}_{\mathscr{G}}$, thanks to the first point of the proof, there exists $C(t) > 0$ uniformly bounded for t lying on bounded intervals, such that

$$\|F(\psi^1)(t) - F(\psi^2)(t)\|_{(2+d)} \leq \left\| \int_0^t e^{-iA(t-s)} u(s) B(\psi^1(s) - \psi^2(s)) ds \right\|_{(2+d)}$$
$$\leq C(t) \|u\|_{L^2((0,t),\mathbb{R})} \| B \|_{L(H^{2+d}_{\mathscr{G}}, H^{2+d})} \|\psi^1 - \psi^2\|_{L^\infty((0,t), H^{2+d}_{\mathscr{G}})}.$$

We refer to the techniques adopted in the proof of [BL10, *Proposition* 2]. If $\|u\|_{L^2((0,t),\mathbb{R})}$ is small enough, then F is a contraction and Banach Fixed Point Theorem implies that there exists $\psi \in C^0([0,T], H^{2+d}_{\mathscr{G}})$ such that $F(\psi) = \psi$. When $\|u\|_{L^2((0,t),\mathbb{R})}$ is not sufficiently small, one considers $\{t_j\}_{0 \leq j \leq n}$ a partition of $[0,t]$ with $n \in \mathbb{N}$. We choose a partition such that each $\|u\|_{L^2([t_{j-1},t_j],\mathbb{R})}$ is so small that the map F, defined on the interval $[t_{j-1}, t_j]$, is a contraction and we apply the Banach Fixed Point Theorem. The remaining claim follows from the proof of [BL10, *relation* (23)]. □

4 Proof of Theorem 2.3

The result is achieved as in the proof of [Ducb, *Proposition* 3.4]. In particular, it is obtained by gathering the local exact controllability and the global approximate controllability (both proved below) thanks to the time reversibility of the (BSE) (see [Ducb, *Apprendix* 1.3]).

4.1 Local exact controllability in $H^s_{\mathscr{G}}$

Let $O^s_{\epsilon,T} := \left\{\psi \in H^s_{\mathscr{G}} \middle|\ \|\psi\| = 1,\ \|\psi - \phi_1(T)\|_{(s)} < \epsilon\right\}$. We ensure the local exact controllability of the (BSE) in $O^s_{\epsilon,T}$ with $s = 2 + d$ and d from Assumptions II, *i.e.* the existence of $T > 0$ and $\epsilon > 0$ such that

$$\forall \psi \in O^s_{\epsilon,T},\ \ \exists u \in L^2((0,T),\mathbb{R}) \qquad : \qquad \psi = \Gamma^u_T \phi_1.$$

Let Assumptions I be verified. We define the application α, the sequence with elements $\alpha_k(u) = \langle \phi_k(T), \Gamma^u_T \phi_1 \rangle$ for $k \in \mathbb{N}$, such that

$$\alpha : L^2((0,T),\mathbb{R}) \longrightarrow Q := \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^s(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

The local exact controllability in $O^s_{\epsilon,T}$ with $T > 0$ is equivalent to the surjectivity of the map $\Gamma^{(\cdot)}_T \phi_1 : u \in L^2((0,T),\mathbb{R}) \longmapsto \psi \in O^s_{\epsilon,T} \subset H^s_{\mathscr{G}}$. As

$$\Gamma^u_t \phi_1 = \sum_{k\in\mathbb{N}} \phi_k(t) \langle \phi_k(t), \Gamma^u_t \phi_1 \rangle, \qquad T > 0,\ \ u \in L^2((0,T),\mathbb{R}),$$

the controllability is equivalent to the local surjectivity of the map α. To this end, we use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of $\gamma(v) := (d_u\alpha(0)) \cdot v$ the Fréchet derivative of α with $\alpha(0) = \delta = \{\delta_{k,1}\}_{k\in\mathbb{N}}$. Let $B_{j,k} := \langle \phi_j, B\phi_k \rangle$ with $j,k \in \mathbb{N}$. As in [Duca, *relation* (6)], the map γ is the sequence of elements $\gamma_k(v) := -i \int_0^T v(\tau) e^{i(\lambda_k - \lambda_1)s} d\tau B_{k,1}$ with $k \in \mathbb{N}$ such that

$$\gamma : L^2((0,T),\mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := \{x_k\}_{k\in\mathbb{N}} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}.$$

The surjectivity of γ corresponds to the solvability of the moments problem

$$x_k/B_{k,1} = -i \int_0^T u(\tau) e^{i(\lambda_k - \lambda_1)\tau} d\tau, \qquad \forall \{x_k\}_{k\in\mathbb{N}} \in T_\delta Q \subset h^s. \tag{17}$$

Proposition $B.5$ leads to the solvability of (17) in $h^{\tilde{d}}$. Now, $B_{1,1} \in \mathbb{R}$ as B is symmetric, $ix_1/B_{1,1} \in \mathbb{R}$ and $\left\{x_k B^{-1}_{k,l}\right\}_{k\in\mathbb{N}} \in h^{d-\eta} \subseteq h^{\tilde{d}}$ thanks to the first point of Assumptions I. Thus, there exists $T > 0$ large enough such that, for every $\{x_k\}_{k\in\mathbb{N}} \in T_\delta Q$, there exists $u \in L^2((0,T),\mathbb{R})$ such that $\{x_k\}_{k\in\mathbb{N}} = \{\gamma_k(u)\}_{k\in\mathbb{N}}$. In conclusion, the map γ is surjective and α is locally surjective, which implies the local exact controllability.

4.2 Global approximate controllability in $H^s_{\mathscr{G}}$

We study the approximate controllability of the (BSE) in $H^s_{\mathscr{G}}$ with $s>0$, *i.e* for every $\psi\in H^s_{\mathscr{G}}$, $\widehat{\Gamma}\in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi\in H^s_{\mathscr{G}}$ and $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\widehat{\Gamma}\psi-\Gamma^u_T\psi\|_{(s)}<\epsilon$.

Let $B: H^{s_1}_{\mathscr{G}}\to H^{s_1}_{\mathscr{G}}$ with $s_1>0$. The claim is due the proof of [Ducb, *Theorem* 3.3] that we retrace by using the norm $\|\cdot\|_{(s)}$ with $s\in[0,s_1+2)$ instead of $\|\cdot\|_{(3)}$ and by considering Lemma C.3. The proof of [Ducb, *relation* (26)] implies

$$\exists\, n\in\mathbb{N}\quad:\quad \|\psi\|^{n+1}_{(s)}\leq\|\psi\|\|\psi\|^n_{(s_1+2)},\qquad\qquad \forall\psi\in H^{s_1}_{\mathscr{G}}. \tag{18}$$

As in [Ducb, *p.* 16], for each $T>0$, $u\in BV((0,T),\mathbb{R})$ and $\psi\in H^{s_1+2}_{\mathscr{G}}$,

$$\exists C(K)>0\quad\text{ with }\quad K=\big(\|u\|_{BV((0,T),\mathbb{R})},\|u\|_{L^\infty((0,T),\mathbb{R})},T\|u\|_{L^\infty((0,T),\mathbb{R})}\big)$$

such that $\|\Gamma^u_T\psi\|_{(s_1+2)}\leq C(K)\|\psi\|_{(s_1+2)}$. This identity and (18) attain the global approximate controllability in $H^s_{\mathscr{G}}$ as in the mentioned proof.

Let d be the parameter introduced by the validity of Assumptions II. If $d<2$, then $B: H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ and the global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<4$. If $d\in[2,5/2)$, then $B: H^{d_1}\to H^{d_1}$ with $d_1\in(d,5/2)$ from Assumptions II. Now, $H^{d_1}_{\mathscr{G}}=H^{d_1}\cap H^2_{\mathscr{G}}$, thanks to Proposition 3.2, and $B: H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ implies $B: H^{d_1}_{\mathscr{G}}\to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<d_1+2$.
If $d\in[5/2,7/2)$, then $B: H^{d_1}_{\mathcal{NK}}\to H^{d_1}_{\mathcal{NK}}$ for $d_1\in(d,7/2)$ and $H^{d_1}_{\mathscr{G}}=H^{d_1}_{\mathcal{NK}}\cap H^2_{\mathscr{G}}$ from Proposition 3.2. Now, $B: H^2_{\mathscr{G}}\to H^2_{\mathscr{G}}$ that implies $B: H^{d_1}_{\mathscr{G}}\to H^{d_1}_{\mathscr{G}}$. The global approximate controllability is verified in $H^{d+2}_{\mathscr{G}}$ since $d+2<d_1+2$.

5 Proofs of Theorem 2.4 and Corollary 2.6

Let $\{\lambda^{\widetilde{\mathscr{G}}}_k\}_{k\in\mathbb{N}}$ denote the eigenvalues of A on a compact quantum graph $\widetilde{\mathscr{G}}$.

Proof of Theorem 2.4. Let $\mathscr{G}$ be a tadpole graph equipped with $(\mathcal{D})$-$(\mathcal{NK})$ where the edge e_1 connects $v\in V_i$ to itself. Let $\mathscr{G}^{\mathcal{D}}$ be the graph obtained from $\mathscr{G}$ by imposing $(\mathcal{D})$ on v. We define $\mathscr{G}^{\mathcal{N}}$ the graph obtained by disconnecting e_1 on one side and by imposing $(\mathcal{N})$ on the new external vertex of e_1 (see the first line of Figure 7 for further details). From Proposition A.3,

$$\lambda^{\mathscr{G}}_k\leq\lambda^{\mathscr{G}^{\mathcal{D}}}_k\leq\lambda^{\mathscr{G}}_{k+1},\qquad \lambda^{\mathscr{G}}_k\leq\lambda^{\mathscr{G}^{\mathcal{N}}}_{k+1}\leq\lambda^{\mathscr{G}}_{k+1},\qquad \forall k\in\mathbb{N}. \tag{19}$$

Now, $\{\lambda^{\mathscr{G}^{\mathcal{D}}}_k\}_{k\in\mathbb{N}}$ and $\{\lambda^{\mathscr{G}^{\mathcal{N}}}_k\}_{k\in\mathbb{N}}$ are the sequences of eigenvalues respectively obtained by reordering $\{\frac{k^2\pi^2}{L_j^2}\}_{\substack{k\in\mathbb{N}\\ j\in\{1,2\}}}$ and $\{\frac{(2k-1)^2\pi^2}{4(L_1+L_2)^2}\}_{k\in\mathbb{N}}$. If $\{L_1,L_2\}\in\mathcal{AL}$,

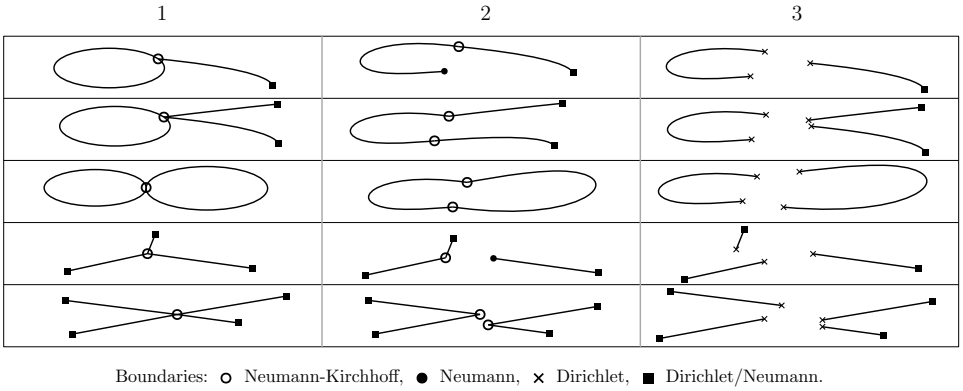

Figure 7: The figure represents the graphs described in the proof of Theorem 2.4. *The column* **1** *shows the considered graphs* $\mathscr{G}$*, which respectively are a tadpole, a two-tails tadpole, a double-rings graph, a star graph with* 3 *edges and a star graph with* 4 *edges. The column* **2** *provides the corresponding graphs* $\mathscr{G}^{\mathcal{N}}$*, while the column* **3** *shows* $\mathscr{G}^{\mathcal{D}}$*.*

then $\{L_1, L_2, L_1 + L_2\} \in \mathcal{AL}$. The techniques of the proof of Proposition $A.2$ lead to the existence of $C > 0$ such that, for every $\epsilon > 0$, there holds

$$|\lambda^{\mathscr{G}}_{k+1} - \lambda^{\mathscr{G}}_k| \geq |\lambda^{\mathscr{G}^{\mathcal{N}}}_{k+1} - \lambda^{\mathscr{G}^{\mathcal{D}}}_k| \geq Ck^{-\epsilon}, \qquad \forall k \in \mathbb{N}.$$

The relation (5) is verified and the claim is guaranteed by Theorem 2.3.

The techniques just introduced lead to the claim when $\mathscr{G}$ is a tadpole graph equipped with $(\mathcal{N})$-$(\mathcal{NK})$, but also when $\mathscr{G}$ is a two-tails tadpole graph, a double-rings graph or a star graph with $N \leq 4$ edges. In every framework, we impose that $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$. In Figure 7, we represent how to define $\mathscr{G}^{\mathcal{N}}$ and $\mathscr{G}^{\mathcal{D}}$ from the corresponding graphs $\mathscr{G}$. □

Remark 5.1. *The techniques leading to Theorem* 2.4 *can be adopted in order to prove Remark* 2.5. *The peculiarity of the proof is that when* $\mathscr{G}$ *is a star graphs, we construct* $\mathscr{G}^{\mathcal{N}}$ *so that the edges of equal length do not belong to the same connected component composing* $\mathscr{G}^{\mathcal{N}}$*.*

Proof of Corollary 2.6. As $\{\lambda_j\}_{j\in\mathbb{N}} \subset \left\{\frac{(k-1)^2\pi^2}{4L_j^2}\right\}_{\substack{k,j\in\mathbb{N}\\ j\leq N}}$, the claim follows from [Rot56]. In fact, thanks to the arguments adopted in the proof of Proposition $A.2$, for every $\epsilon > 0$, there exists $C_1 > 0$ such that $|\lambda_{k+1} - \lambda_k| \geq C_1 k^{-\epsilon}$ for every $k \in \mathbb{N}$. In conclusion, Theorem 2.3 attains the proof. □

6 Proofs of the examples 1.2, 1.3 and 1.4

Proof of Example 1.2. Let $\mathscr{G}$ be a star graph with 4 edges of lengths $\{L_j\}_{j\leq 4}$ equipped $(\mathcal{D})$-$(\mathcal{NK})$. The $(\mathcal{D})$ conditions on the external vertices imply that

each eigenfunction ϕ_j with $j \in \mathbb{N}$ satisfies $\phi_j^l(0) = 0$ for every $l \leq 4$. Then,

$$\phi_j(x) = (a_j^1 \sin(x\sqrt{\lambda_j}), a_j^2 \sin(x\sqrt{\lambda_j}), a_j^3 \sin(x\sqrt{\lambda_j}), a_j^4 \sin(x\sqrt{\lambda_j}))$$

with $\{a_j^l\}_{l\leq 4} \subset \mathbb{C}$ such that $\{\phi_j\}_{j\in\mathbb{N}}$ forms a Hilbert basis of $\mathscr{H}$, *i.e.*

$$1 = \sum_{l\leq 4} \int_0^{L_l} |a_j^l|^2 \sin^2(x\sqrt{\lambda_j})dx = \sum_{l\leq 4} |a_j^l|^2 \Big(\frac{L_l}{2} + \frac{\cos(L_l\sqrt{\lambda_j})\sin(L_l\sqrt{\lambda_j})}{2\sqrt{\lambda_j}}\Big).$$

For every $j \in \mathbb{N}$, the $(\mathcal{NK})$ condition in V_i leads to

$$\begin{aligned} &a_j^1 \sin(\sqrt{\lambda_j}L_1) = ... = a_j^4 \sin(\sqrt{\lambda_j}L_N), \qquad \sum_{l\leq 4} a_j^l \cos(\sqrt{\lambda_j}L_l) = 0, \\ &\sum_{l\leq 4} \cot(\sqrt{\lambda_j}L_l) = 0, \qquad \sum_{l\leq N} |a_j^l|^2 \sin(L_l\sqrt{\lambda_j})\cos(L_l\sqrt{\lambda_j}) = 0. \end{aligned} \tag{20}$$

Now, $1 = \sum_{l=1}^4 |a_j^l|^2 L_l/2$ and the continuity implies $a_j^l = a_j^1 \frac{\sin(\sqrt{\lambda_j}L_1)}{\sin(\sqrt{\lambda_j}L_l)}$ for $l \neq 1$ and $j \in \mathbb{N}$, which ensures $|a_j^1|^2\big(L_1 + \sum_{l=2}^4 L_l \frac{\sin^2(\sqrt{\lambda_j}L_1)}{\sin^2(\sqrt{\lambda_j}L_l)}\big) = 2$. Thus,

$$|a_j^1|^2 = \frac{2\prod_{m\neq 1}\sin^2(\sqrt{\lambda_j}L_m)}{\sum_{k=1}^4 L_k \prod_{m\neq k}\sin^2(\sqrt{\lambda_j}L_m)}, \quad \forall j \in \mathbb{N}. \tag{21}$$

From (20) and (21), we have $\sum_{l=1}^4 \cos(\sqrt{\lambda_k}L_l)\prod_{m\neq l}\sin(\sqrt{\lambda_k}L_m) = 0$. The validity of [DZ06, *Proposition A*.11] and Remark *A*.4 ensure that, for every $\epsilon > 0$, there exist $C_1, C_2 > 0$ such that, for every $j \in \mathbb{N}$,

$$|a_j^1| = \sqrt{\frac{2}{\sum_{l=1}^4 L_l \sin^{-2}(\sqrt{\lambda_j}L_l)}} \geq \sqrt{\frac{2}{\sum_{l=1}^4 L_l C_1^{-2}\lambda_j^{1+\epsilon}}} \geq \frac{C_2}{j^{1+\epsilon}}. \tag{22}$$

Now, $\langle \phi_k^l, B\phi_j^l\rangle_{L^2(e_j,\mathbb{C})} = 0$ for every $2 \leq l \leq 4$ and $k, j \in \mathbb{N}$. Let

$$\begin{aligned} a_j(x) &:= \frac{2\prod_{m\neq 1}\sin^2(\sqrt{\lambda_j}L_m)}{\sum_{k=2}^4 L_k \sin^2(\sqrt{\lambda_j}x)\prod_{m\neq k,1}\sin^2(\sqrt{\lambda_j}L_m) + x\prod_{m\neq 1}\sin^2(\sqrt{\lambda_j}L_m)}, \\ B_1(x) &:= \frac{-30\sqrt{\lambda_1}x + 20\sqrt{\lambda_1}^3 x^3 + 4\sqrt{\lambda_1}^5 x^5 + 15\sin(2\sqrt{\lambda_1}x)}{40\sqrt{\lambda_1}^5}, \\ B_j(x) &:= 2\frac{-6(\sqrt{\lambda_1}-\sqrt{\lambda_j})x + (\sqrt{\lambda_1}-\sqrt{\lambda_j})^3x^3 + 6\sin((\sqrt{\lambda_1}-\sqrt{\lambda_j})x)}{(\sqrt{\lambda_1}-\sqrt{\lambda_j})^5} \\ &\quad - 2\frac{-6(\sqrt{\lambda_1}+\sqrt{\lambda_j})x + (\sqrt{\lambda_1}+\sqrt{\lambda_j})^3x^3 + 6\sin((\sqrt{\lambda_1}+\sqrt{\lambda_j})x)}{(\sqrt{\lambda_1}+\sqrt{\lambda_j})^5} \end{aligned}$$

with $j \in \mathbb{N}$. Each function $\widetilde{B}_j(\cdot) := \sqrt{a_1(\cdot)}\sqrt{a_j(\cdot)}B_j(\cdot)$ is non-constant and analytic in $\mathbb{R}^+$, while we notice that $B_{1,j} = \langle \phi_1, B\phi_j\rangle = \widetilde{B}_j(L_1)$ by

calculation. The set of positive zeros $\tilde{V}_j$ of each $\widetilde{B}_j$ is a discrete subset of $\mathbb{R}^+$ and $\tilde{V} = \bigcup_{j\in\mathbb{N}} \tilde{V}_j$ is countable. For every $\{L_l\}_{l\leq 4} \in \mathcal{AL}(4)$ such that $L_1 \notin \tilde{V}$, we have $|B_{1,j}| \neq 0$ for every $j \in \mathbb{N}$. Now, there holds

$$|B_{1,j}| \sim |a_j| L_1 \sqrt{\lambda_1}\sqrt{\lambda_j}(\lambda_j - \lambda_1)^{-2}, \qquad \forall j \in \mathbb{N}\setminus\{1\}.$$

From Remark $A.4$ and the identity (22), the first point of Assumptions $\mathrm{I}(2+\epsilon)$ is verified since, for each $\epsilon > 0$, there exists $C_3 > 0$ such that $|B_{1,j}| \geq \frac{C_3}{j^{4+\epsilon}}$ for every $j \in \mathbb{N}$.

Let $(k,j),(m,n) \in I,\ (k,j) \neq (m,n)$ for $I := \{(j,k) \in \mathbb{N}^2 : j \neq k\}$ and

$$F_j(x) := a_j(x)\frac{-30\sqrt{\lambda_k}x + 20\sqrt{\lambda_k}^3 x^3 + 4\sqrt{\lambda_k}^5 x^5 + 15\sin(2\sqrt{\lambda_k}x)}{40\sqrt{\lambda_k}^5}.$$

By calculation, we notice that $B_{j,j} = \langle \phi_j, B\phi_j\rangle = F_j(L_1)$. Moreover, for $F_{j,k,l,m}(x) = F_j(x) - F_k(x) - F_l(x) + F_m(x)$, it follows $F_{j,k,l,m}(L_1) = B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m}$ and $F_{j,k,l,m}(x)$ is a non-constant analytic function for $x > 0$. Furthermore $V_{j,k,l,m}$, the set of the positive zeros of $F_{j,k,l,m}(x)$, is discrete and $V := \bigcup_{\substack{j,k,l,m\in\mathbb{N}\\ j\neq k\neq l\neq m}} V_{j,k,l,m}$ is a countable subset of $\mathbb{R}^+$. For each $\{L_l\}_{l\leq 4} \in \mathcal{AL}(4)$ such that $L_1 \notin V \cup \tilde{V}$, Assumptions $\mathrm{I}(2+\epsilon)$ are verified.

The third point of Assumptions $\mathrm{II}(2+\epsilon_1,\epsilon_2)$ is valid for each $\epsilon_1,\epsilon_2 > 0$ such that $\epsilon_1+\epsilon_2 \in (0,1/2)$ since B stabilizes $H^2_{\mathscr{G}}$, H^m and $H^m_{\mathcal{NK}}$ for $m \in (0,9/2)$. Indeed, for every $n \in \mathbb{N}$ such that $n < 5$, we have

$$\forall \psi \in H^n_{\mathcal{NK}} \ \Rightarrow\ \partial_x^{n-1}(B\psi)^1(L_1) = ... = \partial_x^{n-1}(B\psi)^4(L_4) = 0 \ \Rightarrow\ B\psi \in H^n_{\mathcal{NK}}.$$

From Theorem 2.4, the controllability holds in $H^{4+\epsilon}_{\mathscr{G}}$ with $\epsilon \in (0,1/2)$. □

Proof of Example 1.3. Let $\mathscr{G}$ be a tadpole graph containing an edge e_1 self-closing in an internal vertex $v \in V_i$ equipped with $(\mathcal{NK})$. The edge e_2 is connecting v to the external vertex $v_1 \in V_e$ equipped with $(\mathcal{D})$. Let r be the axis passing along e_2 and crossing e_1 in its middle (see Figure 8).

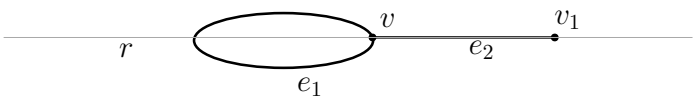

Figure 8: The figure represents the symmetry axis r of the tadpole graph.

The graph $\mathscr{G}$ is symmetric with respect to r and we construct the eigenfunctions $\{\phi_k\}_{k\in\mathbb{N}}$ as a sequence of symmetric or skew-symmetric functions with respect to r. If an eigenfunction $\phi_k = (\phi_k^1, \phi_k^2)$ is skew-symmetric, then

$$\phi_k^2 \equiv 0, \quad \phi_k^1(0) = \phi_k^1(L_1/2) = \phi_k^1(L_1) = 0, \quad \partial_x\phi_k^1(0) = \partial_x\phi_k^1(L_1).$$

We denote $\{f_k\}_{k\in\mathbb{N}}$ the skew-symmetric eigenfunctions belonging to the Hilbert basis $\{\phi_k\}_{k\in\mathbb{N}}$ and $\{\nu_k\}_{k\in\mathbb{N}}$ the corresponding eigenvalues. We set

$$\{f_k\}_{k\in\mathbb{N}} = \left\{\Big(\sqrt{\frac{2}{L_1}}\sin\Big(x\frac{2k\pi}{L_1}\Big),0\Big)\right\}_{k\in\mathbb{N}}, \qquad \{\nu_k\}_{k\in\mathbb{N}} := \left\{\frac{4k^2\pi^2}{L_1^2}\right\}_{k\in\mathbb{N}}.$$

If $\phi_k = (\phi_k^1, \phi_k^2)$ is symmetric, then we have $\partial_x\phi_k^1(L_1/2) = 0$ and $\phi_k^1(\cdot) = \phi_k^1(L_1 - \cdot)$. The $(\mathcal{D})$ conditions on v_1 implies that, for $\{(a_k^1, a_k^2)\}_{k\in\mathbb{N}} \subset \mathbb{C}^2$,

$$\{g_k\}_{k\in\mathbb{N}} := \Big\{\Big(a_k^1\cos\Big(\sqrt{\mu_k}\Big(x-\frac{L_1}{2}\Big)\Big), a_k^2\sin(\sqrt{\mu_k}x)\Big)\}_{k\in\mathbb{N}},$$

is the sequence of symmetric eigenfunctions and corresponding to the eigenvalues $\{\mu_k\}_{k\in\mathbb{N}}$. We characterize $\{\mu_k\}_{k\in\mathbb{N}}$ by considering that the $(\mathcal{NK})$ conditions in v_1 ensure that $a_k^1\cos(\sqrt{\mu_k}(L_1/2)) = a_k^2\sin(\sqrt{\mu_k}L_2))$ and

$$2a_k^1\sin(\sqrt{\mu_k}(L_1/2)) + a_k^2\cos(\sqrt{\mu_k}L_2)) = 0, \tag{23}$$

which imply $2\tan(\sqrt{\mu_k}(L_1/2))+\cot(\sqrt{\mu_k}L_2)) = 0$. We choose $\{(a_k^1, a_k^2)\}_{k\in\mathbb{N}} \subset \mathbb{C}^2$ such that $\{\phi_k\}_{k\in\mathbb{N}} := \{f_k\}_{k\in\mathbb{N}} \cup \{g_k\}_{k\in\mathbb{N}}$ forms an Hilbert basis of $\mathscr{H}$. In particular, the techniques leading to relation (21) in Example 1.2 attain

$$|a_k^1|^2 = \frac{2\cos^2(\sqrt{\mu_k}(L_1/2))\sin^2(\sqrt{\mu_k}L_2)}{a_k}, \quad |a_k^2|^2 = \frac{2\cos^4(\sqrt{\mu_k}(L_1/2))}{a_k}$$

with $a_k := L_1\cos^2(\sqrt{\mu_k}(L_1/2)) + L_2\sin^2(\sqrt{\mu_k}L_2)$ and $k\in\mathbb{N}$. If $\{L_1, L_2\} \in \mathcal{AL}(2)$, then $\{L_1/2, L_2\} \in \mathcal{AL}(2)$. From (23), there holds

$$2\cos\Big(\sqrt{\mu_k}\frac{L_1}{2}\Big)\sin(\sqrt{\mu_k}L_2)\sin\Big(\sqrt{\mu_k}\frac{L_1}{2}\Big) + \cos^2\Big(\sqrt{\mu_k}\frac{L_1}{2}\Big)\cos(\sqrt{\mu_k}L_2) = 0.$$

We underline that $\cos(\sqrt{\mu_k}(L_1/2)) \neq 0$ for every $k\in\mathbb{N}$ and

$$2\sin(\sqrt{\mu_k}L_2)\sin(\sqrt{\mu_k}(L_1/2)) + \cos(\sqrt{\mu_k}(L_1/2))\cos(\sqrt{\mu_k}L_2) = 0,$$

which implies to the validity of the two points of Remark $A.6$ for each $l\in\{1,2\}$ and with $\{L_1/2, L_2\} \in \mathcal{AL}(2)$. The arguments leading to (22) in Example 1.2, applied with the identities (29) and (30), imply that

$$\forall\epsilon>0, \ \ \exists C>0 \ \ : \ \ |a_k^l| \geq Ck^{-1-\epsilon}, \qquad \forall k\in\mathbb{N}, \ \forall l\in\{1,2\}. \tag{24}$$

Let $B_1 : (\psi^1,\psi^2) \mapsto (h\psi^1, 0)$ and $B_2 : (\psi^1,\psi^2) \mapsto (h_1\psi^1, h_2\psi^2)$ with $h(x) := \sin\Big(\frac{2\pi}{L_1}x\Big)$, $h_1(x) := x(x-L_1)$ and $h_2(x) := x^2 - (2L_1+2L_2)x + L_2^2 + 2L_1L_2$. As h is skew-symmetric with respect to r and h_1 is symmetric, we have

$$\langle f_k, B_1 f_k\rangle = \langle g_k, B_1 g_k\rangle = 0, \qquad \langle f_k, B_2 g_k\rangle = \langle g_k, B_2 f_k\rangle = 0.$$

The remaining part of the example is ensured as Example 1.2. We fix $j \in \mathbb{N}$ and we notice by calculation that $|\langle f_j^1, Bf_k^1\rangle| = |\langle f_j^1, B_2 f_k^1\rangle| \sim k^{-3}$,

$$|\langle f_j, Bg_k\rangle| = |\langle f_j, B_1 g_k\rangle| \sim \frac{|a_k|}{\mu_k}\Big|\sin\Big(\Big(\sqrt{\mu_k}-\frac{2j\pi}{L_1}\Big)\frac{L_1}{2}\Big)+\sin\Big(\Big(\sqrt{\mu_k}+\frac{2j\pi}{L_1}\Big)\frac{L_1}{2}\Big)\Big|.$$

From Remark *A*.4, we have $\mu_k \sim k^2$ and $|a_k^2|^{-1}|\langle g_j, Bg_k\rangle| \sim k^{-2}$ as $L_2 > L_1$. As in Example 1.2, there exists $\tilde{V} \subset \mathbb{R}^+$ countable such that, for every $\{L_1, L_2\} \in \mathcal{AL}(2)$ such that $L_1 \notin \tilde{V}$, we have $|B_{1,k}| \neq 0$ for every $j \in \mathbb{N}$. Thanks to (30), for every $\epsilon > 0$, there exists $C_1 > 0$ such that

$$\Big|\sin\Big(\Big(\sqrt{\mu_k}-\frac{2j\pi}{L_1}\Big)\frac{L_1}{2}\Big)+\sin\Big(\Big(\sqrt{\mu_k}+\frac{2j\pi}{L_1}\Big)\frac{L_1}{2}\Big)\Big| = \Big|\sin\Big(\sqrt{\mu_k}\frac{L_1}{2}\Big)\Big| \geq \frac{C}{k^{1+\epsilon}}.$$

From Remark *A*.4, the first point of Assumptions I$(2+\epsilon)$ is attained, *i.e.*

$$\forall \epsilon > 0,\ \exists C_2 > 0 \quad : \quad |B_{1,k}| \geq C_2 k^{-4-\epsilon}, \qquad \forall k \in \mathbb{N}.$$

The second point of Assumptions I$(2+\epsilon)$ is verified as in Example 1.2 and there exists $V \subset \mathbb{R}^+$ countable such that, for each $\{L_1, L_2\} \in \mathcal{AL}(2)$ such that $L_1 \notin V \cup \tilde{V}$, Assumptions I$(2+\epsilon)$ are verified.

The third point of Assumptions II$(2+\epsilon_1, \epsilon_2)$ is valid for $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 \in (0, 1/2)$ since B stabilizes $H^2_{\mathscr{G}}$, H^m and $H^m_{\mathcal{NK}}$ for $m \in \mathbb{N}$ similarly to Example 1.2. From Theorem 2.4, the controllability holds in

$$H^{4+\epsilon}_{\mathscr{G}} \qquad \text{with} \qquad \epsilon \in (0, 1/2). \qquad \square$$

Proof of Example 1.4. The $(\mathcal{D})$ conditions imply that ϕ_k satisfies $\phi_k^l(0) = 0$ and $\phi_k^l(L_l) = 0$ for every $k \in \mathbb{N}$ and $l \leq N$. As $\{L_l\}_{l\leq N} \in \mathcal{AL}(N)$, for each $k \in \mathbb{N}$, there exist $m(k) \in \mathbb{N}$ and $l(k) \leq N$ such that, for every $n \neq l(k)$,

$$\lambda_k = m(k)^2\pi^2 L_{l(k)}^{-2}, \qquad \phi_k^{l(k)}(x) = \sqrt{2L_{l(k)}^{-1}}\sin\left(\sqrt{\lambda_k}x\right), \qquad \phi_k^n \equiv 0.$$

Hence, $\{\lambda_k\}_{k\in\mathbb{N}}$ is obtained by reordering $\left\{\frac{m^2\pi^2}{L_l^2}\right\}_{m\in\mathbb{N}}$ for every $l \leq N$. Now,

$$|B_{1,j}| \geq 2\min\{L_l^2 :\ l \leq N\}\Big|\int_0^1 x^2 \sin(m(j)\pi x)\sin(m(1)\pi x)dx\Big|.$$

This is the integral treated in [Ducb, *Example* 1.1] where it is showed that, for every $j \in \mathbb{N}$, there exists $C_1 > 0$ such that $|B_{j,1}| \geq \frac{C_1}{m(j)^3} \geq \frac{C_1}{j^3}$ for every $j \in \mathbb{N}$ since $m(j) \leq j$. Moreover, there holds

$$B_{j,j} = 2L_{m(j)}^2\int_0^1 x^2\sin^2(m(j)\pi x)dx = \frac{L_{m(j)}^2}{3} - \frac{L_{m(j)}^2}{2m(j)^2\pi^2}.$$

As done in the proof of Example 1.2, there exists a countable set V such that, for each $\{L_l\}_{l\leq N} \in \mathcal{AL}(N) \setminus V$, Assumptions I(1) are verified.

The third point of Assumptions II$(1, \epsilon)$ is valid for each $\epsilon \in (0, 3/2)$ since B stabilizes $H^2_{\mathscr{G}}$ and H^m for $m > 0$ ($H^m \equiv H^m_{\mathcal{NK}}$). Corollary 2.6 achieves the controllability for every $\epsilon \in (0, 3/2)$ in $H^{3+\epsilon}_{\mathscr{G}} = \prod_{j=1}^N H^{3+\epsilon}_{I_j}$. $\square$

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A Appendix: Spectral properties

In the current appendix, we characterize $\{\lambda_k\}_{k\in\mathbb{N}}$, the eigenvalues of the Laplacian A, according to the structure of $\mathscr{G}$ and to the choice of $D(A)$.

Proposition A.1. *(Roth's Theorem; [Rot56]) If z is an algebraic irrational number, then for every $\epsilon > 0$ the inequality $\left|z - \frac{n}{m}\right| \leq \frac{1}{m^{2+\epsilon}}$ is satisfied for at most a finite number of $n, m \in \mathbb{Z}$.*

Lemma A.2. *Let $\left\{\lambda_k^1\right\}_{k\in\mathbb{N}}$ and $\left\{\lambda_k^2\right\}_{k\in\mathbb{N}}$ be obtained by reordering*

$$\left\{\frac{k^2\pi^2}{L_l^2}\right\}_{\substack{k,l\in\mathbb{N}\\ l\leq N_1}}, \qquad \left\{\frac{k^2\pi^2}{\tilde{L}_i^2}\right\}_{\substack{k,i\in\mathbb{N}\\ i\leq N_2}} \qquad \text{for } N_1, N_2 \in \mathbb{N},\ \{L_l\}_{l\leq N_1}, \{\tilde{L}_i\}_{i\leq N_2} \subset \mathbb{R}$$

respectively. If all the ratios $\tilde{L}_i/L_l$ are algebraic irrational numbers, then

$$\forall \epsilon > 0,\ \exists C > 0\ :\ |\lambda_{k+1}^1 - \lambda_k^2| \geq Ck^{-\epsilon}, \qquad \forall k \in \mathbb{N}.$$

Proof. Let z be an algebraic irrational number. From Proposition $A.1$, we have that, for every $\epsilon > 0$, there exists $C > 0$ such that $|z - n/m| \geq Cm^{-2-\epsilon}$ for every $m, n \in \mathbb{N}$. Now, for every $k \in \mathbb{N}$, there exist $m, n \in \mathbb{N}$ and $i, l \leq N$ such that $\lambda_{k+1}^1 = \frac{m^2\pi^2}{L_l^2}$, $\lambda_k^2 = \frac{n^2\pi^2}{\tilde{L}_i^2}$, $\lambda_{k+1}^1 \neq \lambda_k^2$. We suppose $L_l < \tilde{L}_i$. If $m < n$, then, for each $\epsilon > 0$, there exists $C_1 > 0$ such that

$$\left|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\right| = \left|\left(\frac{m\pi}{L_l} + \frac{n\pi}{\tilde{L}_i}\right)\left(\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\right)\right| \geq \frac{2m\pi}{\tilde{L}_i}\left|\frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i}\right| \geq \frac{2C_1\pi^2}{m^\epsilon \tilde{L}_i^2}.$$

If $m \geq n$, then $\left|\frac{m^2\pi^2}{L_l^2} - \frac{n^2\pi^2}{\tilde{L}_i^2}\right| \geq \pi^2(L_l^{-2} - \tilde{L}_i^{-2})$. In conclusion,

$$\forall \epsilon > 0,\ \exists C_2 > 0\ :\ |\lambda_{k+1}^1 - \lambda_k^2| \geq C_2(k+1)^{-\epsilon} \geq C_2 2^{-\epsilon} k^{-\epsilon}, \qquad \forall k \in \mathbb{N}. \quad \square$$

The following proposition rephrases the results of [BK13, *Theorem* 3.1.8] and [BK13, *Theorem* 3.1.10]. Let $\left\{\lambda_k^{\widehat{\mathscr{G}}}\right\}_{k\in\mathbb{N}}$ be the spectrum of A on a generic compact quantum graph $\widehat{\mathscr{G}}$.

Proposition A.3. [*BK13, Theorem* 3.1.8] *&* [*BK13, Theorem* 3.1.10]

1. *Let w, v be two vertices of $\mathscr{G}$ equipped with $(\mathcal{NK})$ or $(\mathcal{N})$. If $\mathscr{G}'$ is the graph obtained by merging in $\mathscr{G}$ the vertices w and v in one unique vertex equipped with $(\mathcal{NK})$, then $\lambda_k^{\mathscr{G}} \leq \lambda_k^{\mathscr{G}'} \leq \lambda_{k+1}^{\mathscr{G}}$ for every $k \in \mathbb{N}$.*

2. *Let w be a vertex of $\mathscr{G}$. If $\mathscr{G}^{\mathcal{D}}$ is the graph obtained by imposing $(\mathcal{D})$ on w, then $\lambda_k^{\mathscr{G}} \le \lambda_k^{\mathscr{G}^{\mathcal{D}}} \le \lambda_{k+1}^{\mathscr{G}}$ for every $k \in \mathbb{N}$.*

Remark A.4. *Let $\mathscr{G}$ be compact quantum graphs made by edges of lengths $\{L_l\}_{l\le N}$. From Proposition A.3, there exist $C_1, C_2 > 0$ such that, for $k \ge 2$,*

$$C_1 k^2 \le \lambda_k^{\mathscr{G}} \le C_2 k^2. \tag{25}$$

Indeed, we define $\mathscr{G}^{\mathcal{D}}$ from $\mathscr{G}$ by imposing $(\mathcal{D})$ in each vertex. We denote $\mathscr{G}^{\mathcal{N}}$ from $\mathscr{G}$ by disconnecting each edge and by imposing $(\mathcal{N})$ in each vertex. From Proposition A.3, we have $\lambda_{k-2N}^{\mathscr{G}^{\mathcal{N}}} \le \lambda_k^{\mathscr{G}} \le \lambda_{k+M}^{\mathscr{G}^{\mathcal{D}}}$ for $k > 2N$. The sequences $\lambda_k^{\mathscr{G}^{\mathcal{N}}}$ and $\lambda_k^{\mathscr{G}^{\mathcal{D}}}$ are respectively obtained by reordering $\big\{\frac{k^2\pi^2}{L_l^2}\big\}_{\substack{k\in\mathbb{N}\\ l\le N}}$ and $\big\{\frac{(k-1)^2\pi^2}{L_i^2}\big\}_{\substack{k\in\mathbb{N}\\ i\le N}}$. For $l > 2N+1$, $\tilde{m} = \max_{j\le N} L_j^2$ and $\hat{m} = \min_{j\le N} L_j^2$,

$$\lambda_{l-2N}^{\mathscr{G}^{\mathcal{N}}} \ge \frac{(l-2N-1)^2\pi^2}{N^2\tilde{m}} \ge \frac{l^2\pi^2}{2^{2(2N+1)}N^2\tilde{m}}, \quad \lambda_{l+M}^{\mathscr{G}^{\mathcal{D}}} \le \frac{(l+M)^2\pi^2}{\hat{m}} \le \frac{l^2 2^{2M}\pi^2}{\hat{m}}.$$

The identity (25) is valid for $k \ge 2$ as $\lambda_k \ne 0$, but also for $k = 1$ if $\lambda_1 \ne 0$.

The techniques developed in [DZ06, *Appendix A*] and adopted in order to prove [DZ06, *Proposition A.11*] lead to following proposition.

Proposition A.5. *Let $\{L_k\}_{k\le N} \in \mathcal{AL}(N)$ with $N \in \mathbb{N}$. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be the unbounded sequence of positive solutions of the equation*

$$\sum_{l\le N} \sin(xL_l) \prod_{m\ne l} \cos(xL_m) = 0, \qquad x \in \mathbb{R}. \tag{26}$$

For every $\epsilon > 0$, there exists $C_\epsilon > 0$ so that, for every $l \le N$,

$$|\cos(\omega_n L_l)| \ge \frac{C_\epsilon}{\omega_n^{1+\epsilon}}, \qquad \forall n \in \mathbb{N}.$$

Proof. We consider the notation introduced in [DZ06, *Appendix A*] as $|||\cdot|||$, $E(\cdot)$ and $F(\cdot)$. For $x \in \mathbb{R}$, $\{L_k\}_{k\le N} \in (\mathbb{R}^+)^N$ and $i \le N$, we also denote

$$n(x) := E(x-1/2), \quad r(x) := F(x-1/2), \quad d(x) := |||\, x-1/2\,|||, \quad \widetilde{m}^i(x) := n\Big(\frac{L_i}{\pi}x\Big).$$

From [DZ06, *relation (A.3)*], for every $x \in \mathbb{R}$, we obtain the identities

$$2d(x) \le |\cos(\pi x)| \le \pi d(x), \qquad 2d\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \le \Big|\cos\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\pi\Big)\Big|. \tag{27}$$

As $\cos(\alpha_1-\alpha_2) = \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)$ for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\widetilde{m}^i(x) + \frac{1}{2} = \frac{L_i}{\pi}x - r\big(\frac{L_i}{\pi}x\big)$ for every $x \in \mathbb{R}$, we have

$$2d\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \le |\cos(L_j x)| + \Big|\sin\Big(\pi\frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big|\Big)\Big|. \tag{28}$$

From [DZ06, *relation* (*A*.3)] and (27), we have the following inequalities $|\sin(\pi|r(\cdot)|)| \leq \pi \,|||\, |r(\cdot)| \,|||\, \leq \pi|r(\cdot)| = \pi d(\cdot) \leq \frac{\pi}{2}|\cos(\pi(\cdot))|$, which imply

$$\left|\sin\left(\pi\frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big|\right)\right| \leq \pi\frac{L_j}{L_i}\Big|r\Big(\frac{L_i}{\pi}x\Big)\Big| \leq \frac{\pi L_j}{2L_i}|\cos(L_i x)|, \quad \forall x \in \mathbb{R}.$$

From (28), there exists $C_1 > 0$ such that, for every $i \leq N$,

$$\prod_{j\neq i} d\Big(\Big(\widetilde{m}^i(x)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) \leq \frac{1}{2^{N-1}}\prod_{j\neq i}|\cos(L_j x)| + C_1|\cos(L_i x)| \quad \forall x \in \mathbb{R}.$$

Thanks to (26), if there exists $\{\omega_{n_k}\}_{k\in\mathbb{N}}$, subsequence of $\{\omega_n\}_{n\in\mathbb{N}}$, such that

$$|\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0 \qquad \text{then} \quad \prod_{j\neq i}|\cos(L_i\omega_{n_k})| \xrightarrow{k\to\infty} 0.$$

Equivalently to [DZ06, *relation* (*A*.10)] (proof of [DZ06, *Proposition A*.11]), there exists a constant $C_2 > 0$ such that, for every $i \in \{0, ..., N\}$, we have

$$C_2|\cos(L_i\omega_n)| \geq \prod_{j\neq i} d\Big(\Big(\widetilde{m}^i(\omega_n)+\frac{1}{2}\Big)\frac{L_j}{L_i}\Big) = \prod_{j\neq i} |||\, \frac{1}{2}\Big(\Big(\widetilde{m}^i(\omega_n)+\frac{1}{2}\Big)\frac{2L_j}{L_i}-1\Big)\,|||\,.$$

Now, we have $|||\, \frac{1}{2}(\cdot)\,||| \geq \frac{1}{2}|||\,\cdot\,|||$ and $|||\,(\cdot)-1\,||| = |||\,\cdot\,|||$. We consider the Schmidt's Theorem [DZ06, *Theorem A*.7] since $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$. For every $\epsilon > 0$, there exist $C_3, C_4 > 0$ such that, for every $n \in \mathbb{N}$, we have

$$\prod_{j\neq i}\frac{1}{2}|||\,\Big(\widetilde{m}^i(\omega_n)+\frac{1}{2}\Big)\frac{2L_j}{L_i}\,||| \geq \frac{C_3}{(2\widetilde{m}^i(\omega_n)+1)^{1+\epsilon}} \geq \frac{C_4}{\omega_n^{1+\epsilon}}. \qquad \square$$

Remark A.6. *The techniques proving [DZ06, Proposition A*.11*] and Proposition A*.5 *lead to the following results. Let* $\{L_k\}_{k\leq N} \in \mathcal{AL}(N)$ *with* $N \in \mathbb{N}$. *Let* $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *be an unbounded sequence and* $l \leq N$.

1) *If the existence of* $\{\omega_{n_k}\}_{k\in\mathbb{N}} \subset \{\omega_n\}_{n\in\mathbb{N}}$, *such that* $|\cos(L_l\omega_{n_k})| \xrightarrow{k\to\infty} 0$ *implies* $\prod_{j\neq l}|\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$ *or* $\prod_{j\neq l}|\sin(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$, *then*

$$(29) \qquad \forall\epsilon > 0,\ \exists C_{\epsilon,l} > 0 \quad : \quad |\cos(\omega_n L_l)| \geq \frac{C_{\epsilon,l}}{\omega_n^{1+\epsilon}}, \qquad \forall n \in \mathbb{N}.$$

2) *If the existence of* $\{\omega_{n_k}\}_{k\in\mathbb{N}} \subset \{\omega_n\}_{n\in\mathbb{N}}$, *such that* $|\sin(L_l\omega_{n_k})| \xrightarrow{k\to\infty} 0$ *implies* $\prod_{j\neq l}|\cos(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$ *or* $\prod_{j\neq l}|\sin(L_j\omega_{n_k})| \xrightarrow{k\to\infty} 0$, *then*

$$(30) \qquad \forall\epsilon > 0,\ \exists C_{\epsilon,l} > 0 \quad : \quad |\sin(\omega_n L_l)| \geq \frac{C_{\epsilon,l}}{\omega_n^{1+\epsilon}}, \qquad \forall n \in \mathbb{N}.$$

B Appendix: Moments problem

Let $\mathscr{H}$ be a Hilbert space over a field $\mathcal{K}$ for $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$ and $\{f_n\}_{n\in\mathbb{Z}} \subset \mathscr{H}$. In this appendix, we study the solvability of the so-called "moments problem", which consists in finding $v \in \mathscr{H}$ such that, for a $\{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathcal{K}')$ with $\mathcal{K}' = \mathbb{C}$ or $\mathbb{R}$, there holds $x_n = \langle v, f_n\rangle_{\mathscr{H}}$ for every $n \in \mathbb{Z}$.

Let $\mathscr{H} = L^2((0,T),\mathbb{R})$ with $T > 0$. Let $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$ and $\mathbf{\Lambda} = \{\lambda_k\}_{k\in\mathbb{Z}^*}$ be pairwise distinct ordered real numbers such that

$$\exists \mathcal{M} \in \mathbb{N},\ \exists \delta > 0 \quad : \quad \inf_{k\in\mathbb{Z}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}. \tag{31}$$

We consider $\{f_n\}_{n\in\mathbb{N}} = \{e^{i\lambda_n(\cdot)}\}_{n\in\mathbb{N}}$ and the following moment problem

$$x_n = \int_0^T e^{i\lambda_n s} u(s)ds, \qquad \text{with} \ \ \{x_n\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{C}), \quad u \in \mathscr{H}.$$

From (31), we notice that there does not exist $\mathcal{M}$ consecutive $k \in \mathbb{Z}^*$ such that $|\lambda_{k+1} - \lambda_k| < \delta$. This fact leads to a partition of $\mathbb{Z}^*$ in subsets that we call E_m with $m \in \mathbb{Z}^*$. By definition, for every $m \in \mathbb{Z}^*$, if $k, n \in E_m$, then $|\lambda_k - \lambda_n| < \delta(\mathcal{M}-1)$, while if $k \in E_m$ and $n \notin E_m$, then $|\lambda_k - \lambda_n| \geq \delta$.

The partition also defines an equivalence relation in $\mathbb{Z}^*$ such that $k, n \in \mathbb{Z}^*$ are equivalent if and only if there exists $m \in \mathbb{Z}^*$ such that $k, n \in E_m$. The sets $\{E_m\}_{m\in\mathbb{Z}^*}$ are the corresponding equivalence classes and $i(m) := |E_m| \leq \mathcal{M} - 1$. For every sequence $\mathbf{x} := \{x_l\}_{l\in\mathbb{Z}^*}$, we define the vectors $\mathbf{x}^m := \{x_l\}_{l\in E_m}$ for $m \in \mathbb{Z}^*$.

Let $\widehat{\mathbf{h}} = \{h_j\}_{j\leq i(m)} \in \mathbb{C}^{i(m)}$ with $m \in \mathbb{Z}^*$. For every $m \in \mathbb{Z}^*$, we denote $F_m(\widehat{\mathbf{h}}) : \mathbb{C}^{i(m)} \to \mathbb{C}^{i(m)}$ the matrix with elements, for every $j, k \leq i(m)$,

$$F_{m;j,k}(\widehat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l\neq j \\ l\leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

For each $k \in \mathbb{Z}^*$, there exists $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. Let $F(\mathbf{\Lambda})$ be the linear operator on $\ell^2(\mathbb{C})$ such that $F(\mathbf{\Lambda}) : D(F(\mathbf{\Lambda})) \to \ell^2(\mathbb{C})$ and

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \left(F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)}\right)_k, \qquad \forall \mathbf{x} = \{x_l\}_{l\in\mathbb{Z}^*} \in D(F(\mathbf{\Lambda})),$$

$$H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \left\{\mathbf{x} := \{x_k\}_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C}) \ : \ F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{C})\right\}.$$

Proposition B.1. *Let $\mathbf{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of real numbers satisfying* (31)*. If there exist $\tilde{d} \geq 0$ and $C > 0$ such that*

$$|\lambda_{k+1} - \lambda_k| \geq C|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}} \qquad \forall k \in \mathbb{Z}^*, \tag{32}$$

then we have $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}(\mathbb{C})$.

Proof. Thanks to (32), we have $|\lambda_j - \lambda_k| \geq C \min_{l\in E_m} |l|^{-\frac{\tilde{d}}{\mathcal{M}-1}}$ for every $m \in \mathbb{Z}^*$ and $j,k \in E_m$. There exists $C_1 > 0$ such that, for $1 < j,k \leq i(m)$,

$$|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_1\big(\max_{l\in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{k-1} \leq C_1\big(\max_{l\in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}}\big)^{\mathcal{M}-1} \leq C_1 2^{\mathcal{M}\tilde{d}} \min_{l\in E_m} |l|^{\tilde{d}}$$

and $|F_{m;1,1}(\mathbf{\Lambda}^m)| = 1$. Then, there exist $C_2, C_3 > 0$ such that, for $j \leq i(m)$,

$$\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)_{j,j} \leq C_2 \min_{l\in E_m} |l|^{2\tilde{d}}, \quad Tr\Big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\Big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}$$

with $F_m(\mathbf{\Lambda}^m)^*$ the transposed matrix of $F_m(\mathbf{\Lambda}^m)$. Let $\rho(M)$ be the spectral radius of a matrix M and we denote $|\!|\!| M |\!|\!| = \sqrt{\rho(M^*M)}$ its euclidean norm. As $\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big)$ is positive-definite, there holds

$$|\!|\!| F_m(\mathbf{\Lambda}^m) |\!|\!|^2 = \rho\big(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)\big) \leq C_3 \min_{l\in E_m} |l|^{2\tilde{d}}, \quad \forall m \in \mathbb{Z}^*.$$

In conclusion, $\|F(\mathbf{\Lambda})\mathbf{x}\|^2_{\ell^2} \leq C_3 \|\mathbf{x}\|^2_{h^{\tilde{d}}} < +\infty$ for $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$ as

$$\|F(\mathbf{\Lambda})\mathbf{x}\|^2_{\ell^2} \leq \sum_{m\in\mathbb{Z}^*} |\!|\!| F_m(\mathbf{\Lambda}^m) |\!|\!|^2 \sum_{l\in E_m} |x_l|^2 \leq C_3 \sum_{m\in\mathbb{Z}^*} \min_{l\in E_m} |l|^{2\tilde{d}} \sum_{l\in E_m} |x_l|^2.$$

□

Corollary B.2. *If $\mathbf{\Lambda} := \{\lambda_k\}_{k\in\mathbb{Z}^*}$ is an ordered sequence of pairwise distinct real numbers satisfying* (31)*, then $F(\mathbf{\Lambda}) : H(\mathbf{\Lambda}) \to Ran(F(\mathbf{\Lambda}))$ is invertible.*

Proof. As in [DZ06, *p.* 48], we define $F_m(\mathbf{\Lambda}^m)^{-1}$ the inverse matrix of $F_m(\mathbf{\Lambda}^m)$ for every $m \in \mathbb{Z}^*$. We call $F(\mathbf{\Lambda})^{-1}$ the operator such that $(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k = \big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)}\big)_k$, for every $\mathbf{x} \in Ran(F(\mathbf{\Lambda}))$ and $k \in \mathbb{Z}^*$, which implies $F(\mathbf{\Lambda})^{-1}F(\mathbf{\Lambda}) = Id_{H(\mathbf{\Lambda})}$ and $F(\mathbf{\Lambda})F(\mathbf{\Lambda})^{-1} = Id_{Ran(F(\mathbf{\Lambda}))}$. □

For every $k \in \mathbb{Z}^*$, we have the existence of $m(k) \in \mathbb{Z}^*$ such that $k \in E_{m(k)}$. We define $F(\mathbf{\Lambda})^*$ the infinite matrix such that $(F(\mathbf{\Lambda})^*\mathbf{x})_k = \big(F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*\mathbf{x}^{m(k)}\big)_k$ for every $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}^*}$ and $k \in \mathbb{Z}^*$, where $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$ is the transposed matrix of $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$. For $T > 0$, we introduce

$$\mathbf{e} := \{e^{i\lambda_j t}\}_{j\in\mathbb{Z}^*} \subset L^2((0,T),\mathbb{C}), \quad \mathbf{\Xi} := \{\xi_k(\cdot)\}_{k\in\mathbb{Z}^*} = F(\mathbf{\Lambda})^*\mathbf{e} \subset L^2((0,T),\mathbb{C}).$$

Remark B.3. *Thanks to Proposition B.*1*, when $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ satisfies* (31)*, the space $H(\mathbf{\Lambda})$ is dense in $\ell^2(\mathbb{C})$ as $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}$ which is dense in ℓ^2. In this case, we can consider the infinite matrix $F(\mathbf{\Lambda})^*$ as the unique adjoint operator of $F(\mathbf{\Lambda})$ with domain $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*) \subseteq \ell^2(\mathbb{C})$.*
*By transposing each $F_m(\mathbf{\Lambda}^m)$ for $m \in \mathbb{Z}^*$, the arguments of the proof of Corollary B.*2 *lead to the invertibility of the map $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \to Ran(F(\mathbf{\Lambda})^*)$ and $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$. Moreover, $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}$ as in Proposition B.*1*.*

In the following theorem, we rephrase a result of Avdonin and Moran [AM01], which is also proved by Baiocchi, Komornik and Loreti in [BKL02].

Theorem B.4 (Theorem 3.29; [DZ06])**.** *Let $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying* (31)*. If $T > 2\pi/\delta$, then $\{\xi_k\}_{k\in\mathbb{Z}^*}$ forms a Riesz Basis in the space $X := \overline{span\{\xi_k|\ k\in\mathbb{Z}^*\}}^{L^2}$.*

Proposition B.5. *Let $\{\omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \cup \{0\}$ be an ordered sequence of real numbers with $\omega_1 = 0$ such that there exist $\tilde{d} \geq 0$, $\delta, C > 0$ and $\mathcal{M} \in \mathbb{N}$ with*

$$\inf_{k\in\mathbb{N}} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta\mathcal{M}, \qquad |\omega_{k+1} - \omega_k| \geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \qquad \forall k \in \mathbb{N}.$$

Then, for $T > 2\pi/\delta$ and for every $\{x_k\}_{k\in\mathbb{N}} \in h^{\tilde{d}}(\mathbb{C})$ with $x_1 \in \mathbb{R}$,

$$\exists u \in L^2((0,T),\mathbb{R}) \quad : \quad x_k = \int_0^T u(\tau)e^{i\omega_k\tau}d\tau \qquad \forall k \in \mathbb{N}. \tag{33}$$

Proof. From the definition of Reisz basis ([BL10, *Appendix B.*1; *Definition* 2]) and [BL10, *Appendix B.*1; *Proposition* 19; 2)], the map $M : g \in X \mapsto \{\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{Z}^*} \in \ell^2(\mathbb{C})$ is invertible and, for every $k \in \mathbb{Z}^*$, we have

$$\langle \xi_k, g\rangle_{L^2((0,T),\mathbb{C})} = (F(\mathbf{\Lambda})^*\langle \mathbf{e}, g\rangle_{L^2((0,T),\mathbb{C})})_k.$$

Let $\tilde{X} := M^{-1}\circ F(\mathbf{\Lambda})^*(h^{\tilde{d}}(\mathbb{C}))$. From Remark $B.3$, we have $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{C})$. The following maps are invertible $(F(\mathbf{\Lambda})^*)^{-1} : Ran(F(\mathbf{\Lambda})^*) \to H(\mathbf{\Lambda})^*$ and

$$(F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle \mathbf{e}, g\rangle_{L^2((0,T),\mathbb{C})}\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C}).$$

For every $\{x_k\}_{k\in\mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$, there exists $u \in L^2((0,T),\mathbb{C})$ such that

$$x_k = \int_0^T u(\tau)e^{i\lambda_k\tau}d\tau, \qquad \forall k \in \mathbb{Z}^*.$$

When $k > 0$, we call $\lambda_k = \omega_k$, while $\lambda_k = -\omega_{-k}$ for $k < 0$ such that $k \neq -1$. The sequence $\{\lambda_k\}_{k\in\mathbb{Z}^*\setminus\{-1\}}$ is such that there exists $C_1 > 0$ satisfying

$$\inf_{k\in\mathbb{Z}*} |\lambda_{k+2\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}, \qquad |\lambda_{k+1} - \lambda_k| \geq C_1|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \quad \forall k \in \mathbb{Z}^* \setminus \{-1\}.$$

Given $\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$, we introduce $\{\tilde{x}_k\}_{k\in\mathbb{Z}^*\setminus\{-1\}} \in \ell^2(\mathbb{C})$ such that $\tilde{x}_k = x_k$ for $k > 0$, while $\tilde{x}_k = \overline{x}_{-k}$ for $k < 0$ and $k \neq -1$. As above, there exists $u \in L^2((0,T),\mathbb{C})$ such that

$$x_1 = \int_0^T u(s)ds, \qquad \tilde{x}_k = \int_0^T u(s)e^{-i\omega_k s}ds, \qquad \forall k \in \mathbb{Z}^* \setminus \{-1\},$$

$$\implies \qquad \int_0^T u(s)e^{i\lambda_k s}ds = x_k = \int_0^T \overline{u}(s)e^{i\lambda_k s}ds, \qquad k \in \mathbb{N} \setminus \{1\}.$$

If $x_1 \in \mathbb{R}$, then u is real and (33) is solvable for $u \in L^2((0,T),\mathbb{R})$. □

Proposition B.6. *Let $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be an ordered sequence of pairwise distinct real numbers satisfying* (31)*. For every $T>0$, there exists $C(T)>0$ uniformly bounded for T lying on bounded intervals such that*

$$\forall g\in L^2((0,T),\mathbb{C}),\qquad \left\|\int_0^T e^{i\lambda_{(\cdot)}s}g(s)ds\right\|_{\ell^2}\leq C(T)\|g\|_{L^2((0,T),\mathbb{C})}.$$

Proof. **1) Uniformly separated numbers.** Let $\{\omega_k\}_{k\in\mathbb{N}}\subset\mathbb{R}$ be such that $\gamma:=\inf_{k\neq j}|\omega_k-\omega_j|>0$ and $L^2:=L^2((0,T),\mathbb{C})$. Thanks to the Ingham's Theorem [KL05, *Theorem* 4.3], the sequence $\{e^{i\omega_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz Basis in

$$X=\overline{span\{e^{i\omega_k(\cdot)}:\ k\in\mathbb{N}\}}^{\,L^2}\subset L^2((0,T),\mathbb{C})\quad\text{when}\quad T>2\pi/\gamma.$$

Now, there exists $C_1(T)>0$ such that $\sum_{k\in\mathbb{N}}|\langle e^{i\omega_k(\cdot)},u\rangle_{L^2}|^2\leq C_1(T)^2\|u\|_{L^2}^2$ for every $u\in X$ thanks to [Duca, *relation* (29)]. Let $P:L^2\longrightarrow X$ be the orthogonal projector. For $g\in L^2$, we have

$$\left\|\{\langle e^{i\omega_k(\cdot)},g\rangle_{L^2}\}_{k\in\mathbb{N}}\right\|_{\ell^2}=\left\|\{\langle e^{i\omega_k(\cdot)},Pg\rangle_{L^2}\}_{k\in\mathbb{N}}\right\|_{\ell^2}\leq C_1(T)\|Pg\|_{L^2}\leq C_1(T)\|g\|_{L^2}.$$

2) Pairwise distinct numbers. Let $\{\lambda_k\}_{k\in\mathbb{Z}^*}$ be as in the hypotheses. We decompose $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathcal{M}$ sequences $\{\lambda_k^j\}_{k\in\mathbb{N}}$ with $j\leq\mathcal{M}$ such that

$$\inf_{k\neq l}|\lambda_k^j-\lambda_l^j|>\delta\mathcal{M},\qquad\forall j\leq\mathcal{M}.$$

Now, for every $j\leq\mathcal{M}$, we apply the point **1)** with $\{\omega_k\}_{k\in\mathbb{N}}=\{\lambda_k^j\}_{k\in\mathbb{N}}$. For every $T>2\pi/\delta\mathcal{M}$ and $g\in L^2$, there exists $C(T)>0$ uniformly bounded for T in bounded intervals such that

$$\left\|\{\langle e^{i\lambda_k(\cdot)},g\rangle_{L^2}\}_{k\in\mathbb{N}}\right\|_{\ell^2}\leq\sum_{j=1}^{\mathcal{M}}\left\|\{\langle e^{i\lambda_k^j(\cdot)},g\rangle_{L^2}\}_{k\in\mathbb{N}}\right\|_{\ell^2}\leq\mathcal{M}C(T)\|g\|_{L^2},.$$

Thus, $\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2}\leq C(T)\|g\|_{L^2}$ for every $g\in L^2$ and, for $T>2\pi/\delta\mathcal{M}$, we choose the smallest value possible for $C(T)$. When $T\leq 2\pi/\delta\mathcal{M}$, for $g\in L^2$, we define $\tilde{g}\in L^2((0,2\pi/\delta\mathcal{M}+1),\mathbb{C})$ such that $\tilde{g}=g$ on $(0,T)$ and $\tilde{g}=0$ in $(T,2\pi/\delta\mathcal{M}+1)$. Then

$$\left\|\int_0^T e^{i\lambda_{(\cdot)}\tau}g(\tau)dt\right\|_{\ell^2}=\left\|\int_0^{2\pi/\delta\mathcal{M}+1}e^{i\lambda_{(\cdot)}\tau}\tilde{g}(\tau)dt\right\|_{\ell^2}\leq\mathcal{M}C(2\pi/\delta\mathcal{M}+1)\|g\|_{L^2}.$$

Let $0<T_1<T_2<+\infty$, $g\in L^2(0,T_1)$ and $\tilde{g}\in L^2(0,T_2)$ be defined as $\tilde{g}=g$ on $(0,T_1)$ and $\tilde{g}=0$ on (T_1,T_2). We apply the last inequality to $\tilde{g}$ that leads to $C(T_1)\leq C(T_2)$. □

C Appendix: Analytic perturbation

The aim of the appendix is to adapt the perturbation theory techniques provided in [Ducb, *Appendix B*], where the (BSE) is considered on $\mathscr{G} = (0,1)$ and A is the Dirichlet Laplacian. As in the mentioned appendix, we decompose $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. Let $A + u(t)B = A + u_0B + u_1(t)B$. We consider u_0B as a perturbative term of A.

Let $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ be the spectrum of $A + u_0B$ corresponding to some eigenfunctions $\{\phi_j^{u_0}\}_{j\in\mathbb{N}}$. We refer to the definition of the equivalence classes $\{E_m\}_{m\in\mathbb{Z}^*}$ provided in the first part of Appendix B.

We denote as $n : \mathbb{N} \to \mathbb{N}$ the application mapping $j \in \mathbb{N}$ in the value $n(j) \in \mathbb{N}$ such that $j \in E_{n(j)}$, while $s : \mathbb{N} \to \mathbb{N}$ is such that $\lambda_{s(j)} = \inf\{\lambda_k > \lambda_j \mid k \notin E_{n(j)}\}$. Moreover, $p : \mathbb{N} \to \mathbb{N}$ is such that $\lambda_{p(j)} = \sup\{k \in E_{n(j)}\}$.

The proofs of [Ducb, *Lemma B.2* & *Lemma B.3*] lead to next lemma. Let $j \in \mathbb{N}$ and $P_j^\perp$ be the projector onto $\overline{span\{\phi_m \ : \ m \notin E_{n(j)}\}}^{L^2}$.

Lemma C.1. *Let the hypotheses of Theorem* 2.3 *be satisfied. There exists a neighborhood $U(0)$ of $u = 0$ in $\mathbb{R}$ such that there exists $c > 0$ so that*

$$\left|\!\left|\!\left| \, (A+u_0B-\nu_k)^{-1} \, \right|\!\right|\!\right| \leq c, \qquad \nu_k := (\lambda_{s(k)}-\lambda_{p(k)})/2, \quad \forall u_0 \in U(0), \ \forall k \in \mathbb{N}.$$

Moreover, for $u_0 \in U(0)$, the operator $(A + u_0P_k^\perp B - \lambda_k^{u_0})$ is invertible with bounded inverse from $D(A) \cap Ran(P_k^\perp)$ to $Ran(P_k^\perp)$ for every $k \in \mathbb{N}$.

Lemma C.2. *Let the hypotheses of Theorem* 2.3 *be satisfied. There exists a neighborhood $U(0)$ of $u = 0$ in $\mathbb{R}$ such that, up to a countable subset Q and for every $(k,j),(m,n) \in I := \{(j,k) \in \mathbb{N}^2 : j \neq k\}$, $(k,j) \neq (m,n)$, we have*

$$\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} \neq 0, \qquad \langle \phi_k^{u_0}, B\phi_j^{u_0} \rangle \neq 0, \qquad \forall u_0 \in U(0) \setminus Q.$$

Proof. For $k \in \mathbb{N}$, we decompose $\phi_k^{u_0} = a_k\phi_k + \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k\phi_j + \eta_k$, where $a_k \in \mathbb{C}$, $\{\beta_j^k\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and η_k is orthogonal to ϕ_l for every $l \in E_{n(k)}$. Moreover, $\lim_{|u_0|\to 0} |a_k| = 1$ and $\lim_{|u_0|\to 0} |\beta_j^k| = 0$ for every $j,k \in \mathbb{N}$ and

$$\lambda_k^{u_0}\phi_k^{u_0} = (A + u_0B)(a_k\phi_k + \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k\phi_j + \eta_k) = Aa_k\phi_k$$
$$+ \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k A\phi_j + A\eta_k + u_0Ba_k\phi_k + u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k B\phi_j + u_0B\eta_k.$$

Now, Lemma $C.1$ leads to the existence of $C_1 > 0$ such that, for every $k \in \mathbb{N}$,

(34)
$$\eta_k = -\left(\left(A + u_0P_k^\perp B - \lambda_k^{u_0}\right)P_k^\perp\right)^{-1} u_0\Big(a_kP_k^\perp B\phi_k + \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k P_k^\perp B\phi_j\Big)$$

and $\|\eta_k\| \leq C_1|u_0|$. We compute $\lambda_k^{u_0} = \langle \phi_k^{u_0}, (A+u_0B)\phi_k^{u_0}\rangle$ and

$$\lambda_k^{u_0} = |a_k|^2\lambda_k + \langle \eta_k, (A+u_0B)\eta_k\rangle + \sum_{j\in E_{n(k)}\setminus\{k\}} \lambda_j|\beta_j^k|^2 + u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2 B_{k,k}$$
$$+ u_0 \sum_{j,l\in E_{n(k)}\setminus\{k\}\ j\neq l} \overline{\beta_j^k}\beta_l^k B_{j,l} + u_0 \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2(B_{j,j}-B_{k,k}) + u_0|a_k|^2 B_{k,k}$$
$$+ 2u_0\Re\Big(\sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k\langle \eta_k, B\phi_j\rangle + \overline{a_k} \sum_{j\in E_{n(k)}\setminus\{k\}} \beta_j^k B_{k,j} + \overline{a_k}\langle \phi_k, B\eta_k\rangle\Big).$$

Thanks to (34), it follows $\langle \eta_k, (A+u_0B)\eta_k\rangle = \lambda_k^{u_0}\|\eta_k\|^2 + O(u_0^2)$. Let

$$\widehat{a}_k := \frac{|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} |\beta_j^k|^2}{1-\|\eta_k\|^2}, \qquad \widetilde{a}_k := \frac{|a_k|^2 + \sum_{j\in E_{n(k)}\setminus\{k\}} \lambda_j/\lambda_k|\beta_j^k|^2}{1-\|\eta_k\|^2}.$$

As $\|\eta_k\| \leq C_1|u_0|$, it follows $\lim_{|u_0|\to 0} |\widehat{a}_k| = 1$ uniformly in k. Thanks to

$$\lim_{k\to+\infty} \inf_{j\in E_{n(k)}\setminus\{k\}} \lambda_j\lambda_k{}^{-1} = \lim_{k\to+\infty} \sup_{j\in E_{n(k)}\setminus\{k\}} \lambda_j\lambda_k{}^{-1} = 1,$$

we have $\lim_{|u_0|\to 0} |\widetilde{a}_k| = 1$ uniformly in k. Now, there exists f_k such that

$$\lambda_k^{u_0} = \widetilde{a}_k\lambda_k + u_0\widehat{a}_k B_{k,k} + u_0 f_k' + O(u_0^2) \tag{35}$$

where $\lim_{|u_0|\to 0} f_k = 0$ uniformly in k. When $\lambda_k = 0$, the identity (35) is still valid. For each $(k,j),(m,n) \in I$ such that $(k,j) \neq (m,n)$, there exists $f_{k,j,m,n}$ such that $\lim_{|u_0|\to 0} f_{k,j,m,n} = 0$ uniformly in k,j,m,n and

$$\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} = \widetilde{a}_k\lambda_k - \widetilde{a}_j\lambda_j - \widetilde{a}_m\lambda_m + \widetilde{a}_n\lambda_n + u_0 f_{k,j,m,n}$$
$$+ u_0(\widehat{a}_k B_{k,k} - \widehat{a}_j B_{j,j} - \widehat{a}_m B_{m,m} + \widehat{a}_n B_{n,n}) = \widetilde{a}_k\lambda_k - \widetilde{a}_j\lambda_j$$
$$- \widetilde{a}_m\lambda_m + \widetilde{a}_n\lambda_n + u_0(\widehat{a}_k B_{k,k} - \widehat{a}_j B_{j,j} - \widehat{a}_m B_{m,m} + \widehat{a}_n B_{n,n}) + O(u_0^2).$$

Thanks to the third point of Assumptions I, there exists $U(0)$ a neighborhood of $u = 0$ in $\mathbb{R}$ small enough such that, for each $u \in U(0)$, we have that every function $\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0}$ is not constant and analytic. Now, $V_{(k,j,m,n)} = \{u \in D \big|\ \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$ is a discrete subset of D and

$$V = \{u \in D \big|\ \exists((k,j),(m,n)) \in I^2 : \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$

is a countable subset of D, which achieves the proof of the first claim. The second relation is proved with the same technique. For $j,k \in \mathbb{N}$, the analytic function $u_0 \to \langle \phi_j^{u_0}, B\phi_k^{u_0}\rangle$ is not constantly zero since $\langle \phi_j, B\phi_k\rangle \neq 0$ and $W = \{u \in D \big|\ \exists(k,j) \in I : \langle \phi_j^{u_0}, B\phi_k^{u_0}\rangle = 0\}$ is a countable subset of D. $\square$

Lemma C.3. *Let the hypotheses of Theorem* 2.3 *be satisfied. Let* $T > 0$ *and* $s = d + 2$ *for* d *introduced in Assumptions II. Let* $c \in \mathbb{R}$ *such that* $0 \notin \sigma(A + u_0 B + c)$ *(the spectrum of* $A + u_0 B + c$*) and such that* $A + u_0 B + c$ *is a positive operator. There exists a neighborhood* $U(0)$ *of* 0 *in* $\mathbb{R}$ *such that,*

$$\forall u_0 \in U(0), \qquad \left\| |A + u_0 B + c|^{\frac{s}{2}} \cdot \right\| \asymp \| \cdot \|_{(s)}. \tag{36}$$

Proof. Let D be the neighborhood provided by Lemma $C.2$. The proof follows the one of [Ducb, *Lemma* $B.6$]. We suppose that $0 \notin \sigma(A + u_0 B)$ and $A + u_0 B$ is positive such that we can assume $c = 0$. If $c \neq 0$, then the proof follows from the same arguments.

Thanks to Remark 2.1, we have $\| \cdot \|_{(s)} \asymp \| |A|^{\frac{s}{2}} \cdot \|$. We prove the existence of $C_1, C_2, C_3 > 0$ such that, for every $\psi \in D(|A + u_0 B|^{\frac{s}{2}}) = D(|A|^{\frac{s}{2}})$,

$$\|(A + u_0 B)^{\frac{s}{2}} \psi\| \leq C_1 \|A^{\frac{s}{2}} \psi\| + C_2 \|\psi\| \leq C_3 \|A^{\frac{s}{2}} \psi\|. \tag{37}$$

Let $s/2 = k \in \mathbb{N}$. The relation (37) is proved by iterative argument. First, it is true for $k = 1$ when $B \in L(D(A))$ as there exists $C > 0$ such that $\|AB\psi\| \leq C \, |\!|\!| B |\!|\!|_{L(D(A))} \|A\psi\|$ for $\psi \in D(A)$. When $k = 2$ if $B \in L(\mathscr{H})$ and $B \in L(D(A^{k_1}))$ for $1 \leq k_1 \leq 2$, then there exist $C_4, C_5 > 0$ such that, for $\psi \in D(A^2)$,

$$\begin{aligned} &\|(A + u_0 B)^2 \psi\| \leq \|A^2 \psi\| + |u_0|^2 \|B^2 \psi\| + |u_0| \|AB\psi\| + |u_0| \|BA\psi\| \\ &\leq \|A^2 \psi\| + |u_0|^2 \, |\!|\!| B^2 |\!|\!| \, \|\psi\| + C_4 |u_0| \, |\!|\!| B |\!|\!|_{L(D(A^{k_1}))} \|\psi\|_{(k_1)} + |u_0| \, |\!|\!| B |\!|\!| \, \|\psi\|_{(2)} \end{aligned}$$

and $\|(A + u_0 B)^2 \psi\| \leq C_5 \|A^2 \psi\|$. Second, we assume (37) be valid for $k \in \mathbb{N}$ when $B \in L(D(A^{k_j}))$ for $k - j - 1 \leq k_j \leq k - j$ and for every $j \in \{0, ..., k-1\}$. We prove (37) for $k+1$ when $B \in L(D(A^{k_j}))$ for $k - j \leq k_j \leq k - j + 1$ and for every $j \in \{0, ..., k\}$. Now, there exists $C > 0$ such that $\|A^k B \psi\| \leq C \, |\!|\!| B |\!|\!|_{D(A^{k_0})} \|A^{k_0} \psi\|$ for every $\psi \in D(A^{k+1})$. Thus, as $\|(A + u_0 B)^{k+1} \psi\| = \|(A + u_0 B)^k (A + u_0 B) \psi\|$, there exist $C_6, C_7 > 0$ such that, for every $\psi \in D(A^{k+1})$,

$$\|(A + u_0 B)^{k+1} \psi\| \leq C_6 (\|A^{k+1} \psi\| + |u_0| \|A^k B \psi\| + \|A\psi\| + |u_0| \|B\psi\|) \leq C_7 \|A^{k+1} \psi\|.$$

As in the proof of [Ducb, *Lemma* $B.6$], the relation (37) is valid for any $s \leq k$ when $B \in L(D(A^{k_0}))$ for $k - 1 \leq k_0 \leq s$ and $B \in L(D(A^{k_j}))$ for $k - j - 1 \leq k_j \leq k - j$ and for every $j \in \{1, ..., k-1\}$. The opposite inequality follows by decomposing $A = A + u_0 B - u_0 B$.

In our framework, Assumptions II ensure that the parameter s is $2 + d$. If the second point of Assumptions II is verified for $s \in [4, 11/2)$, then B preserves $H^{d_1}_{\mathcal{NK}}$ and $H^2_{\mathscr{G}}$ for d_1 introduced in Assumptions II. Proposition 3.2 claims that $B : H^{d_1}_{\mathscr{G}} \to H^{d_1}_{\mathscr{G}}$ and the argument of [Ducb, *Remark* 1.1] implies

$B \in L(H_{\mathscr{G}}^{d_1})$. Thus, the identity (36) is valid because $B \in L(\mathscr{H})$, $B \in L(H_{\mathscr{G}}^2)$ and $B \in L(H_{\mathscr{G}}^{d_1})$ with $d_1 > s - 2$. If the third point of Assumptions II is verified for $s \in [4, 9/2)$, then $B \in L(\mathscr{H})$, $B \in L(H_{\mathscr{G}}^2)$ and $B \in L(H_{\mathscr{G}}^{d_1})$ for $d_1 \in [d, 9, 2)$. The claim follows thanks to Proposition 3.2 since B stabilizes H^{d_1} and $H_{\mathscr{G}}^2$ for d_1 introduced in Assumptions II. If $s < 4$ instead, then the conditions $B \in L(\mathscr{H})$ and $B \in L(H_{\mathscr{G}}^2)$ are sufficient to guarantee (36). □

Remark C.4. *The techniques developed in the proof of Lemma C.3 imply the following claim. Let the hypotheses of Theorem 2.3 be satisfied and $0 < s_1 < d + 2$ for d introduced in Assumptions II. Let $c \in \mathbb{R}$ such that $0 \notin \sigma(A + u_0 B + c)$ and such that $A + u_0 B + c$ is a positive operator. We have There exists a neighborhood $U(0) \subset \mathbb{R}$ of 0 so that, for any $u_0 \in U(0)$, we have $\| |A + u_0 B + c|^{\frac{s_1}{2}} \cdot \| \asymp \| \cdot \|_{(s_1)}$.*

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