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Abstract

The present article is concerned with global subelliptic estimates for Kramers-Fokker-Planck operators with polynomials of degree less than or equal to two. The constants appearing in those estimates are accurately formulated in terms of the coefficients, especially when those are large.

Contents

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1 Introduction and main results

In this work, we consider the Kramers-Fokker-Planck operator given by

$$K_V = p.\partial_q - \partial_q V(q).\partial_p + \frac{1}{2}(-\Delta_p + p^2), \quad (q,p) \in \mathbb{R}^{2d}\,, \tag{1.1}$$

where q denotes the space variable, p denotes the velocity variable, $x.y = \sum_{j=1}^{d} x_j y_j$, $x^2 = \sum_{j=1}^{d} x_j^2$ and the potential $V(q) = \sum_{|\alpha|\leq 2} V_\alpha q^\alpha$ is a real-valued polynomial function on $\mathbb{R}^d$ with $d^\circ V = 2$. After making an orthogonal change of variables one may assume that its Hessian matrix is

$$Hess\, V = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_d \end{pmatrix} .$$

The constant term V_0 does not appear in K_V and can be set to 0 and we distinguish two cases:

- If $Hess\, V$ is non-degenerate, a translation in q reduces the problem to

$$V(q) = \sum_{i=1}^{d} \frac{\nu_i}{2} q_i^2\,. \tag{1.2}$$

- If $Hess\, V$ is degenerate, a good choice of orthonormal basis gives:

$$V(q) = \lambda_1 q_1 + \sum_{i=2}^{d} \frac{\nu_i}{2} q_i^2\,, \tag{1.3}$$

where λ_1 is invariantly defined as $\min_{q\in\mathbb{R}^d} |\nabla\, V(q)| \geq 0$.

As established in [HeNi], the non-selfadjoint operator K_V is maximal accretive when endowed with the domain $D(K_V) = \big\{u \in L^2(\mathbb{R}^{2d}),\quad K_V u \in L^2(\mathbb{R}^{2d})\big\}$. The question about the compactness of the resolvent combined with subelliptic estimates is intimately related with the return to the equilibrium or exponential decay estimates. As pointed out in [HerNi] and [HeNi], the analysis of K_V is also strongly related to the one of the Witten Laplacian $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$ for which maximal hypoelliptic techniques developed by Helffer and Nourrigat in [HeNo] provide accurate criteria for general polynomial potentials $V(q)$.

Within this maximal hypoelliptic analysis of $\Delta_V^{(0)}$ there is a recurrent interplay between qualitative estimates and quantitative estimates in terms of the size of the coefficients of the

polynomial $V(q)$. The general idea is that the study of the operator $\Delta_V^{(0)}$ as $q \to \infty$ when V is a degree r polynomial, is reduced to a quantitative version of subelliptic estimates for $\Delta_{\tau\tilde{V}}^{(0)}$, where $\tilde{V}$ belongs to some family of polynomials related to V with degree less than r, and τ is a large parameter.

"Quantitative estimates" means that we consider subelliptic estimates with a good and optimal control of the constant with respect to the parameter τ. Remember also that the compactness of the resolvent on $\Delta_V^{(0)}$ obtained by maximal hypoelliptic techniques, relies on the fact that no polynomial $\tilde{V}$ of the family associated with V admits a local minimum. It shows in particular that the compactness of the resolvent of $\Delta_{+V}^{(0)}$ and $\Delta_{-V}^{(0)}$ differ and the first non trivial example comes with the potential $\pm V(q_1, q_2) = \pm q_1^2 q_2^2$ in $\mathbb{R}^2$. For the Kramers-Fokker-Planck operator K_V, no sufficient condition until the recent work by W.X. Li [Li2] exhibited such a different behavior.

We hope to develop the same strategy for the non self-adjoint operator K_V as for the Witten Laplacian $\Delta_V^{(0)}$, namely try to get the optimal subelliptic estimates for some class of polynomial functions $V(q)$, by making use of quantitative estimates for some lower degree polynomials. The case $d^\circ V \leq 2$ for which the Weyl symbol of K_V is a polynomial of degree ≤ 2 in the variable (q, p, ξ_q, ξ_p) allows a lot of exact analytic calcultations and was already deeply studied in [Hor][Sjo][HiPr][Vio][Vio2][AlVi]. Nevertheless exploiting those exact analytic expressions for the semigroup kernel or symbol (Mehler's type formulas) or for the spectrum does not solve completely the question of optimal quantitative subelliptic estimates for the non self-adjoint operator K_V. The semiclassical regime which can be handled quite accurately via symbolic calculus gives results after rescaling essentially when the transport part $p.\partial_q - \partial_q V(q).\partial_p$ is small compared to the diffusive–friction part $\frac{-\Delta_p + p^2}{2}$.

Actually, we are mainly interested in the other regime where the Hamiltonian dynamics is stronger than the diffusive and friction part. The difficulty then appears clearly, because understanding the operator K_V requires the understanding of the Hamiltonian dynamics associated with $p.\partial_q - \partial_q V(q).\partial_p$ which, for a general polynomial V exhibits a rich variety of phenomena, and which, for a polynomial of degree ≤ 2, already contains the three types of dynamics: a) elliptic (bounded trajectories when V is a positive definite quadratic form); b) hyperbolic (trajectories escaping exponentially quickly in time to infinity when V is a negative definite quadratic form); and c) parabolic (trajectories escaping polynomially quickly in time to infinity when V is linear).

At a more fundamental level, understanding the operator K_V when the transport term is dominant also proceeds in the same direction as Bismut's program: in [Bis1], Bismut introduced his hypoelliptic Laplacian in order to interpolate Morse theory (in the high diffusion-friction regime via the Witten Laplacian) and the topology of loop spaces (dominant transport term). The difficult part with a dominant transport term was understood only for the geodesic flow on symmetric spaces making use of the specific algebraic structure in [Bis2].

With this respect our simpler case also requires a better understanding of the underlying algebra, and it appeared that after using the general FBI-techniques the Kramers-Fokker-Planck evolution with quadratic potentials, even in dimension $d = 1$, is reduced to some linear dynamics on $\mathbb{C}^4$ which are easily computed after elucidating some quaternionic structure. In this specific case, this also completes the unfruitful attempts in [HeNi], Section 9.1, to

exhibit some useful nilpotent Lie algebra structure for Kramers-Fokker-Planck operators. Actually, quaternions and Pauli matrices are related to the $\mathfrak{su}(2)$ Lie algebra, so the Lie algebra structure decomposition useful to the analysis of Kramers-Fokker-Planck operators with polynomial potentials is certainly not nilpotent.

Denoting

$$O_p = \frac{1}{2}(D_p^2 + p^2)$$

and

$$X_V = p.\partial_q - \partial_q V(q).\partial_p,$$

we can rewrite the Kramers-Fokker-Planck operator K_V defined in (1.2) as $K_V = X_V + O_p$. In this work, we are mainly based on recent publications by Hitrik, Pravda-Starov, Viola, and Aleman [AlVi], [Vio2], and [HPV2] which deal with operators having polynomial symbols of degree less than or equal to two.

Notations:

$$\begin{aligned}
\mathrm{Tr}_+ &= \sum_{\nu_i > 0} \nu_i \,, \\
\mathrm{Tr}_- &= -\sum_{\nu_i \leq 0} \nu_i \,, \\
A &= \max\{(1+\mathrm{Tr}_+)^{2/3}, 1+\mathrm{Tr}_-\} \\
B &= \max\{\lambda_1^{4/3}, \frac{1+\mathrm{Tr}_-}{\log(2+\mathrm{Tr}_-)^2}\},
\end{aligned}$$

The main goal of this work is the following subelliptic estimates.

Theorem 1.1. *Let $V(q)$ be a potential as in (1.2) or (1.3). Then there exists a constant $c > 0$ that does not depend on V such that the subelliptic estimate with a remainder term*

$$||K_V u||^2_{L^2(\mathbb{R}^{2d})} + A||u||^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(||O_p u||^2_{L^2(\mathbb{R}^{2d})} + ||X_V u||^2_{L^2(\mathbb{R}^{2d})}\Big) \tag{1.4}$$

holds for all $u \in D(K_V)$.

Theorem 1.2. *Let $V(q)$ as in (1.2) or (1.3). Then there is a constant $c > 0$ independent of the polynomial V so that the subelliptic estimate without a remainder*

$$||K_V u||^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1+\frac{A}{B}}\Big(||O_p u||^2_{L^2(\mathbb{R}^{2d})} + ||X_V u||^2_{L^2(\mathbb{R}^{2d})}\Big)$$

holds for all $u \in D(K_V)$, under the condition $Tr_- + \lambda_1 \neq 0$.

Remark 1.2.1. *In view of the comparison with compactness criteria for Witten Laplacians with polynomial potentials (see [HeNi]), note that the condition $Tr_- + \lambda_1 \neq 0$ imposed in Theorem 1.2 is equivalent to the fact that the potential V does not have a local minimum.*

The two previous Theorems are both consequences of the following result.

Proposition 1.3. *There exists a constant $c > 0$ such that*

$$\sum_{i=1}^{d} \| |D_{q_i}| e^{-t(K_V+\sqrt{A})} \|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} + \| |\partial_{q_i} V(q_i)| e^{-t(K_V+\sqrt{A})} \|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \frac{c}{t^{\frac{3}{2}}}$$

for all $t > 0$.

Moreover, if $Tr_- + \lambda_1 \neq 0$,

$$\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \int_0^{+\infty} ||e^{-tK_V}||_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} dt \leq \frac{c}{\sqrt{B}}$$

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2 Reduction to a one-dimensional problem

Interpolation results of Lunardi (see [Lun]) show that the first inequality of Proposition 1.3 combined with the fact that

$$|\mathbb{R}\mathrm{e}\langle [O_p, X_V]u, u\rangle| \leq C_\epsilon(|| \, |D_q|^{\frac{2}{3}} u||^2 + || \, |\partial_q V(q)|^{\frac{2}{3}} u||^2) + \epsilon ||O_p u||^2$$

for all $u \in D(K_V)$ (where $\epsilon > 0$ is small enough), implies the subelliptic estimates given in Theorem 1.1. Theorem 1.2 is then a consequence of Theorem 1.1 and the second inequality of Proposition 1.3.
Details are given below.

Proof of Proposition 1.3. Since this result is expressed in terms of the semigroup, it can be studied by a separation of variables for a potential of the form (1.2) or (1.3). Actually e^{-tK_V} is a commutative product of contraction semigroups, and it suffices to write

$$\sum_{i=1}^{d} ||M_i e^{-t(K_V+\sqrt{A})}||_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \sum_{i=1}^{d} ||M_i e^{-t(K_{V(q_i)}+\alpha_i)}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \quad M_i = |D_{q_i}| \text{ or } M_i = |\partial_{q_i} V(q)| \,,$$

where $V(q_i)$ denotes the one-dimensional potential in the q_i variable, with $V(q_1) = \frac{\nu_1 q_1^2}{2}$ or $V(q_1) = \lambda_1 q_1$, $V(q_i) = \frac{\nu_i q_i^2}{2}$ for $i \geq 2$, $\alpha_i = |\nu_i|^{1/2}$ if $\nu_i < 0$, $\alpha_i = \nu_i^{1/3}$ if $\nu_i > 0$ and $\alpha_i = 0$ if $\partial_{q_i}^2 V = 0$. The second estimate of Proposition 1.3 is even simpler. Hence Proposition 1.3 will be the result of a careful analysis of the three one-dimensional potentials $V(q) = \pm\frac{\nu q^2}{2}$, $\nu > 0$, and $V(q) = \lambda_1 q$, $\lambda_1 \in \mathbb{R}$, developed in the next sections. □

Proof of Theorem 1.1. In this proof we use nearly the same notations as in [Lun] (Corollary 5.13 and Remark 5.11). Set

$$\begin{aligned} T(t) &= e^{-t(\sqrt{A}+K_V)}, \\ L^2 &= L^2(\mathbb{R}^{2d}), \\ E &= \{u \in L^2(\mathbb{R}^{2d}), qu, \partial_q u \in L^2(\mathbb{R}^{2d})\} \end{aligned}$$

where E is equipped with the norm

$$||u||_E^2 = \sum_{i=1}^{d} |||D_{q_i}|u||_{L^2(\mathbb{R}^{2d})}^2 + |||\partial_{q_i} V(q_i)|u||_{L^2(\mathbb{R}^{2d})}^2 + ||u||_{L^2(\mathbb{R}^{2d})}^2 \ .$$

Applying Lemma 3.4 and Proposition 3.1, we obtain by separation of variables

$$||T(t)||_{\mathcal{L}(L^2,E)} \leq \frac{c}{t^{\frac{3}{2}}} \quad \text{for all} \quad t > 0 \ .$$

If $m = 3$ and $\beta = \frac{1}{2}$, then one has the following embedding of real interpolation spaces

$$\Big(L^2, D\Big((\sqrt{A} + K_V)^3\Big)\Big)_{\frac{\theta}{2},p} \subset \Big(L^2, E\Big)_{\theta,p}$$

for all $\theta \in (0,1),\ p \in [1, +\infty]$. In particular for $\theta = \frac{2}{3}$,

$$[L^2, E]_{\frac{2}{3}} = (L^2, E)_{\frac{2}{3},2} = \{u \in L^2,\ |D_{q_i}|^{\frac{2}{3}} u \in L^2,\ |\partial_{q_i} V(q_i)|^{\frac{2}{3}} u \in L^2 \ \text{ for all } 1 \leq i \leq d\} \ ,$$

where the complex interpolation space $[L^2, E]_{\frac{2}{3}}$ is equipped with the norm

$$||u||_{[L^2,E]_{\frac{2}{3}}} = \sum_{i=1}^{d} \Big(|| \ |D_{q_i}|^{\frac{2}{3}} u||_{L^2(\mathbb{R}^{2d})}^2 + || \ |\partial_{q_i} V(q_i)|^{\frac{2}{3}} u||_{L^2(\mathbb{R}^{2d})}^2\Big) + ||u||_{L^2(\mathbb{R}^{2d})}^2 \ .$$

Moreover, $\Big(L^2, D\Big((\sqrt{A}+K_V)^3\Big)\Big)_{\frac{1}{3},2} = D(\sqrt{A}+K_V)$ (since L^2 is a Hilbert space and $(\sqrt{A}+K_V)$ is a maximal accretive operator).

Thus

$$D(\sqrt{A} + K_V) \subset \{u \in L^2,\ |D_{q_i}|^{\frac{2}{3}} u \in L^2,\ |\partial_{q_i} V(q_i)|^{\frac{2}{3}} u \in L^2 \ \text{ for all } 1 \leq i \leq d\} \ .$$

Hence there exists a constant $c > 0$ such that

$$\sum_{i=1}^{d} \Big(|| \ |D_{q_i}|^{\frac{2}{3}} u||^2 + || \ |\partial_{q_i} V(q_i)|^{\frac{2}{3}} u||_{L^2}^2\Big) \leq c||(\sqrt{A} + K_V)u||_{L^2}^2 \tag{2.1}$$

holds for all $u \in D(K_V)$.

Write for $u \in D(K_V)$,

$$||(\sqrt{A}+K_V)u||_{L^2}^2 = ||(\sqrt{A}+O_p)u||_{L^2}^2 + ||X_V u||_{L^2}^2 + 2\mathbb{R}\mathrm{e}\langle [O_p, X_V]u, u\rangle \ , \tag{2.2}$$

so

$$\begin{aligned}
|2\mathbb{R}\mathrm{e}\langle [O_p, X_V]u, u\rangle| &\leq \sum_{i=1}^{d} \Big|\mathbb{R}\mathrm{e}\langle u, \Big(D_{p_i}D_{q_i} + p_i\partial_{q_i}V(q)\Big)u\rangle\Big| \\
&\leq \sum_{i=1}^{d} |\mathbb{R}\mathrm{e}\langle u, (D_{p_i}D_{q_i})u\rangle| + |\mathbb{R}\mathrm{e}\langle u, p_i\partial_{q_i}V(q)u\rangle| \\
&\leq \sum_{i=1}^{d} \langle u, |p_i||\partial_{q_i}V(q)|u\rangle + \langle u, |D_{p_i}||D_{q_i}|u\rangle \\
&\leq \sum_{i=1}^{d} \epsilon\langle u, |p_i|^4 u\rangle + c_\epsilon\langle u, |\partial_{q_i}V(q)|^{\frac{4}{3}}u\rangle + \epsilon\langle u, |D_{p_i}|^4 u\rangle + c_\epsilon\langle u, |D_{q_i}|^{\frac{4}{3}}u\rangle \qquad (2.3) \\
&\leq c\Big(\epsilon||O_p u||_{L^2}^2 + c_\epsilon||(\sqrt{A}+K_V)u||_{L^2}^2\Big) \ ,
\end{aligned}$$

where (2.3) is due to the Young inequality $ts \leq \frac{1}{4}t^4 + \frac{3}{4}t^{\frac{3}{4}}$ for all $t, s \geq 0$ and the last line is a consequence of (2.1).

Therefore, combining the last inequality with (2.2), we obtain

$$\begin{aligned}
||(\sqrt{A}+K_V)u||_{L^2}^2 &\geq ||(\sqrt{A}+O_p)u||_{L^2}^2 + ||X_V u||_{L^2}^2 - c\Big[\epsilon||O_p u||_{L^2}^2 + c_\epsilon||(\sqrt{A}+K_V)u||_{L^2}^2\Big] \\
&\geq (1-c\epsilon)||(\sqrt{A}+O_p)u||_{L^2}^2 + ||X_V u||_{L^2}^2 - cc_\epsilon||(\sqrt{A}+K_V)u||_{L^2}^2
\end{aligned}$$

for all $u \in D(K_V)$. To complete the proof, it is enough to use the fact that

$$2\Big(A||u||_{L^2}^2 + ||K_V u||_{L^2}^2\Big) \geq ||(\sqrt{A}+K_V)u||_{L^2}^2$$

for all $u \in D(K_V)$. □

Proof of Theorem 1.2. If $\mathrm{Tr}_- + \lambda_1 \neq 0$, by Proposition 1.3, there exists a constant $c > 0$ such that

$$\begin{aligned}
\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} = \left\| \int_0^{+\infty} e^{-tK_V} dt \right\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} &\leq \int_0^{+\infty} ||e^{-tK_V}||_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} dt \\
&\leq \frac{c}{\sqrt{B}} \ .
\end{aligned}$$

Consequently, for all $u \in D(K_V)$,

$$||u||_{L^2(\mathbb{R}^{2d})}^2 \leq \frac{c}{B}||K_V||_{L^2(\mathbb{R}^{2d})}^2 \ .$$

Combining the above inequality, along with (1.4), one gets immediately the global subelliptic estimates

$$||K_V u||^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1+\frac{A}{B}}\Big(||O_p u||^2_{L^2(\mathbb{R}^{2d})} + ||X_V u||^2_{L^2(\mathbb{R}^{2d})}\Big)$$

for all $u \in D(K_V)$. □

3 Subelliptic estimates with remainder for non-degenerate one-dimensional potentials

The operator K_V with a potential $V(q) = \mp\frac{\nu q^2}{2} = -e^{2i\alpha}\frac{\nu q^2}{2}$ (where $\nu > 0$ is a parameter and $\alpha \in \{0, \frac{\pi}{2}\}$), is unitarily equivalent to

$$\begin{aligned} K_{\nu,\alpha} &= \frac{1}{2}(-\partial_p^2 + p^2) + \Big(e^{i\alpha}\sqrt{\nu}\Big)e^{-i\alpha}\Big(p\partial_q + e^{2i\alpha}q\partial_p\Big) \\ &= O_p + zX_\alpha \end{aligned}$$

where $z := e^{i\alpha}\sqrt{\nu}$ and $X_\alpha := i(e^{-i\alpha}pD_q + e^{i\alpha}qD_p)$. Actually introducing the possibly complex parameter z allows us to use the same computations for both cases because they involve entire functions of $z \in \mathbb{C}$. On the other hand, some identities make sense only when $\alpha \in \left\{0, \frac{\pi}{2}\right\}$, particularly those involving O_q (the harmonic oscillator in q) or the symplectic product. Below we sum up the cases to be studied:

$V(q)$	α	z
$\frac{-\nu q^2}{2}$	0	$\sqrt{\nu}$
$\frac{+\nu q^2}{2}$	$\frac{\pi}{2}$	$i\sqrt{\nu}$

In this one dimensional case, we use the following notations:

$$\begin{aligned} &O_q = \frac{1}{2}(D_q^2 + q^2) \quad , \quad O_{e^{i\alpha}q} = \frac{1}{2}(e^{-2i\alpha}D_q^2 + e^{2i\alpha}q^2) \quad , \quad O_p = \frac{1}{2}(D_p^2 + p^2)\,, \\ &X_\alpha = i(e^{-i\alpha}pD_q + e^{i\alpha}qD_p) \quad , \quad Y_\alpha = i(e^{i\alpha}pq - e^{-i\alpha}D_qD_p)\,, \end{aligned}$$

where $\alpha \in \left\{0, \frac{\pi}{2}\right\}$ and $O_{e^{i\alpha}q} = e^{2i\alpha}O_q$ in the final applications.

The Hamilton map written as a matrix equals

$$H_Q := \begin{pmatrix} \mathbf{q}''_{\xi x} & \mathbf{q}''_{\xi\xi} \\ -\mathbf{q}''_{xx} & -\mathbf{q}''_{x\xi} \end{pmatrix}\,,$$

where $\mathbf{q}(q,p,\xi_q,\xi_p)$ is the Weyl-symbol of the operator Q, meaning $Q = \mathbf{q}^w(q,p,D_q,D_p) = q^w(x,D_x)\,,\ x = (q,p)$:

$$Qu(x) = \int_{\mathbb{R}^{4d}} e^{i(x-x').\xi} q\left(\frac{x+x'}{2},\xi\right) u(x')\ \frac{d\xi}{(2\pi)^{2d}}dx'\,.$$

Noticing that O_p, O_q, $O_{e^{i\alpha}q}$, X_α, Y_α and $K_{\nu,\alpha}$ have quadratic symbols, the corresponding Hamilton maps are written accordingly H_{O_p}, H_{O_q}, $H_{O_{e^{i\alpha}q}}$, H_{X_α}, H_{Y_α} and $H_{K_{\nu,\alpha}}$. Let $E := H_{O_{e^{i\alpha}q}-O_p}$, $I := H_{-O_{e^{i\alpha}q}-O_p}$, $J := H_{-X_\alpha}$, and $K := H_{Y_\alpha}$ denote respectively the Hamiltonian matrices associated to the operators $O_{e^{i\alpha}q} - O_p$, $-O_{e^{i\alpha}q} - O_p$, $-X_\alpha$ and Y_α. Then one has

$$E = \begin{pmatrix} 0 & 0 & e^{-2i\alpha} & 0 \\ 0 & 0 & 0 & -1 \\ -e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & -e^{-2i\alpha} & 0 \\ 0 & 0 & 0 & -1 \\ e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -ie^{-i\alpha} & 0 & 0 \\ -ie^{i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{i\alpha} \\ 0 & 0 & ie^{-i\alpha} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & -ie^{-i\alpha} \\ 0 & 0 & -ie^{-i\alpha} & 0 \\ 0 & -ie^{i\alpha} & 0 & 0 \\ -ie^{i\alpha} & 0 & 0 & 0 \end{pmatrix}.$$

Note that E commutes with I, J, K and $IJ = K$ with the relations

$$E^2 = I^2 = J^2 = K^2 = -1 \text{ for all } \alpha \in \mathbb{R}\,, \tag{3.1}$$

$$\text{and} \quad \overline{E} = E,\ \overline{I} = I,\ \overline{J} = -e^{2i\alpha}J, \overline{K} = -e^{2i\alpha}K \quad \text{when } \alpha \in \{0, \pi/2\}\,. \tag{3.2}$$

These relations, $IJ = K$, and (3.1) ensure that $(1, I, J, K)$ can be considered algebraically as a basis of (bi-)quaternions. Note in particular that

$$H_{O_p} = -\frac{1}{2}(E + I) \quad , \quad H_{X_\alpha} = -J$$

$$H_{Y_\alpha} = K \quad , \quad H_{K_{\nu,\alpha}} = -\frac{1}{2}(E + I + 2zJ)$$

for all $\alpha \in \mathbb{R}$, while the relations

$$\begin{pmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^2} \\ \mathrm{Id}_{\mathbb{R}^2} & 0 \end{pmatrix} = \sin(\alpha)E + \cos(\alpha)I \quad , \quad H_{O_q} = \frac{e^{2i\alpha}}{2}(E - I)$$

hold for $\alpha \in \{0, \frac{\pi}{2}\}$.

The commutation property with the matrix E can be interpreted as follows at the operator level: consider the two commutators $\left[O_p, X_\alpha\right] = iY_\alpha$ and $\left[O_{e^{i\alpha}q}, X_\alpha\right] = iY_\alpha$. Then the operator $O_{e^{i\alpha}q} - O_p$ commutes with O_p and X_α. Once this reduction is done, the quaternionic structure can be guessed as well from the operator level after computing all the commutators of O_p, $O_{e^{i\alpha}q}$,X_α and Y_α.

3.1 General estimate when $V(q) = \pm\frac{\nu q^2}{2}$, $\nu > 0$

Proposition 3.1. *Let $\nu > 0$ be a parameter and $\alpha \in \left\{0, \frac{\pi}{2}\right\}$. There exists a constant $C > 0$, independent of ν, such that*

$$||\sqrt{\nu\, O_q}\, e^{-t(K_{\nu,\alpha}+\sqrt{\nu})}||_{\mathcal{L}(L^2(\mathbb{R}^2)} \leq \frac{C}{t^{\frac{3}{2}}}$$

holds for all $t > 0$.

Lemma 3.2. *One can find a function $\delta_0(t) > 0$, specified below in (3.8)(3.9), defined in $[0, +\infty[$ such that for all $\delta(t) \in [0, \delta_0(t)[$*

$$||e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1$$

is satisfied for all $t > 0$.

Proof. The exact classical quantum correspondence, valid for $Q_j = q_j^w, j = 1, 2, 3$, when q_j are complex-valued quadratic forms with associated Hamilton maps H_{Q_j} and positive Hamilton flows $\exp H_{Q_j}$ (see[Hor][Vio]), says that

$$\exp H_{Q_1} \exp H_{Q_2} = \exp H_{Q_3} \iff e^{-iQ_1} e^{-iQ_2} = \pm e^{-iQ_3}\,.$$

We will determine conditions such that the canonical transformation

$$\exp H_{i\delta(t)O_q} \exp H_{-itK_{\nu,\alpha}}$$

is strictly positive in the sense defined in (3.5). Working from the Hamilton flow, one can therefore compute exactly ([Vio2], Proposition 4.8) a compact operator of the form e^{-iQ_2} for Q_2 quadratic such that

$$e^{-\delta(t)O_q} e^{-iQ_2} = e^{-i\delta(t)(i^{-1}O_q)} e^{-iQ_2} = \pm e^{-it(i^{-1}K_{\nu,\alpha})} = \pm e^{-tK_{\nu,\alpha}}$$

Applying this equality to the dense set of linear combinations of Hermite functions, this shows that $e^{-tK_{\nu,\alpha}}$ takes $L^2(\mathbb{R}^2)$ to the domain of $e^{\delta(t)O_q}$ with the estimate

$$\|e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}\| = \|e^{-iQ_2}\| \leq 1.$$

We will compute

$$e^{i\delta(t)H_{O_q}} e^{-itH_{K_{\nu,\alpha}}}$$

which will be done by using biquatertionic expressions. The compactness of e^{-iQ_2}, and the fact that its norm is bounded by 1, is a consequence of the positivity condition (3.5) which will be checked explicitly.

Set, for all $t \geq 0$, $\kappa(t) = e^{-itH_{K_{\nu,\alpha}}}$ and $\kappa_0(\delta) = e^{i\delta(t)H_{O_q}}$, and consider the canonical transformation

$$\kappa(t) := e^{-itH_{K_{\nu,\alpha}}} = e^{i\frac{t}{2}(E+I+2zJ)}$$

for all $t \geq 0$. Let n_1 denote

$$n_1 = \sqrt{N(I + 2zJ)} = \sqrt{1 + 4z^2} \neq 0 \text{ when } z \neq \pm\frac{i}{2}$$

such that $\hat{v} = \frac{I+2zJ}{n_1}$ satisfies $\hat{v}^2 = -1$.

Using the fact that E commutes with I and J, and the formula (A.1),

$$\begin{aligned}\kappa(t) = e^{i\frac{t}{2}E} e^{i\frac{t}{2}n_1\hat{v}} &= e^{i\frac{t}{2}E}\Big(\operatorname{ch}(\frac{tn_1}{2}) + i\,\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}(I + 2zJ)\Big) \\ &=: e^{i\frac{t}{2}E}\Big(C(t) + i\,S(t)(I + 2zJ)\Big)\,. \end{aligned} \tag{3.3}$$

The functions $\mathbb{R} \ni t \mapsto C(t)$ and $\mathbb{R} \ni t \mapsto S(t)$ do not depend on the choice of the square root $\sqrt{1+4z^2}$, because ch is an even function and sh an odd function. Moreover, they are real when $z \in \mathbb{R} \cup i\mathbb{R}$, which corresponds to $z = e^{i\alpha}\sqrt{\nu}$, $\alpha \in \{0, \frac{\pi}{2}\}$.

On the other hand,

$$\begin{aligned}\kappa_0(\delta) &= e^{-i\delta(t)H_{O_q}} = e^{\frac{i}{2}\delta(t)e^{2i\alpha}E}e^{-\frac{i}{2}\delta(t)e^{2i\alpha}I} \\ &= e^{\frac{i}{2}\delta(t)e^{2i\alpha}E}\Big(\operatorname{ch}(\frac{\delta(t)}{2}e^{2i\alpha}) - i\ \operatorname{sh}(\frac{\delta(t)}{2}e^{2i\alpha})I\Big)\ . \end{aligned} \tag{3.4}$$

When $\sigma = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ denotes the matrix of the symplectic form on $\mathbb{R}^{2\times 2}$, the equality $\sigma = \sin(\alpha)E + \cos(\alpha)I$ holds when $\alpha \in \{0, \frac{\pi}{2}\}$ (and only in those cases $\bmod \pi$). As established in [Vio2], it is possible to write $e^{\delta(t)O_q}e^{-tK_{\nu,\alpha}} = e^{-iQ_2}$ with $Q_2 = q_2^w$, with e^{-iQ_2} a compact operator, when the canonical transformation $\kappa_0\kappa$ satisfies the strict positivity condition

$$i\Big[\sigma\Big(\overline{\kappa_0\kappa z}, \kappa_0\kappa z\Big) - \sigma\Big(\overline{z}, z\Big)\Big] > 0 \text{ for all } \ z \in \mathbb{C}^4 \setminus \{0\}\ . \tag{3.5}$$

This condition is equivalent to the condition that the Hermitian matrix

$$i\Big((\kappa_0\kappa)^*\sigma\kappa_0\kappa - \sigma\Big) = i\Big(\kappa^*\kappa_0^*\sigma\kappa_0\kappa - \sigma\Big) = i\Big(\kappa^*\kappa_0\sigma\kappa_0\kappa - \sigma\Big)\ ,$$

is positive definite, or equivalently that

$$\kappa_0(i\sigma)\kappa_0 - (\kappa^*)^{-1}(i\sigma)(\kappa)^{-1} = \kappa_0(\delta)(i\sigma)\kappa_0(\delta) - \kappa^*(-t)(i\sigma)\kappa(-t)$$

is positive definite.

Since E commutes with I, J and K, the spectral decomposition of E allows us to study 2-by-2 matrices instead of 4-by-4 matrices: $T_\pm^*(iE)T_\pm = \pm\mathrm{Id}$, where

$$T_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \mp ie^{\pm 2i\alpha} & 0 \\ 0 & \pm i \end{pmatrix}, \quad T_\pm^*T_\pm = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ .$$

Letting

$$\tilde{E} := T_\pm^*ET_\pm = \mp i\mathrm{Id} = \begin{pmatrix} \mp i & 0 \\ 0 & \mp i \end{pmatrix}, \quad \tilde{I} := T_\pm^*IT_\pm = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix},$$

$$\tilde{J} := T_\pm^*JT_\pm = \begin{pmatrix} 0 & -ie^{-i\alpha} \\ -ie^{i\alpha} & 0 \end{pmatrix}, \quad \tilde{K} := T_\pm^*KT_\pm = \begin{pmatrix} 0 & \pm e^{-i\alpha} \\ \mp e^{i\alpha} & 0 \end{pmatrix}\ ,$$

we get

$$T_\pm^*\kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm = e^{\pm\delta(t)e^{2i\alpha}}\Big[\pm\sin(\alpha)c(t) - s(t)\cos(\alpha) + i\Big(\cos(\alpha)c(t) \mp \sin(\alpha)s(t)\Big)\tilde{I}\Big]\ , \tag{3.6}$$

where $c(t) = \mathrm{ch}(\delta(t)e^{2i\alpha})$ and $s(t) = \mathrm{sh}(\delta(t)e^{2i\alpha})$. Similarly,

$$\begin{aligned} T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm = e^{\mp t}\Big[&\pm \sin(\alpha)\Big(C^2(t) + (1-(2z)^2)S^2(t)\Big) - 2\cos(\alpha)C(t)S(t) \\ &+ i\cos(\alpha)\Big(C^2(t) + (1-(2z)^2)S^2(t)\Big)\tilde{I} \mp 2i\sin(\alpha)C(t)S(t)\tilde{I} \\ &+ 4zi\cos(\alpha)S^2(t)\tilde{J} \mp 4z\sin(\alpha)S^2(t)\tilde{K}\Big] \ . \end{aligned} \tag{3.7}$$

Taking into account (3.6) and (3.7),

$$\begin{aligned} T_\pm^*\kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm - T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm = &\pm e^{\pm\delta(t)e^{2i\alpha}}\Big(\sin(\alpha)c(t) \mp \cos(\alpha)s(t)\Big) \\ &+ e^{\mp t}\Big(\mp\sin(\alpha)(1+2S^2) + 2\cos(\alpha)CS\Big) \\ &+ i\Big[e^{\pm\delta(t)e^{2i\alpha}}\Big(\cos(\alpha)c(t) \mp \sin(\alpha)s(t)\Big) \\ &\quad - e^{\mp t}\Big(\cos(\alpha)(1+2S^2(t)) \mp 2\sin(\alpha)C(t)S(t)\Big)\Big]\tilde{I} \\ &- 4zie^{\mp t}\cos(\alpha)S^2(t)\tilde{J} \pm 4ze^{\mp t}\sin(\alpha)S^2(t)\tilde{K} \\ =&e^{\mp t}\Big(a + b\tilde{I} + c\tilde{J} + d\tilde{K}\Big) \ . \end{aligned}$$

The determinant of the Hermitian matrix $e^{\pm t}\Big(T_\pm^*\kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm - T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm\Big)$ is equal to

$$a^2+b^2+c^2+d^2 = 1 - e^{\pm t}(2+4S^2-e^{\pm t}) \mp e^{\pm t}(1-e^{\pm 2\delta(t)e^{2i\alpha}})\Big(2CS \mp (1+2S^2-e^{\pm t})\Big) \ .$$

Let $\delta_0(t) > 0$ be the function which cancels the determinant, or equivalently for which one has, for all $t > 0$,

$$2\Big(2S^2 - (\mathrm{ch}(t)-1)\Big) = \mp(1-e^{\pm 2\delta(t)e^{2i\alpha}})\Big(2CS + \mathrm{sh}(t) \mp \Big(2S^2 - (\mathrm{ch}(t)-1)\Big)\Big) \ .$$

After some computation, we find that this function is independent of the sign in the expression above and is given by

$$\delta_0(t) = \frac{e^{-2i\alpha}}{2}\ln\Big(1 - \frac{2A(t)}{2C(t)S(t) + \mathrm{sh}(t) + A(t)}\Big) \ , \tag{3.8}$$

$$\text{where} \qquad A(t) := \Big(2S^2(t) - (\mathrm{ch}(t)-1)\Big) . \tag{3.9}$$

We know that, when $\delta = 0$ and $\alpha \in \{0, \frac{\pi}{2}\}$, the Hamilton flow $\kappa(t)$ is positive because $e^{-tK_{\nu,0}}$ is a compact operator (see [HeNi][HiPr]). By connectedness of the set of positive definite hermitian matrices and because the result holds for $\delta(t) = 0$, the flow $\kappa_0(\delta)\kappa(t)$ is a

positive canonical transformation so long as the determinant is positive on $[0,\delta]$. Therefore $\delta(t) \in [0,\delta_0(t)[$ implies

$$||e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1 \ ,$$

because any such compact Schrödinger evolution has norm less than 1 (see [Vio2]). □

Proof of Proposition 3.1. When $0 < \epsilon_0 < 1$, there exists a constant $c > 0$ independent of ν such that

$$\delta_0(t) \geq c\nu t^3$$

holds for all $0 < t \leq t_0 := \frac{\epsilon_0}{1+|z|} = \frac{\epsilon_0}{1+\sqrt{\nu}}$. This can be seen via the expansion

$$\frac{1}{2}\ln\left(1 - \frac{2A(t)}{A(t)+2C(t)S(t)+\mathrm{sh}(t)}\right) = \frac{z^2}{12}t^3 + \mathcal{O}\left((1+|2z|^2)^2 t^5\right) \ ,$$

which is uniform with respect to the parameter ν for all $t \in]0,t_0]$.

We write the quantity $||\sqrt{\nu\ O_q} e^{-t(K_{\nu,\alpha}+\sqrt{\nu})}||_{\mathcal{L}(L^2(\mathbb{R}^2))}$ in the form

$$\begin{cases} ||\sqrt{\frac{\nu}{\delta(t)}}\sqrt{\delta(t)\ O_q}\ e^{-\delta(t)O_q} e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}} e^{-t\sqrt{\nu}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \quad \text{if} \quad 0 < t \leq t_0 \, , \\ \\ ||\sqrt{\nu} e^{-t\sqrt{\nu}}\sqrt{O_q}\ e^{-\delta(t_0)O_q} e^{\delta(t_0)O_q} e^{-t_0 K_{\nu,\alpha}} e^{-(t-t_0)K_{\nu,\alpha}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \quad \text{if} \quad t \geq t_0 \, , \end{cases}$$

where Lemma 3.2 is applied with $\delta(t) = \frac{\delta_0(t)}{2}$. For both cases we get the upper bounds

$$\begin{cases} \underbrace{\sqrt{\frac{\nu}{\delta(t)}}}_{\leq \frac{\sqrt{2}}{\sqrt{c}t^{\frac{3}{2}}}} \underbrace{||\sqrt{\delta(t)O_q} e^{-\delta(t)O_q}||}_{\leq c} \cdot \underbrace{||e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}||}_{\leq 1} \cdot \underbrace{e^{-t\sqrt{\nu}}}_{\leq 1} \quad \text{if} \quad 0 < t \leq t_0 \, , \\ \\ (1+\sqrt{\nu})^{\frac{3}{2}} e^{-t\sqrt{\nu}} \underbrace{||\frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}}}\sqrt{O_q} e^{-\delta(t_0)O_q}||}_{\leq \frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}}\sqrt{\delta(t_0)}}} \cdot \underbrace{||e^{\delta(t_0)O_q} e^{-t_0 K_{\nu,\alpha}}||}_{\leq 1} \cdot \underbrace{||e^{-(t-t_0)K_{\nu,\alpha}}||}_{\leq e^{-\frac{(t-t_0)}{2}}} \quad \text{if} \quad t \geq t_0 \, . \end{cases}$$

For the second case $t \geq t_0$, we use

$$(1+\sqrt{\nu})^{\frac{3}{2}} e^{-t(\frac{t}{2}+\sqrt{\nu})} \times \frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}}\sqrt{\delta(t_0)}} \times e^{\frac{t_0}{2}} \leq \frac{c_0}{t^{3/2}} \times \frac{\sqrt{2\nu}}{(1+\sqrt{\nu})^{3/2}\sqrt{c\nu\frac{\epsilon_0^3}{(1+\sqrt{\nu})^3}}} \times e^{\frac{\varepsilon_0}{2}} \leq \frac{c_0'}{t^{\frac{3}{2}}} \, .$$

This ends the proof of Proposition 3.1 and gives

$$||\sqrt{\nu}(|D_q| + |q|) e^{-t(K_{\nu,\alpha}+\sqrt{\nu})}||_{L^2} \leq \frac{C}{t^{\frac{3}{2}}} \tag{3.10}$$

for all $t > 0$.

□

3.2 Improved remainder, case $V(q) = \frac{\nu q^2}{2},\ \nu \gg 1$

In this section we follow the explicit methods of Aleman and Viola in [Vio][AlVi]. Following [HSV][HPV] it makes use of an FBI transform, which in this specific case is nothing but the usual Bargmann transform

$$B_2 u(z) = \frac{1}{2^{2/2}\pi^{(3\times 2)/4}} \int_{\mathbb{R}^2} e^{-\frac{(z-y)^2 - z^2/2}{2}} u(y)\, dy$$

with $B_2 : L^2(\mathbb{R}^2, dy) \to L^2(\mathbb{C}^2; e^{-\frac{|z|^2}{2}} L(dz)) \cap \mathrm{Hol}(\mathbb{C}^2)$ unitary.

Lemma 3.3. *For $\nu > \frac{1}{4}$, the adjoint operator*

$$K^*_{\nu,\frac{\pi}{2}} = \frac{1}{2}(-\partial_p^2 + p^2) - \sqrt{\nu}\Big(p\partial_q - q\partial_p\Big) = O_p - \sqrt{\nu} X_{\frac{\pi}{2}}$$

is tranformed via the Bargmann transform B_2 into

$$B_2 (K_{\nu,\frac{\pi}{2}})^* B_2^* = {}^t z M \partial_z \quad , \quad M = \begin{pmatrix} 0 & -\sqrt{\nu} \\ \sqrt{\nu} & 1 \end{pmatrix} .$$

and

$$[B_2(e^{-tK_{\nu,\frac{\pi}{2}}} u)](z) = (B_2 u)(e^{-tM} z)\,.$$

Proof. Although it may be proved by a direct computation, it is instructive as an illustration of the general method to follow the lines of [AlVi] or [Vio], Example 2.7. Remember that it is made in essentially two steps : 1) Write the operator, up to an additive constant, in the "supersymmetric" form ${}^t(D_x - A_+ x)B(D_x - A_+ x)$ after some real canonical transformation in $\mathbb{R}^{2d}$ (here $d = 2$); 2) transform the supersymmetric form into $i^t z M \zeta$ after some linear complex canonical transformation associated with an FBI-transform.
Step 1: The two variables (q, p) are gathered in the notation $x = (q, p) \in \mathbb{R}^2$, with dual variable $\xi = (\xi_q, \xi_p) \in \mathbb{R}^2$. The hamiltonian matrix associated to $K^*_{\nu,\frac{\pi}{2}}$ is given by

$$H_{K^*_{\nu,\frac{\pi}{2}}} = \begin{pmatrix} 0 & -i\sqrt{\nu} & 0 & 0 \\ i\sqrt{\nu} & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\sqrt{\nu} \\ 0 & -1 & i\sqrt{\nu} & 0 \end{pmatrix} .$$

Set $\lambda_{\epsilon_1,\epsilon_2} = \frac{\epsilon_1 i + \epsilon_2 i n_1}{2}$ the eigenvalues of $H_{K^*_{\nu,\frac{\pi}{2}}}$ with their associated eigenvectors

$${}^t X_{\epsilon_1,\epsilon_2} = \Big(1, \frac{i\lambda_{\epsilon_1,\epsilon_2}}{\sqrt{\nu}}, \frac{(\lambda_{\epsilon_1,\epsilon_2})^2 - \nu}{\lambda_{\epsilon_1,\epsilon_2}}, i\frac{(\lambda_{\epsilon_1,\epsilon_2})^2 - \nu}{\sqrt{\nu}}\Big) ,$$

where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. In the case $\alpha = \frac{\pi}{2}$, one has $n_1 = \sqrt{1 - 4\nu} = i\sqrt{4\nu - 1}$ for $\nu > \frac{1}{4}$.

As a first step we need to determine the following two spaces:

$$\Lambda_- = \bigoplus_{\mathrm{Im}\,\lambda < 0} \ker(H_{K^*_{\nu,\frac{\pi}{2}}} - \lambda I) = \left\{ \begin{pmatrix} x \\ A_- x \end{pmatrix}, \quad x \in \mathbb{C}^2 \right\}$$

and

$$\Lambda_+ = \bigoplus_{\mathrm{Im}\,\lambda>0} \ker(H_{K^*_{\nu,\frac{\pi}{2}}} - \lambda I) = \left\{ \begin{pmatrix} x \\ A_+ x \end{pmatrix}, \quad x \in \mathbb{C}^2 \right\},$$

where A_+ and A_- are two matrices in $\mathbb{M}_2(\mathbb{C})$ satisfying ${}^tA_\pm = A_\pm$ and $\pm\mathrm{Im}(A_\pm) > 0$.

The matrix A_+ is given by $A_+ = B_{1+}^{-1}B_{2+}$ where

$$B_{1+} = \begin{pmatrix} 1 & \frac{-1-n_1}{2\sqrt{\nu}} \\ \\ 1 & \frac{-1+n_1}{2\sqrt{\nu}} \end{pmatrix} \quad \text{and} \quad B_{2+} = \begin{pmatrix} i & \frac{-i-in_1}{2\sqrt{\nu}} \\ \\ i & \frac{-i+in_1}{2\sqrt{\nu}} \end{pmatrix},$$

so $A_+ = i\mathrm{Id}$. Similarly, $A_- = B_{1-}^{-1}B_{2-}$ with

$$B_{1-} = \begin{pmatrix} 1 & \frac{1+n_1}{2\sqrt{\nu}} \\ \\ 1 & \frac{1-n_1}{2\sqrt{\nu}} \end{pmatrix} \quad \text{and} \quad B_{2-} = \begin{pmatrix} -i & \frac{-i+in_1}{2\sqrt{\nu}} \\ \\ -i & \frac{-i-in_1}{2\sqrt{\nu}} \end{pmatrix},$$

so $A_- = -i\mathrm{Id}$. This means, after [Vio] formula (2.3), that the real canonical transformation on $\mathbb{R}^4$ is nothing but the identity.

Hence it suffices to write $K^*_{\nu,\frac{\pi}{2}}$ in the form

$$K^*_{\nu,\frac{\pi}{2}} = {}^t(D_x - A_+x)B(D_x - A_-x) \ ,$$

for all $x = (q,p) \in \mathbb{R}^2$, where the matrix B is found by identification of the two sides:

$$B = \begin{pmatrix} 0 & \frac{-\sqrt{\nu}}{2} \\ \frac{\sqrt{\nu}}{2} & \frac{1}{2} \end{pmatrix} \ .$$

Step 2: Once A_+ and A_- are known, the complex canonical transformation is given by

$$\kappa = \begin{pmatrix} 1 & -i \\ -(1-iA_+)^{-1}A_+ & (1-iA_+)^{-1} \end{pmatrix},$$

with associated quadratic phase $\varphi_{A_+} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$

$$\varphi_{A_+}(x,y) = \frac{i(x-y)^2}{2} - \frac{1}{2}\Big(x, (1-iA_+)^{-1}A_+x\Big) = i\left[\frac{(x-y)^2}{2} - \frac{x^2}{4}\right],$$

which is the one entering in the definition of the associated FBI transform (which is B_2) . The computation of $B_2K^*_{\nu,\frac{\pi}{2}}B_2^*$ then comes from Egorov's theorem

$$K^*_{\nu,\frac{\pi}{2}}(\kappa^{-1}Z) =^t Z^t\kappa^{-1}\begin{pmatrix} -A_+ \\ \mathrm{Id} \end{pmatrix} B(-A_-, \mathrm{Id})\kappa^{-1}Z = i^t z M \zeta$$

$$\text{with} \qquad M = (1-iA_+)B = 2B = \begin{pmatrix} 0 & -\sqrt{\nu} \\ \sqrt{\nu} & 1 \end{pmatrix}$$

The weight $e^{-2\phi(z)}L(dz)$ occuring in the range of B_2 is $\phi(z) = \frac{|z|^2}{4}$ which is coherent with the formulas (2.6) and (2.7) of [Vio] , $\phi(x) = \frac{1}{4}\Big(|x|^2 - {}^txCx\Big)$ because $C = (1-iA_+)^{-1}(1+iA_+) = 0$. □

Lemma 3.4.

There exists a constant $c > 0$ independent of $\nu > 1$, such that for all $t > 0$ and all $u \in L^2(\mathbb{R}^2)$, $u_t = e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u$ satisfies

$$\frac{\nu}{2}\left(\|u_t\|^2_{L^2(\mathbb{R}^2)} + \|D_q u_t\|^2_{L^2(\mathbb{R}^2)} + \|qu_t\|^2_{L^2(\mathbb{R}^2)}\right) = \|\sqrt{\nu}(\frac{-\partial_q + q}{\sqrt{2}})e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{\frac{1}{3}})}u\|^2_{L^2(\mathbb{R}^2)} \leq \frac{c}{t^3}\|u\|^2_{L^2(\mathbb{R}^2)}\,. \tag{3.11}$$

Proof. Set $a_q = \frac{\partial_q + q}{\sqrt{2}}$ and $a_q^* = \frac{-\partial_q + q}{\sqrt{2}}$ so that $a_q a_q^* = a_q^* a_q + 1 = \frac{1}{2}(D_q^2 + q^2 + 1)$. The identity

$$\begin{aligned}\nu\|a_q^* e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2(\mathbb{R}^2)} &= \nu\|e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2(\mathbb{R}^2)} + \nu\|a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2} \\ &\leq \nu e^{-t\nu^{1/3}}\|u\|^2_{L^2(\mathbb{R}^2)} + \nu\|a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2}\end{aligned}$$

reduces the problem to that of estimating $\|\sqrt{\nu}a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}\|$. By taking the adjoint, it suffices to prove that

$$\|\sqrt{\nu}e^{-t(K^*_{\nu,\frac{\pi}{2}}+\nu^{1/3})}a_q^* f\|_{L^2(\mathbb{R}^2)} \leq \frac{c}{t^{\frac{3}{2}}}\|f\|_{L^2(\mathbb{R}^2)} \tag{3.12}$$

is satisfied for all $f \in L^2(\mathbb{R}^2, dqdp)$ and for all $t > 0$.

Conjugating by the Bargmann transform B_2, the creation operator $B_2 a_q^* B_2^* = B_2(\frac{-\partial_q+q}{\sqrt{2}})B_2^* = \frac{z_q}{\sqrt{2}}\times$ is nothing but multiplication by the complex component z_q in $\mathbb{C}^2 = \mathbb{C}_q \times \mathbb{C}_p$. The inequality (3.12) is therefore equivalent to

$$||\sqrt{\nu}e^{-t(Mz\partial_z+\nu^{1/3})}z_q u||_{H_\phi} \leq \frac{c}{t^{\frac{3}{2}}}||u||_{H_\phi} \tag{3.13}$$

for all $u \in H_\phi = L^2(\mathbb{C}^2, e^{-\frac{|z|^2}{2}}L(dz)) \cap \mathrm{Hol}(\mathbb{C}^2)$, with $\phi(z) = \frac{|z|^2}{4}$.

Let $u \in H_\phi$, setting $v(z) = z_q u(z)$, one has $e^{-tMz\partial_z}v(z) = v(e^{-tM}z)$ and it follows that

$$\begin{aligned}\|e^{-tMz\partial_z}z_q u\|^2_{H_\phi} &= \int_{\mathbb{C}^2}|v(e^{-tM}z)|^2|(e^{-tM}z)_q|^2 e^{-2\phi(z)}L(dz) \\ &= e^{2t\,\mathrm{Tr}\,M}\int_{\mathbb{C}^2}|v(z')|^2|z'_q|^2 e^{-\phi(z')}e^{-2[\phi(e^{tM}z')-\phi(z')]}L(dz')\,.\end{aligned}$$

So our problem is reduced to the proof of the existence of a constant $c > 0$ that does not depend on ν such that

$$\sup_{z\in\mathbb{C}^2}|z_q|^2 e^{-\frac{1}{2}\left(|e^{tM}z|^2-|z|^2\right)}e^{-t\nu^{1/3}} \leq \frac{c}{\nu t^3}$$

for all $t > 0$.

Let us start by checking that $z \mapsto \phi(e^{tM}z) - \phi(z)$ defines a positive definite hermitian form for $t > 0$.

From the expression given in Lemma 3.3 , M is easily written in terms of Pauli's matrices:

$$M = \frac{1}{2}\mathrm{Id} - \frac{1}{2}\sigma_3 - i\sqrt{\nu}\sigma_2 \ ,$$
$$\text{with} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Recall that Pauli's matrices are involutory:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \mathrm{Id} \ ,$$

and that $(\mathrm{Id}, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ can be interpreted as a basis of (bi)quaternions.

Using formula (A.1), one has for all $t > 0$

$$e^{tM} = e^{\frac{t}{2}}\Big(C(t) + 2S(t)(-\frac{1}{2}\sigma_3 - i\sqrt{\nu}\sigma_2)\Big) \ .$$

From this, we compute

$$\begin{aligned}(e^{tM})^* e^{tM} &= e^t\Big(1 + 2S^2(t) - 2C(t)S(t)\sigma_3 - 4\sqrt{\nu}S^2(t)\sigma_1\Big) \\ &= e^t(a + v) \ ,\end{aligned}$$

with $a = 1 + 2S^2(t)$ and $v = -2C(t)S(t)\sigma_3 - 4\sqrt{\nu}s_1^2(t)\sigma_1$.

The eigenvalues of $(e^{tM})^* e^{tM}$ are given by

$$\lambda_\pm = e^t(a \pm \sqrt{-N(v)}) \ ,$$

where $N(v) = -\Big(2C(t)S(t)\Big)^2 - \Big(4\sqrt{\nu}S^2(t)\Big)^2 = -4S^2 - 4S^4 < 0$ owing to $(4\nu - 1)S^2 + C^2 = 1$, and where $\sqrt{-N(v)}$ is the usual square root.

In order to prove that the hermitian form $z \mapsto \phi(e^{tM}z) - \phi(z) = {}^t\bar{z}\Big((e^{tM})^*(e^{tM}) - \mathrm{Id}\Big)z$ is positive definite, it suffices to check $\lambda_- > 1$ for all $t > 0$, λ_+ being clearly strictly larger than 1 . The eigenvalue λ_- equals

$$\lambda_- = e^t(1 + (1 + S^2) - 2|S|\sqrt{1 + S^2}) = e^t e^{-2\,\mathrm{Argsh}\,|S|}$$

which is larger than 1 if and only if $\mathrm{sh}(t/2) - |S| > 0$ or $[\mathrm{sh}(t/2) - S(t)][\mathrm{sh}(t/2) + S(t)] > 0$ because $\mathrm{sh}(t/2) > 0$. This is true since both factors vanish at $t = 0$ with a positive derivative for $t > 0$ owing to $\mathrm{ch}(t/2) > 1 > \pm\cos(\frac{t}{2}\sqrt{4\nu - 1})$.

Now denote

$$r_1 = \sqrt{4\nu - 1} \ ; \quad Q_t(z) = {}^t\bar{z}\Big[(e^{tM})^*(e^{tM}) - \mathrm{Id}\Big]z \quad \text{and} \quad S_t(z_1, z_2) = {}^t\bar{z_1}\Big[(e^{tM})^*(e^{tM}) - \mathrm{Id}\Big]z_2$$

for all $z=(z_1,z_2)\in\mathbb{C}\times\mathbb{C}$. Writing $z_q=l(z)$ where l is a linear form with kernel $ker\ l=\mathbb{C}e_p$, $\Big(\mathbb{C}^2=\mathbb{C}e_q\oplus\mathbb{C}e_p$ where $e_q=(1,0)$ and $e_p=(0,1)\Big)$, we construct an orthonormal basis (e'_q,e_p) for Q_t with

$$e'_q=e_q-\frac{S_t(e_p,e_q)}{S_t(e_p,e_p)}e_p=\begin{pmatrix}1\\ \\ \frac{4\sqrt{\nu}\ S^2(t)}{(1-e^{-t})+2S^2(t)+2S(t)C(t)}\end{pmatrix}\ ,$$

where

$$\begin{cases} S_t(e_p,e_q)=-4\sqrt{\nu}\ e^t\ S^2(t)\\ \\ S_t(e_p,e_p)=e^t\Big[(1-e^{-t})+2S^2(t)+2S(t)C(t)\Big]\ . \end{cases}$$

In this new basis, $z=\alpha e'_q+\beta e_p$ then $l(z)=\alpha\ l(e'_q)$ and $Q_t(z)=|\alpha|^2Q_t(e'_q)+|\beta|^2Q_t(e_p)$. This gives immediately

$$|z_q|^2e^{\frac{-Q_t(z)}{2}}=|\alpha|^2|l(e'_q)|^2e^{\frac{-|\alpha|^2Q_t(e'_q)-|\beta|^2Q_t(e_p)}{2}}.$$

and then

$$\sup_{z\in\mathbb{C}^2}|z_q|^2e^{-\frac{1}{2}\Big(|e^{tM}z|^2-|z|^2\Big)}=\sup_{s\in\mathbb{R}_+}|l(e'_q)|^2e^{-\frac{sQ_t(e'_q)}{2}}=\frac{2|l(e'_q)|^2}{Q_t(e'_q)}\sup_{\sigma\in\mathbb{R}_+}\sigma e^{-\sigma}=c_0\frac{2|l(e'_q)|^2}{Q_t(e'_q)}=c_0\frac{2}{Q_t(e'_q)}$$

where $c_0=\sup\limits_{\sigma\in\mathbb{R}_+}\sigma e^{-\sigma}$ and

$$Q_t(e'_q)=S_t(e'_q,e'_q)=\frac{4\Big(\mathrm{sh}^2(\frac{t}{2})-S^2(t)\Big)}{(1-e^{-t})+2S^2(t)+2S(t)C(t)}\ .$$

Recall that, in the case $\alpha=\frac{\pi}{2}$ and for $\nu>\frac{1}{4}$, we define $C(t)=\cos(\frac{tr_1}{2})$ and $S(t)=\frac{\sin(\frac{tr_1}{2})}{r_1}$.

All that remains is to control the following quotient for all $t>0$:

$$\frac{1}{Q_t(e'_q)}=\frac{(1-e^{-t})+2S(t)\Big(S(t)+C(t)\Big)}{4\Big[\mathrm{sh}^2(\frac{t}{2})-S^2(t)\Big]}:=\frac{N}{D}\ .$$

• Starting with the case when $t\geq\frac{4}{r_1}$,

$$N=(1-e^{-t})+2S(t)\Big(S(t)+C(t)\Big)\leq 1+\frac{4}{r_1}\leq 2\ .$$

On the other hand,

$$|S(t)|\leq\frac{1}{r_1}\leq\frac{t}{4}\leq\frac{1}{2}\,\mathrm{sh}(\frac{t}{2})\quad\text{ implies }\quad D\geq\mathrm{sh}^2(\frac{t}{2})\ .$$

Then

$$\frac{1}{Q_t(e'_q)} \leq \frac{2}{\mathrm{sh}^2(\frac{t}{2})} \leq 2e^{-t}$$

for all $t \geq \frac{4}{r_1}$.

• Now observe that for $t \leq \frac{4}{r_1}$, one has the following two expansions:

$$\mathrm{sh}(\frac{t}{2}) + S(t) = \sum_{k=0}^{+\infty} (-1)^k (r_1^{2k} + (-1)^k) \frac{t^{2k+1}}{2^{2k+1}(2k+1)!}$$

and

$$\mathrm{sh}(\frac{t}{2}) - S(t) = \sum_{k=0}^{+\infty} (-1)^k (-r_1^{2k} + (-1)^k) \frac{t^{2k+1}}{2^{2k+1}(2k+1)!} \ .$$

Furthermore,

$$\left| \mathrm{sh}(\frac{t}{2}) + S(t) - t - \frac{r_1^2 - 1}{48} t^3 \right| \leq \frac{(r_1^4 + 1)t^5}{2^5 \times 120}$$

which implies

$$\begin{aligned} \frac{1}{t}\Big(\mathrm{sh}(\frac{t}{2}) + S(t) \Big) &\geq 1 - \frac{(r_1^2 - 1)}{48} t^2 - \frac{r_1^4 + 1}{2^5 \times 120} t^4 \\ &\geq 1 - \frac{16}{48} - \frac{2 \times 4^4}{2^5 \times 120} \geq 1 - \frac{1}{3} - \frac{2}{15} = \frac{8}{15} \ . \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} \left| \mathrm{sh}(\frac{t}{2}) - S(t) - \frac{r_1^2 + 1}{48} t^3 \right| &\leq \frac{(r_1^4 - 1)t^5}{2^5 \times 120} = \frac{(r_1^2 + 1)t^3}{48} \times \frac{(r_1^2 - 1)t^3}{4 \times 20} \\ &\leq \frac{(r_1^2 + 1)t^3}{48} \frac{(r_1 t)^2}{4 \times 20} \\ &\leq \frac{(r_1^2 + 1)t^3}{48} \frac{1}{5} \ , \end{aligned}$$

which gives

$$\mathrm{sh}(\frac{t}{2}) - S(t) \geq \frac{(r_1^2 + 1)t^3}{48} \times \frac{4}{5} \ . \tag{3.15}$$

Taking into account (3.14) and (3.15) we get

$$D \geq \Big(\mathrm{sh}(\frac{t}{2}) + S(t) \Big) \Big(\mathrm{sh}(\frac{t}{2}) - S(t) \Big) \geq t \times \frac{8}{15} \times \frac{(r_1^2 + 1)t^3}{48} \times \frac{4}{5} \ .$$

On the other hand,

$$N = (1-e^{-t}) + 2S(t)\Big(S(t)+C(t)\Big) = 2t + (1-\frac{r_1^2}{6})t^3 - \frac{1+r_1^2}{24}t^4 + \mathcal{O}(r_1^4 t^5)$$
$$= t\Big(2 + (1-\frac{r_1^2}{6})t^2 - \frac{1+r_1^2}{24}t^3 + \mathcal{O}((r_1 t)^4)\Big) \ .$$

Hence $N \leq ct$ for all $t \leq \frac{4}{r_1}$ and

$$\frac{1}{Q_t(e_q')} = \frac{N}{D} \leq \frac{c}{\nu t^3} \quad \text{for all} \quad t \leq \frac{4}{r_1} \ .$$

Thus there exists a constant $c > 0$ such that, for all $u \in H_\phi$,

$$||e^{-tMz\partial_z} z_q u||_{H_\phi}^2 \leq \begin{cases} \frac{c}{\nu t^3}||u||_{H_\phi}^2 & \text{for all} \quad t \leq \frac{4}{r_1} \\ ce^{-t}||u||_{H_\phi}^2 & \text{for all} \quad t \geq \frac{4}{r_1} \end{cases}$$

which is equivalent to

$$||e^{-tK^*_{\nu,\frac{\pi}{2}}} a_q^* v||_{L^2} \leq \begin{cases} \frac{c}{\sqrt{\nu t^3}}||v||_{L^2} & \text{for all} \quad t \leq \frac{4}{r_1} \\ ce^{-t}||v||_{L^2} & \text{for all} \quad t \geq \frac{4}{r_1} \end{cases}$$

for all $v \in D(K_{\nu,\frac{\pi}{2}})$.

From this, we deduce that

$$||a_q e^{-t(\nu^{\frac{1}{3}} + K_{\nu,\frac{\pi}{2}})} v||_{L^2} \leq \begin{cases} \frac{c}{\sqrt{\nu t^3}}||v||_{L^2} & \text{if} \quad t \leq \frac{4}{r_1} \\ ce^{-\nu^{\frac{1}{3}}t}||v||_{L^2} & \text{if} \quad t \geq \frac{4}{r_1} \end{cases}$$

for all $v \in D(K_{\nu,\frac{\pi}{2}})$. When $0 < t \leq \frac{4}{r_1}$, we clearly have

$$\|\sqrt{\nu} a_q e^{-t(\nu^{\frac{1}{3}} + K_{\nu,\frac{\pi}{2}})}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \frac{C}{t^{\frac{3}{2}}}.$$

When $t \geq \frac{4}{r_1}$, we obtain the same result by writing

$$c\sqrt{\nu} e^{-\nu^{\frac{1}{3}}t} = \frac{c}{t^{\frac{3}{2}}} \left(\nu^{\frac{1}{3}} t\right)^{\frac{3}{2}} e^{-\nu^{\frac{1}{3}}t}$$

and noting that the function $s^{3/2}e^{-s}$ is bounded on $[0,\infty)$. This establishes the inequality for all $t > 0$ and completes the proof of the lemma. □

4 Resolvent estimates when $V(q) = -\frac{\nu q^2}{2}, \nu \gg 1$

In this section, we use the same notations as in the previous one and we take $\alpha = 0$. Giving the exact norm of the semigroup $e^{-tK_{\nu,0}}$ allows us to control the resolvent of the operator $K_{\nu,0}$. When doing so, a logarithmic factor appears, with optimality up to an exponent.

Lemma 4.1. *For every $t \geq 0$, one has*

$$||e^{-tK_{\nu,0}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} = e^{-\operatorname{Argsh}\Big(S(t)\Big)}$$

where

$$S(t) = \frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1} = \frac{\operatorname{sh}(\frac{t\sqrt{4\nu+1}}{2})}{\sqrt{4\nu+1}} \ .$$

Proof. Using (3.3) and (3.4), we directly compute that

$$\overline{(\kappa(t))^{-1}}\,\kappa(t) := \overline{e^{-itH_{K_{\nu,0}}}}e^{itH_{K_{\nu,0}}} = e^{itE}\Big(a + bI - cJ\Big)\Big(a + bI + cJ\Big) \ ,$$

with $a = C(t),\ b = iS(t)$ and $c = 2izS(t)$.

Note that $\Big(a+bI-cJ\Big)\Big(a+bI+cJ\Big) = a^2-b^2+c^2+v$. Furthermore, $a^2+b^2+c^2 = 1$ and $(a^2-b^2+c^2)^2+N(v) = 1$. It follows that $N(v) = 1-(a^2-b^2+c^2)^2 = 1-(1-2b^2)^2 = 4b^2(1-b^2)$.

Denote $\operatorname{sh}(u) = \sqrt{-b^2}$, so $\sqrt{-N(v)} = 2\operatorname{sh}(u)\operatorname{ch}(u) = \operatorname{sh}(2u)$.

The eigenvalues of $\overline{(\kappa(t))^{-1}}\,\kappa(t)$ are given by

$$\begin{aligned}
\frac{1}{\mu_1} &= e^t(a^2 - b^2 + c^2 + \sqrt{-N(v)}) \\
\mu_1 &= e^{-t}(a^2 - b^2 + c^2 - \sqrt{-N(v)}) \\
\frac{1}{\mu_2} &= e^t(a^2 - b^2 + c^2 - \sqrt{-N(v)}) \\
\mu_2 &= e^{-t}(a^2 - b^2 + c^2 + \sqrt{-N(v)}) \ .
\end{aligned}$$

Therefore (see [Vio2] Theorem 1.3),

$$||e^{-tK_{\nu,0}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} = (\mu_1\frac{1}{\mu_2})^{\frac{1}{4}} = e^{-\frac{1}{2}\operatorname{Argsh}(\sqrt{-N(v)})} = e^{-Argsh(\sqrt{-b^2})} \ ,$$

where

$$-b^2 = \Big(S(t)\Big)^2 = \Big(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\Big)^2 \ .$$

□

Proposition 4.2. *There exists some $c > 0$ such that, for all $\nu > c$,*

$$||K_{\nu,0}^{-1}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq c\frac{\log(\nu)}{\sqrt{\nu}}.$$

Proof. Observing that

$$||K_{\nu,0}^{-1}||_{\mathcal{L}(L^2(\mathbb{R}^2))} = || \int_0^{+\infty} e^{-tK_{\nu,0}} dt||_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \int_0^{+\infty} ||e^{-tK_{\nu,0}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} dt \ ,$$

we aim to obtain an upper bound of the right-hand side.

Using the exact norm of the semigroup generated by $K_{\nu,0}$, we write

$$\begin{aligned}
\int_0^{+\infty} ||e^{-tK_{\nu,0}}||_{\mathcal{L}(L^2(\mathbb{R}^2))} dt &= \int_0^{+\infty} e^{-\operatorname{Argsh}\left(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\right)} dt = \int_0^{+\infty} \frac{1}{\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1} + \sqrt{1+\left(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\right)^2}} dt \\
&= \int_0^{\log(\nu)} \frac{2du}{\operatorname{sh}(u) + \sqrt{n_1^2 + \operatorname{sh}^2(u)}} + \int_{\log(\nu)}^{+\infty} \frac{2du}{\operatorname{sh}(u) + \sqrt{n_1^2 + \operatorname{sh}^2(u)}} \\
&\leq 2\Big(\frac{\log(\nu)}{n_1} + \int_{\log(\nu)}^{+\infty} e^{-u} du\Big) \\
&\leq 2\Big(\frac{\log(\nu)}{n_1} + \frac{1}{\nu}\Big) \leq c\, \frac{\log(\nu)}{\sqrt{\nu}} \ .
\end{aligned}$$

This completes the proof. □

4.1 Optimality with a logarithmic factor

Proposition 4.3.

One can find a function $u \in L^2(\mathbb{R}^2)$ such that

$$||K_{\nu,0}u||_{L^2(\mathbb{R}^2)} \leq c\, \frac{\sqrt{\nu}}{\sqrt{\log(\nu)}} ||u||_{L^2(\mathbb{R}^2)}$$

where $c > 0$ is a constant that does not depend on the parameter $\nu \gg 1$.

Proof. We recall here that

$$K_{\nu,0} = \frac{1}{2}(-\partial_p^2 + p^2) + \sqrt{\nu}\Big(p\partial_q + q\partial_p\Big) = O_p + \sqrt{\nu} X_0.$$

For all $u \in D(K_{\nu,0})$,

$$||K_{\nu,0}u||^2_{L^2(\mathbb{R}^2)} \leq 2\Big(||O_p u||^2_{L^2(\mathbb{R}^2)} + \nu ||X_0 u||^2_{L^2(\mathbb{R}^2)}\Big) \ ,$$

then to prove the Proposition we will look for a function $u \in L^2(\mathbb{R}^2)$ such that

$$\frac{||O_p u||^2_{L^2(\mathbb{R}^2)} + \nu ||X_0 u||^2_{L^2(\mathbb{R}^2)}}{||u||^2_{L^2(\mathbb{R}^2)}} \leq c\, \frac{\nu}{\log(\nu)} \ .$$

Consider the Gaussian

$$\varphi(q,p) = \frac{e^{-\frac{(q^2+p^2)}{2}}}{\sqrt{\pi}}$$

and set

$$u(q,p) = \frac{1}{L}\int_0^L e^{sX_0}\varphi ds = \frac{1}{L}\int_0^L \varphi_s(q,p)ds$$

where $\varphi_s(q,p) = e^{sX_0}\varphi(q,p)$ and $L > 0$ is a constant to be specified at the end of the proof.

One has

$$\frac{d}{ds}\varphi_s = X_0(\varphi_s) = (p\partial_q + q\partial_p)\varphi_s \ .$$

Let $\Big(q(t), p(t)\Big)$ be the solution of the following system:

$$\begin{cases} \frac{d}{dt}q = p \\ \\ \frac{d}{dt}p = q \end{cases}$$

with $(q(0), p(0)) = (q_0, p_0)$. The solution is given by

$$\begin{cases} q(t) = \mathrm{ch}(t)q_0 + \mathrm{sh}(t)p_0 \\ \\ p(t) = \mathrm{sh}(t)q_0 + \mathrm{ch}(t)p_0 \end{cases}$$

The function φ_s verifies

$$\frac{d}{ds}\Big(\varphi_s(q(-s), p(-s))\Big) = \frac{\partial}{\partial s}\varphi_s - \frac{d}{ds}q(-s)\partial_q\varphi_s - \frac{d}{ds}p(-s)\partial_p\varphi_s = 0$$

then

$$\begin{aligned}\varphi_s(q,p) = \varphi_0(q(s), p(s)) &= \varphi\Big(\mathrm{ch}(s)q + \mathrm{sh}(s)p, \mathrm{sh}(s)q + \mathrm{ch}(s)p\Big) \\ &= \frac{1}{\sqrt{\pi}}e^{-\frac{\Big(\mathrm{ch}(s)q+\mathrm{sh}(s)p\Big)^2 + \Big(\mathrm{sh}(s)q+\mathrm{ch}(s)p\Big)^2}{2}} \ .\end{aligned}$$

For all $p \in [0, +\infty]$, $||\varphi_s||_{L^p} = ||\varphi||_{L^p}$. In particular, $||\varphi_s||_{L^2} = ||\varphi||_{L^2} = 1$.

Let's start by calculating $||X_0 u||_{L^2(\mathbb{R}^2)}$:

$$X_0 u = \frac{1}{L}\int_0^L X_0 e^{sX_0}\varphi ds = \frac{1}{L}\int_0^L \frac{d}{ds}\varphi_s ds = \frac{1}{L}(\varphi_L - \varphi) \ .$$

As a result,

$$||X_0 u||^2_{L^2(\mathbb{R}^2)} = \frac{1}{L^2}||\varphi_L - \varphi||^2_{L^2(\mathbb{R}^2)} = \frac{1}{L^2}\Big(\underbrace{||\varphi_L||^2_{L^2(\mathbb{R}^2)}}_{=1} + \underbrace{||\varphi||^2_{L^2(\mathbb{R}^2)}}_{=1} - 2\int_{\mathbb{R}^2} \varphi_L \varphi \; dqdp \Big)$$
$$= \frac{2}{L^2}\Big(1 - \int_{\mathbb{R}^2} \varphi_L \varphi \; dqdp\Big) \; .$$

We directly compute that

$$\int_{\mathbb{R}^2} \varphi_L(q,p)\varphi(q,p)dqdp = \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{\Big(\mathrm{ch}(L)q+\mathrm{sh}(L)p\Big)^2+\Big(\mathrm{sh}(L)q+\mathrm{ch}(L)p\Big)^2}{2}} e^{-\frac{q^2+p^2}{2}} dqdp$$
$$= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\Big[2\,\mathrm{ch}^2(L)q^2+2\,\mathrm{ch}^2(L)p^2+4\,\mathrm{sh}(L)\,\mathrm{ch}(L)qp\Big]} dqdp$$
$$= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{1}{2}(q,p)A\,{}^t(q,p)} dqdp = \frac{1}{\pi}\sqrt{\frac{(2\pi)^2}{det(A)}} = \frac{1}{\mathrm{ch}(L)}$$

where

$$A = \begin{pmatrix} 2\,\mathrm{ch}^2(L) & 2\,\mathrm{ch}(L)\,\mathrm{sh}(L) \\ \\ 2\,\mathrm{ch}(L)\,\mathrm{sh}(L) & 2\,\mathrm{ch}^2(L) \end{pmatrix} \; .$$

Then

$$||X_0 u||^2_{L^2(\mathbb{R}^2)} = \frac{2}{L^2}\Big(1 - \frac{1}{\mathrm{ch}(L)}\Big) \; . \tag{4.1}$$

Now, let's find a lower bound for $||u||^2_{L^2(\mathbb{R}^2)}$:

$$||u||^2_{L^2(\mathbb{R}^2)} = \frac{1}{L^2}\int_0^L \int_0^L \mathbb{R}\mathrm{e}\langle \varphi_{s_1}, \varphi_{s_2}\rangle_{L^2(\mathbb{R}^2)} ds_1 ds_2$$
$$= \frac{2}{L^2}\int_0^L \Big[\int_{s_1}^L \mathbb{R}\mathrm{e}\langle \varphi_{s_1}, \varphi_{s_2}\rangle_{L^2(\mathbb{R}^2)} ds_2\Big] ds_1$$
$$\underset{s_2 = s_1+s}{=} \frac{2}{L^2}\int_0^L \Big[\int_0^{L-s_1} \mathbb{R}\mathrm{e}\langle \varphi_{s_1}, \varphi_{s_1+s}\rangle_{L^2(\mathbb{R}^2)} ds\Big] ds_1 \; .$$

But

$$\mathbb{R}\mathrm{e}\langle \varphi_{s_1+s}, \varphi_{s_1}\rangle_{L^2(\mathbb{R}^2)} = \langle e^{s_1 X_0}\varphi, e^{(s_1+s)X_0}\varphi\rangle_{L^2(\mathbb{R}^2)}$$
$$= \langle e^{s_1 X_0}\varphi, e^{sX_0}\varphi\rangle_{L^2(\mathbb{R}^2)}$$
$$= \int_{\mathbb{R}^2} \varphi_s(q,p)\varphi(q,p)dqdp = \frac{1}{\mathrm{ch}(s)} \; .$$

For $L > 2$ we obtain

$$\begin{aligned} ||u||^2_{L^2(\mathbb{R}^2)} &= \frac{2}{L^2}\int_0^L \Big[\int_0^{L-s_1} \frac{1}{\mathrm{ch}(s)} ds\Big] ds_1 \\ &\geq \frac{2}{L^2}\int_0^{\frac{L}{2}} \Big[\int_0^{\frac{L}{2}} \frac{1}{\mathrm{ch}(s)} ds\Big] ds_1 \geq \frac{2}{L^2}\int_0^{\frac{L}{2}} \Big[\int_0^{1} \frac{1}{\mathrm{ch}(s)} ds\Big] ds_1 \\ &\geq \frac{c}{L} \ . \end{aligned} \tag{4.2}$$

The final step is the upper bound of $||O_p u||^2_{L^2(\mathbb{R}^2)}$:

$$||O_p u||^2_{L^2(\mathbb{R}^2)} = ||O_p\Big(\frac{1}{L}\int_0^L \varphi_s(q,p) ds\Big)||^2_{L^2(\mathbb{R}^2)} \leq \frac{1}{L^2}\int_0^L ||O_p\varphi_s||^2_{L^2(\mathbb{R}^2)} ds \ .$$

With $O_p = \frac{1}{2}(D_p^2 + p^2)$, we want to compute

$$\|O_p\varphi_s\|_{L^2(\mathbb{R}^2)} = \|e^{-sX_0} O_p e^{sX_0}\varphi_0\|_{L^2(\mathbb{R}^2)}$$

(because e^{-sX_0} is unitary and $\varphi_s = e^{sX_0}\varphi_0$).

For any $u \in L^2(\mathbb{R}^2)$, $e^{sX_0}u(q,p) = u(e^{sM}(q,p))$ where

$$e^{sM} = \begin{pmatrix} \mathrm{ch}\, s & \mathrm{sh}\, s \\ \mathrm{sh}\, s & \mathrm{ch}\, s \end{pmatrix}.$$

Egorov's theorem gives that, for any symbol $a(q,p,\xi_q,\xi_p)$,

$$e^{-sX_0} a^w(q,p,D_q,D_p) e^{sX_0} = a^w(e^{-sM}(q,p), e^{sM}(D_q,D_p)).$$

In particular, writing $O_q = \frac{1}{2}(D_q^2 + q^2)$ as well,

$$\begin{aligned} e^{-sX_0}(p^2 + D_p^2)e^{sX_0} &= (-\mathrm{sh}(s)q + \mathrm{ch}(s)p)^2 + (\mathrm{sh}(s)D_q + \mathrm{ch}(s)D_p)^2 \\ &= \mathrm{sh}^2(s)q^2 - 2\,\mathrm{ch}(s)\,\mathrm{sh}(s)qp + \mathrm{ch}^2(s)p^2 \\ &\quad + \mathrm{sh}^2(s)D_q^2 + 2\,\mathrm{ch}(s)\,\mathrm{sh}(s)D_qD_p + \mathrm{ch}^2(s)D_p^2 \\ &= 2\,\mathrm{ch}^2(s)O_q + 2\,\mathrm{sh}^2(s)O_p + 2\,\mathrm{ch}(s)\,\mathrm{sh}(s)(D_qD_p - qp). \end{aligned}$$

We have chosen φ_0 an eigenfunction of both O_p and O_q with eigenvalue $\frac{1}{2}$, and $D_qD_p\varphi_0 = -qp\varphi_0$. Therefore

$$e^{-sX_0} O_p e^{sX_0}\varphi_0 = \left(\frac{1}{2}(\mathrm{ch}^2(s) + \mathrm{sh}^2(s)) - 2\,\mathrm{ch}(s)\,\mathrm{sh}(s)qp\right)\varphi_0.$$

This can be interpreted as the sum of products of the first two orthornormal Hermite functions: if

$$h_0(x) = \pi^{-1/4}e^{-x^2/2}, \quad h_1(x) = \sqrt{2}xh_0(x),$$

then $\varphi_0(q,p) = h_0(q)h_0(p)$ and

$$e^{-sX_0}O_p e^{sX_0}\varphi_0 = \frac{1}{2}(\mathrm{ch}^2(s) + \mathrm{sh}^2(s))h_0(q)h_0(p) - \mathrm{ch}(s)\,\mathrm{sh}(s)h_1(q)h_1(p).$$

This type of tensor product forms an orthonormal family, so by the Pythagorean relation the square of the norm can be computed as the sum of squares of the coefficients:

$$\|O_p\varphi_s\|^2_{L^2(\mathbb{R}^2)} = \|e^{-sX_0}O_p e^{sX_0}\varphi_0\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4}(\mathrm{ch}^2(s) + \mathrm{sh}^2(s))^2 + \mathrm{ch}^2(s)\,\mathrm{sh}^2(s) = \frac{1}{4}\,\mathrm{ch}(4s).$$

Thus we deduce that

$$\begin{aligned} ||O_p u||^2_{L^2} &\leq \frac{1}{L^2}\int_0^L e^{4s}ds = \frac{1}{4L^2}(e^{4L}-1) \\ &\leq \frac{1}{L^2}e^{4L}\ . \end{aligned} \tag{4.3}$$

The estimates in (4.1) and (4.2) taken with (4.3), allow us to establish that

$$\frac{||K_{\nu,0}u||^2_{L^2}}{||u||^2_{L^2}} \leq \frac{||O_p u||^2_{L^2} + \nu||X_0 u||^2_{L^2}}{||u||^2_{L^2}} \leq c\,\frac{e^{4L} + \nu\Big(1 - \frac{1}{\mathrm{ch}(L)}\Big)}{L}\ .$$

Now letting $L = \frac{\log(\nu)}{4}$, we get the desired inequality

$$||K_{\nu,0}u||^2_{L^2} \leq c\,\frac{\nu}{\log(\nu)}||u||^2_{L^2}\ .$$

□

5 Degenerate one-dimensional case

Lemma 5.1. *Let $\lambda_1 \in \mathbb{R}$ be parameter. Consider the operator $K_1 = p.\partial_q - \lambda_1\partial_p + \frac{1}{2}(-\partial_p^2 + p^2 - 1)$ with domain $D(K_1) = \{u \in L^2(\mathbb{R}^2),\quad K_1 u \in L^2(\mathbb{R}^2)\}$. There exists a constant $c > 0$ such that*

$$||(D_q^2 + \lambda_1^2)e^{-t(K_1+1)}||_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \frac{c}{t^3}$$

for all $t > 0$.

Proof. For each ξ_q fixed, there is a metaplectic operator on $L^2(\mathbb{R}_p)$ which, via conjugation, takes $ip.\xi_q - i\lambda_1 D_p$ to $ip\sqrt{\xi_q^2 + \lambda_1^2}$ while leaving O_p invariant. Taking the direct integral of this rotation (whose angle depends on ξ_q) gives a unitary equivalence between the operator K_1 and

$$\widehat{K_1} = \frac{1}{2}\Big(2ip\sqrt{D_q^2 + \lambda_1^2} + (-\partial_p^2 + p^2 - 1)\Big).$$

We also note that $\sqrt{D_q^2+\lambda_1^2}$ is left invariant by the rotation in the variables (p,ξ_p).

It is shown in [Vio2] that

$$||e^{-i(t_1+it_2)P_b}||_{\mathcal{L}(L^2(\mathbb{R}))}=\exp\Big(\frac{\cos(t_1)-\mathrm{ch}(t_2)}{\mathrm{sh}(t_2)}b^2\Big)$$

for all $t_1\in\mathbb{R}$ and all $t_2<0$, where $P_b=\frac{1}{2}\Big(D_x^2+x^2-1+2ibx-b^2\Big),\quad b\in\mathbb{R}$. Applying this result with $t_1=0$, $t_2=-t<0$ and $b=b(\xi_q)=\sqrt{\xi_q^2+\lambda_1^2}$, we obtain

$$\|\sqrt{D_q^2+\lambda_1^2}e^{-t\widehat{K_1}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}\leq\sup_{\xi_q\in\mathbb{R}}\|b^2e^{-t(P_b+\frac{b^2}{2})}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}=\sup_{\xi_q\in\mathbb{R}}b^2e^{-\frac{t}{2}b^2}e^{(\frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)})b^2}\ .$$

(We remark that this inequality can be strengthened to an equality by taking the tensor product of explicit optimisers for the norm of e^{-tP_b} with functions in q localized in phase space near the optimising ξ_q.) For all $t\in[0,1]$, denote $f_b(t)=b^2e^{(\frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)}-\frac{t}{2})b^2}$, and $u(t)=\frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)}-\frac{t}{2}=\mathrm{th}(\frac{t}{2})-\frac{t}{2}<0$.

Since $\max\limits_{x\in\mathbb{R}}\ xe^{-ax}=\frac{e^{-a}}{a}$ when $a>0$, we get

$$b^2t^3\exp\Big(u(t)b^2\Big)\leq\frac{-t^3e^{u(t)}}{u(t)}=:F(t)\,.$$

The expansion $u(t)=\mathrm{th}(\frac{t}{2})-\frac{t}{2}=\frac{-t^3}{24}+\mathcal{O}(t^4)$ yields $\lim_{t\to0}F(t)=24$ and the function F is bounded on the interval $[0,1]$. Replacing b^2 with $D_q^2+\lambda_1^2$, we conclude that, for $t\in[0,1]$,

$$||(D_q^2+\lambda_1^2)e^{-t(K_1+1)}||_{\mathcal{L}(L^2(\mathbb{R}^2))}\leq\frac{c}{t^3}\,.$$

For all $t\geq1$, just write with $t_0=\frac{1}{2}$,

$$||(D_q^2+\lambda_1^2)e^{-t(K_1+1)}||_{\mathcal{L}(L^2(\mathbb{R}^2))}\leq\underbrace{||(D_q^2+\lambda_1^2)e^{-t_0K_1}||_{\mathcal{L}(L^2(\mathbb{R}^2))}}_{\leq\frac{c}{t_0}}\underbrace{||e^{-(t-t_0)K_1}||_{\mathcal{L}(L^2(\mathbb{R}^2))}}_{\leq1}e^{-\frac{t}{2}}\leq\frac{c}{t^3}\,.$$

□

A Biquaternions

We define a biquaternion W as follows:

$$W=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$$

where a,b,c,d are complex numbers and $\mathbf{i},\mathbf{j},\mathbf{k}$ multiply according to the rules

$$\begin{aligned}&\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1\\&\mathbf{ij}=-\mathbf{ji}=\mathbf{k}\\&\mathbf{jk}=-\mathbf{kj}=\mathbf{i}\\&\mathbf{ki}=-\mathbf{ik}=\mathbf{j}\ .\end{aligned}$$

For convenience we use a vector notation for biquaternions as follows:

$$W = a + v \ , \quad v = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \ .$$

The conjugate of a biquaternion W is given by

$$\mathrm{conj}(W) = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \ .$$

The biquaternion ring B_Q is isomorphic to the matrix ring $\mathbb{M}_2(\mathbb{C})$. This can be seen via the following map:

$$\begin{aligned} &f : B_Q \to \mathbb{M}_2(\mathbb{C}) \\ &a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto M = \begin{pmatrix} a + b\mathbf{i} & c + d\mathbf{i} \\ -c + d\mathbf{i} & a - b\mathbf{i} \end{pmatrix} \ . \end{aligned}$$

The "norm" $N(W)$ of a biquaternion W is

$$N(W) = \mathrm{conj}(W)W = det(M) = a^2 + b^2 + c^2 + d^2 \ .$$

Note that the norm is homogeneous of degree 2 and may take complex values. In particular, a biquaternion W is invertible if and only if $N(W) \neq 0$. In this case its inverse is given by

$$\mathrm{inv}(W) = \frac{\mathrm{conj}(W)}{N(W)} \ .$$

Exponential and spectrum.

Let $a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k} = a+v$ be a biquarternion such that $N(v) \neq 0$. In this case $\widehat{v} = \frac{v}{\sqrt{N(v)}}$ verifies $\widehat{v}^2 = -1$.

Hence write

$$\begin{aligned} e^{a+v} = e^a e^v &= e^a e^{\widehat{v}\sqrt{N(v)}} \\ &e^a \sum_{k=0}^{+\infty} \frac{(\widehat{v}\sqrt{N(v)})^k}{k!} \\ &e^a \Big(\sum_{k=0}^{+\infty} \frac{(-1)^k (\sqrt{N(v)})^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(-1)^k (\sqrt{N(v)})^{2k+1}}{(2k+1)!} \widehat{v} \Big) \\ &= e^a \Big(\cos(\sqrt{N(v)}) + \frac{\sin(\sqrt{N(v)})}{\sqrt{N(v))}} v \Big) \ . \end{aligned} \tag{A.1}$$

The above computation do not depend on the choice of $\sqrt{N(v)}$ because cos is even and sin is odd.

Finally the set of $\lambda \in \mathbb{C}$ such that $(a + v - \lambda)$ is non-invertible can be explicitly determined:

$(a + v - \lambda)$ is non-invertible if and only if $0 = N(a + v - \lambda) = (a - \lambda)^2 + N(v)$ if and only if $\lambda \in \Big\{ a \pm \sqrt{-N(v)} \Big\}$.

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