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# Bilinear quantum systems on compact graphs: well-posedness and global exact controllability

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## Abstract

We consider the bilinear Schrödinger equations on compact quantum graphs. We prove the well-posedness and the global exact controllability according to the structure of the graph. We apply the main results to examples involving star graphs and tadpole graphs.

**AMS subject classifications:** 35Q41, 93C20, 93B05, 81Q15.

**Keywords:** Schrödinger equation, global exact controllability, quantum graphs, moment problems, perturbation theory.

## 1 Introduction

In this paper, we study the evolution of a particle confined in a compact graph type structure  $\mathcal{G}$  (*e.g.* Figure 1) and subjected to an external field. Its dynamics is modeled by the bilinear Schrödinger equation in the Hilbert space  $\mathcal{H} := L^2(\mathcal{G}, \mathbb{C})$

$$(BSE) \quad \begin{cases} i\partial_t \psi(t) = A\psi(t) + u(t)B\psi(t), & t \in (0, T), \\ \psi(0) = \psi_0, & T > 0. \end{cases}$$

The term  $u(t)B$  represents the control field, where the symmetric operator  $B$  describes the action of the field and  $u \in L^2((0, T), \mathbb{R})$  its intensity. The operator  $A = -\Delta$  is a self-adjoint Laplacian. When the (BSE) is well-posed, we call  $\Gamma_t^u$  the unitary propagator generated by  $A + u(t)B$ .

A natural question of practical implications is whether, given any couple of states, there exists  $u$  steering the quantum system from the first state in the second one. In other words, when the (BSE) is *exactly controllable*.

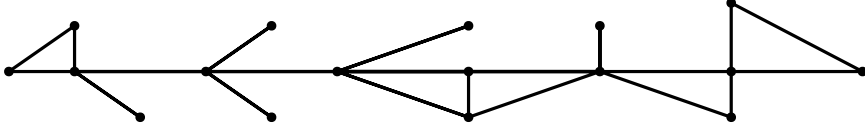


Figure 1: Example of compact graph

The *(BSE)* is said to be *approximately controllable* when, for any couple of states, it is possible to drive the first state as close as desired to the second one with a suitable control  $u$  and in finite time.

Each type of controllability is *simultaneous* when it is simultaneously satisfied between more couples of states with the same control  $u$ .

The use of graph theory in mathematics and in physics is nowadays gaining more and more popularity. In control theory, problems involving graphs have been popularized in the very last decades and many results are still missing. Indeed, a complete theory is far from being formulated as the interaction between the different components of a graph may generate unexpected phenomena. See [DZ06] for further details on the problem and for various controllability results.

Regarding inverse problems, we refer to [ALM10] and [Bel04] for uniqueness outcomes with the boundary control approach and to [BCV11] and [IPR12] for uniqueness and stability results through Carleman estimates.

Despite the mentioned works, the controllability of the bilinear Schrödinger equation on compact graphs is still an open problem. For this reason, we study the well-posedness and the global exact controllability of the *(BSE)* in suitable subspaces of  $D(A)$ .

Considering subspaces of  $D(A)$  is classical for this type of problems and it is due to the seminal work [BMS82] on bilinear systems by Ball, Mardsen and Slemrod. Even though [BMS82] guarantees that the *(BSE)* admits a unique solution in  $\mathcal{H}$ , it provides the following non-controllability result. Let  $S$  be the unit sphere in  $\mathcal{H}$  and  $\Gamma_T^u \psi_0$  be the solution of the *(BSE)* at time  $T > 0$  with initial state  $\psi_0 \in S$ . The set of the attainable states from  $\psi_0$ ,

$$\{\Gamma_T^u \psi_0 : T > 0, u \in L^2((0, T), \mathbb{R})\},$$

is contained in a countable union of compact sets. Then, it has dense complement in  $S$ . As a consequence, the exact controllability of the bilinear Schrödinger equation can not be achieved in  $S$  when  $u \in L_{loc}^2((0, \infty), \mathbb{R})$  (see also [Tur00] by Turinici).

Because of the Ball, Mardsen and Slemrod result, many authors have addressed the problem by weaker notions of controllability when  $\mathcal{G} = (0, 1)$ . Let  $A_D = -\Delta$  be the Dirichlet Laplacian on the interval  $(0, 1)$ , *i.e.*

$$D(A_D) = H^2((0, 1), \mathbb{C}) \cap H_0^1((0, 1), \mathbb{C}).$$

In [BL10], Beauchard and Laurent prove the *well-posedness* and the *local exact controllability* of the bilinear Schrödinger equation in  $H_{(0)}^s := D(A_D^{s/2})$  for  $s = 3$ , when  $B$  is a multiplication operator for suitable  $\mu \in H^3((0, 1), \mathbb{R})$ . In [Mor14], Morancey proves the *simultaneous local exact controllability* of two or three (*BSE*) in  $H_{(0)}^3$  for suitable  $B = \mu \in H^3((0, 1), \mathbb{R})$ . In [MN15], Morancey and Nersesyan extend the previous result and achieve the *simultaneous global exact controllability* of finitely many (*BSE*). In [Duch], the author ensures the *simultaneous global exact controllability in projection* for sequences in  $H_{(0)}^3$ , while he exhibits the *global exact controllability* between eigenstates with explicit controls and times in [Duca]. Both the results are provided for suitable bounded symmetric operators  $B$ .

The *global approximate controllability* of the bilinear Schrödinger equation is proved in literature with different techniques. Adiabatic arguments are considered by Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti in [BCMS12] and [BGRS15]. The result is achieved with Lyapunov techniques by Mirrahimi in [Mir09] and by Nersesyan in [Ner10]. Lie-Galerking arguments are used by Boscain, Boussaïd, Caponigro, Chambrión, Mason and Sigalotti in [CMSB09], [BCCS12], [BdCC13] and [BCS14].

## 1.1 Preliminaries

Let  $\mathcal{G}$  be a compact graph composed by  $N \in \mathbb{N}$  edges  $\{e_j\}_{j \leq N}$  of lengths  $\{L_j\}_{j \leq N}$  and  $M \in \mathbb{N}$  vertices  $\{v_j\}_{j \leq M}$ . We call  $V_e$  and  $V_i$  the external and the internal vertices of  $\mathcal{G}$ , *i.e.*

$$V_e := \{v \in \{v_j\}_{1 \leq j \leq M} \mid \exists! e \in \{e_j\}_{j \leq N} : v \in e\}, \quad V_i := \{v_j\}_{1 \leq j \leq M} \setminus V_e.$$

In the current work, we study graphs equipped with a metric, which allows to parametrize each edge with a coordinate going from 0 to the length of the edge. We recall that a graph is said to be compact when it composed by a finite number of vertices and edges of finite length.

We consider a compact metric graph  $\mathcal{G}$  as domain of functions  $f := (f^1, \dots, f^N) : \mathcal{G} \rightarrow \mathbb{C}$  so that  $f^j : e_j \rightarrow \mathbb{C}$  with  $j \leq N$  and the Hilbert space

$$\mathcal{H} = L^2(\mathcal{G}, \mathbb{C}) = \prod_{j=1}^N L^2(e_j, \mathbb{C}).$$

The space  $\mathcal{H}$  is equipped with the norm  $\|\cdot\|$  induced by the scalar product

$$\langle \psi, \phi \rangle := \langle \psi, \phi \rangle_{\mathcal{H}} = \sum_{j \leq N} \langle \psi^j, \phi^j \rangle_{L^2(e_j, \mathbb{C})} = \sum_{j \leq N} \int_{e_j} \overline{\psi^j(x)} \phi^j(x) dx, \quad \forall \psi, \phi \in \mathcal{H}.$$

We also denote  $H^s := H^s(\mathcal{G}, \mathbb{C}) = \prod_{j=1}^N H^s(e_j, \mathbb{C})$  for every  $s > 0$ .

In the (BSE), the operator  $A$  is a self-adjoint Laplacian such that the functions in  $D(A)$  satisfy the following boundary conditions. Each  $v \in V_i$  is equipped with *Neumann-Kirchhoff* boundary conditions when

$$\begin{cases} f \text{ is continuous in } v, \\ \sum_{e \ni v} \frac{\partial f}{\partial x_e}(v) = 0, \end{cases} \quad \forall f \in D(A).$$

The derivatives are assumed to be taken in the directions away from the vertex (outgoing directions). In addition, the external vertices  $V_e$  are equipped with *Dirichlet* or *Neumann* type boundary conditions.

## 1.2 Novelties of the work

The main difference between studying the controllability of the bilinear Schrödinger equation on  $\mathcal{G} = (0, 1)$  and on generic graph  $\mathcal{G}$  is the following fact. Let  $\{\lambda_k\}_{k \in \mathbb{N}}$  be the ordered sequence of eigenvalues of  $A$  and  $\{\phi_k\}_{k \in \mathbb{N}}$  be a Hilbert basis of  $\mathcal{H}$  made by corresponding eigenfunctions. We know

$$(1) \quad \inf_{k \in \mathbb{N}} |\lambda_{k+1} - \lambda_k| \geq 0 \quad \text{if} \quad \mathcal{G} = (0, 1),$$

which is an important hypothesis for the works [BL10], [Duch], [Duca] and [Mor14]. Unfortunately, the identity (1) is not guaranteed when  $\mathcal{G}$  is a generic compact graphs. Nevertheless, there exist  $\mathcal{M} \in \mathbb{N}$  and  $\delta' > 0$  so that

$$(2) \quad \inf_{k \in \mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| > \delta' \mathcal{M}$$

(see Remark 2.2 for further details). To ensure controllability results, we introduce a weaker assumption on the spectral gap and we assume that there exist  $C > 0$  and  $\tilde{d} \geq 0$  such that

$$(3) \quad |\lambda_{k+1} - \lambda_k| \geq C k^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \quad \forall k \in \mathbb{N}.$$

Proving the validity of the identity (3) is not an easy task as the spectrum of  $A$  is usually unexplicit. In addition, the more the structure of the graph is complicated, the more the spectral behaviour is difficult to characterize.

By using Roth's Theorem [Rot56], we prove the validity of the spectral gap (3) for the following types of graphs.



Figure 2: Respectively a star graph, a double-ring graph, a tadpole graph and a two tails tadpole graph.

The identity spectral gap is valid when  $\{1, \{L_j\}_{j \leq N}\}$  are *linearly independent* over  $\mathbb{Q}$  and all the ratios  $L_k/L_j$  are *algebraic irrational numbers*. The result does not depend from the choice of boundary conditions of  $D(A)$  in the external vertices  $V_e$ , which can be both Dirichlet, or Neumann type.

Moreover, we characterize the spaces  $H_{\mathcal{G}}^s$  with  $s > 0$  and we ensure different interpolation features as the following one.

**Proposition 1.1.** *Let  $\mathcal{G}$  be a compact graph. Let  $D(A)$  be equipped with Dirichlet or Neumann boundary conditions in  $V_e$  and Neumann-Kirchhoff in  $V_i$ . Then*

$$H_{\mathcal{G}}^{s_1+s_2} = H_{\mathcal{G}}^{s_1} \cap H^{s_1+s_2} \quad \forall s_1 \in \mathbb{N} \cup \{0\}, \quad s_2 \in [0, 1/2).$$

Proposition 1.1 holds under generic assumptions on the problem, but stronger outcomes can be guaranteed by imposing more restrictive conditions. We provide the complete result in Proposition 3.2.

The interpolation properties are crucial for *well-posedness* of the bilinear Schrödinger equation in  $H_{\mathcal{G}}^s$  with specific  $s \geq 3$ . In the same space, we prove the *global exact controllability* when the identities (2) and (3) are satisfied for suitable parameters. The complete result is provided in Theorem 2.3.

Two interesting applications of Theorem 2.3 are the following examples that respectively involve a *star graph* and a *tadpole graph*.

Let  $\mathcal{G}$  be a *star graph* composed by  $N \in \mathbb{N}$  edges  $\{e_k\}_{k \leq N}$ . Each  $e_k$  is parametrized with a coordinate going from 0 to the length of the edge  $L_k$ . We set the 0 in the external vertex belonging to  $e_k$ .



Figure 3: The figure represents the parametrization of the star graph edges.

**Example 1.2.** *Let  $\mathcal{G}$  be a star graph with four edges of lengths  $\{L_j\}_{j \leq 4}$  and  $D(A)$  be equipped with Dirichlet boundary conditions in  $V_e$  and Neumann-Kirchhoff in  $V_i$ . Let  $B$*

$$B : \psi = (\psi^1, \psi^2, \psi^3, \psi^4) \mapsto \left( (x - L_1)^4 \psi^1, 0, 0, 0 \right), \quad \forall \psi \in \mathcal{H}.$$

*There exists  $\mathcal{C} \subset (\mathbb{R}^+)^4$  countable such that, for every  $\{L_j\}_{j \leq 4} \in (\mathbb{R}^+)^4 \setminus \mathcal{C}$  such that  $\{1, L_1, L_2, L_3, L_4\}$  are linearly independent over  $\mathbb{Q}$  and all the ratios  $L_k/L_j$  are algebraic irrational numbers, the (BSE) is globally exactly controllable in*

$$H_{\mathcal{G}}^{4+\epsilon} \quad \epsilon \in (0, 1/2).$$

*Proof.* See Section 6. □

Let  $\mathcal{G}$  be a *tadpole graph* composed by  $\{e_1, e_2\}$ . The self-closing edge  $e_1$  is parametrized in the clockwise direction with a coordinate going from 0 to  $L_1$  (the length  $e_1$ ). On the “tail”  $e_2$ , we consider a coordinate going from 0 to  $L_2$  and we associate the 0 to the external vertex.

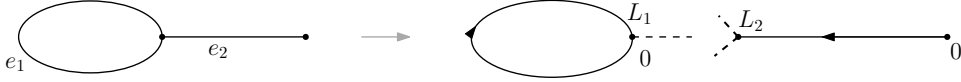


Figure 4: The parametrization of the tadpole graph.

**Example 1.3.** Let  $\mathcal{G}$  be a tadpole graph composed by a self-closing edge  $e_1$  of length  $L_1$  and a second edge  $e_2$  of length  $L_2$ . Let  $D(A)$  be equipped with Neumann boundary conditions in  $V_e$  and Neumann-Kirchhoff in  $V_i$ . Let

$$\mu(x) := \sin\left(\frac{2\pi}{L_1}x\right) + \sin\left(\frac{\pi}{L_1}x\right) + \frac{\pi}{L_1^2}x(x-L_1) - \frac{\pi^3}{6L_1^6}x^3(x-L_1)^3,$$

$$B : \psi = (\psi^1, \psi^2) \mapsto (\mu\psi^1, 0), \quad \forall \psi \in \mathcal{H}.$$

There exists  $\mathcal{C} \subset (\mathbb{R}^+)^2$  countable so that, for each  $\{L_1, L_2\} \in (\mathbb{R}^+)^2 \setminus \mathcal{C}$  such that  $\{1, L_1, L_2\}$  are linearly independent over  $\mathbb{Q}$  and all the ratios  $L_k/L_j$  are algebraic irrational numbers, the (BSE) is globally exactly controllable in

$$H_{\mathcal{G}}^{5+\epsilon} \quad \epsilon \in (0, 1/2).$$

*Proof.* See Section 6. □

In Example 1.2 and Example 1.3, we notice an interesting phenomenon. The controllability holds even if the control fields act on one edge of the graphs. It is due to the choice of the lengths, which are linearly independent over  $\mathbb{Q}$  and such that all the ratios  $L_k/L_j$  are algebraic irrational numbers.

The same techniques adopted in Example 1.2 and Example 1.3 are also valid if we change the boundary condition in the external vertices from Dirichlet type to Neumann and vice versa.

Let  $\{I_j\}_{j \leq N}$  be a set of non-connected intervals with  $N \in \mathbb{N}$  and  $\Gamma_t^{u,j}$  be the propagator generated by  $A_j + u(t)B_j$  with

$$B_j := B|_{L^2(I_j)}, \quad A_j := A|_{L^2(I_j)}, \quad H_{I_j}^s := D\left((A_j)^{\frac{s}{2}}\right), \quad s > 0.$$

The following controllability result, denoted *contemporaneous controllability*, follows from Theorem 2.3 when we consider  $\mathcal{G} = \{I_j\}_{j \leq N}$ .

**Example 1.4.** Let  $\{I_j\}_{j \leq N}$  be a set of bounded unconnected intervals and  $D(A)$  be equipped with Dirichlet boundary conditions for every  $j \leq N$  and  $N \in \mathbb{N}$ . Let  $B$  be such that

$$B : \psi = (\psi^1, \dots, \psi^N) \mapsto \left( \sum_{j=1}^N \frac{L_j^{\frac{1}{2}} x^2}{L_1^{\frac{1}{2}}} \psi^j \left( \frac{L_j}{L_1} x \right), \dots, \sum_{j=1}^N \frac{L_j^{\frac{1}{2}} x^2}{L_N^{\frac{1}{2}}} \psi^j \left( \frac{L_j}{L_N} x \right) \right).$$

There exists  $\mathcal{C} \subset (\mathbb{R}^+)^N$  countable such that, for every  $\{L_j\}_{j \leq N} \in (\mathbb{R}^+)^N \setminus \mathcal{C}$  such that  $\{1, \{L_j\}_{j \leq N}\}$  are linearly independent over  $\mathbb{Q}$  and all the ratios  $L_k/L_j$  are algebraic irrational numbers, the bilinear Schrödinger equations are contemporaneously globally exactly controllable in

$$\prod_{j \leq N} H_{I_j}^{3+\epsilon} \quad \epsilon \in (0, 1/2).$$

In other words, for every  $\psi_1, \psi_2 \in \prod_{j \leq N} H_{I_j}^{3+\epsilon}$  such that  $\|\psi_1\| = \|\psi_2\|$ , there exist  $T > 0$  and  $u \in L^2((0, T), \mathbb{R})$  such that  $\Gamma_T^{u,j} \psi_1^j = \psi_2^j$  for every  $j \leq N$ .

*Proof.* See Section 6. □

We underline that the *contemporaneous global exact controllability* is deeply different from the *simultaneous controllability* results provided by [Mor14], [MN15] and [Ducb]. Indeed, the mentioned works consider vectors (or sequences) of functions each one belonging to the same space.

### 1.3 Scheme of the work

In Section 2, we present the main results of the work. The *global exact controllability* of the (BSE) is ensured in Theorem 2.3. Theorem 2.5 and Corollary 2.6 show types of graphs satisfying the hypothesis of Theorem 2.3. The *contemporaneous controllability* is introduced in Corollary 2.7.

In Section 3, Proposition 3.1 ensures the *well-posedness* of the (BSE). We provide *interpolation features* of the spaces  $H_{\mathcal{G}}^s$  for  $s > 0$  in Proposition 3.2. Section 4 exhibits the proof of Theorem 2.3, while the proofs of Theorem 2.5, Corollary 2.6 and Corollary 2.7 are provided in Section 5.

In Section 6, we explain Example 1.2, Example 1.3 and Example 1.4 and the other examples introduced in Section 2.

In Appendix A, we provide some spectral results by using some classical theorems on the approximation of real numbers by rational ones, while we treat the solvability of the so called “moments problems” in Appendix B.

In Appendix C, we adapt the perturbation theory techniques exposed in [Ducb, Appendix A].



## 2 Main results

Let  $\mathcal{G}$  be a compact graph composed by  $N$  edges  $\{e_j\}_{j \leq N}$  of lengths  $\{L_j\}_{j \leq N}$  connecting  $M$  vertices  $\{v_j\}_{1 \leq j \leq M}$ . For each  $j \leq M$ , we denote

$$(4) \quad N(v_j) := \{l \in \{1, \dots, N\} \mid v_j \in e_l\}, \quad n(v_j) := |N(v_j)|.$$

We respectively call  $(\mathcal{NK})$ ,  $(\mathcal{D})$  and  $(\mathcal{N})$  the *Neumann-Kirchhoff*, *Dirichlet* and *Neumann boundary* conditions for the  $D(A)$ .

When we consider the self-adjoint operator  $A$  on  $\mathcal{G}$ , it is denoted *quantum graph*. By denoting  $\mathcal{G}$  a compact quantum graph, we are implicitly introducing a Laplacian  $A$  equipped with self-adjoint boundary conditions.

We say that a quantum graph  $\mathcal{G}$  is equipped with one of the previous boundary conditions in a vertex  $v$ , when each  $f \in D(A)$  satisfies it in  $v$ .

We say that a quantum graph  $\mathcal{G}$  is equipped with  $(\mathcal{D}/\mathcal{N})$ - $(\mathcal{NK})$  when, for every  $f \in D(A)$  and  $v \in V_e$ , the function  $f$  satisfies  $(\mathcal{D})$  or  $(\mathcal{N})$  in  $v$  and, for every  $v \in V_i$ , the function  $f$  verifies  $(\mathcal{NK})$  in  $v$ .

We say that a quantum graph  $\mathcal{G}$  is equipped with  $(\mathcal{D})$ - $(\mathcal{NK})$  (or  $(\mathcal{N})$ - $(\mathcal{NK})$ ) when, for every  $f \in D(A)$  and  $v \in V_e$ , the function  $f$  satisfies  $(\mathcal{D})$  (or  $(\mathcal{N})$ ) in  $v$  and verifies  $(\mathcal{NK})$  in every  $v \in V_i$ .

For every compact graph  $\mathcal{G}$ , the operator  $A$  admits purely discrete spectrum (see [Kuc04, Theorem 18]). We call  $\{\lambda_j\}_{j \in \mathbb{N}}$  the non-decreasing sequence of eigenvalues of  $A$  and  $\{\phi_j\}_{j \in \mathbb{N}}$  a Hilbert basis of  $\mathcal{H}$  composed by the corresponding eigenfunctions. Let  $\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j$  and  $[r]$  be the entire part of a real number  $r \in \mathbb{R}$ . For  $s > 0$ , we define the spaces

$$H_{\mathcal{NK}}^s := \left\{ \psi \in H^s \mid \partial_x^{2n} \psi \text{ is continuous in } v, \quad \forall n \in \mathbb{N} \cup \{0\}, \quad n < [(s+1)/2]; \right. \\ \left. \sum_{e \in N(v)} \partial_{x_e}^{2n+1} f(v) = 0, \quad \forall n \in \mathbb{N} \cup \{0\}, \quad n < [s/2], \quad \forall v \in V_i \right\},$$

$$H_{\mathcal{G}}^s = H_{\mathcal{G}}^s(\mathcal{G}, \mathbb{C}) := D(A^{s/2}), \quad \|\cdot\|_{(s)} := \|\cdot\|_{H_{\mathcal{G}}^s} = \left( \sum_{k \in \mathbb{N}} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}},$$

$$h^s(\mathbb{C}) := \left\{ \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{C} \mid \sum_{k \in \mathbb{N}} |k^s a_k|^2 < \infty \right\}, \quad \|\cdot\|_{(s)} := \left( \sum_{k \in \mathbb{N}} |k^s \cdot|^2 \right)^{\frac{1}{2}}.$$

**Remark 2.1.** If  $0 \notin \sigma(A)$  (the spectrum of  $A$ ), then  $\|\cdot\|_{(s)} \asymp \| |A|^{\frac{s}{2}} \cdot \|$ , i.e.

$$\exists C_1, C_2 > 0 \quad : \quad C_1 \|\cdot\|_{(s)}^2 \leq \| |A|^{\frac{s}{2}} \cdot \|^2 = \sum_{k \in \mathbb{N}} |\lambda_k^{\frac{s}{2}} \langle \cdot, \phi_k \rangle|^2 \leq C_2 \|\cdot\|_{(s)}^2.$$

Indeed, from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10], there exist  $C_3, C_4 > 0$  such that  $C_3 k^2 \leq \lambda_k \leq C_4 k^2$  for every  $k \geq 2$  and for  $k = 1$  if  $\lambda_1 \neq 0$  (see Remark A.4 for further details). If  $0 \in \sigma(A)$ , then  $\lambda_1 = 0$  and there exists  $c \in \mathbb{R}$  such that  $0 \notin \sigma(A + c)$  and  $\|\cdot\|_{(s)} \asymp \| |A + c|^{\frac{s}{2}} \cdot \|$ .

**Remark 2.2.** For any  $\mathcal{G}$  the only eigenvalue that can be 0 is  $\lambda_1$ . Moreover, the identity (2) follows from [DZ06, relation (6.6)], which implies the existence of  $\mathcal{M} \in \mathbb{N}$  and  $\delta' > 0$  such that  $\inf_{k \in \mathbb{N}} |\sqrt{\lambda_{k+\mathcal{M}}} - \sqrt{\lambda_k}| > \delta' \mathcal{M}$  and

$$\inf_{k \in \mathbb{N}} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \sqrt{\lambda_2} |\sqrt{\lambda_{n+\mathcal{M}}} - \sqrt{\lambda_n}| > \delta' \mathcal{M}.$$

We point out that it is possible to set  $\mathcal{M} \geq M+N+1$  (even though this value is not optimal). This property can be deduced from [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10] adopted as in Remark A.4.

Let  $\eta > 0$  and  $a \geq 0$ . We define the following assumptions on  $(A, B)$  for

$$(5) \quad I := \{(j, k) \in \mathbb{N}^2 : j \neq k\}.$$

**Assumptions (I( $\eta$ )).** The operator  $B$  satisfies the following conditions.

1. There exists  $C > 0$  such that  $|\langle \phi_j, B\phi_1 \rangle| \geq \frac{C}{j^{2+\eta}}$  for every  $j \in \mathbb{N}$ .
2. For every  $(j, k), (l, m) \in I$  such that  $(j, k) \neq (l, m)$  and  $\lambda_j - \lambda_k = \lambda_l - \lambda_m$ , it holds  $\langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0$ .

**Assumptions (II( $\eta, a$ )).** Let  $\text{Ran}(B|_{H_{\mathcal{G}}^2}) \subseteq H_{\mathcal{G}}^2$  and one of the following assumptions be satisfied.

1. When  $\mathcal{G}$  is equipped with  $(\mathcal{D}/\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$  and  $a + \eta \in (0, 3/2)$ , there exists  $d \in [\max\{a + \eta, 1\}, 3/2)$  such that  $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{G}}^2$ .
2. When  $\mathcal{G}$  is equipped with  $(\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$  and  $a + \eta \in (0, 5/2)$ , there exist  $d \in [\max\{a + \eta, 2\}, 5/2)$  and  $d_1 \in (d, 5/2)$  such that  $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^3 \cap H_{\mathcal{G}}^2$  and  $\text{Ran}(B|_{H^{d_1}}) \subseteq H^{d_1}$ .
3. When  $\mathcal{G}$  is equipped with  $(\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$  and  $a + \eta \in (0, 7/2)$ , there exist  $d \in [\max\{a + \eta, 2\}, 7/2)$  and  $d_1 \in (d, 7/2)$  such that  $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{2+d} \cap H_{\mathcal{G}}^2$  and  $\text{Ran}(B|_{H^{d_1} \cap H_{\mathcal{N}\mathcal{K}}^{d_1}}) \subseteq H_{\mathcal{N}\mathcal{K}}^{d_1}$ .
4. When  $\mathcal{G}$  is equipped with  $(\mathcal{D})$ - $(\mathcal{N}\mathcal{K})$  and  $a + \eta \in (0, 5/2)$ , there exists  $d \in [\max\{a + \eta, 1\}, 5/2)$  such that  $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{N}\mathcal{K}}^{2+d} \cap H_{\mathcal{G}}^2$ . If  $a + \eta \geq 2$ , then there exists  $d_1 \in (d, 5/2)$  such that  $\text{Ran}(B|_{H^{d_1}}) \subseteq H^{d_1}$ .

From now on, we omit  $\eta$  and  $a$  from the notations of Assumptions I and Assumptions II when these parameters are not relevant.

**Theorem 2.3.** Let  $\mathcal{G}$  be a compact quantum graph. Let  $\tilde{d} \geq 0$  and  $C > 0$  be such that

$$(6) \quad |\lambda_{k+1} - \lambda_k| \geq Ck^{-\frac{\tilde{d}}{\mathcal{M}-1}} \quad \forall k \in \mathbb{N}.$$

If the couple  $(A, B)$  satisfies Assumptions I( $\eta$ ) and Assumptions II( $\eta, \tilde{d}$ ) for some  $\eta > 0$ , then the (BSE) is globally exactly controllable in  $H_{\mathcal{G}}^s$  for  $s = 2 + d$  and  $d$  from Assumptions II.

*Proof.* See Paragraph 4. □

**Definition 2.4.** For every  $N \in \mathbb{N}$ , we define  $\mathcal{AL}(N) \subset (\mathbb{R}^+)^N$  as follows. For every  $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$ , the numbers  $\{1, \{L_j\}_{j \leq N}\}$  are linearly independent over  $\mathbb{Q}$  and all the ratios  $L_k/L_j$  are algebraic irrational numbers.

In the next theorem, we provide the validity of the spectral hypothesis of Theorem 2.3 when  $\mathcal{G}$  is one of the graphs introduced in Figure 2. The provided result leads to the examples 1.2 and 1.3.

**Theorem 2.5.** *Let  $\{L_j\}_{j \leq N} \in \mathcal{AL}(N)$ . Let  $\mathcal{G}$  be either a tadpole, a two-tails tadpole, a double-rings graph or a star graph of  $N \leq 4$  edges and let  $\mathcal{G}$  be equipped with  $(\mathcal{D}/N)$ - $(\mathcal{NK})$ . If the couple  $(A, B)$  satisfies Assumptions  $I(\eta)$  and Assumptions  $II(\eta, \epsilon)$  for  $\eta, \epsilon > 0$ , then the (BSE) is globally exactly controllable in  $H_{\mathcal{G}}^s$  for  $s = 2 + d$  and  $d$  from Assumptions II.*

*Proof.* See Paragraph 5. □

The techniques leading to Theorem 2.5 can be adopted in order to prove the following corollary.

**Corollary 2.6.** *Let  $\{L_j\}_{j \leq 2} \in \mathcal{AL}(2)$ . Let  $\mathcal{G}$  be one of the following graphs:*

- *a two-tails tadpole with one edge long  $L_1$  and the others  $L_2$ ;*
- *a 3 edges star graph so that one edge is long  $L_1$  and the others  $L_2$ ;*
- *a 4 edges star graph so that two edges are long  $L_1$  and the others  $L_2$ .*

*If the couple  $(A, B)$  satisfies Assumptions  $I(\eta)$  and Assumptions  $II(\eta, \epsilon)$  for  $\eta, \epsilon > 0$ , then the (BSE) is globally exactly controllable in  $H_{\mathcal{G}}^s$  for  $s = 2 + d$  and  $d$  from Assumptions II.*

*Proof.* See Remark Paragraph 5. □

Let  $\{I_j\}_{j \leq N}$  be a set of non-connected intervals with  $N \in \mathbb{N}$  and  $\Gamma_t^{u,j}$  be the propagator generated by  $A_j + u(t)B_j$  with

$$B_j := B|_{L^2(I_j)}, \quad A_j := A|_{L^2(I_j)}, \quad H_{I_j}^s := D((A_j)^{\frac{s}{2}}), \quad s > 0.$$

Theorem 2.3 implies the following corollary that leads to Example 1.4.

**Corollary 2.7.** *Let  $\mathcal{G} = \{I_j\}_{j \leq N}$  be a compact quantum graph composed by a set of bounded unconnected intervals. Let the couple  $(A, B)$  satisfy Assumptions  $I(\eta)$  and Assumptions  $II(\eta, \epsilon)$  for some  $\eta, \epsilon > 0$ . If  $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$ , then the (BSE) is contemporaneously globally exactly controllable in  $\prod_{j \leq N} H_{I_j}^s$  for  $s = 2 + d$  and  $d$  from Assumptions II.*

*Proof.* See Paragraph 5. □

### 3 Well-posedness and interpolation properties of the spaces $H_{\mathcal{G}}^s$

Let introduce the *well-posedness* of the (BSE).

**Proposition 3.1.** *Let  $\mathcal{G}$  a compact quantum graph and the couple  $(A, B)$  satisfies Assumptions I( $\eta$ ) and Assumptions II( $\eta, \tilde{d}$ ) for  $\eta, \tilde{d} > 0$ .*

1) *Let  $T > 0$  and  $f \in L^2((0, T), H^{2+d} \cap H_{\mathcal{G}}^2)$  with  $d$  from Assumptions II. Let  $t \mapsto G(t) = \int_0^t e^{iA\tau} f(\tau) d\tau$ . The map  $G \in C^0([0, T], H_{\mathcal{G}}^{2+d})$  and there exists  $C(T) > 0$  uniformly bounded for  $T$  lying on bounded intervals so that*

$$\|G\|_{L^\infty((0, T), H_{\mathcal{G}}^{2+d})} \leq C(T) \|f\|_{L^2((0, T), H^{2+d} \cap H_{\mathcal{G}}^2)}.$$

2) *Let  $\psi^0 \in H_{\mathcal{G}}^{2+d}$  with  $d$  introduced in Assumptions II and  $u \in L^2((0, T), \mathbb{R})$ . There exists a unique mild solution of (BSE) in  $H_{\mathcal{G}}^{2+d}$ , i.e. a function  $\psi \in C_0([0, T], H_{\mathcal{G}}^{2+d})$  such that for every  $t \in [0, T]$ ,*

$$(7) \quad \psi(t, x) = e^{-iAt} \psi^0(x) - i \int_0^t e^{-iA(t-s)} u(s) B\psi(s, x) ds.$$

Moreover, there exists  $C = C(T, \lambda, \|u\|_{L^2((0, T), \mathbb{R})}) > 0$  so that

$$\|\psi\|_{C^0([0, T], H_{\mathcal{G}}^{2+d})} \leq C \|\psi^0\|_{H_{\mathcal{G}}^{2+d}}, \quad \|\psi(t)\| = \|\psi^0\|, \quad \forall t \in [0, T], \quad \psi_0 \in H_{\mathcal{G}}^{2+d}.$$

Now, we present some *interpolation properties* for the spaces  $H_{\mathcal{G}}^s$  for  $s > 0$ . The proof of Proposition 3.1 is provided in the end of the section.

**Proposition 3.2.**

1) *If the compact quantum graph  $\mathcal{G}$  is equipped with  $(\mathcal{D}/\mathcal{N})$ - $(\mathcal{NK})$ , then*

$$H_{\mathcal{G}}^{s_1+s_2} = H_{\mathcal{G}}^{s_1} \cap H^{s_1+s_2}(\mathcal{G}, \mathbb{C}) \quad \text{for } s_1 \in \mathbb{N} \cup \{0\}, \quad s_2 \in [0, 1/2].$$

2) *If the compact quantum graph  $\mathcal{G}$  is equipped with  $(\mathcal{N})$ - $(\mathcal{NK})$ , then*

$$H_{\mathcal{G}}^{s_1+s_2} = H_{\mathcal{G}}^{s_1} \cap H_{\mathcal{NK}}^{s_1+s_2} \quad \text{for } s_1 \in 2\mathbb{N} \cup \{0\}, \quad s_2 \in [0, 3/2].$$

3) *If the compact quantum graph  $\mathcal{G}$  is equipped with  $(\mathcal{D})$ - $(\mathcal{NK})$ , then*

$$H_{\mathcal{G}}^{s_1+s_2+1} = H_{\mathcal{G}}^{s_1+1} \cap H_{\mathcal{NK}}^{s_1+s_2+1} \quad \text{for } s_1 \in 2\mathbb{N} \cup \{0\}, \quad s_2 \in [0, 3/2].$$

*Proof.* We recall that by defining  $\mathcal{G}$  as a quantum graph, we are implicitly introducing a Laplacian  $A$  equipped with suitable boundary conditions. We refer to the first part of Section 2 for the definitions of  $H_{\mathcal{G}}^s$  and  $H_{\mathcal{NK}}^s$ .

**1) (a) Bounded intervals.** Let  $\mathcal{G} = I^{\mathcal{N}}$  be an interval equipped with  $(\mathcal{N})$  on the external vertices  $V_e$ . From [Gru16, *Definition 2.1*],

$$(8) \quad H_{I^{\mathcal{N}}}^{s_1+s_2} = H_{I^{\mathcal{N}}}^{s_1} \cap H_{I^{\mathcal{N}}}^{s_1+s_2}(I^{\mathcal{N}}, \mathbb{C}), \quad \forall s_1 \in 2\mathbb{N} \cup \{0\}, s_2 \in [0, 3/2).$$

Let  $\mathcal{G} = I^{\mathcal{D}}$  be an interval equipped with  $(\mathcal{D})$  on the external vertices. From [Gru16, *Definition 2.1*], for  $s_1 \in 2\mathbb{N} \cup \{0\}$ ,  $s_2 \in [0, 3/2)$  and  $s_3 \in [0, 1/2)$ ,

$$(9) \quad H_{I^{\mathcal{D}}}^{s_1+s_2+1} = H_{I^{\mathcal{D}}}^{s_1+1} \cap H_{I^{\mathcal{D}}}^{s_1+s_2+1}(I^{\mathcal{D}}, \mathbb{C}), \quad H_{I^{\mathcal{D}}}^{s_3} = H^{s_3}(I^{\mathcal{D}}, \mathbb{C}).$$

Let  $\mathcal{G} = I^{\mathcal{M}}$  be an interval equipped with  $(\mathcal{D})$  on one external vertex  $v_1$  and  $(\mathcal{N})$  on the other  $v_2$ . We prove that

$$(10) \quad H_{I^{\mathcal{M}}}^{s_1+s_2} = H_{I^{\mathcal{M}}}^{s_1} \cap H_{I^{\mathcal{M}}}^{s_1+s_2}(I^{\mathcal{M}}, \mathbb{C}), \quad \forall s_1 \in \mathbb{N} \cup \{0\}, s_2 \in [0, 1/2).$$

We consider the interval  $\tilde{I}^{\mathcal{D}} \subseteq I^{\mathcal{M}}$  of length  $\frac{3}{4}|I^{\mathcal{M}}|$  as a quantum graph containing  $v_1$  and equipped in both the external vertices with  $(\mathcal{D})$ .

Let the quantum graph  $\tilde{I}^{\mathcal{N}} \subseteq I^{\mathcal{M}}$  be an interval of length  $\frac{3}{4}|I^{\mathcal{M}}|$ , containing  $v_2$  and equipped in both the external vertices with  $(\mathcal{N})$ .

Let  $\chi$  be the partition of the unity so that  $\chi(x) = 1$  in  $\tilde{I}$ ,  $\chi(x) = 0$  in  $I^{\mathcal{M}} \setminus I^{\mathcal{D}}$  and  $\chi(x) \in (0, 1)$  in  $I^{\mathcal{D}} \setminus \tilde{I}$ . There holds  $\chi\psi \in H_{I^{\mathcal{D}}}^2$  and  $(1-\chi)\psi \in H_{I^{\mathcal{N}}}^2$  and

$$\psi(x) = \chi(x)\psi(x) + (1-\chi(x))\psi(x).$$

The same property is valid for  $L^2(I^{\mathcal{M}}, \mathbb{C})$  and  $H^s(I^{\mathcal{M}}, \mathbb{C})$ . Thus, for  $s \in (0, 2]$ , we have  $H^s(I^{\mathcal{M}}, \mathbb{C}) = H^s(\tilde{I}^{\mathcal{D}}, \mathbb{C}) \times H^s(\tilde{I}^{\mathcal{N}}, \mathbb{C})$  and

$$H_{I^{\mathcal{M}}}^2 = H_{\tilde{I}^{\mathcal{D}}}^2 \times H_{\tilde{I}^{\mathcal{N}}}^2, \quad L^2(I^{\mathcal{M}}, \mathbb{C}) = L^2(\tilde{I}^{\mathcal{D}}, \mathbb{C}) \times L^2(\tilde{I}^{\mathcal{N}}, \mathbb{C}).$$

Let  $[\cdot, \cdot]_{\theta}$  be the complex interpolation of two spaces for  $0 < \theta < 1$  defined in [Tri95, *Definition, Chapter 1.9.2*]. From [Tri95, *Remark 1, Chapter 1.15.1*] and [Tri95, *Theorem, Chapter 1.15.3*], for  $s_1 \in \mathbb{N} \cup \{0\}$  and  $s_2 \in [0, 1/2)$ ,

$$\left[ L^2(\tilde{I}^{\mathcal{N}}, \mathbb{C}), H_{\tilde{I}^{\mathcal{N}}}^2 \right]_{s_2/2} = H_{\tilde{I}^{\mathcal{N}}}^{s_2}, \quad \left[ L^2(\tilde{I}^{\mathcal{D}}, \mathbb{C}), H_{\tilde{I}^{\mathcal{D}}}^2 \right]_{s_2/2} = H_{\tilde{I}^{\mathcal{D}}}^{s_2}.$$

Thanks to [Tri95, *relation (12), Chapter 1.18.1*],

$$\begin{aligned} H_{I^{\mathcal{M}}}^{s_2} &= \left[ L^2(I^{\mathcal{M}}, \mathbb{C}), H_{I^{\mathcal{M}}}^2 \right]_{s_2/2} = \left[ L^2(\tilde{I}^{\mathcal{N}}, \mathbb{C}) \times L^2(\tilde{I}^{\mathcal{D}}, \mathbb{C}), H_{\tilde{I}^{\mathcal{N}}}^2 \times H_{\tilde{I}^{\mathcal{D}}}^2 \right]_{s_2/2} \\ &= \left[ L^2(\tilde{I}^{\mathcal{N}}, \mathbb{C}), H_{\tilde{I}^{\mathcal{N}}}^2 \right]_{s_2/2} \times \left[ L^2(\tilde{I}^{\mathcal{D}}, \mathbb{C}), H_{\tilde{I}^{\mathcal{D}}}^2 \right]_{s_2/2} = H_{\tilde{I}^{\mathcal{N}}}^{s_2} \times H_{\tilde{I}^{\mathcal{D}}}^{s_2}. \end{aligned}$$

Equally,  $H_{I^{\mathcal{M}}}^{s_1+s_2} = H_{\tilde{I}^{\mathcal{N}}}^{s_1+s_2} \times H_{\tilde{I}^{\mathcal{D}}}^{s_1+s_2}$  that leads to (10) thanks to (8) and (9).

**(b) Star graphs with equal edges.** Let  $A_{\mathcal{N}}$  be a Laplacian on an interval  $I$  of length  $L$  and equipped with  $(\mathcal{N})$ . Let  $I^{\mathcal{N}}$  be the relative quantum graph and  $\{f_j^1\}_{j \in \mathbb{N}}$  be an Hilbert basis of  $L^2(I, \mathbb{C})$  made by eigenfunctions of  $A_{\mathcal{N}}$ . Let  $A_{\mathcal{M}}$  be a Laplacian on  $I$  equipped with  $(\mathcal{D})$  in the external vertex parametrized with 0 and with  $(\mathcal{N})$  in the other. We call  $I^{\mathcal{M}}$  the relative quantum graph and  $\{f_j^2\}_{j \in \mathbb{N}}$  a Hilbert basis of  $L^2(I, \mathbb{C})$  composed by eigenfunctions of  $A_{\mathcal{M}}$ .

Let  $\mathcal{S}$  be a star graph of  $N$  edges long  $L$  and equipped with  $(\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$ . The conditions  $(\mathcal{N})$  on  $V_e$  imply that each  $\phi_k$  is  $(a_k^1 \cos(x\sqrt{\lambda_k}), \dots, a_k^N \cos(x\sqrt{\lambda_k}))$  with  $\lambda_k$  the corresponding eigenvalue and  $\{a_j^l\}_{l \leq N} \subset \mathbb{C}$ . The condition  $(\mathcal{N}\mathcal{K})$  in  $V_i$  ensures  $\sin(\sqrt{\lambda_k}L) \sum_{l \leq N} a_k^l = 0$  and

$$a_k^1 \cos(\sqrt{\lambda_k}L) = \dots = a_k^N \cos(\sqrt{\lambda_k}L), \quad \forall k \in \mathbb{N}.$$

Each eigenvalues is either of the form  $\frac{(n-1)^2\pi^2}{L^2}$ , or  $\frac{(2n-1)^2\pi^2}{4L^2}$  when  $\sum_{l \leq N} a_k^l = 0$  with  $n \in \mathbb{N}$ . Hence, for every  $k \in \mathbb{N}$ , there exists  $j(k) \in \mathbb{N}$  such that

$$(11) \quad \begin{aligned} \text{either } \phi_k^l &= c_k^l f_{j(k)}^1 & \text{for } c_k^l \in \mathbb{C}, \quad |c_k^l| \leq 1, \quad \forall l \in \{1, \dots, N\}, \\ \text{or } \phi_k^l &= c_k^l f_{j(k)}^2 & \text{for } c_k^l \in \mathbb{C}, \quad |c_k^l| \leq 1, \quad \forall l \in \{1, \dots, N\}. \end{aligned}$$

After, for each  $k \in \mathbb{N}$  and  $m \in \{1, 2\}$ , there exist  $\tilde{j} \in \mathbb{N}$  and  $l \leq N$  such that  $f_k^m = c_j^l \phi_j^l$  with  $c_j^l \in \mathbb{C}$  and  $|c_j^l| \leq 1$ . Thanks to the last identity and (11),

$$(12) \quad \psi = (\psi^1, \dots, \psi^N) \in H_{\mathcal{S}}^s \iff \psi^l \in H_{I^{\mathcal{N}}}^s \cap H_{I^{\mathcal{M}}}^s, \quad \forall l \leq N.$$

Now, we can consider each edge  $e_j$  composing  $\mathcal{S}$  as  $I$  (introduced above) as its length is  $L$ . Let  $I^{\mathcal{M}}$  and  $I^{\mathcal{N}}$  be defined above and  $H^s(\mathcal{S}, \mathbb{C}) = (H^s(I, \mathbb{C}))^N$ . For  $s_1 \in \mathbb{N} \cup \{0\}$  and  $s_2 \in [0, 1/2)$ , from (12), we have

$$\psi \in H^{s_1+s_2}(\mathcal{S}, \mathbb{C}) \cap H_{\mathcal{S}}^{s_1} \iff \psi^l \in H^{s_1+s_2}(I, \mathbb{C}) \cap H_{I^{\mathcal{N}}}^{s_1} \cap H_{I^{\mathcal{M}}}^{s_1}, \quad \forall l \leq N.$$

From **1) (a)**, it follows  $\psi^l \in H_{I^{\mathcal{N}}}^{s_1+s_2} \cap H_{I^{\mathcal{M}}}^{s_1+s_2}$  for every  $l \leq N$ , which is valid if and only if  $\psi \in H_{\mathcal{S}}^{s_1+s_2}$  thanks to (12). In conclusion, we have

$$H_{\mathcal{S}}^{s_1+s_2} = H_{\mathcal{S}}^{s_1} \cap H^{s_1+s_2}(\mathcal{S}, \mathbb{C}).$$

**(c) Generic graphs.** Let  $\mathcal{G}$  be equipped with  $(\mathcal{D}/\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$  and  $\tilde{L} < \min\{L_k/2 : k \in \{1, \dots, N\}\}$ . We define  $\mathcal{G}(v)$  for every  $v \in V_i \cup V_e$  and a family of intervals  $\{I_j\}_{j \leq N}$  as follows (see Figure 5). Let  $n(v)$  be defined in (4) for every  $v \in V_e \cup V_i$ .

If  $v \in V_i$ , then  $\mathcal{G}(v)$  is a star sub-graph of  $\mathcal{G}$  equipped with  $(\mathcal{N})$ - $(\mathcal{N}\mathcal{K})$ , with  $n(v)$  edges of equal length  $\tilde{L}$ , with internal vertex  $v$ .

If  $v \in V_e$ , then  $\mathcal{G}(v)$  is an interval of length  $\tilde{L}$  with external vertex  $v$  equipped with the same boundary conditions that  $v$  has in  $\mathcal{G}$ . We impose  $(\mathcal{N})$  on the other vertex.

For each  $v, \hat{v}$  such that  $\tilde{\mathcal{G}}(v)$  and  $\tilde{\mathcal{G}}(\hat{v})$  have respectively two external vertices  $w_1$  and  $w_2$  lying on the same edge  $e$  and such that  $w_1 \notin \tilde{\mathcal{G}}(\hat{v})$ , we construct an interval strictly containing  $w_1$  and  $w_2$ , strictly contained in  $e$  and equipped with  $(\mathcal{N})$ . We collect those intervals in  $\{I_j\}_{j \leq N}$ .

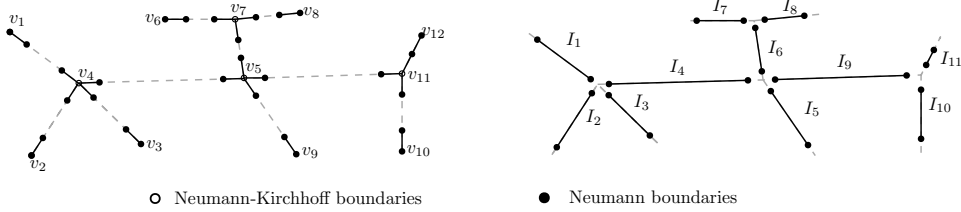


Figure 5: The left and the right figure respectively represent the graphs  $\{\tilde{\mathcal{G}}(v)\}_{v \in V_i \cup V_e}$  and the intervals  $\{I_j\}_{j \leq N}$  defined in the point **1) (c)** of the proof of Proposition 3.2.

From **1) (a)** and **1) (b)**, for  $v \in V_i \cup V_e$ ,  $j \leq N$ ,  $s_1 \in \mathbb{N} \cup \{0\}$  and  $s_2 \in [0, 1/2)$ ,

$$H_{\tilde{\mathcal{G}}(v)}^{s_1+s_2} = H_{\tilde{\mathcal{G}}(v)}^{s_1} \cap H^{s_1+s_2}(\tilde{\mathcal{G}}(v), \mathbb{C}), \quad H_{I_j}^{s_1+s_2} = H_{I_j}^{s_1} \cap H^{s_1+s_2}(I_j, \mathbb{C}).$$

We notice that  $G := \{\tilde{\mathcal{G}}(v_j)\}_{j \leq M} \cup \{I_j\}_{j \leq N}$  covers  $\mathcal{G}$ . As in **1) (a)**, we see each function of domain  $\mathcal{G}$  as a vector of functions of domain  $G_j$  for a suitable  $j \leq M + N$ . We use [Tri95, relation (12), Chapter 1.18.1] as in **1) (a)**. The interpolation between two products of spaces is the product of the respective interpolations, which leads to

$$H_{\mathcal{G}}^{s_1+s_2} = H_{\mathcal{G}}^{s_1} \cap H^{s_1+s_2}(\mathcal{G}, \mathbb{C}) \quad \text{for } s_1 \in \mathbb{N} \cup \{0\}, s_2 \in [0, 1/2).$$

**2)** Let  $\mathcal{G}$  be equipped with  $(\mathcal{N})$ - $(\mathcal{NK})$ . We consider  $\{\tilde{\mathcal{G}}(v)\}_{v \in V_e}$  introduced in **1) (c)** and we define  $\tilde{\mathcal{G}}$  as follows (see Figure 6). For every  $v \in V_e$ , we remove from the edge including  $v$ , a section of length  $\tilde{L}/2$  containing  $v$ . We equip the new external vertex with  $(\mathcal{N})$  and we call  $N_e \in \mathbb{N}$  the number of external vertices of  $\tilde{\mathcal{G}}$ .

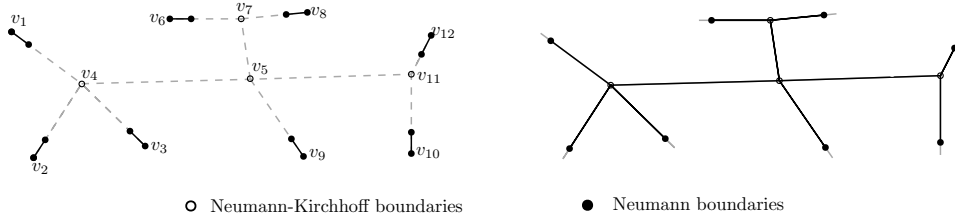


Figure 6: The left and the right figure respectively represent the graphs  $\{\tilde{\mathcal{G}}(v)\}_{v \in V_e}$  and the graph  $\tilde{\mathcal{G}}$  considered in the point **2)** of the proof of Proposition 3.2.

We call  $G' := \{G'_j\}_{j \leq N_e+1} := \{\tilde{\mathcal{G}}(v)\}_{v \in V_e} \cup \{\tilde{\mathcal{G}}\}$  which covers  $\mathcal{G}$ . For every  $s_1 \in 2\mathbb{N} \cup \{0\}$ ,  $s_2 \in [0, 3/2)$ , we have  $H_{\tilde{\mathcal{G}}(v)}^{s_1+s_2} = H_{\tilde{\mathcal{G}}(v)}^{s_1} \cap H^{s_1+s_2}$  from (8) and

$$H_{NK}^{s_1+s_2}(\mathcal{G}, \mathbb{C}) = H_{\tilde{\mathcal{G}}}^{s_1+s_2} \times \prod_{v \in V_e} H^{s_1+s_2}(\tilde{\mathcal{G}}(v), \mathbb{C}).$$

The arguments of **1) (a)**, also adopted in **1) (c)**, lead to the proof.

**3)** As in **2)**, the claim follows by considering  $\{\tilde{\mathcal{G}}(v)\}_{v \in V_e}$  as intervals equipped with  $(\mathcal{D})$  and  $\tilde{\mathcal{G}}$  equipped with  $(\mathcal{D})$  in its external vertices.  $\square$

*Proof of Proposition 3.1.*

**1) (a) Assumptions II.1 .** Let  $f(s) \in H^3 \cap H_{\mathcal{G}}^2$  for almost every  $s \in (0, t)$ ,  $t \in (0, T)$  and  $f(s) = (f^1(s), \dots, f^N(s))$ . We prove that  $G \in C^0([0, T], H_{\mathcal{G}}^3)$ . The definition of  $G(t)$  implies  $G(t) = \sum_{k=1}^{\infty} \phi_k \int_0^t e^{i\lambda_k s} \langle \phi_k, f(s) \rangle ds$  and

$$(13) \quad \|G(t)\|_{(3)} = \left( \sum_{k \in \mathbb{N}} \left| k^3 \int_0^t e^{i\lambda_k s} \langle \phi_k, f(s) \rangle ds \right|^2 \right)^{\frac{1}{2}}.$$

We estimate  $\langle \phi_k, f(s, \cdot) \rangle$  for each  $k \in \mathbb{N}$  and  $s \in (0, t)$ . We suppose that  $\lambda_1 \neq 0$ . Let  $\partial_x f(s) = (\partial_x f^1(s), \dots, \partial_x f^N(s))$  be the derivative of  $f(s)$  and  $P(\phi_k) = (P(\phi_k^1), \dots, P(\phi_k^N))$  be the primitive of  $\phi_k$  such that  $P(\phi_k) = -\frac{1}{\lambda_k} \partial_x \phi_k$ . We call  $\partial e$  the two points composing the boundaries of an edge  $e$ . For every  $v \in V_e$ ,  $\tilde{v} \in V_i$  and  $j \in N(\tilde{v})$ , there exist  $a(v), a^j(\tilde{v}) \in \{-1, +1\}$  such that

$$(14) \quad \begin{aligned} \langle \phi_k, f(s) \rangle &= \frac{1}{\lambda_k} \langle \phi_k, \partial_x^2 f(s) \rangle = \frac{1}{\lambda_k^2} \sum_{v \in V_e} a(v) \partial_x \phi_k(v) \partial_x^2 f(s, v) \\ &+ \frac{1}{\lambda_k^2} \sum_{v \in V_i} \sum_{j \in N(v)} a^j(v) \partial_x \phi_k^j(v) \partial_x^2 f^j(s, v) + \frac{1}{\lambda_k^2} \int_{\mathcal{G}} \partial_x \phi_k(y) \partial_x^3 f(s, y) dy. \end{aligned}$$

From [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10], there exist  $C_1 > 0$  such that  $\lambda_k^{-2} \leq C_1 k^{-4}$  for every  $k \in \mathbb{N}$  (see Remark A.4), then

$$(15) \quad \begin{aligned} \left| k^3 \int_0^t e^{i\lambda_k s} \langle \phi_k, f(s) \rangle ds \right| &\leq \frac{C_1}{k} \left( \sum_{v \in V_e} \left| \partial_x \phi_k(v) \int_0^t e^{i\lambda_k s} \partial_x^2 f(s, v) ds \right| \right. \\ &\left. + \sum_{v \in V_i} \sum_{j \in N(v)} \left| \partial_x \phi_k^j(v) \int_0^t e^{i\lambda_k s} \partial_x^2 f^j(s, v) ds \right| + \left| \int_0^t e^{i\lambda_k s} \int_{\mathcal{G}} \partial_x \phi_k(y) \partial_x^3 f(s, y) dy ds \right| \right). \end{aligned}$$

**Remark 3.3.** We point out that  $A' \lambda_k^{-1/2} \partial_x \phi_k = \lambda_k \lambda_k^{-1/2} \partial_x \phi_k$  for every  $k \in \mathbb{N}$ , where  $A' = -\Delta$  is a self-adjoint Laplacian with compact resolvent. Thus,  $\|\lambda_k^{-1/2} \partial_x \phi_k\|^2 = \langle \lambda_k^{-1/2} \partial_x \phi_k, \lambda_k^{-1/2} \partial_x \phi_k \rangle = \langle \phi_k, \lambda_k^{-1} A \phi_k \rangle = 1$  and then  $\{\lambda_k^{-1/2} \partial_x \phi_k\}_{k \in \mathbb{N}}$  is a Hilbert basis of  $\mathcal{H}$ .



Let  $\mathbf{a}^l = \{a_k^l\} \subset \mathbb{C}$  and  $\mathbf{b}^l = \{b_k^l\} \subset \mathbb{C}$  for  $l \in \{1, \dots, N\}$  be so that  $\phi_k^l(x) = a_k^l \cos(\sqrt{\lambda_k}x) + b_k^l \sin(\sqrt{\lambda_k}x)$ . Hence,  $-a_k^l \sin(\sqrt{\lambda_k}x) + b_k^l \cos(\sqrt{\lambda_k}x) = \lambda_k^{-1/2} \partial_x \phi_k^l(x)$ . Now,  $2 \geq \|\lambda_k^{-1/2} \partial_x \phi_k^l\|_{L^2(e^l)}^2 + \|\phi_k^l\|_{L^2(e^l)}^2 = (|a_k^l|^2 + |b_k^l|^2) |e^l|$  for every  $k \in \mathbb{N}$  and  $l \in \{1, \dots, N\}$ . Thus,  $\mathbf{a}^1, \mathbf{b}^1 \in \ell^\infty(\mathbb{C})$  and there exists  $C_2 > 0$  such that, for every  $k \in \mathbb{N}$  and  $v \in V_e \cup V_i$ , we have  $|\lambda_k^{-1/2} \partial_x \phi_k(v)| \leq C_2$ . Thanks to the identities (13) and (15), it follows

$$(16) \quad \begin{aligned} \|G(t)\|_{(3)} &\leq C_1 C_2 \sum_{v \in V_e \cup V_i} \sum_{j \in N(v)} \left\| \int_0^t \partial_x^2 f^j(s, v) e^{i\lambda(\cdot)s} ds \right\|_{\ell^2} \\ &\quad + C_1 \left\| \int_0^t \langle \lambda(\cdot)^{-1/2} \partial_x \phi(\cdot)(s), \partial_x^3 f(s) \rangle e^{i\lambda(\cdot)s} ds \right\|_{\ell^2}. \end{aligned}$$

From Proposition B.6 and (16), there exist  $C_3(t), C_4(t) > 0$  uniformly bounded for  $t$  in bounded intervals such that

$$(17) \quad \|G\|_{H_{\mathcal{G}}^3} \leq C_3(t) \sum_{v \in V_e \cup V_i} \sum_{j \in N(v)} \|\partial_x^2 f^j(\cdot, v)\|_{L^2((0,t), \mathbb{C})} + \sqrt{t} \|f\|_{L^2((0,t), H^3)}$$

and  $\|G\|_{H_{\mathcal{G}}^3} \leq C_4(t) \|f(\cdot, \cdot)\|_{L^2((0,t), H^3 \cap H_{\mathcal{G}}^2)}$ . If  $\lambda_1 = 0$ , then there exists  $C_5(t) > 0$  uniformly bounded for  $t$  lying on bounded intervals such that  $|\int_0^t \langle \phi_1, f(s) \rangle ds| \leq C_5(t) \|f(\cdot, \cdot)\|_{L^2((0,t), H^3 \cap H_{\mathcal{G}}^2)}$ . Equivalently to (15),

$$\|G(t)\|_{(3)} \leq C_1 \left( \sum_{k \in \mathbb{N}} \left| k \int_0^t e^{i\lambda_k s} \langle \phi_k, \partial_x^2 f \rangle ds \right|^2 \right)^{\frac{1}{2}} + \left( \left| \int_0^t \langle \phi_1(\cdot), f(t, \cdot) \rangle ds \right|^2 \right)^{\frac{1}{2}}.$$

Then, (17) is proved as above. For every  $t \in [0, T]$ , the inequality (17) shows that  $G(t) \in H^3 \cap H_{\mathcal{G}}^2$ . The provided upper bounds are uniform and the Dominated Convergence Theorem leads to  $G \in C^0([0, T], H_{\mathcal{G}}^3)$ .

Let  $f(s) \in H^5 \cap H_{\mathcal{G}}^4$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ . The same techniques adopted above shows that  $G \in C^0([0, T], H_{\mathcal{G}}^5)$ .

We denote  $F(f)(t) := \int_0^t e^{iA\tau} f(\tau) d\tau$  for  $f \in \mathcal{H}$  and  $t \in (0, T)$ . Let  $X(B)$  be the space of functions  $f$  so that  $f(s)$  belongs to a Banach space  $B$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ . The first part of the proof implies

$$F : X(H^3 \cap H_{\mathcal{G}}^2) \longrightarrow C^0([0, T], H_{\mathcal{G}}^3), \quad F : X(H^5 \cap H_{\mathcal{G}}^4) \longrightarrow C^0([0, T], H_{\mathcal{G}}^5).$$

From a classical interpolation result (see [BL76, Theorem 4.4.1] with  $n = 1$ ), we have  $F : X(H^{2+d} \cap H_{\mathcal{G}}^{1+d}) \longrightarrow C^0([0, T], H_{\mathcal{G}}^{2+d})$  with  $d \in [1, 3]$ . Thanks to Proposition 3.2, if  $d \in [1, 3/2)$  and  $f(s) \in H^{2+d} \cap H_{\mathcal{G}}^2 = H^{2+d} \cap H_{\mathcal{G}}^{1+d}$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then  $G \in C^0([0, T], H_{\mathcal{G}}^{2+d})$ . The proof is achieved when the first point Assumptions II is verified.

**(b) Assumptions II.4 .** If  $\mathcal{G}$  is equipped with  $(\mathcal{D})$ - $(\mathcal{NK})$ , then  $H_{\mathcal{G}}^2 = H_{\mathcal{NK}}^2 \cap H_{\mathcal{G}}^1$  and  $H_{\mathcal{G}}^4 = H_{\mathcal{NK}}^4 \cap H_{\mathcal{G}}^3$  from Proposition 3.2. The previous part of the proof implies that if  $f(s) \in H^3 \cap H_{\mathcal{NK}}^2 \cap H_{\mathcal{G}}^1$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then  $G \in C^0([0, T], H_{\mathcal{G}}^3)$ , while if  $f(s) \in H^5 \cap H_{\mathcal{NK}}^4 \cap H_{\mathcal{G}}^3$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then  $G \in C^0([0, T], H_{\mathcal{G}}^5)$ .

Thanks to interpolation technique previously adopted, if  $d \in [1, 5/2)$  and  $f(s) \in H^{2+d} \cap H_{\mathcal{NK}}^{1+d} \cap H_{\mathcal{G}}^d$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then  $G \in C^0([0, T], H_{\mathcal{G}}^{2+d})$ . The proof is attained under the hypotheses of the fourth point Assumptions II by considering that  $H_{\mathcal{NK}}^{2+d} \cap H_{\mathcal{G}}^2 = H_{\mathcal{NK}}^{2+d} \cap H_{\mathcal{G}}^d \subseteq H^{2+d} \cap H_{\mathcal{NK}}^{1+d} \cap H_{\mathcal{G}}^d$  thanks to Proposition 3.2.

**(c) Assumptions II.2 .** Let  $f(s) \in H^4 \cap H_{\mathcal{NK}}^3 \cap H_{\mathcal{G}}^2$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$  and  $\mathcal{G}$  be equipped with  $(\mathcal{N})$ . We proceed as in (14) and we notice that the first two terms of the last line are equal to zero. Indeed,  $\partial_x^2 f(s) \in C^0$  as  $f(s) \in H_{\mathcal{NK}}^3$  and, for  $v \in V_e$ , we have  $\partial_x \phi_k(v) = 0$  thanks to the  $(\mathcal{N})$  boundary conditions. After, for every  $v \in V_i$ , thanks to the  $(\mathcal{NK})$  in  $v \in V_i$ , we have  $\sum_{j \in N(v)} a^j(v) \partial_x \phi_k^j(v) = 0$  as the terms  $a^j(v)$  assume different signs according to the orientation of the edges connected in  $v$ . In (14), the integration by parts leads to

$$\begin{aligned} \langle \phi_k, f(s) \rangle &= -\frac{1}{\lambda_k^2} \int_{\mathcal{G}} \partial_x \phi_k(y) \partial_x^3 f(s, y) dy = -\frac{1}{\lambda_k^2} \sum_{v \in V_e} a(v) \phi_k(v) \partial_x^3 f(s, v) \\ &\quad - \frac{1}{\lambda_k^2} \sum_{v \in V_i} \sum_{j \in N(v)} a^j(v) \phi_k^j(v) \partial_x^3 f^j(s, v) + \frac{1}{\lambda_k^2} \int_{\mathcal{G}} \phi_k(y) \partial_x^4 f(s, y) dy. \end{aligned}$$

Now,  $\{\phi_k\}_{k \in \mathbb{N}}$  is a Hilbert basis of  $\mathcal{H}$  and we proceed as in (15), (16) and (17). From Proposition B.6, there exists  $C_6(t) > 0$  uniformly bounded for  $t$  lying in bounded intervals such that  $\|G\|_{H_{\mathcal{G}}^4} \leq C_1(t) \|f(\cdot, \cdot)\|_{L^2((0, t), H^4 \cap H_{\mathcal{G}}^3)}$ .

As above, when  $d \in [2, 5/2)$  and  $f(s) \in H^{2+d} \cap H_{\mathcal{G}}^2 \cap H_{\mathcal{NK}}^3$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then we have  $G \in C^0([0, T], H_{\mathcal{G}}^{2+d})$ , thanks to  $H^{2+d} \cap H_{\mathcal{G}}^4 = H_{\mathcal{G}}^{2+d}$  for  $d \in [2, 5/2)$  due to Proposition 3.2. It achieves the proof under the hypotheses of the second point of Assumptions II.

**(d) Assumptions II.3 .** If  $d \in [2, 7/2)$  and  $f(s) \in H_{\mathcal{NK}}^{2+d} \cap H_{\mathcal{G}}^2$  for almost every  $s \in (0, t)$  and  $t \in (0, T)$ , then  $G \in C^0([0, T], H_{\mathcal{G}}^{2+d})$ , thanks to  $H_{\mathcal{NK}}^{2+d} \cap H_{\mathcal{G}}^4 = H_{\mathcal{G}}^{2+d}$  for  $d \in [2, 7/2)$  due to Proposition 3.2. This attains the proof under the hypotheses of the third point of Assumptions II.

**2)** The arguments of [Ducb, Remark 1.1] and  $\text{Ran}(B|_{H_{\mathcal{G}}^{2+d}}) \subseteq H^{2+d} \cap H_{\mathcal{G}}^2$  lead to  $B \in L(H_{\mathcal{G}}^{2+d}, H^{2+d} \cap H_{\mathcal{G}}^2)$ . For every  $\psi \in H_{\mathcal{G}}^{2+d}$ , let

$$t \mapsto F(\psi)(t) = e^{-iAt} - \int_0^t e^{-iA(t-s)} u(s) B \psi(s) ds \in C^0([0, T], H_{\mathcal{G}}^{2+d}).$$

For every  $\psi^1, \psi^2 \in H_{\mathcal{G}}^{2+d}$ , thanks to the first point of the proof, there exists  $C(t) > 0$  uniformly bounded for  $t$  lying on bounded intervals, such that

$$\begin{aligned} \|F(\psi^1)(t) - F(\psi^2)(t)\|_{(2+d)} &\leq \left\| \int_0^t e^{-iA(t-s)} u(s) B(\psi^1(s) - \psi^2(s)) ds \right\|_{(2+d)} \\ &\leq C(t) \|u\|_{L^2((0,t),\mathbb{R})} \|B\|_{L(H_{\mathcal{G}}^{2+d}, H^{2+d} \cap H_{\mathcal{G}}^2)} \|\psi^1 - \psi^2\|_{L^\infty((0,t), H_{\mathcal{G}}^{2+d})}. \end{aligned}$$

We refer to the techniques adopted in the proof of [BL10, *Proposition 2*]. If  $\|u\|_{L^2((0,t),\mathbb{R})}$  is small enough, then  $F$  is a contraction and Banach Fixed Point Theorem implies that there exists  $\psi \in C^0([0, T], H_{\mathcal{G}}^{2+d})$  such that  $F(\psi) = \psi$ . When  $\|u\|_{L^2((0,t),\mathbb{R})}$  is not sufficiently small, one considers  $\{t_j\}_{0 \leq j \leq n}$  a partition of  $[0, t]$  for  $n \in \mathbb{N}$ . We choose a partition such that each  $\|u\|_{L^2([t_{j-1}, t_j], \mathbb{R})}$  is so small that the map  $F$ , defined on the interval  $[t_{j-1}, t_j]$ , is a contraction and proceed as before. The remaining claim follows from the proof of [BL10, *relation (23)*].  $\square$

## 4 Proof of Theorem 2.3

The result is achieved as in the proof of [Duch, *Proposition 3.4*]. In particular, it is obtained by gathering the local exact controllability and the global approximate controllability (both proved below) thanks to the time reversibility of the (*BSE*) (see [Duch, *Appendix 1.3*]).

### 4.1 Local exact controllability in $H_{\mathcal{G}}^s$

Let  $O_{\epsilon, T}^s := \{\psi \in H_{\mathcal{G}}^s \mid \|\psi\| = 1, \|\psi - \phi_1(T)\|_{(s)} < \epsilon\}$  and Assumptions I be verified. We define the application  $\alpha$ , the sequence with elements  $\alpha_k(u) = \langle \phi_k(T), \Gamma_T^u \phi_1 \rangle$  for  $k \in \mathbb{N}$ , such that

$$\alpha : L^2((0, T), \mathbb{R}) \longrightarrow Q := \{\mathbf{x} := \{x_k\}_{k \in \mathbb{N}} \in h^s(\mathbb{C}) \mid \|\mathbf{x}\|_{\ell^2} = 1\}.$$

The local exact controllability in  $O_{\epsilon, T}^s$  with  $T > 0$  is equivalent to the local surjectivity of  $\Gamma_T^{(\cdot)} \phi_1 : u \in L^2((0, T), \mathbb{R}) \longmapsto \psi \in O_{\epsilon, T}^s \subset H_{\mathcal{G}}^s$ . Thanks to

$$\Gamma_t^u \phi_1 = \sum_{k \in \mathbb{N}} \phi_k(t) \langle \phi_k(t), \Gamma_t^u \phi_1 \rangle, \quad T > 0, \quad u \in L^2((0, T), \mathbb{R}),$$

the controllability is equivalent to the local surjectivity of the map  $\alpha$ . To this end, we use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of  $\gamma(v) := (d_u \alpha(0)) \cdot v$  the Fréchet derivative of  $\alpha$  with  $\alpha(0) = \delta = \{\delta_{k,1}\}_{k \in \mathbb{N}}$ . Let  $B_{j,k} := \langle \phi_j, B \phi_k \rangle$  with  $j, k \in \mathbb{N}$ . As in [Duca, *relation (6)*], the map  $\gamma$  is the sequence of elements  $\gamma_k(v) := -i \int_0^T v(\tau) e^{i(\lambda_k - \lambda_1)s} d\tau B_{k,1}$  with  $k \in \mathbb{N}$  such that

$$\gamma : L^2((0, T), \mathbb{R}) \longrightarrow T_\delta Q = \{\mathbf{x} := \{x_k\}_{k \in \mathbb{N}} \in h^s(\mathbb{C}) \mid ix_1 \in \mathbb{R}\}.$$

Thanks to Proposition 3.1, the well-posedness of the  $(BSE)$  is guaranteed in  $H_{\mathcal{G}}^s$ . Hence,  $\alpha$  and  $\gamma$  take both value in  $h^s$ . Thus, the local surjectivity of  $\alpha$  can be proved by ensuring the solvability of the moment problem

$$(18) \quad x_k/B_{k,1} = -i \int_0^T u(\tau) e^{i(\lambda_k - \lambda_1)\tau} d\tau, \quad \forall \{x_k\}_{k \in \mathbb{N}} \in h^s.$$

Now, Proposition B.5 leads to the solvability of (18) in  $h^{\tilde{d}}$ . Indeed, if  $\{x_k\}_{k \in \mathbb{N}} \in h^{\tilde{d}+2+\eta}$ , then the hypotheses of Proposition B.5 are satisfied since  $B_{1,1} \in \mathbb{R}$  as  $B$  is symmetric, the element  $ix_1/B_{1,1} \in \mathbb{R}$  and  $\{x_k B_{k,l}^{-1}\}_{k \in \mathbb{N}} \in h^{\tilde{d}}$  thanks to the first point of Assumptions I. In conclusion,  $\{\gamma_k(u)\}_{k \in \mathbb{N}} \in h^s$  for every  $u \in L^2((0, T), \mathbb{R})$  and the moment problem (18) is solvable for  $\{x_k\}_{k \in \mathbb{N}} \in h^s \subseteq h^{2+\tilde{d}+\eta}$  with  $s = d + 2$  since  $d \geq \tilde{d} + \eta$ .

## 4.2 Global approximate controllability in $H_{\mathcal{G}}^s$

Let  $s = d + 2$  for  $d$  introduced in Assumptions II. The approximate controllability of the  $(BSE)$  in  $H_{\mathcal{G}}^s$  follows from the proof of [Duch, Theorem 3.3]. In other words, for every  $\psi \in H_{\mathcal{G}}^s$ ,  $\hat{\Gamma} \in U(\mathcal{H})$  such that  $\hat{\Gamma}\psi \in H_{\mathcal{G}}^s$  and  $\epsilon > 0$ , there exist  $T > 0$  and  $u \in L^2((0, T), \mathbb{R})$  such that  $\|\hat{\Gamma}\psi - \Gamma_T^u \psi\|_{(s)} < \epsilon$ .

The only difference with the mentioned proof is that the propagation of regularity from Kato [Kat53] has to be applied by considering different spaces. Let  $B : H_{\mathcal{G}}^{s_1} \rightarrow H_{\mathcal{G}}^{s_1}$  for  $s_1 \geq 0$ . As in [Duch, p. 16], for every  $T > 0$ ,  $u \in BV((0, T), \mathbb{R})$  and  $\psi \in H_{\mathcal{G}}^{s_1+2}$ , there exists  $C(K) > 0$  depending on  $K = (\|u\|_{BV((0, T), \mathbb{R})}, \|u\|_{L^\infty((0, T), \mathbb{R})}, T\|u\|_{L^\infty((0, T), \mathbb{R})})$  such that  $\|\Gamma_T^u \psi\|_{(s_1+2)} \leq C(K)\|\psi\|_{(s_1+2)}$ . This result leads to the global approximate controllability in  $H_{\mathcal{G}}^s$  with  $s \in [s_1, s_1 + 2)$  when  $B : H_{\mathcal{G}}^{s_1} \rightarrow H_{\mathcal{G}}^{s_1}$ .

Let  $d$  be the parameter introduced by the validity of Assumptions II. If  $d < 2$ , then  $B : H_{\mathcal{G}}^2 \rightarrow H_{\mathcal{G}}^2$  and the global approximate controllability is verified in  $H_{\mathcal{G}}^{d+2}$  since  $d + 2 < 4$ .

If  $d \in [2, 5/2)$  and the second or the fourth point of Assumptions II is verified, then  $B : H^{d_1} \rightarrow H^{d_1}$  for  $d_1 \in (d, 5/2)$  from Assumptions II. Now,  $H_{\mathcal{G}}^{d_1} = H^{d_1} \cap H_{\mathcal{G}}^2$ , thanks to Proposition 3.2, and  $B : H_{\mathcal{G}}^2 \rightarrow H_{\mathcal{G}}^2$  implies  $B : H_{\mathcal{G}}^{d_1} \rightarrow H_{\mathcal{G}}^{d_1}$ . The global approximate controllability is verified in  $H_{\mathcal{G}}^{d+2}$  since  $d + 2 < d_1 + 2$ .

If  $d \in [5/2, 7/2)$ , then  $B : H_{\mathcal{N}\mathcal{K}}^{d_1} \rightarrow H_{\mathcal{N}\mathcal{K}}^{d_1}$  for  $d_1 \in (d, 7/2)$  and  $H_{\mathcal{G}}^{d_1} = H_{\mathcal{N}\mathcal{K}}^{d_1} \cap H_{\mathcal{G}}^2$  from Proposition 3.2. Now,  $B : H_{\mathcal{G}}^2 \rightarrow H_{\mathcal{G}}^2$  that implies  $B : H_{\mathcal{G}}^{d_1} \rightarrow H_{\mathcal{G}}^{d_1}$ . The global approximate controllability is verified in  $H_{\mathcal{G}}^{d+2}$  since  $d+2 < d_1+2$ .

## 5 Proofs of Theorem 2.5, Corollary 2.6 and Corollary 2.7

Let  $\{\lambda_k^{\tilde{\mathcal{G}}}\}_{k \in \mathbb{N}}$  denote the eigenvalues  $A$  on a compact quantum graph  $\tilde{\mathcal{G}}$ .

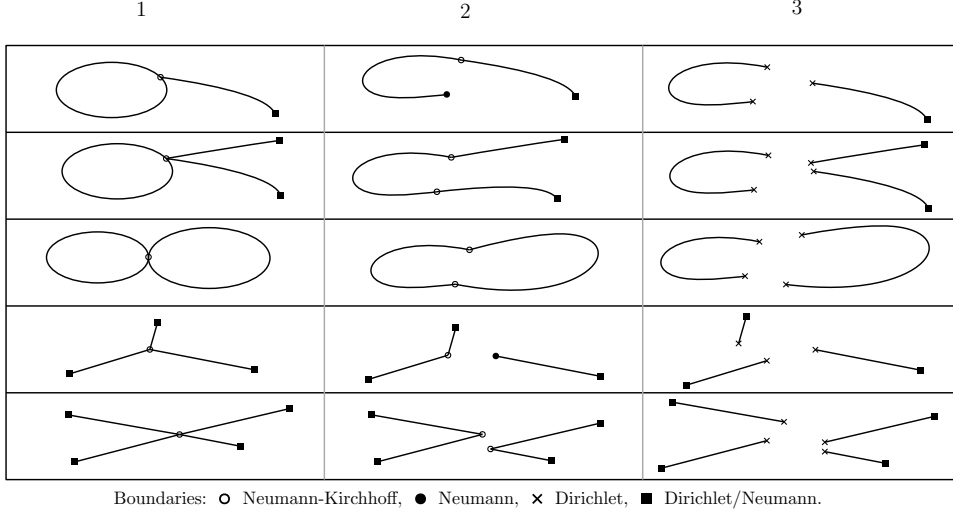


Figure 7: The figure represents the graphs described in the proof of Theorem 2.5. The column **1** shows the considered graphs  $\mathcal{G}$ . The **2** provides the corresponding graphs  $\mathcal{G}^N$ , while the column **3** shows  $\mathcal{G}^D$ .

*Proof of Theorem 2.5.* Let  $\mathcal{G}$  be a *tadpole graph* equipped with  $(\mathcal{D})$ - $(\mathcal{NK})$ . Let  $\mathcal{G}^D$  be the graph obtained from  $\mathcal{G}$  by imposing  $(\mathcal{D})$  on  $v \in V_i$  and  $e_1$  be the edge connecting  $v$  to itself. We define  $\mathcal{G}^N$  the graph obtained by disconnecting  $e_1$  in one side and by imposing  $(\mathcal{N})$  on the new external vertex of  $e_1$  (see the first line of Figure 7 for further details). Thanks to Proposition A.3, for  $k \in \mathbb{N}$ , it holds

$$(19) \quad \dots \leq \lambda_k^{\mathcal{G}} \leq \lambda_k^{\mathcal{G}^D} \leq \lambda_{k+1}^{\mathcal{G}} \leq \dots, \quad \dots \leq \lambda_k^{\mathcal{G}} \leq \lambda_{k+1}^{\mathcal{G}^N} \leq \lambda_{k+1}^{\mathcal{G}} \leq \dots$$

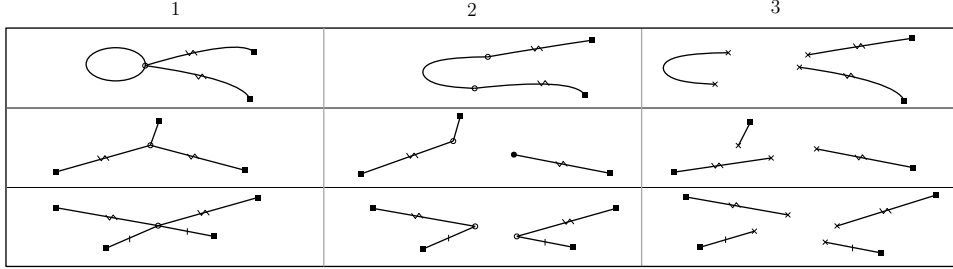
Now,  $\{\lambda_k^{\mathcal{G}^D}\}_{k \in \mathbb{N}}$  and  $\{\lambda_k^{\mathcal{G}^N}\}_{k \in \mathbb{N}}$  are the sequences of eigenvalues respectively obtained by reordering  $\left\{ \frac{k^2 \pi^2}{L_j^2} \right\}_{\substack{k \in \mathbb{N} \\ j \in \{1,2\}}} and  $\left\{ \frac{(2k-1)^2 \pi^2}{4(L_1+L_2)^2} \right\}_{k \in \mathbb{N}}$ . If  $\{L_1, L_2\} \in \mathcal{AL}$ , then  $\{L_1, L_2, L_1 + L_2\} \in \mathcal{AL}$ . The techniques of the proof of Proposition A.2 lead to the existence of  $C > 0$  such that, for every  $\epsilon > 0$ , there holds$

$$|\lambda_{k+1}^{\mathcal{G}} - \lambda_k^{\mathcal{G}}| \geq |\lambda_{k+1}^{\mathcal{G}^N} - \lambda_k^{\mathcal{G}^D}| \geq Ck^{-\epsilon}, \quad \forall k \in \mathbb{N}.$$

The relation (6) is verified and the claim is guaranteed by Theorem 2.3.

The same techniques lead to the claim when  $\mathcal{G}$  is a *tadpole graph* equipped with  $(\mathcal{N})$ - $(\mathcal{NK})$ , but also when  $\mathcal{G}$  is a *two-tails tadpole graph*, *double-rings graph* or *star graph* with  $N \leq 4$  edges. In every framework, we impose that  $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$ . In Figure 7, we represent how to define  $\mathcal{G}^N$  and  $\mathcal{G}^D$  from the corresponding graphs  $\mathcal{G}$ .  $\square$

*Proof of Corollary 2.6.* We decompose the considered graphs  $\mathcal{G}$  in  $\mathcal{G}^N$  and  $\mathcal{G}^D$  as in the proof of Theorem 2.5 (by following the instructions of Figure 8). The techniques proving Theorem 2.5 attain the claim.  $\square$



Boundaries:  $\circ$  Neumann-Kirchhoff,  $\bullet$  Neumann,  $\times$  Dirichlet,  $\blacksquare$  Dirichlet/Neumann.

Figure 8: The figure represents the graphs considered by Corollary 2.6. The columns **2** and **3** respectively show how to define  $\mathcal{G}^N$  and  $\mathcal{G}^D$  from the considered graphs  $\mathcal{G}$  represented in the column **1**. We sign with tildes and hyphens the edges of equal length.

*Proof of Corollary 2.7.* As  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \left\{ \frac{(k-1)^2 \pi^2}{4L_j^2} \right\}_{\substack{k, j \in \mathbb{N} \\ j \leq N}}$ , the claim follows from [Rot56]. In fact, thanks to the arguments adopted in the proof of Proposition A.2, for every  $\epsilon > 0$ , there exists  $C_1 > 0$  such that

$$|\lambda_{k+1} - \lambda_k| \geq C_1 k^{-\epsilon}, \quad \forall k \in \mathbb{N}.$$

In conclusion, the proof is achieved thanks to Theorem 2.3.  $\square$

## 6 Proofs of the examples 1.2, 1.3 and 1.4

*Proof of Example 1.2.* Let  $\mathcal{G}$  be a star graph with 4 edges of lengths  $\{L_j\}_{j \leq 4}$  equipped  $(\mathcal{D})$ - $(\mathcal{NK})$ . The conditions  $(\mathcal{D})$  on the external vertices imply that each eigenfunction  $\phi_j$  with  $j \in \mathbb{N}$  satisfies  $\phi_j^l(0) = 0$  for every  $l \leq 4$ . Then,

$$\phi_j(x) = (a_j^1 \sin(x\sqrt{\lambda_j}), a_j^2 \sin(x\sqrt{\lambda_j}), a_j^3 \sin(x\sqrt{\lambda_j}), a_j^4 \sin(x\sqrt{\lambda_j}))$$

with  $\{a_j^l\}_{l \leq 4} \subset \mathbb{C}$  such that  $\{\phi_j\}_{j \in \mathbb{N}}$  forms a Hilbert basis of  $\mathcal{H}$ , *i.e.*

$$1 = \sum_{l \leq 4} \int_0^{L_l} |a_j^l|^2 \sin^2(x\sqrt{\lambda_j}) dx = \sum_{l \leq 4} |a_j^l|^2 \left( \frac{L_l}{2} + \frac{\cos(L_l \sqrt{\lambda_j}) \sin(L_l \sqrt{\lambda_j})}{2\sqrt{\lambda_j}} \right).$$

For every  $j \in \mathbb{N}$ , the condition  $(\mathcal{NK})$  in  $V_i$  leads to

$$(20) \quad \begin{aligned} a_j^1 \sin(\sqrt{\lambda_j} L_1) &= \dots = a_j^4 \sin(\sqrt{\lambda_j} L_N), & \sum_{l \leq 4} a_j^l \cos(\sqrt{\lambda_j} L_l) &= 0, \\ \sum_{l \leq 4} \cot(\sqrt{\lambda_j} L_l) &= 0, & \sum_{l \leq N} |a_j^l|^2 \sin(L_l \sqrt{\lambda_j}) \cos(L_l \sqrt{\lambda_j}) &= 0. \end{aligned}$$

Now,  $1 = \sum_{l=1}^4 |a_j^l|^2 L_l / 2$  and the continuity implies  $a_j^l = a_j^1 \frac{\sin(\sqrt{\lambda_j} L_1)}{\sin(\sqrt{\lambda_j} L_l)}$  for  $l \neq 1$  and  $j \in \mathbb{N}$ , which ensures  $|a_j^1|^2 (L_1 + \sum_{l=2}^4 L_l \frac{\sin^2(\sqrt{\lambda_j} L_1)}{\sin^2(\sqrt{\lambda_j} L_l)}) = 2$ . Thus,

$$(21) \quad |a_j^1|^2 = \frac{2 \prod_{m \neq 1} \sin^2(\sqrt{\lambda_j} L_m)}{\sum_{k=1}^4 L_k \prod_{m \neq k} \sin^2(\sqrt{\lambda_j} L_m)}, \quad \forall j \in \mathbb{N}.$$

From (20) and (21), we have  $\sum_{l=1}^4 \cos(\sqrt{\lambda_k} L_l) \prod_{m \neq l} \sin(\sqrt{\lambda_k} L_m) = 0$ . The validity of [DZ06, Proposition A.11] and Remark A.4 ensure that, for every  $\epsilon > 0$ , there exist  $C_1, C_2 > 0$  such that

$$(22) \quad |a_j^1| = \sqrt{\frac{2}{\sum_{l=1}^4 L_l \sin^{-2}(\sqrt{\lambda_j} L_l)}} \geq \sqrt{\frac{2}{\sum_{l=1}^4 L_l C_1^{-2} \lambda_j^{1+\epsilon}}} \geq \frac{C_2}{j^{1+\epsilon}}, \quad \forall j \in \mathbb{N}.$$

Now,  $\langle \phi_k^l, B \phi_j^l \rangle_{L^2(e_j, \mathbb{C})} = 0$  for every  $2 \leq l \leq 4$  and  $k, j \in \mathbb{N}$ . Let

$$\begin{aligned} a_j(x) &:= \frac{2 \prod_{m \neq 1} \sin^2(\sqrt{\lambda_j} L_m)}{\sum_{k=2}^4 L_k \sin^2(\sqrt{\lambda_j} x) \prod_{m \neq k, 1} \sin^2(\sqrt{\lambda_j} L_m) + x \prod_{m \neq 1} \sin^2(\sqrt{\lambda_j} L_m)}, \\ B_1(x) &:= \frac{-30\sqrt{\lambda_1}x + 20\sqrt{\lambda_1}^3 x^3 + 4\sqrt{\lambda_1}^5 x^5 + 15 \sin(2\sqrt{\lambda_1}x)}{40\sqrt{\lambda_1}^5}, \\ B_j(x) &:= 2 \frac{-6(\sqrt{\lambda_1} - \sqrt{\lambda_j})x + (\sqrt{\lambda_1} - \sqrt{\lambda_j})^3 x^3 + 6 \sin((\sqrt{\lambda_1} - \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} - \sqrt{\lambda_j})^5} \\ &\quad - 2 \frac{-6(\sqrt{\lambda_1} + \sqrt{\lambda_j})x + (\sqrt{\lambda_1} + \sqrt{\lambda_j})^3 x^3 + 6 \sin((\sqrt{\lambda_1} + \sqrt{\lambda_j})x)}{(\sqrt{\lambda_1} + \sqrt{\lambda_j})^5} \end{aligned}$$

with  $j \in \mathbb{N}$ . Each function  $\tilde{B}_j(\cdot) := \sqrt{a_1(\cdot)} \sqrt{a_j(\cdot)} B_j(\cdot)$  is non-constant and analytic in  $\mathbb{R}^+$ , while we notice that  $B_{1,j} = \langle \phi_1, B \phi_j \rangle = \tilde{B}_j(L_1)$  by calculation. The set of positive zeros  $\tilde{V}_j$  of each  $\tilde{B}_j$  is a discrete subset of  $\mathbb{R}^+$  and  $\tilde{V} = \bigcup_{j \in \mathbb{N}} \tilde{V}_j$  is countable. For every  $\{L_l\}_{l \leq 4} \in \mathcal{AL}(4)$  such that  $L_1 \notin \tilde{V}$ , we have  $|B_{1,j}| \neq 0$  for every  $j \in \mathbb{N}$ . From Remark A.4 and the identity (22), the first point of Assumptions I(2 +  $\epsilon$ ) is verified since, for each  $\epsilon > 0$ , there exists  $C_3 > 0$  such that  $|B_{1,j}| \geq \frac{C_3}{j^{4+\epsilon}}$  for every  $j \in \mathbb{N}$ .

Let  $(k, j), (m, n) \in I$ ,  $(k, j) \neq (m, n)$  for  $I$  denote in (5). We define

$$F_j(x) := a_j(x) \frac{-30\sqrt{\lambda_k}x + 20\sqrt{\lambda_k}^3 x^3 + 4\sqrt{\lambda_k}^5 x^5 + 15 \sin(2\sqrt{\lambda_k}x)}{40\sqrt{\lambda_k}^5}.$$

By calculation, we notice that  $B_{j,j} = \langle \phi_j, B \phi_j \rangle = F_j(L_1)$ . Moreover, for  $F_{j,k,l,m}(x) = F_j(x) - F_k(x) - F_l(x) + F_m(x)$ , it follows  $F_{j,k,l,m}(L_1) = B_{j,j} - B_{k,k} - B_{l,l} + B_{m,m}$  and  $F_{j,k,l,m}(x)$  is a non-constant analytic function for

$x > 0$ . Furthermore  $V_{j,k,l,m}$ , the set of the positive zeros of  $F_{j,k,l,m}(x)$ , is discrete and  $V := \bigcup_{\substack{j,k,l,m \in \mathbb{N} \\ j \neq k \neq l \neq m}} V_{j,k,l,m}$  is a countable subset of  $\mathbb{R}^+$ . For each  $\{L_l\}_{l \leq 4} \in \mathcal{AL}(4)$  such that  $L_1 \notin V \cup \tilde{V}$ , Assumptions I( $2 + \epsilon$ ) are verified.

The fourth point of Assumptions II( $2 + \epsilon_1, \epsilon_2$ ) is valid for each  $\epsilon_1, \epsilon_2 > 0$  such that  $\epsilon_1 + \epsilon_2 \in (0, 1/2)$  since  $B$  stabilizes  $H_{\mathcal{G}}^2$ ,  $H^m$  and  $H_{\mathcal{NK}}^m$  for  $m \in (0, 9/2)$ . Indeed, for every  $n \in \mathbb{N}$  such that  $n < 5$ , we have

$$\forall \psi \in H_{\mathcal{NK}}^n \Rightarrow \partial_x^{n-1}(B\psi)^1(L_1) = \dots = \partial_x^{n-1}(B\psi)^4(L_4) = 0 \Rightarrow B\psi \in H_{\mathcal{NK}}^n.$$

From Theorem 2.5, the controllability holds in  $H_{\mathcal{G}}^{4+\epsilon}$  with  $\epsilon \in (0, 1/2)$ .  $\square$

*Proof of Example 1.3.* Let  $\mathcal{G}$  be a tadpole graph containing an edge  $e_1$  self-closing in an internal vertex  $v \in V_i$  equipped with  $(\mathcal{NK})$ . The edge  $e_2$  is connecting  $v$  to the external vertex  $v_1 \in V_e$  equipped with  $(\mathcal{N})$ . Let  $r$  be the axis passing along  $e_2$  and crossing  $e_1$  in its middle (see Figure 9).

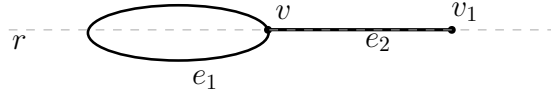


Figure 9: The figure represents the symmetry axis  $r$  of the tadpole graph.

We construct the eigenfunctions  $\{\phi_k\}_{k \in \mathbb{N}}$  as a sequence of symmetric or skew-symmetric functions with respect to  $r$  since  $\mathcal{G}$  is symmetric with respect to  $r$ . If an eigenfunction  $\phi_k = (\phi_k^1, \phi_k^2)$  is skew-symmetric, then

$$\phi_k^2 \equiv 0, \quad \phi_k^1(0) = \phi_k^1(L_1/2) = \phi_k^1(L_1) = 0, \quad \partial_x \phi_k^1(0) = \partial_x \phi_k^1(L_1).$$

We denote  $\{f_k\}_{k \in \mathbb{N}}$  the skew-symmetric eigenfunctions belonging to the Hilbert basis  $\{\phi_k\}_{k \in \mathbb{N}}$  and  $\{\mu_k\}_{k \in \mathbb{N}}$  the corresponding eigenvalues. We set

$$\{f_k\}_{k \in \mathbb{N}} = \left\{ \left( \frac{2}{\sqrt{L_1}} \sin \left( x \frac{2k\pi}{L_1} \right), 0 \right) \right\}_{k \in \mathbb{N}}, \quad \{\mu_k\}_{k \in \mathbb{N}} := \left\{ \frac{4k^2\pi^2}{L_1^2} \right\}_{k \in \mathbb{N}}.$$

If  $\phi_k = (\phi_k^1, \phi_k^2)$  is symmetric, then we have  $\partial_x \phi_k^1(L_1/2) = 0$  and  $\phi_k^1(\cdot) = \phi_k^1(L_1 - \cdot)$ . The  $(\mathcal{N})$  conditions on  $v_1$  implies that

$$\{g_k\}_{k \in \mathbb{N}} := \{(a_k^1 \cos(\sqrt{\mu_k}(x - L_1/2)), a_k^2 \cos(\sqrt{\mu_k}x))\}_{k \in \mathbb{N}}, \quad \text{with } \{(a_k^1, a_k^2)\}_{k \in \mathbb{N}} \subset \mathbb{C}^2,$$

is the sequence of eigenfunctions symmetric and corresponding to the eigenvalues  $\{\mu_k\}_{k \in \mathbb{N}}$ . We characterize  $\{\mu_k\}_{k \in \mathbb{N}}$  by considering that the  $(\mathcal{NK})$  conditions in  $v_1$  ensure  $a_k^1 \cos(\sqrt{\mu_k}(L_1/2)) = a_k^2 \cos(\sqrt{\mu_k}L_2)$  and

$$(23) \quad 2a_k^1 \sin(\sqrt{\mu_k}(L_1/2)) + a_k^2 \sin(\sqrt{\mu_k}L_2) = 0,$$



which imply  $2 \tan(\sqrt{\mu_k}(L_1/2)) + \tan(\sqrt{\mu_k}L_2) = 0$ . We choose  $\{(a_k^1, a_k^2)\}_{k \in \mathbb{N}} \subset \mathbb{C}^2$  such that  $\{\phi_k\}_{k \in \mathbb{N}} := \{f_k\}_{k \in \mathbb{N}} \cup \{g_k\}_{k \in \mathbb{N}}$  forms an Hilbert basis of  $\mathcal{H}$ . In particular, the techniques leading to relation (21) in Example 1.2 imply

$$|a_k^1|^2 = \frac{2 \cos^2(\sqrt{\mu_k}(L_1/2)) \cos^2(\sqrt{\mu_k}L_2)}{a_k}, \quad |a_k^2|^2 = \frac{2 \cos^4(\sqrt{\mu_k}(L_1/2))}{a_k}$$

with  $a_k := 2L_1 \cos^2(\sqrt{\mu_k}(L_1/2)) + L_2 \cos^2(\sqrt{\mu_k}L_2)$  and  $k \in \mathbb{N}$ . From (23),

$$(24) \quad 2 \cos\left(\sqrt{\mu_k} \frac{L_1}{2}\right) \cos(\sqrt{\mu_k}L_2) \sin\left(\sqrt{\mu_k} \frac{L_1}{2}\right) + \cos^2\left(\sqrt{\mu_k} \frac{L_1}{2}\right) \sin(\sqrt{\mu_k}L_2) = 0.$$

We underline that  $\cos(\sqrt{\mu_k}(L_1/2)) \neq 0$  for every  $k \in \mathbb{N}$ . Indeed, if there exists  $k \in \mathbb{N}$  such that  $\cos(\sqrt{\mu_k}(L_1/2)) = 0$ , then there exists  $l \in \mathbb{N}$  so that  $\mu_k = \frac{l^2 \pi^2}{L_1^2}$  and  $g_k(v_1) = 0$ . Now,  $g_k^2 \equiv 0$  since  $\mu_k$  is not an eigenvalue of a Laplacians on  $e_2$  equipped with  $(\mathcal{D})$  in  $v$  and  $(\mathcal{N})$  in  $v_1$  as  $\{L_1, L_2\} \in \mathcal{AL}(2)$ . The contradiction follows because the  $(\mathcal{NK})$  conditions fail in  $v$  since

$$\partial_x g_k^1(0) = -\partial_x g_k^1(L_1) \neq 0, \implies \partial_x g_k^1(0) - \partial_x g_k^1(L_1) - \partial_x g_k^2(L_2) \neq 0.$$

If  $\{L_1, L_2\} \in \mathcal{AL}(2)$ , then  $\{L_1/2, L_2\} \in \mathcal{AL}(2)$ . From (24), we have

$$2 \cos(\sqrt{\mu_k}L_2) \sin(\sqrt{\mu_k}(L_1/2)) + \cos(\sqrt{\mu_k}L_1/2) \sin(\sqrt{\mu_k}L_2) = 0$$

which leads to the validity of the two points of Remark A.6. We consider the relation (30) and the arguments leading to (22) in Example 1.2. For every  $\epsilon > 0$ , there exists  $C > 0$  such that, for every  $l \in \{1, 2\}$ ,

$$(25) \quad |a_k^l| \geq Ck^{-1-\epsilon}, \quad \forall k \in \mathbb{N}.$$

Let  $B_1 : \psi = (\psi^1, \psi^2) \mapsto (\mu_1 \psi^1, 0)$  and  $B_2 : \psi = (\psi^1, \psi^2) \mapsto (\mu_2 \psi^1, 0)$  with

$$\mu_1(x) := \sin\left(\frac{2\pi}{L_1}x\right), \quad \mu_2(x) := \sin\left(\frac{\pi}{L_1}x\right) + \frac{\pi}{L_1^2}x(x-L_1) - \frac{\pi^3}{6L_1^5}x^3(x-L_1)^3.$$

As  $\mu_1$  is skew-symmetric with respect to  $r$  and  $\mu_2$  is symmetric, we have

$$\langle f_k, B_1 f_k \rangle = \langle g_k, B_1 f_k \rangle = 0, \quad \langle f_k, B_2 g_k \rangle = \langle g_k, B_2 f_k \rangle = 0.$$

The remaining part of the example is ensured as Example 1.2. We fix  $j \in \mathbb{N}$  and we notice by calculation that  $|\langle f_j^1, B f_k^1 \rangle| = |\langle f_j^1, B_2 f_k^1 \rangle| \sim k^{-5}$ ,

$$|\langle g_j, B g_k \rangle| = |\langle g_j, B_2 g_k \rangle| \sim \frac{|a_k|}{\mu_k^{3/2}} \left| \sin\left(\left(\sqrt{\mu_k} - \sqrt{\mu_j}\right)L_1\right) + \sin\left(\left(\sqrt{\mu_k} + \sqrt{\mu_j}\right)L_1\right) \right|,$$

$$|\langle f_j, B g_k \rangle| = |\langle f_j, B_1 g_k \rangle| \sim \frac{|a_k|}{\mu_k} \left| \sin\left(\left(\sqrt{\mu_k} - \frac{2j\pi}{L_1}\right)L_1\right) + \sin\left(\left(\sqrt{\mu_k} + \frac{2j\pi}{L_1}\right)L_1\right) \right|.$$

As in Example 1.2, there exists  $\tilde{V} \subset \mathbb{R}^+$  countable such that, for every  $\{L_1, L_2\} \in \mathcal{AL}(2)$  such that  $L_1 \notin \tilde{V}$ , we have  $|B_{1,k}| \neq 0$  for every  $j \in \mathbb{N}$ . We consider that  $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin(\alpha) \cos(\beta)$  with  $\alpha, \beta \in \mathbb{R}$ . In (25), we apply Remark A.4 and the relation (31) (from Remark A.6), which leads to the validity of first point of Assumptions I(3 +  $\epsilon$ ). For every  $\epsilon > 0$ , there exists  $C_2 > 0$  such that  $|B_{1,k}| \geq C_2 k^{-5-\epsilon}$  for every  $k \in \mathbb{N}$ .

The second point of Assumptions I(3 +  $\epsilon$ ) is verified as in Example 1.2 and there exists  $V \subset \mathbb{R}^+$  countable such that, for each  $\{L_1, L_2\} \in \mathcal{AL}(2)$  such that  $L_1 \notin V \cup \tilde{V}$ , Assumptions I(3 +  $\epsilon$ ) are verified.

The third point of Assumptions II(3 +  $\epsilon_1, \epsilon_2$ ) is valid for each  $\epsilon_1, \epsilon_2 > 0$  such that  $\epsilon_1 + \epsilon_2 \in (0, 1/2)$  since  $B$  stabilizes  $H_{\mathcal{G}}^2$ ,  $H^m$  and  $H_{\mathcal{NK}}^m$  for  $m \in (0, 11/2)$ . Indeed, for every  $n \in \mathbb{N} \setminus \{2, 4\}$ , we have

$$\begin{aligned} \forall \psi \in H_{\mathcal{NK}}^n &\Rightarrow \partial_x^{n-1}(B\psi)^1(L_1) = \partial_x^{n-1}(B\psi)^4(L_2) = 0; \\ \forall \psi \in H_{\mathcal{NK}}^2 &\Rightarrow \partial_x(B\psi)^1(0) = \partial_x(B\psi)^1(L_1) = 2\pi/L_1, \quad \partial_x(B\psi)^2(L_2) = 0; \\ \psi \in H_{\mathcal{NK}}^4 &\Rightarrow \partial_x^3(B\psi)^1(0) = \partial_x^3(B\psi)^1(L_1) = -8\pi^3/L_1^3, \quad \partial_x^3(B\psi)^2(L_2) = 0. \end{aligned}$$

From Theorem 2.5, the controllability holds in  $H_{\mathcal{G}}^{5+\epsilon}$  with  $\epsilon \in (0, 1/2)$ .  $\square$

*Proof of Example 1.4.* The conditions ( $\mathcal{D}$ ) imply that  $\phi_k$  satisfies  $\phi_k^l(0) = 0$  and  $\phi_k^l(L_l) = 0$  for every  $k \in \mathbb{N}$  and  $l \leq N$ . As  $\{L_l\}_{l \leq N} \in \mathcal{AL}(N)$ , for each  $k \in \mathbb{N}$ , there exist  $m(k) \in \mathbb{N}$  and  $l(k) \leq N$  such that, for every  $n \neq l(k)$ ,

$$\lambda_k = m(k)^2 \pi^2 L_{l(k)}^{-2}, \quad \phi_k^{l(k)}(x) = \sqrt{2L_{l(k)}^{-1}} \sin(\sqrt{\lambda_k} x), \quad \phi_k^n \equiv 0.$$

Hence,  $\{\lambda_k\}_{k \in \mathbb{N}}$  is obtained by reordering  $\{\frac{m^2 \pi^2}{L_l^2}\}_{\substack{m, l \in \mathbb{N} \\ l \leq N}}$ . By calculations,

$$|B_{1,j}| \geq 2 \min\{L_l^2 : l \leq N\} \left| \int_0^1 x^2 \sin(m(j)\pi x) \sin(m(1)\pi x) dx \right|,$$

which is the integral treated in [Duch, Example 1.1]. As in the mentioned example, for every  $j \in \mathbb{N}$ , there exists  $C_1 > 0$  such that  $|B_{j,1}| \geq \frac{C_1}{m(j)^3} \geq \frac{C_1}{j^3}$  for every  $j \in \mathbb{N}$  since  $m(j) \leq j$ . Now

$$B_{j,j} = 2L_{m(j)}^2 \int_0^1 x^2 \sin^2(m(j)\pi x) dx = \frac{L_{m(j)}^2}{3} - \frac{L_{m(j)}^2}{2m(j)^2 \pi^2}.$$

As done in the proof of Example 1.2, there exists a countable set  $V$  such that, for each  $\{L_l\}_{l \leq N} \in \mathcal{AL}(N) \setminus V$ , Assumptions I(1) are verified.

The fourth point of Assumptions II(1,  $\epsilon$ ) is valid for each  $\epsilon \in (0, 3/2)$  since  $B$  stabilizes  $H_{\mathcal{G}}^2$  and  $H^m$  for  $m > 0$  ( $H^m \equiv H_{\mathcal{NK}}^m$  as there are not internal vertices in  $\mathcal{G}$ ). Moreover,  $H_{\mathcal{G}}^{2+d} \rightarrow H^{2+d}$ ,  $H_{\mathcal{G}}^{2+d} \subset H_{\mathcal{G}}^2 \rightarrow H_{\mathcal{G}}^2$  and  $B$  maps  $H_{\mathcal{G}}^{2+d}$  in  $H^{2+d} \cap H_{\mathcal{NK}}^{2+d} \cap H_{\mathcal{G}}^2$  for every  $d > 0$ . In conclusion, Corollary 2.7 achieves the controllability for every  $\epsilon \in (0, 3/2)$  in  $H_{\mathcal{G}}^{3+\epsilon} = \prod_{j=1}^N H_{I_j}^{3+\epsilon}$ .  $\square$

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## A Spectral properties

In the current chapter, we characterize  $\{\lambda_k\}_{k \in \mathbb{N}}$ , the eigenvalues of the Laplacian  $A$ , according to the structure of  $\mathcal{G}$  and to the boundary conditions defining  $D(A)$ .

**Proposition A.1.** (*Roth's Theorem; [Rot56]*) *If  $z$  is an algebraic irrational number, then for every  $\epsilon > 0$  the inequality  $|z - \frac{n}{m}| \leq \frac{1}{m^{2+\epsilon}}$  is satisfied for at most a finite number of  $n, m \in \mathbb{Z}$ .*

**Lemma A.2.** *Let  $\{\lambda_k^1\}_{k \in \mathbb{N}}$  and  $\{\lambda_k^2\}_{k \in \mathbb{N}}$  be obtained by reordering*

$$\left\{ \frac{k^2 \pi^2}{L_l^2} \right\}_{\substack{k, l \in \mathbb{N} \\ l \leq N_1}}, \quad \left\{ \frac{k^2 \pi^2}{\tilde{L}_i^2} \right\}_{\substack{k, i \in \mathbb{N} \\ i \leq N_2}} \quad \text{for } N_1, N_2 \in \mathbb{N}, \{L_l\}_{l \leq N_1}, \{\tilde{L}_i\}_{i \leq N_2} \subset \mathbb{R}$$

respectively. If all the ratios  $\tilde{L}_i/L_l$  are algebraic irrational numbers, then

$$\forall \epsilon > 0, \exists C > 0 : |\lambda_{k+1}^1 - \lambda_k^2| \geq Ck^{-\epsilon}, \quad \forall k \in \mathbb{N}.$$

*Proof.* Let  $z$  be an algebraic irrational number. From Proposition A.1,

$$\forall \epsilon > 0, \exists C > 0 : \left| z - \frac{n}{m} \right| \geq \frac{C}{m^{2+\epsilon}}, \quad \forall m, n \in \mathbb{N}.$$

Now, for every  $k \in \mathbb{N}$ , there exist  $m, n \in \mathbb{N}$  and  $i, l \leq N$  such that  $\lambda_{k+1}^1 = \frac{m^2 \pi^2}{L_l^2}$ ,  $\lambda_k^2 = \frac{n^2 \pi^2}{\tilde{L}_i^2}$ ,  $\lambda_{k+1}^1 \neq \lambda_k^2$ . We suppose  $L_l < \tilde{L}_i$ . If  $m < n$ , then for each  $\epsilon > 0$ , there exists  $C_1 > 0$  small enough

$$\left| \frac{m^2 \pi^2}{L_l^2} - \frac{n^2 \pi^2}{\tilde{L}_i^2} \right| = \left| \left( \frac{m\pi}{L_l} + \frac{n\pi}{\tilde{L}_i} \right) \left( \frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i} \right) \right| \geq \frac{2m\pi}{\tilde{L}_i} \left| \frac{m\pi}{L_l} - \frac{n\pi}{\tilde{L}_i} \right| \geq \frac{2C_1 \pi^2}{m^\epsilon \tilde{L}_i^2}.$$

If  $m \geq n$ , then  $\left| \frac{m^2 \pi^2}{L_l^2} - \frac{n^2 \pi^2}{\tilde{L}_i^2} \right| \geq \pi^2 (L_l^{-2} - \tilde{L}_i^{-2})$ , which implies

$$\exists C_2 > 0 : |\lambda_{k+1}^1 - \lambda_k^2| \geq C_2 (k+1)^{-\epsilon} \geq C_2 2^{-\epsilon} k^{-\epsilon}, \quad \forall k \in \mathbb{N}. \quad \square$$

The following proposition rephrases the results of [BK13, Theorem 3.1.8] and [BK13, Theorem 3.1.10]. Let  $\{\lambda_k^{\hat{\mathcal{G}}}\}_{k \in \mathbb{N}}$  be the spectrum of  $A$  on a generic compact quantum graph  $\hat{\mathcal{G}}$ .

**Proposition A.3.** [BK13, Theorem 3.1.8] & [BK13, Theorem 3.1.10]

1) Let  $w, v$  be two vertices of  $\mathcal{G}$  equipped with  $(\mathcal{NK})$  or  $(\mathcal{N})$ . If  $\mathcal{G}'$  is the graph obtained by merging in  $\mathcal{G}$  the vertices  $w$  and  $v$  in one unique vertex equipped with  $(\mathcal{NK})$ , then  $\lambda_k^{\mathcal{G}} \leq \lambda_k^{\mathcal{G}'} \leq \lambda_{k+1}^{\mathcal{G}}$  for every  $k \in \mathbb{N}$ .

2) Let  $w$  be a vertex of  $\mathcal{G}$ . If  $\mathcal{G}^{\mathcal{D}}$  is the graph obtained by imposing  $(\mathcal{D})$  on  $w$ , then  $\lambda_k^{\mathcal{G}} \leq \lambda_k^{\mathcal{G}^{\mathcal{D}}} \leq \lambda_{k+1}^{\mathcal{G}}$  for every  $k \in \mathbb{N}$ .

**Remark A.4.** Let  $\mathcal{G}$  be compact quantum graphs made by edges of lengths  $\{L_l\}_{l \leq N}$ . From Proposition A.3, there exist  $C_1, C_2 > 0$  such that, for  $k \geq 2$ ,

$$(26) \quad C_1 k^2 \leq \lambda_k^{\mathcal{G}} \leq C_2 k^2.$$

Indeed, we define  $\mathcal{G}^{\mathcal{D}}$  from  $\mathcal{G}$  by imposing  $(\mathcal{D})$  in each vertex. We denote  $\mathcal{G}^{\mathcal{N}}$  from  $\mathcal{G}$  by disconnecting each edge and by imposing  $(\mathcal{N})$  in each vertex. The graphs  $\mathcal{G}^{\mathcal{D}}$  and  $\mathcal{G}^{\mathcal{N}}$  are respectively obtained in at most  $M$  and  $2N$  steps from  $\mathcal{G}$  ( $M$  and  $N$  are respectively the numbers of vertices and edges). From Proposition A.3, we have  $\lambda_{k-2N}^{\mathcal{G}^{\mathcal{N}}} \leq \lambda_k^{\mathcal{G}} \leq \lambda_{k+M}^{\mathcal{G}^{\mathcal{D}}}$  for  $k > 2N$ . The sequences  $\lambda_k^{\mathcal{G}^{\mathcal{N}}}$  and  $\lambda_k^{\mathcal{G}^{\mathcal{D}}}$  are respectively obtained by reordering  $\left\{ \frac{k^2 \pi^2}{L_l^2} \right\}_{\substack{k \in \mathbb{N} \\ l \leq N}}$  and  $\left\{ \frac{(k-1)^2 \pi^2}{L_i^2} \right\}_{\substack{k \in \mathbb{N} \\ i \leq N}}$ . For  $l > 2N + 1$ ,  $\tilde{m} = \max_{j \leq N} L_j^2$  and  $\hat{m} = \min_{j \leq N} L_j^2$ ,

$$\lambda_{l-2N}^{\mathcal{G}^{\mathcal{N}}} \geq \frac{(l-2N-1)^2 \pi^2}{N^2 \tilde{m}} \geq \frac{l^2 \pi^2}{2^{2(2N+1)} N^2 \tilde{m}}, \quad \lambda_{l+M}^{\mathcal{G}^{\mathcal{D}}} \leq \frac{(l+M)^2 \pi^2}{\hat{m}} \leq \frac{l^2 2^{2M} \pi^2}{\hat{m}}.$$

The identity (26) is valid for  $k \geq 2$  as  $\lambda_k \neq 0$ , but also for  $k = 1$  if  $\lambda_1 \neq 0$ .

The techniques from [DZ06, Appendix A] lead to following proposition.

**Proposition A.5.** Let  $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$  with  $N \in \mathbb{N}$ . Let  $\{\omega_n\}_{n \in \mathbb{N}}$  be the unbounded sequence of positive solutions of the equation

$$(27) \quad \sum_{l \leq N} \sin(xL_l) \prod_{m \neq l} \cos(xL_m) = 0, \quad x \in \mathbb{R}.$$

For every  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  such that, for every  $l \leq N$ ,

$$|\cos(\omega_n L_l)| \geq \frac{C_\epsilon}{\omega_n^{1+\epsilon}}, \quad \forall n \in \mathbb{N}.$$

*Proof.* We consider the notation introduced in [DZ06, Appendix A] as  $\|\cdot\|$ ,  $E(\cdot)$  and  $F(\cdot)$ . For  $x \in \mathbb{R}$ ,  $\{L_k\}_{k \leq N} \in (\mathbb{R}^+)^N$  and  $i \leq N$ , we also denote

$$n(x) := E(x-1/2), \quad r(x) := F(x-1/2), \quad d(x) := \|x-1/2\|, \quad \tilde{m}^i(x) := n\left(\frac{L_i}{\pi} x\right).$$

From [DZ06, relation (A.3)], for every  $x \in \mathbb{R}$ , we obtain the identities  
(28)

$$2d(x) \leq |\cos(\pi x)| \leq \pi d(x), \quad 2d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq \left|\cos\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\pi\right)\right|.$$

As  $\cos(\alpha_1 - \alpha_2) = \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)$  for  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $\tilde{m}^i(x) + \frac{1}{2} = \frac{L_i}{\pi}x - r\left(\frac{L_i}{\pi}x\right)$  for every  $x \in \mathbb{R}$ , we have

$$(29) \quad 2d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq |\cos(L_j x)| + \left|\sin\left(\pi\frac{L_j}{L_i}\left|r\left(\frac{L_i}{\pi}x\right)\right|\right)\right|.$$

From [DZ06, relation (A.3)] and (28), we have the following inequalities  $|\sin(\pi|r(\cdot)|)| \leq \pi \|\|r(\cdot)\|\| \leq \pi|r(\cdot)| = \pi d(\cdot) \leq \frac{\pi}{2}|\cos(\pi(\cdot))|$ , which imply

$$\left|\sin\left(\pi\frac{L_j}{L_i}\left|r\left(\frac{L_i}{\pi}x\right)\right|\right)\right| \leq \pi\frac{L_j}{L_i}\left|r\left(\frac{L_i}{\pi}x\right)\right| \leq \frac{\pi L_j}{2L_i}|\cos(L_i x)|, \quad \forall x \in \mathbb{R}.$$

From (29), there exists  $C_1 > 0$  such that, for every  $i \leq N$ ,

$$\prod_{j \neq i} d\left(\left(\tilde{m}^i(x) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) \leq \frac{1}{2^{N-1}} \prod_{j \neq i} |\cos(L_j x)| + C_1 |\cos(L_i x)| \quad \forall x \in \mathbb{R}.$$

Thanks to (27), if there exists  $\{\omega_{n_k}\}_{k \in \mathbb{N}}$ , subsequence of  $\{\omega_n\}_{n \in \mathbb{N}}$ , such that

$$|\cos(L_j \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0 \quad \text{then} \quad \prod_{j \neq i} |\cos(L_i \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0.$$

Equivalently to [DZ06, relation (A.10)] (proof of [DZ06, Proposition A.11]), there exists a constant  $C_2 > 0$  such that, for every  $i \in \{0, \dots, N\}$ , we have

$$C_2 |\cos(L_i \omega_n)| \geq \prod_{j \neq i} d\left(\left(\tilde{m}^i(\omega_n) + \frac{1}{2}\right)\frac{L_j}{L_i}\right) = \prod_{j \neq i} \left\| \frac{1}{2} \left( \left(\tilde{m}^i(\omega_n) + \frac{1}{2}\right)\frac{2L_j}{L_i} - 1 \right) \right\|.$$

Now, we have  $\|\| \frac{1}{2}(\cdot) \|\| \geq \frac{1}{2} \|\| \cdot \|\|$  and  $\|\| (\cdot) - 1 \|\| = \|\| \cdot \|\|$ . We consider the Schmidt's Theorem [DZ06, Theorem A.7] since  $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$ . For every  $\epsilon > 0$ , there exist  $C_3, C_4 > 0$  such that, for every  $n \in \mathbb{N}$ , we have

$$\begin{aligned} C_2 |\cos(L_i \omega_n)| &\geq \prod_{j \neq i} \frac{1}{2} \left\| \left(\tilde{m}^i(\omega_n) + \frac{1}{2}\right)\frac{2L_j}{L_i} - 1 \right\| \geq \frac{C_3}{(2\tilde{m}^i(\omega_n) + 1)^{1+\epsilon}} \\ &\geq \frac{C_3}{2^{1+\epsilon} \left(\frac{L_i}{\pi}\omega_n + 1\right)^{1+\epsilon}} \geq \frac{C_4}{\omega_n^{1+\epsilon}}. \quad \square \end{aligned}$$

**Remark A.6.** *The techniques adopted in the proofs of Proposition A.5 and [DZ06, Proposition A.11] lead to the following results. Let  $\{L_k\}_{k \leq N} \in \mathcal{AL}(N)$  with  $N \in \mathbb{N}$ . Let  $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$  be an unbounded sequence.*

- If, for every  $j \leq N$ , the existence of  $\{\omega_{n_k}\}_{k \in \mathbb{N}}$ , subsequence of  $\{\omega_n\}_{n \in \mathbb{N}}$ , such that  $|\cos(L_j \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0$  implies  $\prod_{j \neq i} |\cos(L_i \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0$ , then, for every  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  such that, for every  $l \leq N$ ,

$$(30) \quad |\cos(\omega_n L_l)| \geq \frac{C_\epsilon}{\omega_n^{1+\epsilon}}, \quad \forall n \in \mathbb{N}.$$

- If for every  $j \leq N$  the existence of  $\{\omega_{n_k}\}_{k \in \mathbb{N}}$ , subsequence of  $\{\omega_n\}_{n \in \mathbb{N}}$ , such that  $|\sin(L_j \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0$  implies  $\prod_{j \neq i} |\sin(L_i \omega_{n_k})| \xrightarrow{k \rightarrow \infty} 0$ , then, for every  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  such that, for every  $l \leq N$ ,

$$(31) \quad |\sin(\omega_n L_l)| \geq \frac{C_\epsilon}{\omega_n^{1+\epsilon}}, \quad \forall n \in \mathbb{N}.$$

## B Moments problems

Let  $\mathcal{H}$  be a Hilbert space over a field  $\mathcal{K}$  for  $\mathcal{K} = \mathbb{C}$  or  $\mathbb{R}$  and  $\{f_n\}_{n \in \mathbb{Z}} \subset \mathcal{H}$ . In this appendix, we study the so-called ‘‘moments problem’’, which consists in finding  $v \in \mathcal{H}$  such that, for a  $\{x_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathcal{K})$ , there holds

$$x_n = \langle v, f_n \rangle_{\mathcal{H}}, \quad n \in \mathbb{Z}.$$

We assume  $\mathcal{H} = L^2((0, T), \mathbb{R})$  and  $\{f_n\}_{n \in \mathbb{N}} = \{e^{i\lambda_n(\cdot)}\}_{n \in \mathbb{N}}$  that lead to

$$x_n = \int_0^T e^{i\lambda_n s} u(s) ds, \quad \{x_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{C}), \quad u \in \mathcal{H}.$$

Let  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ . We assume the numbers  $\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}^*}$  are not uniformly separated but pairwise distinct and there exist  $\mathcal{M} \in \mathbb{N}$  and  $\delta > 0$  such that

$$(32) \quad \inf_{k \in \mathbb{Z}^*} |\lambda_{k+\mathcal{M}} - \lambda_k| \geq \delta \mathcal{M}.$$

From (32), we notice that there does not exist  $\mathcal{M}$  consecutive  $k \in \mathbb{Z}^*$  such that  $|\lambda_{k+1} - \lambda_k| < \delta$ . This fact leads to a partition of  $\mathbb{Z}^*$  in subsets that we call  $E_m$  with  $m \in \mathbb{Z}^*$ . By definition, for every  $m \in \mathbb{Z}^*$ , if  $k, n \in E_m$ , then  $|\lambda_k - \lambda_n| < \delta(\mathcal{M} - 1)$ , while if  $k \in E_m$  and  $n \notin E_m$ , then  $|\lambda_k - \lambda_n| \geq \delta$ .

The partition also defines an equivalence relation in  $\mathbb{Z}^*$  such that  $k, n \in \mathbb{Z}^*$  are equivalent if and only if there exists  $m \in \mathbb{Z}^*$  such that  $k, n \in E_m$ . The sets  $\{E_m\}_{m \in \mathbb{Z}^*}$  are the corresponding equivalence classes and  $i(m) := |E_m| \leq \mathcal{M} - 1$ . For every sequence  $\mathbf{x} := \{x_l\}_{l \in \mathbb{Z}^*}$ , we define the vectors  $\mathbf{x}^m := \{x_l\}_{l \in E_m}$  for  $m \in \mathbb{Z}^*$ .

Let  $\hat{\mathbf{h}} = \{h_j\}_{j \leq i(m)} \in \mathbb{C}^{i(m)}$  with  $m \in \mathbb{Z}^*$ . For every  $m \in \mathbb{Z}^*$ , we denote  $F_m(\hat{\mathbf{h}}) : \mathbb{C}^{i(m)} \rightarrow \mathbb{C}^{i(m)}$  the matrix with elements, for every  $j, k \leq i(m)$ ,

$$F_{m;j,k}(\hat{\mathbf{h}}) := \begin{cases} \prod_{\substack{l \neq j \\ 1 \leq l \leq k}} (h_j - h_l)^{-1}, & j \leq k, \\ 1, & j = k = 1, \\ 0, & j > k. \end{cases}$$

For each  $k \in \mathbb{Z}^*$ , there exists  $m(k) \in \mathbb{Z}^*$  such that  $k \in E_{m(k)}$ . Let  $F(\mathbf{\Lambda})$  be the linear operator on  $\ell^2(\mathbb{C})$  such that  $F(\mathbf{\Lambda}) : D(F(\mathbf{\Lambda})) \rightarrow \ell^2(\mathbb{C})$  and

$$(F(\mathbf{\Lambda})\mathbf{x})_k = \left( F_{m(k)}(\mathbf{\Lambda}^{m(k)})\mathbf{x}^{m(k)} \right)_k, \quad \forall \mathbf{x} = \{x_l\}_{l \in \mathbb{Z}^*} \in D(F(\mathbf{\Lambda})),$$

$$H(\mathbf{\Lambda}) := D(F(\mathbf{\Lambda})) = \{ \mathbf{x} := \{x_k\}_{k \in \mathbb{Z}^*} \in \ell^2(\mathbb{C}) : F(\mathbf{\Lambda})\mathbf{x} \in \ell^2(\mathbb{C}) \}.$$

**Proposition B.1.** *Let  $\mathbf{\Lambda} := \{\lambda_k\}_{k \in \mathbb{Z}^*}$  be an ordered sequence of real numbers satisfying (32). If there exist  $\tilde{d} \geq 0$  and  $C > 0$  such that*

$$(33) \quad |\lambda_{k+1} - \lambda_k| \geq C|k|^{-\frac{\tilde{d}}{\mathcal{M}-1}} \quad \forall k \in \mathbb{Z}^*,$$

then we have  $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}(\mathbb{C})$ .

*Proof.* Thanks to (33), we have  $|\lambda_j - \lambda_k| \geq C \min_{l \in E_m} |l|^{-\frac{\tilde{d}}{\mathcal{M}-1}}$  for every  $m \in \mathbb{Z}^*$  and  $j, k \in E_m$ . There exists  $C_1 > 0$  such that, for  $1 < j, k \leq i(m)$ ,

$$|F_{m;j,k}(\mathbf{\Lambda}^m)| \leq C_1 \left( \max_{l \in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}} \right)^{k-1} \leq C_1 \left( \max_{l \in E_m} |l|^{\frac{\tilde{d}}{\mathcal{M}-1}} \right)^{\mathcal{M}-1} \leq C_1 2^{|E_m| \tilde{d}} \min_{l \in E_m} |l|^{\tilde{d}}$$

and  $|F_{m;1,1}(\mathbf{\Lambda}^m)| = 1$ . Then, there exist  $C_2, C_3 > 0$  such that, for  $j \leq i(m)$ ,

$$(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m))_{j,j} \leq C_2 \min_{l \in E_m} |l|^{2\tilde{d}}, \quad \text{Tr} \left( F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m) \right) \leq C_3 \min_{l \in E_m} |l|^{2\tilde{d}}$$

with  $F_m(\mathbf{\Lambda}^m)^*$  the transposed matrix of  $F_m(\mathbf{\Lambda}^m)$ . Let  $\rho(M)$  be the spectral radius of a matrix  $M$  and we denote  $\| \| M \| \| = \sqrt{\rho(M^* M)}$  its euclidean norm. As  $(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m))$  is positive-definite, there holds

$$\| \| F_m(\mathbf{\Lambda}^m) \| \|^2 = \rho(F_m(\mathbf{\Lambda}^m)^* F_m(\mathbf{\Lambda}^m)) \leq C_3 \min_{l \in E_m} |l|^{2\tilde{d}}, \quad \forall m \in \mathbb{Z}^*.$$

In conclusion,  $\| F(\mathbf{\Lambda})\mathbf{x} \|_{\ell^2}^2 \leq C_3 \| \mathbf{x} \|_{h^{\tilde{d}}}^2 < +\infty$  for  $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$  as

$$\| F(\mathbf{\Lambda})\mathbf{x} \|_{\ell^2}^2 \leq \sum_{m \in \mathbb{Z}^*} \| \| F_m(\mathbf{\Lambda}^m) \| \|^2 \sum_{l \in E_m} |x_l|^2 \leq C_3 \sum_{m \in \mathbb{Z}^*} \min_{l \in E_m} |l|^{2\tilde{d}} \sum_{l \in E_m} |x_l|^2. \quad \square$$

**Corollary B.2.** *If  $\mathbf{\Lambda} := \{\lambda_k\}_{k \in \mathbb{Z}^*}$  is an ordered sequence of pairwise distinct real numbers satisfying (32), then  $F(\mathbf{\Lambda}) : H(\mathbf{\Lambda}) \rightarrow \text{Ran}(F(\mathbf{\Lambda}))$  is invertible.*

*Proof.* As in [DZ06, p. 48], we define  $F_m(\mathbf{\Lambda}^m)^{-1}$  the inverse matrix of  $F_m(\mathbf{\Lambda}^m)$  for every  $m \in \mathbb{Z}^*$ . We call  $F(\mathbf{\Lambda})^{-1}$  the operator such that  $(F(\mathbf{\Lambda})^{-1}\mathbf{x})_k = (F_{m(k)}(\mathbf{\Lambda}^{m(k)})^{-1}\mathbf{x}^{m(k)})_k$ , for every  $\mathbf{x} \in \text{Ran}(F(\mathbf{\Lambda}))$  and  $k \in \mathbb{Z}^*$ , which implies  $F(\mathbf{\Lambda})^{-1}F(\mathbf{\Lambda}) = \text{Id}_{H(\mathbf{\Lambda})}$  and  $F(\mathbf{\Lambda})F(\mathbf{\Lambda})^{-1} = \text{Id}_{\text{Ran}(F(\mathbf{\Lambda}))}$ .  $\square$

For every  $k \in \mathbb{Z}^*$ , we have the existence of  $m(k) \in \mathbb{Z}^*$  such that  $k \in E_{m(k)}$ . We define  $F(\mathbf{\Lambda})^*$  the infinite matrix such that  $(F(\mathbf{\Lambda})^* \mathbf{x})_k = (F_{m(k)}(\mathbf{\Lambda}^{m(k)})^* \mathbf{x}^{m(k)})_k$  for every  $\mathbf{x} = \{x_k\}_{k \in \mathbb{Z}^*}$  and  $k \in \mathbb{Z}^*$ , where  $F_{m(k)}(\mathbf{\Lambda}^{m(k)})^*$  is the transposed matrix of  $F_{m(k)}(\mathbf{\Lambda}^{m(k)})$ . For  $T > 0$ , we introduce

$$\mathbf{e} := \{e^{i\lambda_j t}\}_{j \in \mathbb{Z}^*} \subset L^2((0, T), \mathbb{C}).$$

Let  $t \in (0, T)$  with  $T > 0$ . We call  $\xi_k(t) = (F(\mathbf{\Lambda})^* \{e^{i\lambda_j t}\}_{j \in \mathbb{Z}^*})_k$  for every  $k \in \mathbb{Z}^*$ . By considering each  $\xi_k(t)$  as time-dependent function, we denote

$$\Xi := \{\xi_k(\cdot)\}_{k \in \mathbb{Z}^*} = F(\mathbf{\Lambda})^* \mathbf{e} \subset L^2((0, T), \mathbb{C}).$$

**Remark B.3.** Thanks to Proposition B.1, when  $\{\lambda_k\}_{k \in \mathbb{Z}^*}$  satisfies (32), the space  $H(\mathbf{\Lambda})$  is dense in  $\ell^2(\mathbb{C})$  as  $H(\mathbf{\Lambda}) \supseteq h^{\tilde{d}}$  which is dense in  $\ell^2$ . In this case, we can consider the infinite matrix  $F(\mathbf{\Lambda})^*$  as the unique adjoint operator of  $F(\mathbf{\Lambda})$  with domain  $H(\mathbf{\Lambda})^* := D(F(\mathbf{\Lambda})^*) \subseteq \ell^2(\mathbb{C})$ .

By transposing each  $F_m(\mathbf{\Lambda}^m)$  for  $m \in \mathbb{Z}^*$ , the arguments of the proof of Corollary B.2 lead to the invertibility of the map  $F(\mathbf{\Lambda})^* : H(\mathbf{\Lambda})^* \rightarrow \text{Ran}(F(\mathbf{\Lambda})^*)$  and  $(F(\mathbf{\Lambda})^*)^{-1} = (F(\mathbf{\Lambda})^{-1})^*$ . Moreover,  $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}$  as in Proposition B.1.

In the following theorem, we rephrase a result of Avdonin and Moran [AM01], which is also proved by Baiocchi, Komornik and Loreti in [BKL02].

**Theorem B.4** (Theorem 3.29; [DZ06]). Let  $\{\lambda_k\}_{k \in \mathbb{Z}^*}$  be an ordered sequence of pairwise distinct real numbers satisfying (32). If  $T > 2\pi/\delta$ , then  $\{\xi_k\}_{k \in \mathbb{Z}^*}$  forms a Riesz Basis in the space  $X := \overline{\text{span}\{\xi_k \mid k \in \mathbb{Z}^*\}}^{L^2}$ .

**Proposition B.5.** Let  $\{\omega_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+ \cup \{0\}$  be an ordered sequence of real numbers with  $\omega_1 = 0$  such that there exist  $\tilde{d}, \delta, C > 0$  and  $\mathcal{M} \in \mathbb{N}$  with

$$\inf_{k \in \mathbb{N}} |\omega_{k+\mathcal{M}} - \omega_k| \geq \delta \mathcal{M}, \quad |\omega_{k+1} - \omega_k| \geq C k^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \quad \forall k \in \mathbb{N}.$$

Then, for  $T > 2\pi/\delta$  and  $\{x_k\}_{k \in \mathbb{N}} \in h^{\tilde{d}}(\mathbb{C})$  with  $x_1 \in \mathbb{R}$ , there exists  $u \in L^2((0, T), \mathbb{R})$  such that

$$(34) \quad x_k = \int_0^T u(\tau) e^{i\omega_k \tau} d\tau \quad \forall k \in \mathbb{N}.$$

*Proof.* From the definition of Riesz basis ([BL10, Appendix B.1; Definition 2]) and [BL10, Appendix B.1; Proposition 19; 2]), the map  $M : g \in X \mapsto \{\langle \xi_k, g \rangle_{L^2((0, T), \mathbb{C})}\}_{k \in \mathbb{Z}^*} \in \ell^2(\mathbb{C})$  is invertible and, for every  $k \in \mathbb{Z}^*$ , we have

$$\langle \xi_k, g \rangle_{L^2((0, T), \mathbb{C})} = (F(\mathbf{\Lambda})^* \langle \mathbf{e}, g \rangle_{L^2((0, T), \mathbb{C})})_k.$$

Let  $\tilde{X} := M^{-1} \circ F(\mathbf{\Lambda})^* (h^{\tilde{d}}(\mathbb{C}))$ . From Remark B.3, we have  $H(\mathbf{\Lambda})^* \supseteq h^{\tilde{d}}(\mathbb{C})$ . The following maps are invertible  $(F(\mathbf{\Lambda})^*)^{-1} : \text{Ran}(F(\mathbf{\Lambda})^*) \rightarrow H(\mathbf{\Lambda})^*$  and

$$(F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle \mathbf{e}, g \rangle_{L^2((0, T), \mathbb{C})}\}_{k \in \mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C}).$$



Let  $I : \mathbf{x} \in \ell^2(\mathbb{C}) \mapsto \bar{\mathbf{x}} \in \ell^2(\mathbb{C})$  be the complex conjugation. The map

$$I \circ (F(\mathbf{\Lambda})^*)^{-1} \circ M : g \in \tilde{X} \mapsto \{\langle g, \mathbf{e} \rangle_{L^2((0,T),\mathbb{C})}\}_{k \in \mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$$

is invertible. For every  $\{x_k\}_{k \in \mathbb{Z}^*} \in h^{\tilde{d}}(\mathbb{C})$ , there exists  $g \in \tilde{X}$  such that  $x_k = \int_0^T \bar{g}(\tau) e^{i\lambda_k \tau} d\tau$  for every  $k \in \mathbb{Z}^*$ . For  $u = \bar{g} \in L^2((0,T),\mathbb{C})$ , we have

$$x_k = \int_0^T u(\tau) e^{i\lambda_k \tau} d\tau, \quad \forall k \in \mathbb{Z}^*.$$

When  $k > 0$ , we call  $\lambda_k = \omega_k$ , while  $\lambda_k = -\omega_{-k}$  for  $k < 0$  such that  $k \neq -1$ . The sequence  $\{\lambda_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}}$  is such that there exists  $C_1 > 0$  satisfying

$$\inf_{k \in \mathbb{Z}^*} |\lambda_{k+2\mathcal{M}} - \lambda_k| \geq \delta\mathcal{M}, \quad |\lambda_{k+1} - \lambda_k| \geq C_1 |k|^{-\frac{\tilde{d}}{\mathcal{M}-1}}, \quad \forall k \in \mathbb{Z}^* \setminus \{-1\}.$$

Given  $\{x_k\}_{k \in \mathbb{N}} \in \ell^2(\mathbb{C})$ , we introduce  $\{\tilde{x}_k\}_{k \in \mathbb{Z}^* \setminus \{-1\}} \in \ell^2(\mathbb{C})$  such that  $\tilde{x}_k = x_k$  for  $k > 0$ , while  $\tilde{x}_k = \bar{x}_{-k}$  for  $k < 0$  and  $k \neq -1$ . As above, there exists  $u \in L^2((0,T),\mathbb{C})$  such that  $x_1 = \int_0^T u(s) ds$  and  $\tilde{x}_k = \int_0^T u(s) e^{-i\omega_k s} ds$  for every  $k \in \mathbb{Z}^* \setminus \{-1\}$ , which implies and

$$\int_0^T u(s) e^{i\lambda_k s} ds = x_k = \int_0^T \bar{u}(s) e^{i\lambda_k s} ds, \quad k \in \mathbb{N} \setminus \{1\}.$$

If  $x_1 \in \mathbb{R}$ , then  $u$  is real and (34) is solvable for  $u \in L^2((0,T),\mathbb{R})$ .  $\square$

**Proposition B.6.** *Let  $\{\lambda_k\}_{k \in \mathbb{Z}^*}$  be an ordered sequence of pairwise distinct real numbers satisfying (32). For every  $T > 0$ , there exists  $C(T) > 0$  uniformly bounded for  $T$  lying on bounded intervals such that, for every  $g \in L^2((0,T),\mathbb{C})$ , we have*

$$\left\| \int_0^T e^{i\lambda(\cdot)s} g(s) ds \right\|_{\ell^2} \leq C(T) \|g\|_{L^2((0,T),\mathbb{C})}.$$

*Proof.* Let  $\{\omega_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$  be such that  $\gamma := \inf_{k \neq j} |\omega_k - \omega_j| > 0$  and  $L^2 := L^2((0,T),\mathbb{C})$ . Thanks to the Ingham's Theorem [KL05, Theorem 4.3], for  $T > \frac{2\pi}{\gamma}$ , the family of functions  $\{e^{i\omega_k(\cdot)}\}_{k \in \mathbb{Z}}$  is a Riesz Basis in

$$X = \overline{\text{span}\{e^{i\omega_k(\cdot)} : k \in \mathbb{N}\}}^{L^2}.$$

From [Duca, relation (29)], there exists  $C_1(T) > 0$  such that

$$\sum_{k \in \mathbb{N}} |\langle e^{i\omega_k(\cdot)}, u \rangle_{L^2}|^2 \leq C_1(T)^2 \|u\|_{L^2}^2, \quad \forall u \in X.$$

Let  $P : L^2 \rightarrow X$  be the orthogonal projector. For  $g \in L^2$ , we have

$$\|\{\langle e^{i\omega_k(\cdot)}, g \rangle_{L^2}\}_{k \in \mathbb{N}}\|_{\ell^2} = \|\{\langle e^{i\omega_k(\cdot)}, Pg \rangle_{L^2}\}_{k \in \mathbb{N}}\|_{\ell^2} \leq C_1(T) \|Pg\|_{L^2} \leq C_1(T) \|g\|_{L^2}.$$

Let  $\{\lambda_k\}_{k \in \mathbb{Z}^*}$  be introduced in the hypotheses. We decompose  $\{\lambda_k\}_{k \in \mathbb{N}}$  in  $\mathcal{M}$  sequences  $\{\lambda_k^j\}_{k \in \mathbb{N}}$  with  $j \leq \mathcal{M}$  such that

$$\inf_{k \neq l} |\lambda_k^j - \lambda_l^j| > \delta \mathcal{M}, \quad \forall j \leq \mathcal{M}.$$

Now, for every  $j \leq \mathcal{M}$ , we apply the point **1**) with  $\{\omega_k\}_{k \in \mathbb{N}} = \{\lambda_k^j\}_{k \in \mathbb{N}}$ . For every  $T > 2\pi/\delta \mathcal{M}$  and  $g \in L^2$ , there exists  $C(T) > 0$  uniformly bounded for  $T$  in bounded intervals such that

$$\left\| \{ \langle e^{i\lambda_k(\cdot)}, g \rangle_{L^2} \}_{k \in \mathbb{N}} \right\|_{\ell^2} \leq \sum_{j=1}^{\mathcal{M}} \left\| \{ \langle e^{i\lambda_k^j(\cdot)}, g \rangle_{L^2} \}_{k \in \mathbb{N}} \right\|_{\ell^2} \leq \mathcal{M} C(T) \|g\|_{L^2}.$$

Thus,  $\left\| \int_0^T e^{i\lambda(\cdot)\tau} \bar{g}(\tau) dt \right\|_{\ell^2} \leq C(T) \|\bar{g}\|_{L^2}$  and, for every  $g \in L^2$ , we obtain

$$\left\| \int_0^T e^{i\lambda(\cdot)\tau} g(\tau) dt \right\|_{\ell^2} \leq \mathcal{M} C(T) \|g\|_{L^2}.$$

In conclusion, for  $T > 2\pi/\delta \mathcal{M}$ , we choose the smallest value possible for  $C(T)$ . When  $T \leq 2\pi/\delta \mathcal{M}$ , for  $g \in L^2$ , we define  $\tilde{g} \in L^2((0, 2\pi/\delta \mathcal{M} + 1), \mathbb{C})$  such that  $\tilde{g} = g$  on  $(0, T)$  and  $\tilde{g} = 0$  in  $(T, 2\pi/\delta \mathcal{M} + 1)$ . Then

$$\left\| \int_0^T e^{i\lambda(\cdot)\tau} g(\tau) dt \right\|_{\ell^2} = \left\| \int_0^{2\pi/\delta \mathcal{M} + 1} e^{i\lambda(\cdot)\tau} \tilde{g}(\tau) dt \right\|_{\ell^2} \leq \mathcal{M} C(2\pi/\delta \mathcal{M} + 1) \|g\|_{L^2}.$$

Let  $0 < T_1 < T_2 < +\infty$ ,  $g \in L^2(0, T_1)$  and  $\tilde{g} \in L^2(0, T_2)$  be defined as  $\tilde{g} = g$  on  $(0, T_1)$  and  $\tilde{g} = 0$  on  $(T_1, T_2)$ . We apply the last inequality to  $\tilde{g}$  that leads to  $C(T_1) \leq C(T_2)$ .  $\square$

## C Analytic perturbation

The aim of the appendix is to adapt the perturbation theory techniques provided in [Duch, *Appendix B*], where the (BSE) is considered on  $\mathcal{G} = (0, 1)$  and  $A$  is the Dirichlet Laplacian. As in the mentioned appendix, we decompose  $u(t) = u_0 + u_1(t)$ , for  $u_0$  and  $u_1(t)$  real. Let  $A + u(t)B = A + u_0B + u_1(t)B$ . We consider  $u_0B$  as a perturbative term of  $A$ .

Let  $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$  be the spectrum of  $A + u_0B$  corresponding to some eigenfunctions  $\{\phi_j^{u_0}\}_{j \in \mathbb{N}}$ . We refer to the definition of the equivalence classes  $\{E_m\}_{m \in \mathbb{Z}^*}$  provided in the first part of Appendix B.

We denote as  $n : \mathbb{N} \rightarrow \mathbb{N}$  the application mapping  $j \in \mathbb{N}$  in the value  $n(j)$  such that  $j \in E_{n(j)}$ , while  $s : \mathbb{N} \rightarrow \mathbb{N}$  is such that  $\lambda_{s(j)} = \inf\{\lambda_k > \lambda_j \mid k \notin E_{n(j)}\}$ . Moreover,  $p : \mathbb{N} \rightarrow \mathbb{N}$  is such that  $\lambda_{p(j)} = \sup\{\lambda_k \in E_{n(j)}\}$ .

The next lemma follows from the techniques adopted in the proofs of [Duch, *Lemma B.2*] and [Duch, *Lemma B.3*].

**Lemma C.1.** *Let the hypotheses of Theorem 2.3 be satisfied. Let  $j \in \mathbb{N}$  and  $P_j^\perp$  be the projector onto  $\overline{\text{span}\{\phi_m : m \notin E_{n(j)}\}}^{L^2}$ . There exists a neighborhood  $U(0)$  small enough of  $u = 0$  in  $\mathbb{R}$  such that there exists  $c > 0$  such that, for every  $u_0 \in U(0)$  and  $k \in \mathbb{N}$ ,*

$$\| (A + u_0 B - \nu_k)^{-1} \| \leq c, \quad \nu_k := (\lambda_{s(k)} - \lambda_{p(k)})/2.$$

Moreover, for  $u_0 \in U(0)$ , the operator  $(A + u_0 P_k^\perp B - \lambda_k^{u_0})$  is invertible with bounded inverse from  $D(A) \cap \text{Ran}(P_k^\perp)$  to  $\text{Ran}(P_k^\perp)$  for every  $k \in \mathbb{N}$ .

**Lemma C.2.** *Let the hypotheses of Theorem 2.3 be satisfied. There exists a neighborhood  $U(0)$  small enough of  $u = 0$  in  $\mathbb{R}$  such that, up to a countable subset  $Q$  and for every  $(k, j), (m, n) \in I$ ,  $(k, j) \neq (m, n)$  (see (5)), we have*

$$\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} \neq 0, \quad \langle \phi_k^{u_0}, B \phi_j^{u_0} \rangle \neq 0, \quad \forall u_0 \in U(0) \setminus Q.$$

*Proof.* For  $k \in \mathbb{N}$ , we decompose the perturbed eigenfunction as follows

$$(35) \quad \phi_k^{u_0} = a_k \phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k \phi_j + \eta_k,$$

where  $a_k \in \mathbb{C}$ ,  $\{\beta_j^k\} \subset \mathbb{C}$  and  $\eta_k$  is orthogonal to  $\phi_l$  for every  $l \in E_{n(k)}$ . Moreover,  $\lim_{|u_0| \rightarrow 0} |a_k| = 1$  and  $\lim_{|u_0| \rightarrow 0} |\beta_j^k| = 0$  for every  $j, k \in \mathbb{N}$  and

$$\begin{aligned} \lambda_k^{u_0} \phi_k^{u_0} &= (A + u_0 B)(a_k \phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k \phi_j + \eta_k) = A a_k \phi_k \\ &+ \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k A \phi_j + A \eta_k + u_0 B a_k \phi_k + u_0 \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k B \phi_j + u_0 B \eta_k. \end{aligned}$$

Now, Lemma C.1 leads to the existence of  $C_1 > 0$  such that, for every  $k \in \mathbb{N}$ ,

$$(36) \quad \eta_k = - \left( (A + u_0 P_k^\perp B - \lambda_k^{u_0}) P_k^\perp \right)^{-1} u_0 \left( a_k P_k^\perp B \phi_k + \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k P_k^\perp B \phi_j \right)$$

and  $\|\eta_k\| \leq C_1 |u_0|$ . We compute  $\lambda_k^{u_0} = \langle \phi_k^{u_0}, (A + u_0 B) \phi_k^{u_0} \rangle$  and

$$\begin{aligned} \lambda_k^{u_0} &= |a_k|^2 \lambda_k + \langle \eta_k, (A + u_0 B) \eta_k \rangle + \sum_{j \in E_{n(k)} \setminus \{k\}} \lambda_j |\beta_j^k|^2 + u_0 \sum_{j \in E_{n(k)} \setminus \{k\}} |\beta_j^k|^2 B_{k,k} \\ &+ u_0 \sum_{j, l \in E_{n(k)} \setminus \{k\}, j \neq l} \overline{\beta_j^k} \beta_l^k B_{j,l} + u_0 \sum_{j \in E_{n(k)} \setminus \{k\}} |\beta_j^k|^2 (B_{j,j} - B_{k,k}) + u_0 |a_k|^2 B_{k,k} \\ &+ 2u_0 \Re \left( \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k \langle \eta_k, B \phi_j \rangle + \overline{a_k} \sum_{j \in E_{n(k)} \setminus \{k\}} \beta_j^k B_{k,j} + \overline{a_k} \langle \phi_k, B \eta_k \rangle \right). \end{aligned}$$

Thanks to (36), it follows  $\langle \eta_k, (A + u_0 B) \eta_k \rangle = \lambda_k^{u_0} \|\eta_k\|^2 + O(u_0^2)$  and there exists  $f_k$  such that  $\lim_{|u_0| \rightarrow 0} f_k = 0$  uniformly in  $k$  and

$$\begin{aligned} \lambda_k^{u_0} &= (1 - \|\eta_k\|^2)^{-1} \left( |a_k|^2 + \sum_{j \in E_{n(k)} \setminus \{k\}} \lambda_j / \lambda_k |\beta_j^k|^2 \right) \lambda_k \\ &+ u_0 (1 - \|\eta_k\|^2)^{-1} \left( |a_k|^2 + \sum_{j \in E_{n(k)} \setminus \{k\}} |\beta_j^k|^2 \right) B_{k,k} + u_0 f'_k + O(u_0^2). \end{aligned}$$

For  $\hat{a}_k := (1 - \|\eta_k\|^2)^{-1} (|a_k|^2 + \sum_{j \in E_{n(k)} \setminus \{k\}} |\beta_j^k|^2)$ , as  $\|\eta_k\| \leq C_1 |u_0|$ , it follows  $\lim_{|u_0| \rightarrow 0} |\hat{a}_k| = 1$  uniformly in  $k$ . From [DZ06, Proposition 6.2; 5)], we have  $\lim_{n \rightarrow +\infty} \lambda_n / n^2 = \pi^2 / (\sum_{l=1}^N L_l)^2$  and, thanks to  $\sup_{j \in \mathbb{N}} |E_j| < \mathcal{M}$ ,

$$\lim_{k \rightarrow +\infty} \inf_{j \in E_{n(k)} \setminus \{k\}} \lambda_j \lambda_k^{-1} = \lim_{k \rightarrow +\infty} \sup_{j \in E_{n(k)} \setminus \{k\}} \lambda_j \lambda_k^{-1} = 1.$$

For  $\tilde{a}_k := (1 - \|\eta_k\|^2)^{-1} (|a_k|^2 + \sum_{j \in E_{n(k)} \setminus \{k\}} \lambda_j / \lambda_k |\beta_j^k|^2)$ , we have  $\lim_{|u_0| \rightarrow 0} |\tilde{a}_k| = 1$  uniformly in  $k$  and

$$(37) \quad \lambda_k^{u_0} = \tilde{a}_k \lambda_k + u_0 \hat{a}_k B_{k,k} + u_0 f'_k + O(u_0^2).$$

When  $\lambda_k = 0$ , the result is still valid. For each  $(k, j), (m, n) \in I$  such that  $(k, j) \neq (m, n)$ , there exists  $f_{k,j,m,n}$  such that  $\lim_{|u_0| \rightarrow 0} f_{k,j,m,n} = 0$  uniformly in  $k, j, m, n$  and

$$\begin{aligned} \lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} &= \tilde{a}_k \lambda_k - \tilde{a}_j \lambda_j - \tilde{a}_m \lambda_m + \tilde{a}_n \lambda_n + u_0 f_{k,j,m,n} \\ &+ u_0 (\hat{a}_k B_{k,k} - \hat{a}_j B_{j,j} - \hat{a}_m B_{m,m} + \hat{a}_n B_{n,n}) = \tilde{a}_k \lambda_k - \tilde{a}_j \lambda_j \\ &- \tilde{a}_m \lambda_m + \tilde{a}_n \lambda_n + u_0 (\hat{a}_k B_{k,k} - \hat{a}_j B_{j,j} - \hat{a}_m B_{m,m} + \hat{a}_n B_{n,n}) + O(u_0^2). \end{aligned}$$

Thanks to the third point of Assumptions I, there exists  $U(0)$  a neighborhood of  $u = 0$  in  $\mathbb{R}$  small enough such that, for each  $u \in U(0)$ , we have that every function  $\lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u$  is not constant and analytic. Now,  $V_{(k,j,m,n)} = \{u \in D \mid \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$  is a discrete subset of  $D$  and

$$V = \{u \in D \mid \exists ((k, j), (m, n)) \in I^2 : \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$

is a countable subset of  $D$ , which achieves the proof of the first claim. The second relation is proved with the same technique by considering that, for every  $j, k \in \mathbb{N}$ , the analytic functions  $u_0 \rightarrow \langle \phi_j^{u_0}, B \phi_k^{u_0} \rangle$  can not be constantly zero since  $\langle \phi_j, B \phi_k \rangle \neq 0$  and  $W = \{u \in D \mid \exists (k, j) \in I : \langle \phi_j^u, B \phi_k^u \rangle = 0\}$  is a countable subset of  $D$ .  $\square$

**Lemma C.3.** *Let the hypotheses of Theorem 2.3 be satisfied. Let  $T > 0$  and  $s = d + 2$  for  $d$  introduced in Assumptions II. Let  $c \in \mathbb{R}$  such that  $0 \notin \sigma(A + u_0 B + c)$  (the spectrum of  $A + u_0 B + c$ ) and such that  $A + u_0 B + c$*

is a positive operator. There exists a neighborhood  $U(0)$  of 0 in  $\mathbb{R}$  such that, for any  $u_0 \in U(0)$ ,

$$\left\| |A + u_0B + c|^{\frac{s}{2}} \cdot \right\| \asymp \|\cdot\|_{(s)}.$$

*Proof.* Let  $D$  be the neighborhood provided by Lemma C.2. We define a neighborhood  $U(0) \subseteq D$  such that the claim is achieved. The proof follows the one of [Ducb, Lemma B.6]. We suppose that  $0 \notin \sigma(A + u_0B)$  and  $A + u_0B$  is positive such that we can assume  $c = 0$ . If  $c \neq 0$ , then the proof follows from the same arguments.

Thanks to Remark 2.1, we have  $\|\cdot\|_{(s)} \asymp \| |A|^{\frac{s}{2}} \cdot \|$ . We prove the existence of  $C_1, C_2, C_3 > 0$  such that, for every  $\psi \in D(|A + u_0B|^{\frac{s}{2}}) = D(|A|^{\frac{s}{2}})$ ,

(38)

$$\| |A + u_0B|^{\frac{s}{2}} \psi \| = \|(A + u_0B)^{\frac{s}{2}} \psi \| \leq C_1 \| |A|^{\frac{s}{2}} \psi \| + C_2 \|\psi\| \leq C_3 \| |A|^{\frac{s}{2}} \psi \|.$$

Let  $s/2 = k \in \mathbb{N}$ . The relation (38) is proved by iterative argument. First, it is true for  $k = 1$  and  $k = 2$  since if  $B \in L(D(A^{k_1}))$  for  $1 \leq k_1 \leq 2$ , then there exists  $C > 0$  such that  $\|AB\psi\| \leq C \| \|B\|_{D(A^{k_1})} \| |A^{k_1} \psi \|$  for  $\psi \in D(A)$ . As  $B \in L(\mathcal{H})$ , there exist  $C_4, C_5 > 0$  such that, for  $\psi \in D(A^2)$ ,

$$\begin{aligned} \|(A + u_0B)^2 \psi \| &\leq \|A^2 \psi \| + |u_0|^2 \|B^2 \psi \| + |u_0| \|AB\psi \| + |u_0| \|BA\psi \| \\ &\leq \|A^2 \psi \| + |u_0|^2 \| \|B\|^2 \|\psi\| + C_4 |u_0| \| \|B\|_{L(D(A^{k_1}))} \|\psi\|_{(k_1)} + |u_0| \| \|B\| \|\psi\|_{(2)} \end{aligned}$$

and  $\|(A + u_0B)^2 \psi \| \leq C_5 \|A^2 \psi \|$ . Second, we assume (38) be valid for  $k \in \mathbb{N}$  when  $B \in L(D(A^{k_j}))$  for  $k - j - 1 \leq k_j \leq k - j$  and for every  $j \in \{0, \dots, k - 1\}$ . We prove (38) for  $k + 1$  when  $B \in L(D(A^{k_j}))$  for  $k - j \leq k_j \leq k - j + 1$  and for every  $j \in \{0, \dots, k\}$ . There exists  $C > 0$  such that  $\|A^k B \psi \| \leq C \| \|B\|_{D(A^{k_0})} \| |A^{k_0} \psi \|$  for every  $\psi \in D(A^{k+1})$ . Thus, as  $\|(A + u_0B)^{k+1} \psi \| = \|(A + u_0B)^k (A + u_0B) \psi \|$ , there exist  $C_6, C_7 > 0$  such that, for every  $\psi \in D(A^{k+1})$ ,

$$\|(A + u_0B)^{k+1} \psi \| \leq C_6 (\| |A^{k+1} \psi \| + |u_0| \| |A^k B \psi \| + \|A\psi \| + |u_0| \|B\psi \|) \leq C_7 \| |A^{k+1} \psi \|.$$

As in the proof of [Ducb, Lemma B.6], the relation (38) is valid for any  $s \leq k$  when  $B \in L(D(A^{k_0}))$  for  $k - 1 \leq k_0 \leq s$  and  $B \in L(D(A^{k_j}))$  for  $k - j - 1 \leq k_j \leq k - j$  and for every  $j \in \{1, \dots, k - 1\}$ . The opposite inequality follows by decomposing  $A = A + u_0B - u_0B$ .

In our framework, Assumptions II ensure that the parameter  $s$  is  $2 + d$ . If the third point of Assumptions II is verified for  $s \in [4, 11/2)$ , then  $B$  preserves  $H_{\mathcal{NK}}^{d_1}$  and  $H_{\mathcal{G}}^2$  for  $d_1$  introduced in Assumptions II. Proposition 3.2 claims that  $B : H_{\mathcal{G}}^{d_1} \rightarrow H_{\mathcal{G}}^{d_1}$  and the argument of [Ducb, Remark 1.1] implies  $B \in L(H_{\mathcal{G}}^{d_1})$ .

If the second or the fourth point of Assumptions II is verified for  $s \in [4, 9/2)$ , then  $B \in L(\mathcal{H})$ ,  $B \in L(H_{\mathcal{G}}^2)$  and  $B \in L(H_{\mathcal{G}}^{d_1})$  for  $d_1 \in [d, 9, 2)$  since  $B$  stabilizes  $H^{d_1}$  and  $H_{\mathcal{G}}^2$  for  $d_1$  introduced in Assumptions II. Thanks to Proposition 3.2,  $B : H_{\mathcal{G}}^{d_1} \rightarrow H_{\mathcal{G}}^{d_1}$  and the argument of [Ducb, Remark 1.1] implies  $B \in L(H_{\mathcal{G}}^{d_1})$ .

If  $s < 4$  instead, then the conditions  $B \in L(\mathcal{H})$  and  $B \in L(H_{\mathcal{G}}^2)$  are sufficient (see [Ducb, Remark 1.1]).  $\square$

**Remark C.4.** *The techniques developed in the proof of Lemma C.3 allow to prove the following claim. Let the hypotheses of Theorem 2.3 be satisfied and  $0 < s_1 < d + 2$  for  $d$  introduced in Assumptions II. Let  $c \in \mathbb{R}$  such that  $0 \notin \sigma(A + u_0B + c)$  and such that  $A + u_0B + c$  is a positive operator. We have There exists a neighborhood  $U(0) \subset \mathbb{R}$  of 0 so that, for any  $u_0 \in U(0)$ , we have  $\| |A + u_0B + c|^{\frac{s_1}{2}} \cdot \| \asymp \| \cdot \|_{(s_1)}$ .*

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