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GENERAL SELF-SIMILARITY PROPERTIES FOR MARKOV PROCESSES AND EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES

GRÉGOIRE VÉCHAMBRE

ABSTRACT. In this work, we are interested in Markov processes that satisfy self-similarity properties of a very general form (we call them general self-similar Markov processes, or gssMp's for short) and we prove a generalized Lamperti representation for these processes. More precisely we show that, in dimension 1, a gssMp can be represented as a function of a time-changed Lévy process, which shows some kind of universality for the classical Lamperti representation in dimension 1. In dimension 2, we show that a gssMp can be represented in term of the exponential functional of a bivariate Lévy process, and we can see that processes which can be represented as functions of time-changed Lévy processes form a strict subclass of gssMp's in dimension 2. In other words, we show that the classical Lamperti representation is not universal in dimension 2. We also study the case of more general state spaces and show that, under some conditions, we can exhibit a topological group structure on the state space of a gssMp which allows to write a Lamperti type representation for the gssMp in term of a Lévy process on this group.

1. INTRODUCTION

We consider X , a strongly Markovian càd-làg process on a locally compact separable metric space E , and we will assume that X satisfies some self-similarity properties. For $y \in E$, let us denote X_y for the process X starting from y . We denote by P_y the law of the process X_y .

We do not assume that X is a Feller process (this restriction would not be natural for the processes we are considering) so, even if for simplicity we do not authorize instantaneous killing for X_y , X_y may have finite life-time if, in finite time, every compact subset of E containing y has been left by X_y . Let us denote by $\zeta(X_y)$ the life-time of X_y , that can possibly be finite or infinite. In the case where $\mathbb{P}(\zeta(X_y) < +\infty) > 0$ we consider the usual compactification $E \cup \{\Delta\}$ of E by adding a cemetery point Δ of which the neighborhoods are $K^c \cup \{\Delta\}$, where K goes along compact sets of E , and we put $X_y(t) = \Delta$ for all $t \geq \zeta(X_y)$. Because of the definition of $\zeta(X_y)$, we have that Δ is an absorbing state for X_y and can only be reached continuously. Thanks to this, we systematically consider that our processes are defined on the time interval $[0, +\infty[$. Note that any homeomorphism f of E is naturally extended to an homeomorphism of $E \cup \{\Delta\}$ by imposing $f(\Delta) = \Delta$. When we compose our processes by homeomorphisms on E we implicitly consider the extension of the homeomorphisms to $E \cup \{\Delta\}$, if necessary.

Classically, a strongly Markovian càd-làg process X on $]0, +\infty[$ is called a *positive self-similar Markovian process* (pssMp) if there is $\alpha \in \mathbb{R}$ such that

$$\forall y > 0, \lambda > 0, (X_{\lambda y}(t), t \geq 0) \stackrel{\mathcal{L}}{=} (\lambda X_y(\lambda^{-\alpha}t), t \geq 0), \quad (1.1)$$

that is, the process starting from λy is equal in law to a scaled version of the process starting from y . In the above we use the convention $\lambda\Delta = \Delta$. These processes appear as the scaling limits

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of Markov processes (see Lamperti [8]) and in many examples of processes built from stable Lévy processes (stable Lévy processes conditioned to stay positive, stable Lévy processes killed when entering $] -\infty, 0[$, norm of an isotropic d -dimensional stable Lévy processes,...). A famous result of Lamperti [9] characterizes and gives a representation, the so-called *Lamperti representation*, of a pssMp as the exponential of a time-changed Lévy process.

In the recent decades, some generalizations to pssMp's have been introduced. *Real self-similar Markovian processes* (rssMp's) are strongly Markovian càd-làg processes on \mathbb{R} for which 0 is an absorbing state and that satisfy (1.1) for all $y \in \mathbb{R} \setminus \{0\}, \lambda > 0$. It is known that a rssMp can be represented in term of a time-changed Markov additive process. This representation is called Lamperti-Kiu representation. It is attributed to Kiu [6], a complete proof is given in [4].

More generally, *self-similar Markovian processes on \mathbb{R}^d* , commonly denoted ssMp's, are strongly Markovian càd-làg processes on \mathbb{R}^d that satisfy (1.1) for all $y \in \mathbb{R}^d \setminus \{0\}, \lambda > 0$. The generalized Lamperti-Kiu representation allows to express a ssMp in term of a time-changed Markov additive process that is a little more complicated than the one in the case of a rssMp. This representation is attributed to Kiu [6], a complete proof is given in [1], see also [5]. See Pardo, Rivero [10] and Kyprianou [7] for recent and complete accounts on pssMp's and rssMp's (for [10]), and on pssMp's, rssMp's and ssMp's (for [7]).

1.1. General self-similar Markov processes. We are interested in self-similarity properties that are the most general possible and in characterizing and representing the processes that satisfy such self-similarity properties. (1.1) says that the process starting from the point λy is equal in law to the process starting from the point y , linearly time-changed by $(t \mapsto \lambda^{-\alpha} t)$ and space-changed by composition of the linear function $(z \mapsto \lambda z)$. It is an open problem to determine the processes that satisfy self-similarity properties given by more general space-time-changes. The linearity of the time change can not be relaxed in the context of homogenous Markov processes. In order to define more general self-similarity relations, we thus allow arbitrary time-changes as long as they are linear and arbitrary space-changes. We thus define a general notion of self-similarity in the following way:

Definition 1.1 (gssMp, invariance components). *We say that a strongly Markovian càd-làg process X on a locally compact separable metric space E is a **general self-similar Markovian process** (gssMp) if for some point $y_0 \in E$ we have*

$$\forall y \in E, (X_y(t), t \geq 0) \stackrel{\mathcal{L}}{=} (f_y(X_{y_0}(c_y t)), t \geq 0), \quad (1.2)$$

where $(f_y, y \in E)$ is a family of homeomorphisms of E such that $(y, x) \mapsto f_y(x)$ and $(y, x) \mapsto f_y^{-1}(x)$ are continuous from $E \times E$ to E , and where $(c_y, y \in E)$ is a family of positive constants such that $y \mapsto c_y$ is continuous from E to \mathbb{R}_+^* . Then, we say that $((f_y, c_y), y \in E)$ is a family of **invariance components** for X , relatively to the reference point y_0 .

Remark 1.2. *Note that, in the above definition, for the case where the processes involved reach the cemetery point Δ in finite time, we have assumed that (1.2) holds for the extension of $f_y(\cdot)$ to $E \cup \{\Delta\}$ defined by $f_y(\Delta) = \Delta$. It is not difficult to see that the extensions to $E \times E \cup \{\Delta\}$ of the functions $(y, x) \mapsto f_y(x)$ and $(y, x) \mapsto f_y^{-1}(x)$ are continuous.*

Remark 1.3. *The continuity of the applications $(y, x) \mapsto f_y(x)$ and $(y, x) \mapsto f_y^{-1}(x)$, assumed in Definition 1.1, is equivalent to the continuity of the applications $y \mapsto f_y(\cdot)$ and $y \mapsto f_y^{-1}(\cdot)$ from E to $\mathcal{C}^0(E, E)$ (respectively E to $\mathcal{C}^0(E \cup \{\Delta\}, E \cup \{\Delta\})$ for the case where it is needed to consider the extension to $E \cup \{\Delta\}$), the set of continuous functions from E to E (respectively from $E \cup \{\Delta\}$ to $E \cup \{\Delta\}$) equipped with the natural topology of uniform convergence on every compact sets of E (respectively uniform convergence on $E \cup \{\Delta\}$).*

As mentioned above, natural examples of ssMp's can be built from stable Lévy processes. For example the norm of an isotropic stable Lévy process in \mathbb{R}^d is a pssMp, a stable Lévy process in \mathbb{R}^d is itself a ssMp, etc... For gssMp's, natural examples of these processes can be built from exponential functionals of Lévy processes. For example, let (ξ_1, ξ_2, ξ_3) be a Lévy process in \mathbb{R}^3 , and for any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, let us define

$$X_y(t) := \left(y_1 + \xi_1(t), y_2 + e^{y_1} \int_0^t e^{\xi_1(s^-)} d\xi_2(s), y_3 + e^{2y_1} \int_0^t e^{2\xi_1(s^-)} d\xi_3(s) \right).$$

Then X is a gssMp in \mathbb{R}^3 and a family of invariance components $((f_y, c_y), y \in \mathbb{R}^3)$, with respect to the reference point $(0, 0, 0)$, is given by

$$\forall x = (x_1, x_2, x_3) \in \mathbb{R}^3, f_y(x) = (y_1 + x_1, y_2 + e^{y_1} x_2, y_3 + e^{2y_1} x_3) \quad \text{and} \quad c_y = 1.$$

As we will see, exponential functionals of Lévy processes play a very important role in the context of gssMp's and, at least in dimension 2, they even allow to represent all gssMp's that satisfy some regularity conditions. More generally, it seems that the methods that we develop in this paper could be applied to study some complicated processes built from exponential functionals of Lévy processes.

A natural problem is to characterize the generalized self-similar Markovian processes that satisfy Definition 1.1. 1) Can they all, similarly to pssMp's, be represented as a function of a time-changed Lévy process? In other words, do they satisfy some kind of Lamperti representation? If not, 2) how can Lamperti representation be generalized? The object of the present paper is to characterize gssMp's that satisfy some reasonable assumptions and to provide a Lamperti type representation for them. In particular we will see that the answer to 1) is positive in dimension 1 and negative in dimension 2. In dimension 2, there are two cases: the case of processes that can be represented as a function of a time-changed Lévy process, and another case for which we provided a generalized Lamperti representation in term of an exponential functional of a bivariate Lévy process. The two cases are unified in one general representation. We also provide a Lamperti type representation for gssMp's when the state space E is a differential manifold. We mention in Subsection 1.5 what kind of results can be expected in dimension greater or equal to 3.

gssMp's can be seen as a generalization of pssMp's, but in a completely different direction than ssMp's. Indeed, there are two important differences between ssMp's and the gssMp's that we have just defined. On one hand the assumption of self-similarity that we make is very strong and says that the processes, starting from any point, can be obtained from the process starting at y_0 . Such a property is not true in general for rssMp's and ssMp's, but it is for pssMp's, and we will see that it implies some similarities between gssMp's and pssMp's like the appearance of Lévy processes in the representation of these processes, while Markov additive processes appear in the representation of rssMp's and ssMp's, or the fact that the life-time of the process is always the exponential functional of a Lévy process. On the other hand, the type of self-similarity that we assume is much more general than a scaling property. In particular, the higher the dimension is, the more complicated may the "structure" of the self-similarity be. We will see that in dimension 1 this structure is always simple, which leads us to extend Lamperti representation to gssMp's of dimension 1 that satisfy some regularity assumptions. In dimension 2, there are two possible structures, and they can be unified in a generalized Lamperti representation where the Lévy process is replaced by a process involving the exponential functional of a bivariate Lévy process. For higher dimensions, there are even more possible structures, as explained in Subsection 1.5.

Let us first mention some simple consequences of Definition 1.1.

Combination with Strong Markov Property: For any starting point z , if $\tilde{X}_{y_0} \sim P_{y_0}$ is independent from X_z and S is a stopping time for X_z , then, according to the strong Markov

property at S and Definition 1.1 we have

$$(X_z(S+t), t \geq 0) \stackrel{\mathcal{L}}{=} \left(f_{X_z(S)} \left(\tilde{X}_{y_0}(c_{X_z(S)}t) \right), t \geq 0 \right).$$

Stability by homeomorphism: Let E and F be locally compact separable metric spaces, and X be a gssMp on E with invariance components $((f_y, c_y), y \in E)$, relatively to some reference point y_0 . Then, if $h : E \rightarrow F$ is an homeomorphism, we have that $h(X)$ is a gssMp on F with invariance components $((h \circ f_{h^{-1}(z)} \circ h^{-1}, c_{h^{-1}(z)}), z \in F)$, relatively to the reference point $h(y_0)$.

Change of reference point: For a gssMp, the reference point y_0 can be chosen arbitrary. Indeed, let X be a gssMp and let $((f_y, c_y), y \in E)$ be a family of invariance components for X , relatively to some reference point y_1 . Let y_2 be any other point in E , then it is easy to see that $((f_y \circ f_{y_2}^{-1}, c_y/c_{y_2}), y \in E)$ is a family of invariance components for X , relatively to the reference point y_2 .

Non-uniqueness of invariance components: For a given reference point y_0 there is not, in general, unicity for the choice of the family of invariance components. Indeed, let X be a gssMp and let $((f_y, c_y), y \in E)$ be a family of invariance components for X , relatively to some reference point y_0 . If there is an homeomorphism Ψ from E to E and a constant λ such that

$$(X_{y_0}(t), t \geq 0) \stackrel{\mathcal{L}}{=} (\Psi(X_{y_0}(\lambda t)), t \geq 0), \quad (1.3)$$

then clearly $((f_y \circ \Psi, c_y \lambda), y \in E)$ is also a family of invariance components for X , relatively to the reference point y_0 . Note that it is also possible to let Ψ and λ vary smoothly relatively to y in the set of (Ψ, λ) that satisfy (1.3), to produce a family of invariance components $((f_y \circ \Psi_y, c_y \lambda_y), y \in E)$. In particular, the set of all possible families of invariance components of a gssMp can possibly be quite complicated. Among all possible choices of families of invariance components it will be convenient to work with those that satisfy some nice properties. In the logic of Remark 1.2, note that in (1.3) (and also in (1.5) below, and everywhere where such equalities in law for processes appear), for the case where the processes involved reach the cemetery point Δ in finite time, we actually consider the extension of Ψ (respectively h) to $E \cup \{\Delta\}$ that satisfies $\Psi(\Delta) = \Delta$ (respectively $h(\Delta) = \Delta$), but the extension of Ψ (respectively h) is still denoted by Ψ (respectively h) for simplicity.

An important part of this work consists in showing the existence of what we will call *good invariance components*. For this, some assumptions on E and X are needed. For $y \in E$, let $Supp(X_y)$ denote the support of X_y in E :

$$\begin{aligned} Supp(X_y) &:= \overline{\bigcup_{t \geq 0} Supp(X_y(t))} \\ &= \{z \in E, \text{ s.t. } \forall \epsilon > 0, \exists t \geq 0 \text{ for which } \mathbb{P}(X_y(t) \in B(z, \epsilon)) > 0\}. \end{aligned} \quad (1.4)$$

By reducing E if necessary, we can assume that $E = Supp(X_{y_0})$. Therefore, we define the following assumption

Assumption 1.

$$E = Supp(X_{y_0}).$$

Because of the use of continuity arguments it is often convenient to assume that E is connected. Moreover, when E is connected, Assumption 1 is implied by a simple assumption of non-degeneracy:

Assumption 2.

$$\exists y \in E \text{ s.t. } y \in \widehat{Supp(X_y)}.$$

Assumption 2 can be motivated by the need to avoid some degenerate cases, for example arithmetic processes on $E = \mathbb{R}$. The fact that Assumption 2 implies Assumption 1 is proved in Lemma 7.1 of Section 7. We also need to take into account the symmetries of X . For any $y \in E$ let us define $Sym(X_y) \subset \mathcal{C}^0(E, E)$ to be the group of symmetries of X_y , i.e. the group of homeomorphisms that leave invariant the law of X_y :

$$Sym(X_y) := \left\{ h \in Hom(E), X_y \stackrel{\mathcal{L}}{=} h(X_y) \right\}. \quad (1.5)$$

If $((f_y, c_y), y \in E)$ is a family of invariance components for X , relatively to some reference point y_0 , then it is easy to see that we have $Sym(X_y) = f_y \circ Sym(X_{y_0}) \circ f_y^{-1}$. Note that, if $y \mapsto h_y$ is a continuous application from E to $Sym(X_{y_0}) \subset \mathcal{C}^0(E, E)$, then $((f_y \circ h_y, c_y), y \in E)$ is still a family of invariance components for X , relatively the reference point y_0 (the required assumptions of continuity are satisfied because of Remark 1.3). Therefore, the symmetries of X can interfere with the self-similarity property, in the sens that they can make the set of families of invariance components very complicated. In order to avoid too much problems, we will often need to prove or assume that $Sym(X_{y_0})$ is discrete (as a subset of $\mathcal{C}^0(E, E)$ equipped with the topology of uniform convergence on every compact sets). Note that, since $Sym(X_y) = f_y \circ Sym(X_{y_0}) \circ f_y^{-1}$, " $Sym(X_{y_0})$ is discrete" is always equivalent to " $Sym(X_y)$ is discrete for some $y \in E$ ". It will be shown in Lemma 7.2 of Section 7 that, under Assumption 1, " $Sym(X_{y_0})$ is discrete" is always true in dimension 1.

Finally, we need assumptions of regularity in order to obtain the existence of what we will call *good invariance components*. When the state space E is equipped with a differential structure of class \mathcal{C}^k , we can define \mathcal{C}^k invariance components in the following way:

Definition 1.4 (\mathcal{C}^k invariance components, \mathcal{C}^k -gssMp). *For $k \geq 1$, let X be a gssMp on a \mathcal{C}^k -differentiable manifold \mathcal{M} . Let $((f_y, c_y), y \in \mathcal{M})$ be a family of invariance components for X , relatively to some reference point y_0 . We say that $((f_y, c_y), y \in E)$ is a family of \mathcal{C}^k **invariance components** if the applications $((y, x) \mapsto f_y(x))$, $((y, x) \mapsto f_y^{-1}(x))$, and $(y \mapsto c_y)$ are of class \mathcal{C}^k .*

If a gssMp X on a \mathcal{C}^k -differentiable manifold \mathcal{M} admits a family of \mathcal{C}^k invariance components, then X is said to be a \mathcal{C}^k -general self-similar Markovian process (\mathcal{C}^k -gssMp).

Note that the procedure of "change of reference point" described above transforms a family of \mathcal{C}^k invariance components into a family of \mathcal{C}^k invariance components. In the remainder, when we consider a \mathcal{C}^k -gssMp on an open interval $I \subset \mathbb{R}$ (respectively on an open simply connected domain $\mathcal{D} \subset \mathbb{R}^2$), it is implicit that the differential structure on I (respectively on \mathcal{D}) is the natural one, arising from the fact that it is an open subset of \mathbb{R} (respectively of \mathbb{R}^2), and it induces the usual differentiation on \mathbb{R} (respectively on \mathbb{R}^2).

1.2. General results. We now state Lamperti type representations for gssMp's in dimension 1 and 2. In dimension 1, it turns out that \mathcal{C}^k -gssMp's can be expressed as the image by some function of a time-changed Lévy process (or equivalently as the image by some function of a pssMp). This shows some kind of universality for Lamperti representation, since it is shared by \mathcal{C}^k -gssMp's in dimension 1, no matter what is the shape of their invariance components.

Recall that the starting point of a Lévy process on \mathbb{R} (or more generally on a group) is always 0 (respectively the neutral element of the group).

In dimension 1 our result is the following:

Theorem 1.5 (Dimension 1). *For $k \geq 1$, let X be a \mathcal{C}^k -gssMp on I , an open interval of \mathbb{R} . Let us fix a point $y_0 \in I$ to be the reference point. We assume that either Assumption 1 or*

Assumption 2 is satisfied for $E = I$. Then, there is a \mathcal{C}^k -diffeomorphism $\psi : \mathbb{R} \rightarrow I$, a real Lévy process ξ and $\alpha \in \mathbb{R}$ such that if we set

$$\forall t \in [0, +\infty], \varphi(t) := \int_0^t e^{\alpha\xi(s)} ds, \quad (1.6)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) = \int_0^{+\infty} e^{\alpha\xi(s)} ds \quad (1.7)$$

and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = \psi(\xi(\varphi^{-1}(t))). \quad (1.8)$$

Reciprocally, for an open interval $I \subset \mathbb{R}$, an homeomorphism $\psi : \mathbb{R} \rightarrow I$, a real Lévy process ξ , and $\alpha \in \mathbb{R}$, let us fix $y \in I$ and define $\forall t \in [0, +\infty]$, $\varphi_y(t) := \int_0^t e^{\alpha(\psi^{-1}(y) + \xi(s))} ds$ and $\forall 0 \leq t < \varphi_y(+\infty)$, $X_y(t) := \psi(\psi^{-1}(y) + \xi(\varphi_y^{-1}(t)))$. If $\varphi_y(+\infty) < +\infty$ then $\forall t \geq \varphi_y(+\infty)$, $X_y(t) := \Delta$, where Δ is a cemetery point. Then we have $\zeta(X_y) = \varphi_y(+\infty)$ a.s. and X is a gssMp on I . A family of invariance components $((f_y, c_y), y \in I)$, relatively to the reference point $\psi(0)$, is given by $f_y(\cdot) := \psi(\psi^{-1}(y) + \psi^{-1}(\cdot))$ and $c_y := e^{-\alpha\psi^{-1}(y)}$.

In dimension 2, it turns out that the class of \mathcal{C}^k -gssMp's is larger than (i.e. contains strictly) the class of processes that can be expressed as the image by some function of a time-changed two-dimensional Lévy process (i.e. processes that satisfy a classical Lamperti representation). \mathcal{C}^k -gssMp's are identified with the class of processes that can be expressed as the image by some function of a time-changed process built from the exponential functional of a bivariate Lévy process, and this class contains, as a subclass, processes that can be expressed as the image by some function of a time-changed two-dimensional Lévy process. For $i \in \{1, 2\}$, let $\pi_i(x)$ denote the linear projection of an element $x \in \mathbb{R}^2$ to its i^{th} coordinate. Our result is the following:

Theorem 1.6 (Dimension 2). *For $k \geq 2$, let X be a \mathcal{C}^k -gssMp on \mathcal{D} , an open simply connected domain of \mathbb{R}^2 . Let us fix a point $y_0 \in \mathcal{D}$ to be the reference point. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{D}$ and that $\text{Sym}(X_{y_0})$ is discrete. Then, there is a \mathcal{C}^k -diffeomorphism $\psi : \mathbb{R}^2 \rightarrow \mathcal{D}$, a Lévy process (ξ, η) on \mathbb{R}^2 , and $\alpha, \beta \in \mathbb{R}$ such that if we set*

$$\forall t \in [0, +\infty], \varphi(t) := \int_0^t e^{\alpha\xi(s)} ds, \quad (1.9)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) = \int_0^{+\infty} e^{\alpha\xi(s)} ds \quad (1.10)$$

and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = \psi \left(\xi(\varphi^{-1}(t)), \int_0^{\varphi^{-1}(t)} e^{\beta\xi(s-)} d\eta(s) \right). \quad (1.11)$$

Reciprocally, for an open simply connected domain $\mathcal{D} \subset \mathbb{R}^2$, an homeomorphism $\psi : \mathbb{R}^2 \rightarrow \mathcal{D}$, a \mathbb{R}^2 -valued Lévy process (ξ, η) , and $\alpha, \beta \in \mathbb{R}$, let us fix $y \in \mathcal{D}$ and define $\forall t \in [0, +\infty]$, $\varphi_y(t) := \int_0^t e^{\alpha(\pi_1(\psi^{-1}(y)) + \xi(s))} ds$, and $\forall 0 \leq t < \varphi_y(+\infty)$,

$$X_y(t) := \psi \left(\pi_1(\psi^{-1}(y)) + \xi(\varphi_y^{-1}(t)), \pi_2(\psi^{-1}(y)) + \int_0^{\varphi_y^{-1}(t)} e^{\beta(\pi_1(\psi^{-1}(y)) + \xi(s-))} d\eta(s) \right).$$

If $\varphi_y(+\infty) < +\infty$ then $\forall t \geq \varphi_y(+\infty), X_y(t) := \Delta$, where Δ is a cemetery point. Then we have $\zeta(X_y) = \varphi_y(+\infty)$ a.s. and X is a gssMp on \mathcal{D} . A family of invariance components $((f_y, c_y), y \in \mathcal{D})$, relatively to the reference point $\psi(0,0)$, is given by $f_y(\cdot) := \psi(\pi_1(\psi^{-1}(y) + \psi^{-1}(\cdot)), \pi_2(\psi^{-1}(y) + e^{\beta\pi_1(\psi^{-1}(y))}\psi^{-1}(\cdot)))$ and $c_y := e^{-\alpha\pi_1(\psi^{-1}(y))}$.

In the above theorem, a gssMp is expressed as a function of a time-changed process of the form $(\xi(\cdot), \int_0^\cdot e^{\beta\xi(s^-)}d\eta(s))$, where (ξ, η) is a two-dimensional Lévy process and $\beta \in \mathbb{R}$. The particular case where $\beta = 0$ corresponds to the case where X can be expressed as a function of a time-changed two-dimensional Lévy process.

Remark 1.7. In the above two theorems, α plays a similar role as the self-similarity index in Lamperti representation, and in the case of a pssMp, our α even coincides with the classical self-similarity index, as it will be seen in Subsection 1.6. It is possible to have $\alpha = 0$ which corresponds to the case where the change of time $\varphi^{-1}(\cdot)$ is trivial.

For a \mathcal{C}^k -gssMp in a more general setting, when the state space is equipped with a differential structure, we can establish a generalized Lamperti representation that involves Lévy processes. This can seem surprising since there is no natural meaning for a Lévy process on a general state space. However, we show that the state space can be equipped with a structure of Lie group such that the \mathcal{C}^k -gssMp is represented in term of a left Lévy process on this Lie group (in the remainder, when we work on a non-commutative group, we only deal with left Lévy processes and we call these simply Lévy processes). Our result is the following.

Theorem 1.8. For $k \geq 1$, let X be a \mathcal{C}^k -gssMp on a connected \mathcal{C}^k -differentiable manifold \mathcal{M} . Let us fix a point $y_0 \in \mathcal{M}$ to be the reference point. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{M}$ and that $\text{Sym}(X_{y_0})$ is discrete. Then, there is an intern composition law \star on \mathcal{M} and a \mathcal{C}^k -function $h : \mathcal{M} \rightarrow \mathbb{R}_+$ such that

- (\mathcal{M}, \star) is a \mathcal{C}^k -Lie group with neutral element y_0 ;
- h is a \mathcal{C}^k -Lie group homomorphism from (\mathcal{M}, \star) to (\mathbb{R}_+^*, \times) ;
- There is a Lévy process L (starting from y_0) on (\mathcal{M}, \star) such that, if we define $\forall t \in [0, +\infty], \varphi(t) := \int_0^t 1/h(L(s))ds$ then $\zeta(X_{y_0}) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (1.12)$$

Moreover, note that if we let $\xi(\cdot) := \log(1/h(L(\cdot)))$, then ξ is a real Lévy process and we have $\zeta(X_{y_0}) = \int_0^{+\infty} e^{\xi(s)} ds$.

The above theorem says that, under some assumptions, \mathcal{C}^k -gssMp's can be identified as time-changed Lévy processes on the state space equipped with some group structure. This can be seen as a generalization of Lamperti representation for general state spaces. Indeed, since exponentials of Lévy processes on $(\mathbb{R}, +)$ coincide with Lévy processes on (\mathbb{R}_+^*, \times) , the classical Lamperti representation can be reformulated in the following way:

Theorem 1.9 (Reformulated classical Lamperti representation). Let X be a pssMp of index $\alpha \in \mathbb{R}$. Let us define the continuous group homomorphism h from (\mathbb{R}_+^*, \times) to (\mathbb{R}_+^*, \times) by $h(x) = x^{-\alpha}$. Then, there is a Lévy process L (starting from 1) on (\mathbb{R}_+^*, \times) such that, if we define $\forall t \in [0, +\infty], \varphi(t) := \int_0^t 1/h(L(s))ds$ then $\zeta(X_1) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty), X_1(t) = L(\varphi^{-1}(t)). \quad (1.13)$$

Remark 1.10. Thanks to the above reformulation of Lamperti representation we see that, in Theorem 1.8, the group homomorphism h generalizes the self-similarity index from Lamperti representation, and the α from Theorems 1.5 and 1.6. It is possible to have $h \equiv 1$ (this means

$\alpha = 0$ in the case of a pssMp) which corresponds to the case where the change of time, $\varphi^{-1}(\cdot)$, is trivial, in that case X_{y_0} is a Lévy process on \mathcal{M} , equipped with the group structure constructed by the theorem.

The interest of Theorem 1.8 is that, not only the state space is quite general, but, besides regularity assumption, we do not assume any particular form for the invariance components, that is, for the type of self-similarity satisfied by the gssMp. It is not difficult to prove that the reciprocal of Theorem 1.8 is true: time-changed Lévy processes on topological groups give rise to gssMp's. We state this in the following proposition.

Proposition 1.11. *Let (E, \star) be a topological group with neutral element denoted by e_0 , and where E is a locally compact separable metric space. Let h be a continuous group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) , and L be a Lévy process on (E, \star) . For a fixed $y \in E$ let us define*

$$\forall t \in [0, +\infty], \varphi_y(t) := \frac{1}{h(y)} \int_0^t \frac{1}{h(L(s))} ds, \quad (1.14)$$

and

$$\forall 0 \leq t < \varphi_y(+\infty), X_y(t) := y \star L(\varphi_y^{-1}(t)). \quad (1.15)$$

If $\varphi_y(+\infty) < +\infty$ then

$$\forall t \geq \varphi_y(+\infty), X_y(t) := \Delta,$$

where Δ is a cemetery point. Then we have $\zeta(X_y) = \varphi_y(+\infty)$ a.s. and X is a gssMp on E . A family of invariance components $((f_y, c_y), y \in E)$ is given by $f_y(\cdot) := y \star \cdot$ and $c_y := h(y)$, for the reference point e_0 . Moreover, note that if we let $\xi(\cdot) := \log(1/h(L(\cdot)))$, then ξ is a real Lévy process and we have $\zeta(X_y) = \frac{1}{h(y)} \int_0^{+\infty} e^{\xi(s)} ds$.

Remark 1.12. *In the representations (1.8), (1.11), (1.12), and (1.13) from the above theorems, if one replaces ξ , $(\xi(\cdot), \int_0^\cdot e^{\beta\xi(s^-)} d\eta(s))$, L , and L by respectively $\psi^{-1}(y) + \xi$, $(\pi_1(\psi^{-1}(y)) + \xi(\cdot), \pi_2(\psi^{-1}(y)) + \int_0^\cdot e^{\beta(\pi_1(\psi^{-1}(y)) + \xi(s^-))} d\eta(s))$, $y \star L$, and $y \times L$ (where, respectively, $y \in I$, $y \in \mathcal{D}$, $y \in E$, and $y \in \mathbb{R}_+^*$), then one obtains a version of X_y .*

Remark 1.13. *An interesting consequence of our results is that, for any gssMp X to which one of our theorems is applicable, the exit time of X from its domain, no matter how complicated this domain is, is always equal in law to the exponential functional of a real Lévy process, just as in the case of a pssMp (an heuristic explanation for this similarity is proposed a little after Remark 1.3). This may seem surprising since, according to (generalized) Kiu-Lamperti representation, the life-time of a rssMp and more generally of a ssMp is in general not the exponential functional of a Lévy process but of a Markov additive process (see for example the end of Section 3.2 in [10]). Also, a pssMp that has finite life-time always leaves the domain $]0, +\infty[$ at $\{0\}$ (when $\alpha > 0$) or at $\{+\infty\}$ (when $\alpha < 0$). In the case of a gssMp, the way to leave the domain can be very different, since the exit from the domain does not have to be made at a particular point of the boundary. It is thus remarkable that the law of the exit time remains the same as in the case of a pssMp. The results of the next subsection make explicite, in some cases, the decompositions from the above theorems so in particular the Lévy process ξ , of which the exponential functional is the the life-time of X , can be constructed explicitly from X and its good invariance components (see Remarks 1.17 and 1.21 below).*

The relation between pssMp's and exponential functionals of Lévy processes has proved to be quite fruitful to study the latter (see for example [11] and [12]). We believe that relating exponential functionals to a larger class of processes, as we have done in the results of this section, may also be interesting for the study of exponential functionals of Lévy processes.

1.3. Good components and explicit results.

Definition 1.14 (Good invariance components). *Let X be a gssMp on a locally compact separable metric space E . Let $((f_y, c_y), y \in E)$ be a family of invariance components for X , relatively to some reference point y_0 . We say that $((f_y, c_y), y \in E)$ are **good invariance components** if they have the following properties:*

$$f_{y_0} = id_E \text{ and } c_{y_0} = 1, \quad (1.16)$$

$$\forall y, z \in E, \quad c_{f_y(z)} = c_y \times c_z. \quad (1.17)$$

For example we see that the invariance components defined in Proposition 1.11 and in the reciprocals of Theorems 1.5-1.6, for the gssMp's X defined there, are good invariance components. As we will see, good invariance components are exactly those that can be naturally related to a group structure on the state space E , as in Proposition 1.11. In the proofs of our results, making appear a group structure on the state space is a key point to represent a gssMp in term of a Lévy process, but for this we need to work with good invariance components. Therefore, the existence of good invariance components has a theoretical interest, for example for the proofs of Theorems 1.5, 1.6 and 1.8, and on the other hand, having good invariance components is also useful to get explicite relations between the invariance components and the representations of X given in Theorems 1.5, 1.6 and 1.8. In particular, in the context of good invariance components, the group structure appearing in Theorem 1.8 is constructed naturally in term of these components, as we will see in Theorem 1.20.

Note that if X is a gssMp, and $((f_y, c_y), y \in E)$ is a family of invariance components for X , relatively to some reference point y_0 , then $((f_y \circ f_{y_0}^{-1}, c_y/c_{y_0}), y \in E)$ is a family of invariance components for X , relatively to the reference point y_0 , and it satisfies (1.16). However, the existence of a family of invariance components that satisfies (1.17) is much more difficult to prove, and is established in the following proposition.

Proposition 1.15 (Existence of \mathcal{C}^k good invariance components). *For $k \geq 1$, let X be a \mathcal{C}^k -gssMp on a connected \mathcal{C}^k -differentiable manifold \mathcal{M} and let y_0 be an element of \mathcal{M} . We assume that either Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{M}$ and that $Sym(X_{y_0})$ is discrete. Then X admits a family of \mathcal{C}^k good invariance components relatively to the reference point y_0 .*

Thanks to this proposition, we will often assume that a gssMp is given along with a family of good invariance components.

For gssMp's, given along with good invariance components, we have the following results that can be thought of as explicit versions of Theorems 1.5, 1.6, and 1.8.

Theorem 1.16 (Explicit result in dimension 1). *For $k \geq 1$, let X be a \mathcal{C}^k -gssMp on I , an open interval of \mathbb{R} , and let $((f_y, c_y), y \in I)$ be a family of \mathcal{C}^k **good invariance components** for X , relatively to some reference point $y_0 \in I$. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = I$. Let $g : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ be defined via*

$$\forall y \in I, \quad g(y) := \int_{y_0}^y \frac{1}{f'_y(y_0)} dy, \quad \alpha := -\log(c_{g^{-1}(1)}), \quad (1.18)$$

where f'_y denotes the derivative of the function $f_y : I \rightarrow I$. Then, g and α are well-defined, g is a \mathcal{C}^k -diffeomorphism from I to \mathbb{R} , and there is a real Lévy process ξ such that if we set

$$\forall t \in [0, +\infty], \quad \varphi(t) := \int_0^t e^{\alpha \xi(s)} ds, \quad (1.19)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) = \int_0^{+\infty} e^{\alpha\xi(s)} ds \quad (1.20)$$

and

$$\forall 0 \leq t < \varphi(+\infty), \quad X_{y_0}(t) = g^{-1}(\xi(\varphi^{-1}(t))). \quad (1.21)$$

Remark 1.17. In the above theorem g is constructed directly from the family of good invariance components of X and we see that the Lévy process ξ can be expressed as $\xi(t) = g(X_{y_0}(\varphi(t)))$. Moreover, as we will justify in the end of the proof of the theorem, $\varphi^{-1}(t)$ has an alternative expression in term of X_{y_0} and of the good invariance components: for all $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$. This, together with the expression $\xi(t) = g(X_{y_0}(\varphi(t)))$, allows to express explicitly ξ in term of X_{y_0} and of the good invariance components, as mentioned in Remark 1.13. Similarly, the ξ in Theorem 1.19 below can also be expressed explicitly in term of X_{y_0} and of the good invariance components.

Before stating the result in dimension 2, let us introduce the notion of *commutative invariance components*.

Definition 1.18 (Commutative invariance components). Let X be a gssMp on a locally compact separable metric space E . Let $((f_y, c_y), y \in E)$ be a family of invariance components for X , relatively to some reference point y_0 . We say that $((f_y, c_y), y \in E)$ are **commutative invariance components** if they have the following property:

$$\forall y, z \in E, \quad f_y(z) = f_z(y). \quad (1.22)$$

Theorem 1.19 (Explicit result in dimension 2). For $k \geq 2$, let X be a \mathcal{C}^k -gssMp on \mathcal{D} , an open simply connected domain of \mathbb{R}^2 , and let $((f_y, c_y), y \in \mathcal{D})$ be a family of \mathcal{C}^k **good invariance components** associated with X , relatively to some reference point $y_0 \in \mathcal{D}$. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{D}$ and that $\text{Sym}(X_{y_0})$ is discrete.

- If $((f_y, c_y), y \in \mathcal{D})$ are **commutative** invariance components, then let Y_0 be the gradient at y_0 of the application $(y \mapsto c_y)$. If $Y_0 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ let $M := \begin{pmatrix} \pi_1(Y_0) & \pi_2(Y_0) \\ -\pi_2(Y_0) & \pi_1(Y_0) \end{pmatrix}$, if $Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ let M be the identity 2×2 matrix. Finally, let $g : \mathcal{D} \rightarrow \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ be defined via

$$\forall y \in \mathcal{D}, \quad g(y) := \int_a^b M \cdot [Jf_{\gamma(s)}(y_0)]^{-1} \cdot \gamma'(s) ds, \quad \alpha := -\log(c_{g^{-1}(1,0)}), \quad (1.23)$$

where Jf_z denotes the Jacobian matrix of the function $f_z(\cdot) : \mathcal{D} \rightarrow \mathcal{D}$ and where $\gamma : [a, b] \rightarrow \mathcal{D}$ is any path, locally \mathcal{C}^1 , with $\gamma(a) = y_0$ and $\gamma(b) = y$. Then, g and α are well-defined, g is a \mathcal{C}^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 , and there is a Lévy process (ξ, η) on \mathbb{R}^2 such that if we set

$$\forall t \in [0, +\infty], \quad \varphi(t) := \int_0^t e^{\alpha\xi(s)} ds, \quad (1.24)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) = \int_0^{+\infty} e^{\alpha\xi(s)} ds \quad (1.25)$$

and

$$\forall 0 \leq t < \varphi(+\infty), \quad X_{y_0}(t) = g^{-1}(\xi(\varphi^{-1}(t)), \eta(\varphi^{-1}(t))). \quad (1.26)$$

- If $((f_y, c_y), y \in \mathcal{D})$ are **not commutative** invariance components, then let $g : \mathcal{D} \rightarrow \mathbb{R}^2$ be defined as in Lemma 4.4, and $\alpha := -\log(c_{g^{-1}(1,0)}) \in \mathbb{R}$. Then, g and α are well-defined, g is a C^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 , and there is a Lévy process (ξ, η) on \mathbb{R}^2 such that if we set

$$\forall t \in [0, +\infty], \varphi(t) := \int_0^t e^{\alpha\xi(s)} ds, \quad (1.27)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) = \int_0^{+\infty} e^{\alpha\xi(s)} ds \quad (1.28)$$

and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = g^{-1} \left(\xi(\varphi^{-1}(t)), \int_0^{\varphi^{-1}(t)} e^{\xi(s^-)} d\eta(s) \right). \quad (1.29)$$

Theorem 1.20. *Let X be a gssMp on a connected locally compact separable metric space E , and let $((f_y, c_y), y \in E)$ be a family of **good invariance components** for X , relatively to some reference point y_0 . We assume that either Assumption 1 or Assumption 2 is satisfied for E and that $\text{Sym}(X_{y_0})$ is discrete. Let us define an interne composition law \star on E by $y \star x := f_y(x)$. Then:*

- (E, \star) is a topological group with neutral element y_0 ;
- $(y \mapsto c_y)$ is a continuous group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) ;
- There is a Lévy process L (starting from y_0) on (E, \star) such that if we define $\forall t \in [0, +\infty], \varphi(t) := \int_0^t 1/c_{L(s)} ds$ then $\zeta(X_{y_0}) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (1.30)$$

Let $\xi(\cdot) := \log(1/c_{L(\cdot)})$, then ξ is clearly a real Lévy process and we have $\zeta(X_{y_0}) = \int_0^{+\infty} e^{\xi(s)} ds$.

If moreover, for $k \geq 1$, E is a C^k -differentiable manifold and $((f_y, c_y), y \in E)$ are C^k good invariance components, then (E, \star) is even a C^k -Lie group and $(y \mapsto c_y)$ is a C^k -Lie group homomorphism.

Remark 1.21. *As in Remark 1.17, let us mention that in the above theorem we see that the Lévy process L can be expressed as $L(t) = X_{y_0}(\varphi(t))$ and, as we will justify in the end of the proof of the theorem, $\varphi^{-1}(t)$ has an alternative expression in term of X_{y_0} and of the good invariance components: for all $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$. This, together with the expression $\xi(\cdot) := \log(1/c_{L(\cdot)})$, allows to express explicitly ξ in term of X_{y_0} and of the good invariance components, as mentioned in Remark 1.13.*

Recall that the family of invariance components defined in Proposition 1.11 always satisfies Definition 1.14 so it is actually a family of good invariance components. Therefore, Proposition 1.11 can be seen as the reciprocal of the above theorem.

Remark 1.22. *In the representations (1.21), (1.26), (1.29), and (1.30) from the above theorems, if one replaces ξ , (ξ, η) , $(\xi(\cdot), \int_0^{\cdot} e^{\xi(s^-)} d\eta(s))$, and L by respectively $\psi^{-1}(y) + \xi$, $\psi^{-1}(y) + (\xi, \eta)$, $(\pi_1(\psi^{-1}(y)) + \xi(\cdot), \pi_2(\psi^{-1}(y)) + \int_0^{\cdot} e^{\pi_1(\psi^{-1}(y)) + \xi(s^-)} d\eta(s))$, and $y \star L$ (where, respectively, $y \in I$, $y \in \mathcal{D}$, $y \in \mathcal{D}$ and $y \in E$) then one obtains a version of X_y .*

As we said above, the results of this subsection are not only explicit versions of Theorems 1.5, 1.6 and 1.8, but they are also the main ingredients for their proofs. Indeed, as we will see, the direct part of Theorem respectively 1.5, 1.6 and 1.8 follows from Proposition 1.15 together with Theorem respectively 1.16, 1.19 and 1.20.

1.4. Sketch of proof and organization of the paper.

In the remaining part of this section we discuss the case of dimension greater or equal to 3 and then we present some examples and particular cases of gssMp's, for which the results above take a simple or nice form.

In Section 2 we prove some preliminary results that are crucial for the rest of the paper. More precisely we prove Proposition 2.1 which is a compatibility relation that is satisfied by invariance components of a gssMp. We then use that relation to prove Proposition 1.15 (the existence of good invariance components).

In Section 3 we first prove Proposition 1.11. Then, we use the compatibility relation from Proposition 2.1 to prove Proposition 3.1 which makes appear a group structure on the state space of a gssMp X that admits good invariance components. Then, we prove Proposition 3.3 which identifies X with a time-changed Lévy process on its state space equipped with the above-mentioned group structure. We then prove Theorems 1.20 and 1.8.

In Section 4 we construct some explicit Lie group isomorphisms between the Lie groups constructed in the previous section (when the state space is an open interval of \mathbb{R} or an open simply connected domain of \mathbb{R}^2) and some canonical Lie groups in dimension 1 and 2. Thanks to that, a gssMp on an open interval of \mathbb{R} , or on an open simply connected domain of \mathbb{R}^2 , can be expressed explicitly as a function of a time-changed Lévy process on one of these canonical groups.

In Section 5 we prove Proposition 5.1 that gives a convenient expression of Lévy processes on the most complicated of these canonical Lie groups, in term of exponential functionals of bivariate Lévy processes.

In Section 6 we put the pieces together to prove Theorems 1.16 and 1.19. Then, Theorems 1.5 and 1.6 follow from those theorems together with Proposition 1.15.

In Section 7 we prove some technical lemmas about the assumptions discussed in the end of Subsection 1.1.

1.5. The case of dimension greater or equal to 3. We now briefly describe how the present methodology can be generalized in order to obtain results analogous to Theorems 1.16, 1.5, 1.19 and 1.6 in dimension $d \geq 3$.

In the case where the domain of a gssMp X is \mathcal{D} , a simply connected open subset of \mathbb{R}^d for some $d \geq 3$, one can still apply the same methodology: under some assumptions, Proposition 1.15 allows to build good invariance components and Theorem 1.20 to identify a Lie group structure (\mathcal{D}, \star) of dimension d on \mathcal{D} for which X is a time-changed Lévy process. Then, one has to choose a 'canonical' representative for each isomorphism class of Lie group structure of dimension d , to build an explicit isomorphism between (\mathcal{D}, \star) and its representative (in particular one needs to be able to identify the isomorphism class of (\mathcal{D}, \star) from the good invariance components of X) and to obtain a convenient expression for Lévy processes on each of these representative Lie groups.

We thus see that in dimension d , there are as many possibilities as the number of Lie group structures of dimension d , up to isomorphism. For $d = 1$ this number is 1 (which is why there is only one case in Theorem 1.16), for $d = 2$ this number is 2 (which corresponds to the two cases in Theorem 1.19).

1.6. Some examples and particular cases. A basic example is the one of a pssMp X with index $\alpha \in \mathbb{R}$. In that case, the state space is the interval $E =]0, +\infty[$. By definition of a pssMp of index α , a family of invariance components $((f_y, c_y), y \in]0, +\infty[)$, with respect to the reference

point 1, is given by $f_y(x) = yx$ and $c_y = y^{-\alpha}$. It is immediate that this family of invariance components is *good* (i.e. it satisfies Definition 1.14) and of class \mathcal{C}^∞ . Then, the function g defined in (1.18) is equal to $\log(\cdot)$ and the real number α defined in (1.18) coincides with the similarity index α of the pssMp X . The application of Theorem 1.16 yields that, provided that Assumption 1 or Assumption 2 is satisfied for $E =]0, +\infty[$, we have

$$\zeta(X_{y_0}) = \varphi(+\infty) \quad \text{and} \quad \forall 0 \leq t < \varphi(+\infty), \quad X_1(t) = \exp(\xi(\varphi^{-1}(t))),$$

where ξ is a real Lévy process and $\varphi(t) := \int_0^t e^{\alpha\xi(s)} ds$. We have thus retrieved Lamperti's representation for X .

Another interesting particular case of gssMp's is the case where the time-changes are trivial, i.e. there exists a family of invariance components $((f_y, c_y), y \in E)$ for X such that $\forall y \in E, c_y = 1$. In this case the self-similarity relation simply becomes $\forall y \in E, X_y \stackrel{\mathcal{L}}{=} f_y(X_{y_0})$. Replacing f_y by $f_y \circ f_{y_0}^{-1}$ if necessary, it is plain that the family $((f_y, 1), y \in E)$ satisfies Definition 1.14, so it is a family of good invariance components.

- When $E = I$, an open interval of \mathbb{R} . If the family of good invariance components $((f_y, 1), y \in I)$ is \mathcal{C}^k for some $k \geq 1$, and if Assumption 1 or Assumption 2 is satisfied for $E = I$, then Theorem 1.16 applies, and the α defined in (1.18) clearly equals 0 so that the time-change function is trivial : $\varphi(t) = t$. Therefore, X_{y_0} is only the image by g^{-1} (where g is defined in (1.18)) of a real Lévy process ξ .
- When $E = \mathcal{D}$, an open simply connected domain of \mathbb{R}^2 . If the family of good invariance components $((f_y, 1), y \in \mathcal{D})$ is \mathcal{C}^k for some $k \geq 2$, if Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{D}$, and if $Sym(X_{y_0})$ is discrete, then Theorem 1.19 applies, and the α defined there equals 0 so that the time-change function is trivial : $\varphi(t) = t$. Therefore, if $((f_y, 1), y \in \mathcal{D})$ are *commutative*, X_{y_0} is the image by g^{-1} (where g is defined in (1.23)) of (ξ, η) , that is a Lévy process on \mathbb{R}^2 . If $((f_y, 1), y \in I)$ are *not commutative*, X_{y_0} is the image by g^{-1} (where g is defined in Lemma 4.4) of $(\xi(\cdot), \int \cdot e^{\xi(s^-)} d\eta(s))$, where (ξ, η) is a Lévy process on \mathbb{R}^2 .
- In a more general context, if the assumptions of Theorem 1.20 are satisfied (E is connected, Assumption 1 or Assumption 2 is satisfied, and $Sym(X_{y_0})$ is discrete), then the theorem applies and we see that the time-change function is trivial : $\varphi(t) = t$. Therefore X_{y_0} is a Lévy process on E , equipped with the group structure defined in the theorem.

2. COMPATIBILITY RELATION AND CONSTRUCTION OF GOOD COMPONENTS

We now prove a key result that yields a compatibility relation satisfied by invariance components of a gssMp. This result is important to establish Proposition 1.15 and then to make appear the structure of group in Proposition 3.1. In the logic of Remark 1.2, note that in the next two propositions, their proofs, and where they are applied, for the case where the processes involved reach the cemetery point Δ in finite time, we actually work with the extension of homeomorphisms to $E \cup \{\Delta\}$ that send the cemetery point Δ on itself (but, for example, the extension of $f_y(\cdot)$ is still denoted by $f_y(\cdot)$ for simplicity).

Proposition 2.1. *Let E be a locally compact separable metric space, and let X be a gssMp on E with invariance components $((f_y, c_y), y \in E)$ relatively to some reference point $y_0 \in E$. We assume that Assumption 1 is satisfied. Then we have*

$$\forall y_1, y_2 \in E, \quad f_{f_{y_1}(y_2)} \left(X_{y_0}(c_{f_{y_1}(y_2)} \times \cdot) \right) \stackrel{\mathcal{L}}{=} f_{y_1} \circ f_{y_2} \left(X_{y_0}(c_{y_1} \times c_{y_2} \times \cdot) \right).$$

Proof. Assume that X and $((f_y, c_y), y \in E)$ are as in the statement of the proposition. Let $y_1, y_2 \in E$ be arbitrary. By self-similarity we have $Supp(X_{y_1}) = f_{y_1}(Supp(X_{y_0}))$. Because of

Assumption 1 $Supp(X_{y_0}) = E$ so, since f_{y_1} is an homeomorphism, we get $Supp(X_{y_1}) = E$. Let us define $a := f_{y_1}(y_2)$ and fix $\epsilon > 0$. Since $a \in E = Supp(X_{y_1})$, and by definition of $Supp(X_{y_1})$, there exists $s_\epsilon \geq 0$ such that $P(X_{y_1}(s_\epsilon) \in B(a, \epsilon)) > 0$. Let us fix such a $s_\epsilon \geq 0$. We consider the process $(X_{y_1}(s_\epsilon + t), t \geq 0)$ conditionally on the event $\{X_{y_1}(s_\epsilon) \in B(a, \epsilon)\}$ that has positive probability. Let \tilde{X}_{y_0} and \hat{X}_{y_0} be independent with law P_{y_0} , and be independent of X_{y_1} . On one hand, by the Markov property at s_ϵ and the self-similarity we have

$$\mathcal{L}(X_{y_1}(s_\epsilon + \cdot) | X_{y_1}(s_\epsilon) \in B(a, \epsilon)) = \mathcal{L}\left(f_{X_{y_1}(s_\epsilon)}\left(\tilde{X}_{y_0}(c_{X_{y_1}(s_\epsilon)} \times \cdot)\right) | X_{y_1}(s_\epsilon) \in B(a, \epsilon)\right). \quad (2.31)$$

On the other hand, by applying the self-similarity at the starting point y_1 , and then the Markov property at time $c_{y_1}s_\epsilon$ and the self-similarity at $\hat{X}_{y_0}(c_{y_1}s_\epsilon)$, we have

$$\begin{aligned} & \mathcal{L}(X_{y_1}(s_\epsilon + \cdot) | X_{y_1}(s_\epsilon) \in B(a, \epsilon)) \\ &= \mathcal{L}\left(f_{y_1}\left(\hat{X}_{y_0}(c_{y_1} \times (s_\epsilon + \cdot))\right) | f_{y_1}\left(\hat{X}_{y_0}(c_{y_1}s_\epsilon)\right) \in B(a, \epsilon)\right) \\ &= \mathcal{L}\left(f_{y_1}\left(\hat{X}_{y_0}(c_{y_1} \times (s_\epsilon + \cdot))\right) | \hat{X}_{y_0}(c_{y_1}s_\epsilon) \in f_{y_1}^{-1}(B(a, \epsilon))\right) \\ &= \mathcal{L}\left(f_{y_1} \circ f_{\hat{X}_{y_0}(c_{y_1}s_\epsilon)}\left(\tilde{X}_{y_0}(c_{y_1} \times c_{\hat{X}_{y_0}(c_{y_1}s_\epsilon)} \times \cdot)\right) | \hat{X}_{y_0}(c_{y_1}s_\epsilon) \in f_{y_1}^{-1}(B(a, \epsilon))\right). \end{aligned} \quad (2.32)$$

Let ϵ decrease to 0 via a countable sequence (for each of these ϵ we can chose an $s_\epsilon \geq 0$ such that $P(X_{y_1}(s_\epsilon) \in B(a, \epsilon)) > 0$, and the above procedure shows that (2.31) and (2.32) hold). In the right hand side of (2.31), we have the law of a function of the process \tilde{X}_{y_0} and of an independent random variable A_ϵ where $\mathcal{L}(A_\epsilon) = \mathcal{L}(X_{y_1}(s_\epsilon) | X_{y_1}(s_\epsilon) \in B(a, \epsilon))$. Clearly, A_ϵ converges in distribution to a when ϵ goes to 0. Using Slutsky's Lemma and the continuity of $(y, x) \mapsto f_y(x)$ and of $y \mapsto c_y$ assumed in Definition 1.1, we deduce that the right hand side of (2.31) converges (in the sens of finite-dimensional distributions) to $\mathcal{L}(f_a(\tilde{X}_{y_0}(c_a \times \cdot)))$.

Similarly, in the right hand side of (2.32), we have the law of a function of the process \tilde{X}_{y_0} and of an independent random variable B_ϵ where $\mathcal{L}(B_\epsilon) = \mathcal{L}(\hat{X}_{y_0}(c_{y_1}s_\epsilon) | \hat{X}_{y_0}(c_{y_1}s_\epsilon) \in f_{y_1}^{-1}(B(a, \epsilon)))$. By continuity of $f_{y_1}^{-1}$, B_ϵ converges in distribution to $f_{y_1}^{-1}(a)$ when ϵ goes to 0. Using Slutsky's Lemma and the continuity of $(y, x) \mapsto f_y(x)$ and of $y \mapsto c_y$ assumed in Definition 1.1, we deduce that the right hand side of (2.32) converges (in the sens of finite-dimensional distributions) to $\mathcal{L}(f_{y_1} \circ f_{f_{y_1}^{-1}(a)}(\tilde{X}_{y_0}(c_{y_1} \times c_{f_{y_1}^{-1}(a)} \times \cdot)))$.

Since the laws in (2.31) and (2.32) are equal, we can identify the limits of the right hand sides of both expressions. We get

$$\mathcal{L}\left(f_a\left(\tilde{X}_{y_0}(c_a \times \cdot)\right)\right) = \mathcal{L}\left(f_{y_1} \circ f_{f_{y_1}^{-1}(a)}\left(\tilde{X}_{y_0}(c_{y_1} \times c_{f_{y_1}^{-1}(a)} \times \cdot)\right)\right).$$

Then, recalling that $a = f_{y_1}(y_2)$ and \tilde{X}_{y_0} follows the law P_{y_0} , we obtain the asserted result. \square

We can now prove Proposition 1.15 that provides the existence of good invariance components.

Proof. of Proposition 1.15

For $k \geq 1$, X is a \mathcal{C}^k -gssMp. Then, let $((f_y, c_y), y \in \mathcal{M})$ be a family of \mathcal{C}^k invariance components relatively to the reference point y_0 (a family of \mathcal{C}^k invariance components exists by definition of X being a \mathcal{C}^k -gssMp, and the reference point can be set at y_0 thanks to the property of change of reference point mentioned in the Introduction, the change of reference point indeed preserves the fact that the family of invariance components is \mathcal{C}^k). Replacing, if necessary, $((f_y, c_y), y \in \mathcal{M})$ by $((f_y \circ f_{y_0}^{-1}, c_y/c_{y_0}), y \in \mathcal{M})$, we can assume that the family of

invariance components satisfy (1.16). Let G be the subgroup of $Hom(\mathcal{M}) \times \mathbb{R}_+^*$ that contains the couples $(\Psi, \lambda) \in Hom(\mathcal{M}) \times \mathbb{R}_+^*$ for which (1.3) is satisfied, and let H be the subgroup of \mathbb{R}_+^* defined by $H := \{\lambda \in \mathbb{R}_+^*, \exists \Psi \in Hom(\mathcal{M}) \text{ s.t. } (\Psi, \lambda) \in G\}$. We have assumed that Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{M}$. Thanks to Lemma 7.1 we have that, in any case, Assumption 1 is satisfied, so Proposition 2.1 applies and yields that

$$\forall y_1, y_2 \in E, U(y_1, y_2) := \left(f_{f_{y_1}(y_2)}^{-1} \circ f_{y_1} \circ f_{y_2}, c_{y_1} \times c_{y_2} / c_{f_{y_1}(y_2)} \right) \in G.$$

In particular the range of the function $U_2 : (y_1, y_2) \mapsto c_{y_1} \times c_{y_2} / c_{f_{y_1}(y_2)}$ is included in H . Recall from Definition 1.1 and Remark 1.3 that $y \mapsto c_y$, $(y, x) \mapsto f_y(x)$, $y \mapsto f_y(\cdot)$, and $y \mapsto f_y^{-1}(\cdot)$ are continuous, and since the composition is continuous on $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$, we get in particular that U is continuous from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{C}^0(\mathcal{M}, \mathcal{M}) \times \mathbb{R}_+^*$. We now distinguish two cases :

Case 1: U_2 is constant. In that case, evaluating U_2 at a point (x, y_0) (where x is arbitrary) and using (1.16), we obtain that $U_2 \equiv 1$. This yields that (1.17) is satisfied so $((f_y, c_y), y \in \mathcal{M})$ are good invariance components.

Case 2: U_2 is non-constant. In that case, since $\mathcal{M} \times \mathcal{M}$ is connected and U_2 is continuous, we have that $U_2(\mathcal{M} \times \mathcal{M}) \subset H$ is a subinterval of \mathbb{R}_+^* , non-reduced to a single point. Then, H is a subgroup of \mathbb{R}_+^* that contains an interval non reduced to a single point so $H = \mathbb{R}_+^*$. We now construct a continuous function $L : \mathbb{R}_+^* \rightarrow \mathcal{C}^0(\mathcal{M}, \mathcal{M})$ such that $L[1] = id_{\mathcal{M}}$, $(L[\lambda], \lambda) \in G$ for all $\lambda \in \mathbb{R}_+^*$, and $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k .

Since $((f_y, c_y), y \in \mathcal{M})$ are \mathcal{C}^k invariance components, $U_2 : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}_+^*$ is \mathcal{C}^k . Since it is non-constant, there exists $\lambda_0 \in \mathbb{R}_+^*, \epsilon > 0$ and a \mathcal{C}^k function $l_1 :]\lambda_0 e^{-\epsilon}, \lambda_0 e^{\epsilon}[\rightarrow \mathcal{M} \times \mathcal{M}$ such that $\forall \lambda \in]\lambda_0 e^{-\epsilon}, \lambda_0 e^{\epsilon}[$, $U_2(l_1[\lambda]) = \lambda$ (just apply the *Local Inversion Theorem* for a local coordinate of the \mathcal{C}^k manifold $\mathcal{M} \times \mathcal{M}$ along which U_2 is nonconstant, the other coordinates being fixed). Let us define the function $l_2 :]\lambda_0 e^{-\epsilon}, \lambda_0 e^{\epsilon}[\rightarrow \mathcal{C}^0(\mathcal{M}, \mathcal{M})$ by $l_2 := \lambda \mapsto U_1(l_1[\lambda])$, where $U_1(y_1, y_2) = f_{f_{y_1}(y_2)}^{-1} \circ f_{y_1} \circ f_{y_2}$. Then l_2 is continuous, moreover $(x, \lambda) \mapsto l_2[\lambda](x)$ is of class \mathcal{C}^k , and clearly for any $\lambda \in]\lambda_0 e^{-\epsilon}, \lambda_0 e^{\epsilon}[$ we have $U(l_1[\lambda]) = (U_1(l_1[\lambda]), U_2(l_1[\lambda])) = (l_2[\lambda], \lambda)$, so in particular $(l_2[\lambda], \lambda) \in G$.

Then let us define $l_3 :]e^{-\epsilon}, e^{\epsilon}[\rightarrow \mathcal{C}^0(\mathcal{M}, \mathcal{M})$ by $l_3[\lambda] := l_2(\lambda_0 \lambda) \circ (l_2(\lambda_0))^{-1}$. It is not difficult to see that l_3 is continuous, that $(x, \lambda) \mapsto l_3[\lambda](x)$ is of class \mathcal{C}^k , that $l_3[1] = id_{\mathcal{M}}$, and that $(l_3[\lambda], \lambda) \in G$ for any $\lambda \in]e^{-\epsilon}, e^{\epsilon}[$. Let us now define $L_\epsilon : \mathbb{R}_+^* \rightarrow \mathcal{C}^0(E, E)$ by

$$\begin{aligned} L_\epsilon[\lambda] &:= l_3 \left[\exp \left(\log(\lambda) - \epsilon \lfloor 2 \log(\lambda) / \epsilon \rfloor / 2 \right) \right] \circ (l_3[e^{\epsilon/2}])^{\circ \lfloor 2 \log(\lambda) / \epsilon \rfloor} \quad \text{if } \lambda \geq 1, \\ L_\epsilon[\lambda] &:= l_3 \left[\exp \left(\log(\lambda) + \epsilon \lfloor -2 \log(\lambda) / \epsilon \rfloor / 2 \right) \right] \circ (l_3[e^{-\epsilon/2}])^{\circ \lfloor -2 \log(\lambda) / \epsilon \rfloor} \quad \text{if } \lambda \leq 1. \end{aligned}$$

In the above, $(l_3[e^{\epsilon/2}])^{on}$ (respectively $(l_3[e^{-\epsilon/2}])^{on}$) means that the function $l_3[e^{\epsilon/2}] \in \mathcal{C}^0(\mathcal{M}, \mathcal{M})$ (respectively $l_3[e^{-\epsilon/2}] \in \mathcal{C}^0(\mathcal{M}, \mathcal{M})$) is composed n times by itself. Then, clearly, L_ϵ is continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$, $L_\epsilon[1] = id_{\mathcal{M}}$, and $(x, \lambda) \mapsto L_\epsilon[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times]e^{-\epsilon/2}, e^{\epsilon/2}[$ (since it coincides with l_3 on this set) and on each set of the form $\mathcal{M} \times]e^{n\epsilon/2}, e^{(n+1)\epsilon/2}[$ for some $n \in \mathbb{Z}$. Moreover, since $(l_3[r], r) \in G$ for any $r \in]e^{-\epsilon}, e^{\epsilon}[$ and since G is a group, we see that for $\lambda \geq 1$,

$$(L_\epsilon[\lambda], \exp(\log(\lambda) - \epsilon \lfloor 2 \log(\lambda) / \epsilon \rfloor / 2) \times (e^{\epsilon/2})^{\lfloor 2 \log(\lambda) / \epsilon \rfloor}) \in G,$$

that is, $(L_\epsilon[\lambda], \lambda) \in G$. Similarly, for $\lambda < 1$,

$$(L_\epsilon[\lambda], \exp(\log(\lambda) + \epsilon \lfloor -2 \log(\lambda) / \epsilon \rfloor / 2) \times (e^{-\epsilon/2})^{\lfloor -2 \log(\lambda) / \epsilon \rfloor}) \in G,$$

that is, $(L_\epsilon[\lambda], \lambda) \in G$.

Let us now justify that there is a unique function $L : \mathbb{R}_+^* \rightarrow \mathcal{C}^0(\mathcal{M}, \mathcal{M})$ that satisfies

- A) L is continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$,
- B) $L[1] = id_{\mathcal{M}}$,
- C) $\forall \lambda \in \mathbb{R}_+^*, (L[\lambda], \lambda) \in G$.

The function L_ϵ satisfies the required A), B) and C) so we only need to prove uniqueness.

First, let us prove that a function L that satisfies A), B) and C) also satisfies $L[\lambda^{-1}] = L[\lambda]^{-1}$ for all $\lambda \in \mathbb{R}_+^*$. Indeed, let L satisfy A), B) and C). Using C) and the fact that G is a group we get

$$\forall \lambda \in \mathbb{R}_+^*, (L[\lambda] \circ L[\lambda^{-1}], 1) \in G \text{ and } (L[\lambda^{-1}] \circ L[\lambda], 1) \in G.$$

Therefore, we have that $L[\lambda] \circ L[\lambda^{-1}]$ and $L[\lambda^{-1}] \circ L[\lambda]$ are in $Sym(X_{y_0})$ for any $\lambda \in \mathbb{R}_+^*$. Moreover, $(\lambda \mapsto L[\lambda] \circ L[\lambda^{-1}])$ and $(\lambda \mapsto L[\lambda^{-1}] \circ L[\lambda])$ are continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$ because of A). Since, by assumption, $Sym(X_{y_0})$ is discrete, we deduce that the functions $(\lambda \mapsto L[\lambda] \circ L[\lambda^{-1}])$ and $(\lambda \mapsto L[\lambda^{-1}] \circ L[\lambda])$ are constant. Evaluating at $\lambda = 1$ and using B), we deduce that $(\lambda \mapsto L[\lambda] \circ L[\lambda^{-1}]) = (\lambda \mapsto L[\lambda^{-1}] \circ L[\lambda]) \equiv id_{\mathcal{M}}$. This proves the claim that $L[\lambda^{-1}] = L[\lambda]^{-1}$. In particular, $(\lambda \mapsto L[\lambda]^{-1})$ is continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$. Note that the above argument can be adapted to show that $(\lambda \mapsto L[\lambda])$ is a group homomorphism from (\mathbb{R}_+^*, \times) to $(Hom(\mathcal{M}), \circ)$.

We now prove uniqueness of the function satisfying A), B) and C). Let L and \tilde{L} be two functions from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$ that both satisfy A), B) and C). Using C) and the fact that G is a group we get $\forall \lambda \in \mathbb{R}_+^*, (\tilde{L}[\lambda] \circ L[\lambda]^{-1}, 1) \in G$, so $\tilde{L}[\lambda] \circ L[\lambda]^{-1} \in Sym(X_{y_0})$ for any $\lambda \in \mathbb{R}_+^*$. We have seen that, since L satisfies A), B) and C), $(\lambda \mapsto L[\lambda]^{-1})$ is continuous. Combining with the fact that \tilde{L} satisfies A) we get that $\lambda \mapsto \tilde{L}[\lambda] \circ L[\lambda]^{-1}$ is continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$. Since, by assumption, $Sym(X_{y_0})$ is discrete, we deduce that $(\lambda \mapsto \tilde{L}[\lambda] \circ L[\lambda]^{-1})$ is constant. Evaluating at $\lambda = 1$ and using B), we deduce that $(\lambda \mapsto \tilde{L}[\lambda] \circ L[\lambda]^{-1}) \equiv id_{\mathcal{M}}$. Therefore $L = \tilde{L}$ so the uniqueness follows.

Now, let L be the unique continuous function from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$ that satisfies $L[1] = id_{\mathcal{M}}$, and $(L[\lambda], \lambda) \in G$ for all $\lambda \in \mathbb{R}_+^*$. We have seen that $L = L_\epsilon$ so in particular $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times]e^{-\epsilon/2}, e^{\epsilon/2}[$ and on each set of the form $\mathcal{M} \times]e^{n\epsilon/2}, e^{(n+1)\epsilon/2}[$ for some $n \in \mathbb{Z}$. To prove that $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times \mathbb{R}_+^*$, we thus only need to justify that for each $n \in \mathbb{Z} \setminus \{0\}$, there is an open interval $\mathcal{I}_n \ni e^{n\epsilon/2}$ such that $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times \mathcal{I}_n$. Let us chose $\epsilon' \in]0, \epsilon[$ such that ϵ'/ϵ is irrational. We can build $L_{\epsilon'}$ similarly as we built L_ϵ . In particular $L_{\epsilon'}$ is continuous from \mathbb{R}_+^* to $\mathcal{C}^0(\mathcal{M}, \mathcal{M})$, $L_{\epsilon'}[1] = id_{\mathcal{M}}$, $\forall \lambda \in \mathbb{R}_+^*, (L_{\epsilon'}[\lambda], \lambda) \in G$, and $(x, \lambda) \mapsto L_{\epsilon'}[\lambda](x)$ is of class \mathcal{C}^k on each set of the form $\mathcal{M} \times]e^{m\epsilon'/2}, e^{(m+1)\epsilon'/2}[$ for some $m \in \mathbb{Z}$. Since ϵ'/ϵ is irrational, we have that for any $n \in \mathbb{Z} \setminus \{0\}$, there is $m_n \in \mathbb{Z}$ such that $e^{n\epsilon/2} \in]e^{m_n\epsilon'/2}, e^{(m_n+1)\epsilon'/2}[=: \mathcal{I}_n$. Clearly $(x, \lambda) \mapsto L_{\epsilon'}[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times \mathcal{I}_n$, but by uniqueness we have $L = L_{\epsilon'}$, so $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times \mathcal{I}_n$. We conclude that $(x, \lambda) \mapsto L[\lambda](x)$ is of class \mathcal{C}^k on $\mathcal{M} \times \mathbb{R}_+^*$.

Then, for all $y \in \mathcal{M}$, let us define $\tilde{f}_y := f_y \circ L[1/c_y]$. We now justify that the family $((\tilde{f}_y, 1), y \in \mathcal{M})$ defines a family of \mathcal{C}^k invariance components for X . Let us fix $y \in \mathcal{M}$, we have

$$\tilde{f}_y(X_{y_0}) = f_y(L[1/c_y](X_{y_0})) \stackrel{\mathcal{L}}{=} f_y(X_{y_0}(c_y \times \cdot)) \stackrel{\mathcal{L}}{=} X_y,$$

where we have used that $(L[1/c_y], 1/c_y) \in G$ for the first equality in law, and that $((f_z, c_z), z \in \mathcal{M})$ is a family of invariance components for X for the second equality in law. Moreover, since $\tilde{f}_y(x) = f_y(L[1/c_y](x))$ where $(y \mapsto c_y)$, $((y, x) \mapsto f_y(x))$ and $(x, \lambda) \mapsto L[\lambda](x)$ are of class \mathcal{C}^k , we see that $((y, x) \mapsto \tilde{f}_y(x))$ is of class \mathcal{C}^k . Similarly, $\tilde{f}_y^{-1}(x) = L[1/c_y]^{-1}(f_y^{-1}(x)) = L[c_y](f_y^{-1}(x))$ where $(y \mapsto c_y)$, $((y, x) \mapsto f_y^{-1}(x))$ and $(x, \lambda) \mapsto L[\lambda](x)$ are of class \mathcal{C}^k , so $((y, x) \mapsto \tilde{f}_y^{-1}(x))$ is also of class \mathcal{C}^k . We conclude that $((\tilde{f}_y, 1), y \in \mathcal{M})$ is a family of \mathcal{C}^k

invariance components for X . Note that $\tilde{f}_{y_0} = f_{y_0} \circ L[1/c_{y_0}] = id_{\mathcal{M}} \circ L[1] = id_{\mathcal{M}} \circ id_{\mathcal{M}} = id_{\mathcal{M}}$ (we have used that $((f_y, c_y), y \in \mathcal{M})$ satisfies (1.16), as mentioned in the beginning of the proof). Moreover, the time-dilatation constants in $((\tilde{f}_y, 1), y \in \mathcal{M})$ are all equal to 1. Therefore $((\tilde{f}_y, 1), y \in \mathcal{M})$ satisfies Definition 1.14, so $((\tilde{f}_y, 1), y \in \mathcal{M})$ is even a family of \mathcal{C}^k good invariance components for X . □

Remark 2.2. *The above proof actually shows a little more than the statement of Proposition 1.15 and can possibly be used to produce good invariance components in practice. It shows that, for X a gssMp that satisfies the assumptions of the proposition, if we pick a family $((f_y, c_y), y \in \mathcal{M})$ of \mathcal{C}^k invariance components, then we have the following dichotomy: either this family is a family of **good** invariance components, either this family can be canonically modified (via composition by some functions $L[\lambda](\cdot)$ that are uniquely determined) to produce a family of \mathcal{C}^k **good** invariance components $((\tilde{f}_y, 1), y \in \mathcal{M})$ for which the changes of time are trivial. Recall that, as mentioned in Subsection 1.6, the case where changes of times are trivial is particularly interesting, because our results show that in that case X_{y_0} is only a Lévy process on some group (in the context of Theorem 1.20), or the image by some function of a canonical Lévy process or of a process built from the exponential functional of a bivariate Lévy process (in the context of Theorems 1.16 and 1.19), with no time-change. In particular, if X satisfies the assumptions of the proposition and is given along with a family of \mathcal{C}^k invariance components, then, if the family is not good we can already conclude that X_{y_0} is as just described.*

3. RELATION WITH GROUPS

First, let us prove Proposition 1.11 which says that it is possible to construct gssMp's when the state space is already equipped with a group structure.

Proof. of Proposition 1.11

Let us fix $y \in E$ and let φ_y be defined by (1.14). Since $(s \mapsto L(s))$ is càd-làg and h is continuous and positive, the function φ_y is well-defined, continuous, increasing and defines a bijection from $[0, +\infty[$ onto $[0, \varphi_y(+\infty)[$. As a consequence φ_y^{-1} is well-defined on $[0, \varphi_y(+\infty)[$ and continuous (we set $\varphi_y^{-1}(t) := +\infty$ for $t \geq \varphi_y(+\infty)$). This implies in particular that X_y , defined by (1.15), is well-defined and càd-làg on $[0, \varphi_y(+\infty)[$.

Let us now justify that $\zeta(X_y) = \varphi_y(+\infty)$ a.s. Since φ_y^{-1} is well-defined and continuous on $[0, \varphi_y(+\infty)[$, we have that for any $t \in [0, \varphi_y(+\infty)[$, the interval $[0, \varphi_y^{-1}(t)]$ is compact. Since L is càd-làg, and since $(z \mapsto y \star z)$ is an homeomorphism, the set $y \star L([0, \varphi_y^{-1}(t)])$ is relatively compact, which show that the trajectory of X_y on $[0, t]$ is contained inside a compact set. Therefore, $t < \zeta(X_y)$, and since this is true a.s. for any $t \in [0, \varphi_y(+\infty)[$ we get $\varphi_y(+\infty) \leq \zeta(X_y)$ a.s. This proves that we have a.s. $\varphi_y(+\infty) = \zeta(X_y)$ whenever $\varphi_y(+\infty) < +\infty$. Let $(K_n)_{n \geq 1}$ be an increasing sequence of compact sets $K_1 \subset K_2 \subset \dots$ such that any compact subset of E is included in K_n for some $n \geq 1$. If $\varphi_y(+\infty) < +\infty$, then $h(y \star L(\cdot))$ takes arbitrary large values on $[0, +\infty[$. Therefore, for any $n \geq 1$ there is a.s. $u_n \in [0, +\infty[$ such that $h(y \star L(u_n)) > \sup_{z \in K_n} h(z)$. We thus have $\tau(y \star L(\cdot), K_n^c) \leq u_n < +\infty$ a.s. Since φ_y^{-1} is increasing we deduce that $\tau(X_y, K_n^c) = \tau(y \star L(\varphi_y^{-1}(\cdot)), K_n^c) = \varphi_y(\tau(y \star L(\cdot), K_n^c)) \leq \varphi_y(u_n) < \varphi_y(+\infty)$. We deduce that $\zeta(X_y) = \lim_{n \rightarrow +\infty} \tau(X_y, K_n^c) \leq \varphi_y(+\infty)$ a.s. In conclusion $\varphi_y(+\infty) = \zeta(X_y)$ a.s. which proves in particular that Δ can only be reached continuously by X_y , so X_y is càd-làg on $[0, +\infty[$.

We denote by P_y the law of the process X_y . Let respectively $(\mathcal{G}_t, t \geq 0)$ and $(\mathcal{F}_t, t \geq 0)$ be the right continuous filtrations generated by the processes L and X_y . Note that for any $t \geq 0$, $\varphi_y^{-1}(t)$ is a (possibly infinite) stopping time for L and that $\mathcal{G}_{\varphi_y^{-1}(t)} = \mathcal{F}_t$.

We now justify that X_y satisfies the Markov property at every instant $t \geq 0$. Let us fix an instant $t \geq 0$ and work on $\{\varphi_y^{-1}(t) < +\infty\} = \{X_y(t) \neq \Delta\}$ (since Δ is an absorbing state, the case where $X_y(t) = \Delta$ is trivial). Since $\varphi_y^{-1}(t)$ is a stopping time for the Lévy process L on (E, \star) , the process $\tilde{L}(\cdot) := (L(\varphi_y^{-1}(t)))^{-1} \star L(\varphi_y^{-1}(t) + \cdot)$ is independent from $\mathcal{G}_{\varphi_y^{-1}(t)} = \mathcal{F}_t$ and has the same law as L . Now let $\tilde{\varphi}_{e_0}$ and \tilde{X}_{e_0} be constructed from \tilde{L} just as φ_y and X_y are constructed from L : for all $s \geq 0$, $\tilde{\varphi}_{e_0}(s) := \int_0^s \frac{1}{h(\tilde{L}(u))} du$ and for all $s \in [0, \tilde{\varphi}_{e_0}(+\infty)[$, $\tilde{X}_{e_0}(s) := \tilde{L}(\tilde{\varphi}_{e_0}^{-1}(s))$ (and $\tilde{X}_{e_0}(s) := \Delta$ for $s \geq \tilde{\varphi}_{e_0}(+\infty)$). Then, since it is a function of \tilde{L} , \tilde{X}_{e_0} is independent from \mathcal{F}_t and, since \tilde{L} has the same law as L , \tilde{X}_{e_0} has law P_{e_0} . We need to link φ_y and $\tilde{\varphi}_{e_0}$. For any $u \geq 0$ we have from the definition of \tilde{L} and the homomorphism property for h that $h(\tilde{L}(u)) = h(L(\varphi_y^{-1}(t) + u))/h(L(\varphi_y^{-1}(t)))$. Therefore, for any $s \in [0, \tilde{\varphi}_{e_0}(+\infty)[$ we have

$$\begin{aligned} s &= \int_0^{\tilde{\varphi}_{e_0}^{-1}(s)} \frac{1}{h(\tilde{L}(u))} du = \int_0^{\tilde{\varphi}_{e_0}^{-1}(s)} \frac{h(L(\varphi_y^{-1}(t)))}{h(L(\varphi_y^{-1}(t) + u))} du \\ &= \frac{h(y \star L(\varphi_y^{-1}(t)))}{h(y)} \int_0^{\tilde{\varphi}_{e_0}^{-1}(s)} \frac{1}{h(L(\varphi_y^{-1}(t) + u))} du \\ &= \frac{h(X_y(t))}{h(y)} \int_{\varphi_y^{-1}(t)}^{\varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(s)} \frac{1}{h(L(u))} du \\ &= h(X_y(t)) \times [\varphi_y(\varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(s)) - t]. \end{aligned}$$

We thus get $\varphi_y(\varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(s)) = t + s/h(X_y(t))$ from which we deduce that $\varphi_y^{-1}(t + s/h(X_y(t))) = \varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(s)$. As a consequence, for any $s \in [0, \tilde{\varphi}_{e_0}(+\infty)/h(X_y(t))]$ we have that $\varphi_y^{-1}(t + s)$ is finite and equals $\varphi_y^{-1}(t + h(X_y(t))s/h(X_y(t))) = \varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(h(X_y(t)) \times s)$. Then,

$$\begin{aligned} X_y(t + s) &= y \star L(\varphi_y^{-1}(t + s)) = y \star L(\varphi_y^{-1}(t) + \tilde{\varphi}_{e_0}^{-1}(h(X_y(t)) \times s)) \\ &= y \star L(\varphi_y^{-1}(t)) \star \tilde{L}(\tilde{\varphi}_{e_0}^{-1}(h(X_y(t)) \times s)) \\ &= X_y(t) \star \tilde{X}_{e_0}(h(X_y(t)) \times s). \end{aligned}$$

We have obtained that, on $[0, \tilde{\varphi}_{e_0}(+\infty)[$, $X_y(t + \cdot) = X_y(t) \star \tilde{X}_{e_0}(h(X_y(t)) \times \cdot)$, with \tilde{X}_{e_0} independent of \mathcal{F}_t and having law P_{e_0} . We can also see that $\varphi_y(+\infty) = t + \tilde{\varphi}_{e_0}(+\infty)/h(X_y(t))$ so $X_y(t + s) = \Delta$ if and only if $\tilde{X}_{e_0}(h(X_y(t)) \times s) = \Delta$. Combing all this we deduce the Markov property for X_y at time t . The same procedure can be done in the case where t is a stopping time so X_y is strongly Markovian.

We now justify the self-similarity of X . Let φ_{e_0} and X_{e_0} be constructed from L as in (1.14) and (1.15). Then, X_{e_0} has law P_{e_0} . Let $y \in E$ be arbitrary, and define φ_y and X_y from L as in (1.14) and (1.15). Then, X_y has law P_y . We have

$$\forall t \geq 0, \varphi_y(t) = \frac{1}{h(y)} \int_0^t \frac{1}{h(L(s))} ds = \frac{1}{h(y)} \varphi_{e_0}(t),$$

so that $\varphi_y^{-1}(\cdot) = \varphi_{e_0}^{-1}(h(y) \times \cdot)$ and $\varphi_y(+\infty) = \varphi_{e_0}(+\infty)/h(y)$. We then have for all $0 \leq t < \varphi_y(+\infty) = \varphi_{e_0}(+\infty)/h(y)$,

$$X_y(t) = y \star L(\varphi_y^{-1}(t)) = y \star L(\varphi_{e_0}^{-1}(h(y) \times t)) = y \star X_{e_0}(h(y) \times t).$$

As a consequence we have $X_y(\cdot) = y \star X_{e_0}(h(y) \times \cdot)$ on $[0, \varphi_y(+\infty)[= [0, \varphi_{e_0}(+\infty)/h(y)[$ and these two processes reach the state Δ at the same time $\varphi_y(+\infty) = \varphi_{e_0}(+\infty)/h(y)$. We deduce

that

$$(X_y(t), t \geq 0) = (y \star X_{e_0}(h(y)t), t \geq 0),$$

where X_y has law P_y while X_{e_0} has law P_{e_0} (note that we have used the convention $y \star \Delta = \Delta$). Since the above is true for any $y \in E$ we deduce that X is a gssMp with invariance components $((f_y, c_y), y \in E)$ given by $f_y(\cdot) := y \star \cdot$ and $c_y := h(y)$, for the reference point e_0 . Since e_0 is the neutral element of (E, \star) , and since h is an homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) , we see that (1.16) and (1.17) are satisfied so that the family $((f_y, c_y), y \in E)$ actually defines good invariance components. \square

We are now interested in proving a key point for our purpose: for X a gssMp, given along with a family of good invariance components, we want to make appear a group structure on the state space such that X is a time-changed Lévy process on this group. In the following proposition we make appear the group structure.

Proposition 3.1. *Let X be a gssMp on a connected locally compact separable metric space E and let $((f_y, c_y), y \in E)$ be a family of good invariance components for X , relatively to some reference point y_0 . We assume that either Assumption 1 or Assumption 2 is satisfied and that $Sym(X_{y_0})$ is discrete. Let us define an interne composition law \star on E by $y \star x := f_y(x)$. Then (E, \star) is a topological group with neutral element y_0 and $(y \mapsto c_y)$ is a continuous group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) .*

If moreover, for $k \geq 1$, E is a C^k -differentiable manifold and $((f_y, c_y), y \in E)$ are C^k good invariance components, then (E, \star) is even a C^k -Lie group and $(y \mapsto c_y)$ is a C^k -Lie group homomorphism.

Proof. Associativity: This is the key point. According to Proposition 2.1, we have that,

$$\forall y_1, y_2 \in E, f_{f_{y_1}(y_2)}(X_{y_0}) \stackrel{\mathcal{L}}{=} f_{y_1} \circ f_{y_2} \left(X_{y_0} \left(\frac{c_{y_1} \times c_{y_2}}{c_{f_{y_1}(y_2)}} \times \cdot \right) \right).$$

Since $((f_y, c_y), y \in E)$ is a family of good invariance components it satisfies (1.17), so $c_{y_1} \times c_{y_2} / c_{f_{y_1}(y_2)} = 1$. We thus get $f_{f_{y_1}(y_2)}(X_{y_0}(\cdot)) \stackrel{\mathcal{L}}{=} f_{y_1} \circ f_{y_2}(X_{y_0}(\cdot))$, so that

$$\forall y_1, y_2 \in E, U_1(y_1, y_2) := f_{f_{y_1}(y_2)}^{-1} \circ f_{y_1} \circ f_{y_2} \in Sym(X_{y_0}).$$

Recall from Definition 1.1 and Remark 1.3 that $(y, x) \mapsto f_y(x)$, $y \mapsto f_y(\cdot)$, and $y \mapsto f_y^{-1}(\cdot)$ are continuous, and since the composition is continuous on $\mathcal{C}^0(E, E)$, we get in particular that $U_1 : E \times E \rightarrow \mathcal{C}^0(E, E)$ is continuous. Moreover, $E \times E$ is connected. By assumption, $Sym(X_{y_0})$ is discrete so U_1 is constant on $E \times E$. Moreover, since $((f_y, c_y), y \in E)$ is a family of good invariance components it satisfies (1.16) from which we deduce $U_1(y_0, y_0) = id_E$. Therefore $U_1 \equiv id_E$ so we get

$$\forall y_1, y_2 \in E, f_{f_{y_1}(y_2)} = f_{y_1} \circ f_{y_2}.$$

Evaluating the above functions at a point $y_3 \in E$ we obtain exactly $(y_1 \star y_2) \star y_3 = y_1 \star (y_2 \star y_3)$, and the associativity follows.

Neutral element: Using (1.16) we see that for any $y \in E$, $y_0 \star y = f_{y_0}(y) = id_E(y) = y$. Also, evaluating (1.2) at $t = 0$ and using that $X_{y_0}(0)$ and $X_y(0)$ are almost surely equal to respectively y_0 and y , we see that $f_y(y_0) = y$, that is, $y \star y_0 = y$. This proves that y_0 is a neutral element for \star .

Inverse: Clearly we only need to justify the existence of a right-inverse for every $y \in E$. Since f_y is bijective, $f_y^{-1}(y_0)$ is well-defined and it clearly satisfies $y \star f_y^{-1}(y_0) = y_0$.

Continuity: Let us denote by y^{-1} the inverse of y . The continuity of $((y, x) \mapsto y \star x)$ and of $(y \mapsto y^{-1})$ follows from the continuity of $((y, x) \mapsto f_y(x))$ and of $((y, x) \mapsto f_y^{-1}(x))$ assumed in Definition 1.1.

Group homomorphism: The fact that $(y \mapsto c_y)$ is a group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) follows from (1.17), which is satisfied since, by assumption, $((f_y, c_y), y \in E)$ are good invariance components. The continuity of $(y \mapsto c_y)$ is satisfied by definition of invariance components in Definition 1.1.

Smoothness: If E is a \mathcal{C}^k -differentiable manifold and $((f_y, c_y), y \in E)$ are \mathcal{C}^k good invariance components, then it only remains to justify that the applications $((y, x) \mapsto y \star x)$, $(y \mapsto y^{-1})$, and $(y \mapsto c_y)$ are \mathcal{C}^k , but this follows from Definition 1.4. □

Definition 3.2. Let X be a gssMp on a locally compact separable metric space E . Let $((f_y, c_y), y \in E)$ be a family of good invariance components for X , relatively to some reference point y_0 . If the interne composition law \star , defined on E by $y \star x := f_y(x)$, gives rise to a topological group structure on E , then we say that (E, \star) is the **bearing group** of X associated with the good invariance components $((f_y, c_y), y \in E)$.

If (E, \star) is a bearing group for X , then the self-similarity relation can be re-expressed as

$$\forall y \in E, (X_y(t), t \geq 0) \stackrel{\mathcal{L}}{=} (y \star X_{y_0}(c_y t), t \geq 0).$$

In particular, for any starting point z , if $\tilde{X}_{y_0} \sim P_{y_0}$ is independent from X_z and S is a stopping time for X_z , then

$$(X_z(S+t), t \geq 0) \stackrel{\mathcal{L}}{=} (X_z(S) \star \tilde{X}_{y_0}(c_{X_z(S)} t), t \geq 0). \quad (3.33)$$

We can now prove that a gssMp can be identified with a time-changed Lévy process on the group constructed in Proposition 3.1.

Proposition 3.3. Let X be a gssMp on a connected locally compact separable metric space E and let $((f_y, c_y), y \in E)$ be a family of good invariance components for X , relatively to some reference point y_0 . We assume that either Assumption 1 or Assumption 2 is satisfied for E and that $\text{Sym}(X_{y_0})$ is discrete. Let (E, \star) be the bearing group associated with $((f_y, c_y), y \in E)$. Then there is a Lévy process L on (E, \star) such that if we set

$$\forall t \in [0, +\infty], \varphi(t) := \int_0^t \frac{1}{c_{L(s)}} ds, \quad (3.34)$$

then

$$\zeta(X_{y_0}) = \varphi(+\infty) \quad (3.35)$$

and

$$\forall 0 \leq t < \varphi(+\infty), X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (3.36)$$

Moreover, if we let $\xi(\cdot) := \log(1/c_{L(\cdot)})$, then ξ is a real Lévy process and we have $\zeta(X_{y_0}) = \int_0^{+\infty} e^{\xi(s)} ds$.

In the particular case where $c_y = 1, \forall y \in E$, then $X_{y_0} = L$, so X_{y_0} is itself a Lévy process on (E, \star) .

Proof. We define for all $0 \leq t < \zeta(X_{y_0})$, $\phi(t) := \int_0^t c_{X_{y_0}(u)} du$ and $L(t) := X_{y_0}(\phi^{-1}(t))$.

We first justify that L is well-defined. Since $(s \mapsto X_{y_0}(s))$ is càd-làg and $(y \mapsto c_y)$ is continuous and positive, the function ϕ is well-defined, continuous, increasing and defines a bijection from $[0, \zeta(X_{y_0})[$ onto its range, $[0, \phi(\zeta(X_{y_0}))]$, where $\phi(\zeta(X_{y_0}))$ has to be considered as possibly finite (we will prove later that it is actually a.s. infinite). As a consequence ϕ^{-1} is well-defined on $[0, \phi(\zeta(X_{y_0}))]$ and continuous. This implies in particular that L is well-defined on $[0, \phi(\zeta(X_{y_0}))]$ and càd-làg. We see from $L(t) := X_{y_0}(\phi^{-1}(t))$ that $\phi(\zeta(X_{y_0})) = \zeta(L)$. If $\phi(\zeta(X_{y_0})) < +\infty$, let us put $\phi^{-1}(t) = +\infty$ and $L(t) := \Delta$ for $t \geq \phi(\zeta(X_{y_0}))$, so that L is well-defined on $[0, +\infty[$ and càd-làg (we see that the state Δ is a.s. reached continuously). Let respectively $(\mathcal{G}_t, t \geq 0)$ and $(\mathcal{F}_t, t \geq 0)$ be the right continuous filtrations generated by the processes L and X_{y_0} . Note that for any $t \geq 0$, $\phi^{-1}(t)$ is a stopping time for X_{y_0} and that for any $t \geq 0$, $\mathcal{G}_t = \mathcal{F}_{\phi^{-1}(t)}$.

We now prove that L is a Lévy process on (E, \star) , possibly absorbed at Δ . Let us fix an instant $t \geq 0$ and work on $\{L(t) \neq \Delta\} = \{\phi^{-1}(t) < +\infty\}$. According to (3.33), the process $\tilde{X}_{y_0}(\cdot) := (X_{y_0}(\phi^{-1}(t)))^{-1} \star X_{y_0}(\phi^{-1}(t) + \cdot / c_{X_{y_0}(\phi^{-1}(t))})$ is independent from $\mathcal{F}_{\phi^{-1}(t)} = \mathcal{G}_t$ and has law P_{y_0} . Now let $\tilde{\phi}$ and \tilde{L} be constructed from \tilde{X}_{y_0} just as ϕ and L are constructed from X_{y_0} : for all $0 \leq s < \zeta(\tilde{X}_{y_0})$, $\tilde{\phi}(s) := \int_0^s c_{\tilde{X}_{y_0}(u)} du$ and $\tilde{L}(s) := \tilde{X}_{y_0}(\tilde{\phi}^{-1}(s))$. Then, since it is a function of \tilde{X}_{y_0} , \tilde{L} is independent from \mathcal{G}_t and, since \tilde{X}_{y_0} has law P_{y_0} , \tilde{L} has the same law as L . We need to link ϕ and $\tilde{\phi}$. For any $0 \leq u < \zeta(\tilde{X}_{y_0})$ we have from the definition of \tilde{X}_{y_0} and the homomorphism property of $(y \mapsto c_y)$ that $c_{\tilde{X}_{y_0}(u)} = c_{X_{y_0}(\phi^{-1}(t)+u/c_{X_{y_0}(\phi^{-1}(t))})} / c_{X_{y_0}(\phi^{-1}(t))}$. For any $0 \leq s < \tilde{\phi}(\zeta(\tilde{X}_{y_0}))$ we have

$$\begin{aligned} s &= \int_0^{\tilde{\phi}^{-1}(s)} c_{\tilde{X}_{y_0}(u)} du = \int_0^{\tilde{\phi}^{-1}(s)} \frac{c_{X_{y_0}(\phi^{-1}(t)+u/c_{X_{y_0}(\phi^{-1}(t))})}}{c_{X_{y_0}(\phi^{-1}(t))}} du = \int_0^{\tilde{\phi}^{-1}(s)/c_{X_{y_0}(\phi^{-1}(t))}} c_{X_{y_0}(\phi^{-1}(t)+v)} dv \\ &= \int_{\phi^{-1}(t)}^{\phi^{-1}(t)+\tilde{\phi}^{-1}(s)/c_{X_{y_0}(\phi^{-1}(t))}} c_{X_{y_0}(u)} du = \phi\left(\phi^{-1}(t) + \tilde{\phi}^{-1}(s)/c_{X_{y_0}(\phi^{-1}(t))}\right) - t. \end{aligned}$$

We thus get $\phi\left(\phi^{-1}(t) + \tilde{\phi}^{-1}(s)/c_{X_{y_0}(\phi^{-1}(t))}\right) = t + s$ from which we deduce that $\phi^{-1}(t + s)$ is finite and satisfies $c_{X_{y_0}(\phi^{-1}(t))}[\phi^{-1}(t + s) - \phi^{-1}(t)] = \tilde{\phi}^{-1}(s)$. As a consequence, for any $0 \leq s < \tilde{\phi}(\zeta(\tilde{X}_{y_0}))$ we have

$$\begin{aligned} L(t + s) &= X_{y_0}(\phi^{-1}(t + s)) = X_{y_0}\left(\phi^{-1}(t) + c_{X_{y_0}(\phi^{-1}(t))}[\phi^{-1}(t + s) - \phi^{-1}(t)]/c_{X_{y_0}(\phi^{-1}(t))}\right) \\ &= X_{y_0}(\phi^{-1}(t)) \star \tilde{X}_{y_0}\left(c_{X_{y_0}(\phi^{-1}(t))}[\phi^{-1}(t + s) - \phi^{-1}(t)]\right) = X_{y_0}(\phi^{-1}(t)) \star \tilde{X}_{y_0}\left(\tilde{\phi}^{-1}(s)\right) \\ &= L(t) \star \tilde{L}(s). \end{aligned}$$

We have obtained that, on $[0, \tilde{\phi}(\zeta(\tilde{X}_{y_0}))]$, $(L(t))^{-1} \star L(t + \cdot) = \tilde{L}(\cdot)$. We can also see that $\phi(\zeta(X_{y_0})) = t + \tilde{\phi}(\zeta(\tilde{X}_{y_0}))$ so $L(t + s) = \Delta$ if and only if $\tilde{L}(s) = \Delta$. Therefore we have that, on $[0, +\infty[$, $(L(t))^{-1} \star L(t + \cdot) = \tilde{L}(\cdot)$, with \tilde{L} independent of \mathcal{G}_t and having the same law as L . Since we have such a relation for any $t \geq 0$, and combining with the fact that L is càd-làg, we deduce that it is a Lévy process on E , possibly absorbed at Δ .

Let us assume that absorption at Δ can occur with a positive probability: $\mathbb{P}(\zeta(L) < +\infty) > 0$, and recall that, in that case, Δ is reached continuously a.s. on $\{\zeta(L) < +\infty\}$. Note that since $L(t + \cdot)$ reaches Δ at s if and only if \tilde{L} reaches Δ at s , the absorption at Δ occurs independently from the past trajectory (i.e. the fact that the absorption occurs on $[t, t + h[$ is independent from the trajectory of L on $[0, t]$) so in particular, given a compact neighborhood K of y_0 , we can see that L may remain in K on $[0, \zeta(L)[$ with positive probability. This means that, with positive probability, Δ may be not reached continuously which is a contradiction. Therefore L is a.s. never absorbed so it is a regular Lévy process on (E, \star) and we have a.s. $\phi(\zeta(X_{y_0})) = \zeta(L) = +\infty$. In

particular, $\phi^{-1}(t)$ is defined and finite for all $t \geq 0$ (it is a bijection from $[0, +\infty[$ to $[0, \zeta(X_{y_0})]$) so the expression $L(t) = X_{y_0}(\phi^{-1}(t))$ holds for all $t \geq 0$.

We now need to prove that $\phi^{-1} = \varphi$, for φ defined from L as in (3.34). For any $t \geq 0$ we have

$$\phi^{-1}(t) = \int_0^{\phi^{-1}(t)} 1 du = \int_0^t \frac{1}{\phi'(\phi^{-1}(v))} dv = \int_0^t \frac{1}{c_{X_{y_0}(\phi^{-1}(v))}} dv = \int_0^t \frac{1}{c_{L(v)}} dv = \varphi(t), \quad (3.37)$$

where we have used the change of variable $v = \phi(u)$, the definition of ϕ , the definition of L in term of X_{y_0} and ϕ^{-1} , and the definition of φ in (3.34). Taking the limit on both sides in (3.37) when t goes to infinity we get $\zeta(X_{y_0}) = \varphi(+\infty)$, which is (3.35). Combining (3.37) with the expression $L(t) = X_{y_0}(\phi^{-1}(t))$ we obtain (3.36).

Let us now justify the representation of $\zeta(X_{y_0})$ in term of an exponential functional of a real Lévy process. Recall from Proposition 3.1 that $(y \mapsto c_y)$ is a continuous group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) , so $(y \mapsto \log(1/c_y))$ is a continuous group homomorphism from (E, \star) to $(\mathbb{R}, +)$. Therefore, $\xi(\cdot) := \log(1/c_{L(\cdot)})$ is a real valued Lévy process and, re-writing (3.35) in term of ξ we get $\zeta(X_{y_0}) = \int_0^{+\infty} e^{\xi(s)} ds$. \square

Remark 3.4. We note from the previous proof that the change of time $\varphi^{-1}(t)$ in (3.36) has an alternative expression in term of X_{y_0} : for all $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \phi(t) = \int_0^t c_{X_{y_0}(u)} du$.

Putting together Proposition 3.1 and Proposition 3.3 we obtain Theorem 1.20 and, combining with Proposition 1.15, we deduce Theorem 1.8. Let us write all the details for the sake of clarity.

Proof. of Theorem 1.20

Let E be a connected locally compact separable metric space, and X be a gssMp on E . $((f_y, c_y), y \in E)$ is a family of good invariance components associated with X , relatively to some reference point $y_0 \in E$. We assume that either Assumption 1 or Assumption 2 is satisfied for E and that $Sym(X_{y_0})$ is discrete.

Let us define an interne composition law \star on E by $y \star x := f_y(x)$. According to Proposition 3.1, (E, \star) is a topological group with neutral element y_0 , and $(y \mapsto c_y)$ is a continuous group homomorphism from (E, \star) to (\mathbb{R}_+^*, \times) . Therefore, (E, \star) is the **bearing group** for X associated with the good invariance components $((f_y, c_y), y \in E)$. According to Proposition 3.3, there is a Lévy process L , on (E, \star) , such that if we set $\forall t \in [0, +\infty]$, $\varphi(t) := \int_0^t 1/c_{L(s)} ds$, then $\zeta(X_{y_0}) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty) = \zeta(X_{y_0}), \quad X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (3.38)$$

Finally, if moreover, for $k \geq 1$, E is a \mathcal{C}^k -differentiable manifold and $((f_y, c_y), y \in E)$ are \mathcal{C}^k good invariance components, then the last statement in Proposition 3.1 guaranties that (E, \star) is even a \mathcal{C}^k -Lie group and $(y \mapsto c_y)$ is a \mathcal{C}^k -Lie group homomorphism.

Let us now justify Remark 1.21. Clearly we only need to justify that for all $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$. In the above proof, an application of Proposition 3.3 yielded the existence of a process L satisfying the relation (3.38), with $\forall t \in [0, +\infty]$, $\varphi(t) = \int_0^t 1/c_{L(s)} ds$. According to Remark 3.4, this implies that we have $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$, which is the claim. \square

Proof. of Theorem 1.8

Under the assumptions of the theorem, Proposition 1.15 applies so we can produce a family $((f_y, c_y), y \in E)$ of \mathcal{C}^k good invariance components associated with X , relatively to the reference point y_0 . Let us define an interne composition law \star on \mathcal{M} by $y \star x := f_y(x)$ and a \mathcal{C}^k -function $h : \mathcal{M} \rightarrow \mathbb{R}_+$ by $h(y) := c_y$. Then, Theorem 1.20 guaranties that the three points in the statement of Theorem 1.8 are true for this \star and this h .

□

4. EXPLICIT ISOMORPHISMS BETWEEN SOME LIE GROUPS

In the previous section we have made appear a group structure on the space E where a gssMp is defined. If $E = I$ (respectively \mathcal{D}), an open interval of \mathbb{R} (respectively an open simply connected domain of \mathbb{R}^2), this gives rise to a Lie group structure of dimension 1 (respectively 2). In this section we provide some explicit group isomorphisms between the Lie groups obtained and canonical examples of Lie groups in dimension 1 and 2, on which Lévy processes are more classical. This allows to construct the explicit diffeomorphisms appearing in Theorems 1.16 and 1.19.

We consider an open set $\mathcal{O} \subset \mathbb{R}^n$ (typically $I \subset \mathbb{R}$ or $\mathcal{D} \subset \mathbb{R}^2$), together with an intern composition law \star such that (\mathcal{O}, \star) is a group. If the group operations (multiplication and inversion) are of class \mathcal{C}^k with respect to the usual differentiation on \mathbb{R}^n , then we say that (\mathcal{O}, \star) is a \mathcal{C}^k -Lie group. In other words, the differential manifold structure attached with the Lie group (\mathcal{O}, \star) is always the natural one, arising from the fact that \mathcal{O} is an open subset of \mathbb{R}^n .

We first need a technical result. For a function $f : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$, $J_1 f$ (respectively $J_2 f$) denotes the Jacobian matrix of f when it is differentiated with respect to the first (respectively the second) entry. In the particular case where $\mathcal{O} = I$, an open interval of \mathbb{R} , we shall only write $\partial_1 f$ (respectively $\partial_2 f$) for the first (respectively second) partial derivative of f . For a function $g : \mathcal{O} \rightarrow \mathcal{O}$, Jg denotes the Jacobian matrix of g .

Lemma 4.1. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open, let $f : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ be a function and let \star be the interne composition law on \mathcal{O} induced by f . If (\mathcal{O}, \star) is a \mathcal{C}^k -Lie group (let us denote its neutral element by y_0), then $J_2 f(u, v)$ and $[J_2 f(u, v)]^{-1}$ are defined and \mathcal{C}^{k-1} in $(u, v) \in \mathcal{O} \times \mathcal{O}$ and we have*

$$\forall u, v \in \mathcal{O}, [J_2 f(f(u, v), y_0)]^{-1} \cdot J_2 f(u, v) = [J_2 f(v, y_0)]^{-1}. \quad (4.39)$$

Proof. Since, by assumption, (\mathcal{O}, \star) is a \mathcal{C}^k -Lie group, the application $((u, v) \mapsto u \star v) = ((u, v) \mapsto f(u, v))$ is \mathcal{C}^k so its differential with respect to the second entry, $J_2 f(u, v)$, is well-defined at any $(u, v) \in \mathcal{O} \times \mathcal{O}$ and \mathcal{C}^{k-1} . Now let us fix an arbitrary $u \in \mathcal{O}$. The application $l_u := (v \mapsto u \star v) = (v \mapsto f(u, v))$ is bijective with inverse application $l_{u^{-1}} = (v \mapsto u^{-1} \star v)$, where u^{-1} denotes the inverse of u for the group law \star . $l_{u^{-1}}$ is \mathcal{C}^k just as l_u so we get $Id = J_2 f(u^{-1}, f(u, v)) \cdot J_2 f(u, v)$. This justifies that $((u, v) \mapsto [J_2 f(u, v)]^{-1})$ is defined at any $(u, v) \in \mathcal{O} \times \mathcal{O}$ and \mathcal{C}^{k-1} .

The associativity property for \star reads

$$\forall u, v, w \in \mathcal{O}, f(f(u, v), w) = f(u, f(v, w)).$$

Differentiating with respect to w we get

$$\forall u, v, w \in \mathcal{O}, J_2 f(f(u, v), w) = J_2 f(u, f(v, w)) \cdot J_2 f(v, w).$$

Evaluating the latter relation at $w = y_0$ and using that $f(\cdot, y_0) = id_{\mathcal{O}}$ we obtain

$$\forall u, v \in \mathcal{O}, J_2 f(f(u, v), y_0) = J_2 f(u, v) \cdot J_2 f(v, y_0).$$

which yields (4.39).

□

Let I be an open interval of \mathbb{R} and let \star be an intern composition law on I such that (I, \star) is a Lie group. Since we will need to differentiate the group operations, it is convenient to consider the function $f : I \times I \rightarrow I$ associated with the interne composition law \star ($f(y, z) = y \star z$). The following Lemma builds an explicit group isomorphism between (I, \star) and the canonical group $(\mathbb{R}, +)$.

Lemma 4.2. *Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \times I \rightarrow I$ define a \mathcal{C}^k -Lie group structure (I, \star) on I , for some $k \geq 1$. Define $g : I \rightarrow \mathbb{R}$ via*

$$\forall y \in I, g(y) := \int_{y_0}^y \frac{1}{\partial_2 f(y, y_0)} dy, \quad (4.40)$$

where $y_0 \in I$ is the neutral element of (I, \star) . Then g is well-defined, it is a \mathcal{C}^k -diffeomorphism from I to \mathbb{R} , and a \mathcal{C}^k -Lie group isomorphism from (I, \star) to $(\mathbb{R}, +)$.

Proof. If g is the sought isomorphism, then we should have

$$\forall y, z \in I, g(f(y, z)) = g(y) + g(z),$$

so differentiating with respect to z and evaluating at $z = y_0$ we get:

$$g'(y) = g'(y_0) / \partial_2 f(y, y_0).$$

We see that the derivative g' has to be proportional to the function $y \mapsto 1/\partial_2 f(y, y_0)$. This explains why the expression (4.40) is a natural candidate for g . Moreover, note that for any $y \in I$, the application $(z \mapsto f(y, z)) = (z \mapsto y \star z)$ is \mathcal{C}^k and it has an inverse application $(z \mapsto y^{-1} \star z)$ which is also \mathcal{C}^k . Therefore, the derivative of $(z \mapsto f(y, z))$ does not vanish at y_0 for all $y \in I$: $\forall y \in I, \partial_2 f(y, y_0) \neq 0$ and, since $((y, z) \mapsto f(y, z))$ is \mathcal{C}^k , we have that $y \mapsto \partial_2 f(y, y_0)$ is \mathcal{C}^{k-1} . Therefore, the expression of g in (4.40) is well-defined and defines a \mathcal{C}^k mapping from I to \mathbb{R} . Let us thus consider g defined by (4.40).

We now justify that g is a group homomorphism from (I, \star) to $(\mathbb{R}^2, +)$. Let $y, x \in I$, we have

$$\begin{aligned} g(f(y, x)) &= \int_{y_0}^{f(y, x)} \frac{1}{\partial_2 f(z, y_0)} dz \\ &= \int_{y_0}^y \frac{1}{\partial_2 f(z, y_0)} dz + \int_y^{f(y, x)} \frac{1}{\partial_2 f(z, y_0)} dz \\ &= g(y) + \int_y^{f(y, x)} \frac{1}{\partial_2 f(z, y_0)} dz \\ &= g(y) + \int_{y_0}^x \frac{\partial_2 f(y, v)}{\partial_2 f(f(y, v), y_0)} dv, \end{aligned} \quad (4.41)$$

where we have made the change of variable $z = f(y, v)$. Here, the relation (4.39) from Lemma 4.1 reads $\frac{\partial_2 f(y, v)}{\partial_2 f(f(y, v), y_0)} = \frac{1}{\partial_2 f(v, y_0)}$. Therefore, the right hand side of (4.41) equals

$$g(y) + \int_{y_0}^x \frac{1}{\partial_2 f(v, y_0)} dv = g(y) + g(x),$$

where we have used (4.40) to make appear $g(x)$ in the last equality. We have obtained $g(f(y, x)) = g(y) + g(x)$ so g is indeed a group homomorphism from (I, \star) to $(\mathbb{R}, +)$.

We now justify that g is a \mathcal{C}^k diffeomorphism. Clearly, $g'(\cdot) = 1/\partial_2 f(\cdot, y_0)$ is continuous and never vanishes so it has constant sign so g is bijective from I to $g(I)$ and $g(I)$ is an open interval of \mathbb{R} containing $g(y_0) = 0$. Since moreover $g(I)$ is a subgroup of $(\mathbb{R}, +)$ we conclude that

$g(I) = \mathbb{R}$. Therefore, g is bijective from I to \mathbb{R} . Since g is \mathcal{C}^k and its derivative never vanishes, it is even a \mathcal{C}^k diffeomorphism from I to \mathbb{R} . Since g is also a group homomorphism from (I, \star) to $(\mathbb{R}, +)$, it is even a \mathcal{C}^k -Lie group isomorphism from (I, \star) to $(\mathbb{R}, +)$, which concludes the proof. \square

We now consider the two-dimensional case. Let \mathcal{D} be an open simply connected domain of \mathbb{R}^2 and let \star be an intern composition law on \mathcal{D} such that (\mathcal{D}, \star) is a Lie group. Since we will need to differentiate the group operations, it is convenient to introduce $f : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ associated with the interne composition law \star ($f(y, z) := y \star z$).

We now recall some classic facts and notations about Lie groups and their associated Lie algebras, but we formulate everything in the context of (\mathcal{D}, \star) . First let us assume for convenience that (\mathcal{D}, \star) is a \mathcal{C}^∞ -Lie group. Given a \mathcal{C}^∞ -vector field $(V_y, y \in \mathcal{D})$ on \mathcal{D} and a \mathcal{C}^∞ function $h : \mathcal{D} \rightarrow \mathbb{R}$ we define the \mathcal{C}^∞ function $V.h : \mathcal{D} \rightarrow \mathbb{R}$ by $(V.h)(y) := Dh(y)[V_y]$. The commutator of two vector fields U, V is still a vector field that we denote $[U, V]$:

$$\forall h \in \mathcal{C}^\infty(\mathcal{D}, \mathbb{R}), [U, V].h := U.(V.h) - V.(U.h). \quad (4.42)$$

The vector space of \mathcal{C}^∞ -vector fields on \mathcal{D} equipped with $[\cdot, \cdot]$ is a Lie algebra. A vector field $(V_y, y \in \mathcal{D})$ on \mathcal{D} is called left-invariant if $\forall y, z \in \mathcal{D}, V_{y \star z} = J_2 f(y, z)[V_z]$. For any $u \in \mathbb{R}^2$, we can define a left-invariant vector field $V(u)$ by $V(u)_y := J_2 f(y, y_0)[u]$. It can be seen that $(u \mapsto V(u))$ is a linear isomorphism (we will denote its inverse by V^{-1}) so the vector space of left-invariant vector fields is isomorphic to \mathbb{R}^2 . Moreover, it is a well known property of Lie groups that the commutator of two left-invariant vector fields is still a left-invariant vector field. In particular we have

$$[V(u), V(v)]_y = J_2 f(y, y_0).[V(u), V(v)]_{y_0}, \quad \forall y \in \mathcal{D}, u, v \in \mathbb{R}^2. \quad (4.43)$$

Using the correspondance between \mathbb{R}^2 and left-invariant vector fields we can define a Lie bracket on \mathbb{R}^2 by

$$\forall u, v \in \mathbb{R}^2, [u, v]_L := V^{-1}([V(u), V(v)]). \quad (4.44)$$

Then, by construction, the Lie algebra $(\mathbb{R}^2, [\cdot, \cdot]_L)$ is isomorphic to the Lie algebra of left-invariant vector fields, and is usually called the Lie algebra associated with the Lie group \mathcal{D} .

Also, using the definition of the vector fields $V(u), V(v)$ and the definition (4.42) of the commutator, we can give an explicit expression for the vector field $[V(u), V(v)]$ in term of $f(\cdot, \cdot)$: for any $y \in \mathcal{D}$ and $u, v \in \mathbb{R}^2$ we have

$$[V(u), V(v)]_y = J_1 J_2 f(y, y_0) [J_2 f(y, y_0).u, v] - J_1 J_2 f(y, y_0) [J_2 f(y, y_0).v, u]. \quad (4.45)$$

In the context where (\mathcal{D}, \star) is only a \mathcal{C}^k -Lie group for some $k \geq 2$, the left-invariant vector fields $V(u)$ can still be defined by $V(u)_y := J_2 f(y, y_0)[u]$ and they are \mathcal{C}^{k-1} . The commutator $[V(u), V(v)]$ can still be defined via (4.42) (and its action on $\mathcal{C}^k(\mathcal{D}, \mathbb{R})$) and we still have (4.45) by direct computation. Finally, it is still possible to see that $[V(u), V(v)]$ is left-invariant so the relation (4.43) still holds and the Lie algebra $(\mathbb{R}^2, [\cdot, \cdot]_L)$ can still be defined by (4.44).

Using the so-called *exponential map* it can be seen that the Lie group (\mathcal{D}, \star) is commutative if and only if the Lie bracket $[\cdot, \cdot]_L$ of its associated Lie algebra $(\mathbb{R}^2, [\cdot, \cdot]_L)$ is trivial (i.e. $[u, v]_L = 0$ for all $u, v \in \mathbb{R}^2$).

Let us now investigate the possible forms of the Lie bracket $[\cdot, \cdot]_L$, to deduce another expression for $[V(u), V(v)]_{y_0}$. Let $\{e_1, e_2\}$ be the canonical basis in \mathbb{R}^2 , so that every u and v in \mathbb{R}^2 can be

decomposed as $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$. Then we have

$$\begin{aligned} [u, v]_L &= [u_1e_1 + u_2e_2, v_1e_1 + v_2e_2]_L \\ &= u_1v_1[e_1, e_1]_L + u_1v_2[e_1, e_2]_L + u_2v_1[e_2, e_1]_L + u_2v_2[e_2, e_2]_L \\ &= (u_1v_2 - u_2v_1)[e_1, e_2]_L \\ &= \det(u, v)[e_1, e_2]_L, \end{aligned}$$

where we have used the bi-linearity of $[\cdot, \cdot]_L$ and the fact that it is anti-symmetric. Let us set $Y_0 := [V(e_1), V(e_2)]_{y_0}$ which can be computed explicitly using (4.45). Then for any $u, v \in \mathbb{R}^2$ we have $[V(u), V(v)]_{y_0} = V([u, v]_L)_{y_0} = \det(u, v)V([e_1, e_2]_L)_{y_0} = \det(u, v)[V(e_1), V(e_2)]_{y_0} = \det(u, v)Y_0$. Thus,

$$\forall u, v \in \mathbb{R}^2, [V(u), V(v)]_{y_0} = \det(u, v)Y_0. \quad (4.46)$$

Note in particular that we have $Y_0 = 0$ if and only if (\mathcal{D}, \star) is commutative.

We have now all the tools required to prove the following Lemmas 4.3 and 4.4 that build an explicit group isomorphism between (\mathcal{D}, \star) and one of the two canonical Lie group structures on \mathbb{R}^2 . The first canonical Lie group structure on \mathbb{R}^2 is of course $(\mathbb{R}^2, +)$, and the second is (\mathbb{R}^2, T) , defined as follows:

$$\forall \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2, yTz := \begin{pmatrix} y_1 + z_1 \\ y_2 + e^{y_1}z_2 \end{pmatrix}. \quad (4.47)$$

It is easy to see that T gives rise to a group structure on \mathbb{R}^2 and that (\mathbb{R}^2, T) is not commutative. As a consequence, the two-dimensional Lie groups (\mathcal{D}, \star) that we are considering will be isomorphic to $(\mathbb{R}^2, +)$ or (\mathbb{R}^2, T) depending on whether they are commutative or not.

Lemma 4.3. *Let $\mathcal{D} \subset \mathbb{R}^2$ be an open simply connected domain, and let $f : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ define a commutative \mathcal{C}^k -Lie group structure (\mathcal{D}, \star) on \mathcal{D} , for some $k \geq 2$. Fix $M \in GL_2(\mathbb{R})$ and define $g : \mathcal{D} \rightarrow \mathbb{R}^2$ via*

$$\forall y \in \mathcal{D}, g(y) := \int_a^b M \cdot [J_2f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds, \quad (4.48)$$

where $y_0 \in \mathcal{D}$ is the neutral element of (\mathcal{D}, \star) , $J_2f(z, \cdot)$ denotes the Jacobian matrix of the function $f(z, \cdot) : \mathcal{D} \rightarrow \mathcal{D}$, and where $\gamma : [a, b] \rightarrow \mathcal{D}$ is any path, locally \mathcal{C}^1 , with $\gamma(a) = y_0$ and $\gamma(b) = y$. Then g is well-defined, it is a \mathcal{C}^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 , we have $Jg(y_0) = M$, and g is a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$.

Proof. If g is the sought isomorphism, then, reasoning as in the beginning of the proof of Lemma 4.2, we see that the Jacobian of g has to be $y \mapsto Q \cdot (J_2f(y, y_0))^{-1}$ where Q is an invertible 2×2 matrix. This explains why the expression (4.48) is a natural candidate for g . Moreover, thanks to Lemma 4.1 applied with $(u, v) = (y, y_0)$, we can see that $J_2f(y, y_0)$ is invertible for any $y \in \mathcal{D}$ and that $y \mapsto (J_2f(y, y_0))^{-1}$ is of class \mathcal{C}^{k-1} . To prove that g is well-defined and \mathcal{C}^k , we thus only need to prove the existence of a primitive function for $y \mapsto M \cdot (J_2f(y, y_0))^{-1}$. The idea is to apply Poincaré's Lemma.

For any $y \in \mathcal{D}$, let us denote by L_y the linear application that sends $z \in \mathbb{R}^2$ to $M \cdot (J_2f(y, y_0))^{-1} \cdot z \in \mathbb{R}^2$. We wish to apply Poincaré's Lemma to the differential forms $y \mapsto \pi_1 \circ L_y$ and $y \mapsto \pi_2 \circ L_y$.

Let us show that these differential forms are closed. For $i \in \{1, 2\}$ and $y \in \mathcal{D}$ we have

$$\begin{aligned}
 & \frac{d}{dy_2} [\pi_i \circ L_y(e_1)] - \frac{d}{dy_1} [\pi_i \circ L_y(e_2)] = \pi_i \left[\frac{d}{dy_2} L_y(e_1) - \frac{d}{dy_1} L_y(e_2) \right] \\
 &= \pi_i \left(M. \left[\frac{d}{dy_2} (J_2 f(y, y_0))^{-1} \cdot e_1 - \frac{d}{dy_1} (J_2 f(y, y_0))^{-1} \cdot e_2 \right] \right) \\
 &= -\pi_i \left(M. \left[(J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1} \cdot e_1] \right. \right. \\
 &\quad \left. \left. - (J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1} \cdot e_2] \right] \right) \\
 &= \pi_i \left(M. (J_2 f(y, y_0))^{-1} \cdot [J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1} \cdot e_2] - J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1} \cdot e_1]] \right) \\
 &= \pi_i \left(M. (J_2 f(y, y_0))^{-1} \cdot [V((J_2 f(y, y_0))^{-1} \cdot e_1), V((J_2 f(y, y_0))^{-1} \cdot e_2)]_y \right) = 0.
 \end{aligned}$$

In the last line we have recognized the expression of the commutator in (4.45) and used that it is equal to 0 since (\mathcal{D}, \star) is commutative. Therefore the differential forms $y \mapsto \pi_1 \circ L_y$ and $y \mapsto \pi_2 \circ L_y$ are closed. Then, since \mathcal{D} is simply connected, Poincaré's Lemma applies and shows that there are two functions $g_1 : \mathcal{D} \rightarrow \mathbb{R}$ and $g_2 : \mathcal{D} \rightarrow \mathbb{R}$ such that $g_i(y_0) = 0$ and for any $y \in \mathcal{D}$, $dg_i(y)$, the differential of g_i at y , is $\pi_i \circ L_y = \pi_i(M.(J_2 f(y, y_0))^{-1})$.

Now, defining the function $g : \mathcal{D} \rightarrow \mathbb{R}^2$ by $g := (g_1, g_2)$, we have obviously that $g(y_0) = 0$ and that $\forall y \in \mathcal{D}$, $Jg(y) = M.(J_2 f(y, y_0))^{-1}$. Therefore g is \mathcal{C}^k , and for any $y \in \mathcal{D}$ and any locally \mathcal{C}^1 path $\gamma : [a, b] \rightarrow \mathcal{D}$ with $\gamma(a) = y_0, \gamma(b) = y$, the derivative of the function $g \circ \gamma : [a, b] \rightarrow \mathbb{R}^2$ at $s \in [a, b]$ is clearly $M.(J_2 f(\gamma(s), y_0))^{-1} \cdot \gamma'(s)$. We thus have

$$g(y) = g(y) - g(y_0) = g(\gamma(b)) - g(\gamma(a)) = \int_a^b M. [J_2 f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds,$$

so that (4.48) holds for any $y \in \mathcal{D}$ for the g that we have just defined. Therefore g is well-defined.

Since g is a primitive function of $y \mapsto M.(J_2 f(y, y_0))^{-1}$ we have $Jg(y_0) = M.(J_2 f(y_0, y_0))^{-1}$, but $f(y_0, \cdot)$ is the identity function on \mathcal{D} so $J_2 f(y_0, y_0)$ is the identity matrix so $Jg(y_0) = M$.

We now justify that g is a group homomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$. Let $y, x \in \mathcal{D}$ and choose a locally \mathcal{C}^1 path $\gamma : [0, 2] \rightarrow \mathcal{D}$ with $\gamma(0) = y_0, \gamma(1) = y, \gamma(2) = f(y, x)$. Using two times (4.48), we have

$$\begin{aligned}
 g(f(y, x)) &= \int_0^2 M. [J_2 f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds \\
 &= \int_0^1 M. [J_2 f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds + \int_1^2 M. [J_2 f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds \\
 &= g(y) + \int_1^2 M. [J_2 f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) ds. \tag{4.49}
 \end{aligned}$$

Now, let h_y denote the inverse function of $f(y, \cdot)$ ($h_y := f(y^{-1}, \cdot)$) and let us consider the locally \mathcal{C}^1 path $\tilde{\gamma} : [1, 2] \rightarrow \mathcal{D}$ defined by $\forall s \in [1, 2], \tilde{\gamma}(s) := h_y(\gamma(s))$. In particular we have $\tilde{\gamma}(1) = y_0, \tilde{\gamma}(2) = x$ and

$$\forall s \in [1, 2], \gamma(s) = f(y, \tilde{\gamma}(s)), \gamma'(s) = J_2 f(y, \tilde{\gamma}(s)) \cdot \tilde{\gamma}'(s).$$

As a consequence, the second term in the right hand side of (4.49) equals

$$\int_1^2 M. [J_2 f(f(y, \tilde{\gamma}(s)), y_0)]^{-1} \cdot J_2 f(y, \tilde{\gamma}(s)) \cdot \tilde{\gamma}'(s) ds = \int_1^2 M. [J_2 f(\tilde{\gamma}(s), y_0)]^{-1} \cdot \tilde{\gamma}'(s) ds = g(x),$$

where we have used (4.39) with $u = y, v = \tilde{\gamma}(s)$, and then (4.48). Putting into (4.49) we obtain $g(f(y, x)) = g(y) + g(x)$ so g is indeed a group homomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$.

We now justify that g is onto \mathbb{R}^2 . $Jg(y_0) = M$ is invertible so, by the *local inversion theorem*, $g(\mathcal{D})$ contains a neighborhood in \mathbb{R}^2 of $g(y_0) = (0, 0)$. Since moreover $g(\mathcal{D})$ is a subgroup of $(\mathbb{R}^2, +)$ we conclude that $g(\mathcal{D}) = \mathbb{R}^2$.

Let us now justify that g is injective. Assume the contrary, that is, $\ker g \neq \{y_0\}$. We have that $Jg(y) = M \cdot (J_2f(y, y_0))^{-1}$ is invertible for any $y \in \mathcal{D}$ so, thanks to the *local inversion theorem*, we get that $\ker g$ is a discrete group. Let (Q, \star) be the quotient group of (\mathcal{D}, \star) by $\ker g$, equipped with the quotient topology (we still denote the group composition law on Q by \star). Then, let $\hat{g} : Q \rightarrow \mathbb{R}^2$ denote the quotient application of g . Then \hat{g} is a continuous injective group homomorphism between (Q, \star) and $(\mathbb{R}^2, +)$. \hat{g} is also onto because g is. As a consequence \hat{g} is bijective. Let us justify that \hat{g}^{-1} is continuous: let $(x_n)_{n \geq 1}$ be a sequence converging to some $x \in \mathbb{R}^2$, and let us justify that $\hat{g}^{-1}(x_n)$ converges to $\hat{g}^{-1}(x)$. Let $y \in \mathcal{D}$ be an antecedent of x by g . Then, the local inversion theorem gives the existence of a neighborhood \mathcal{U} of x and of a neighborhood \mathcal{V} of y such that g is a \mathcal{C}^k -diffeomorphism from \mathcal{V} to \mathcal{U} . Let us choose n_0 such that $x_n \in \mathcal{U}$ for all $n \geq n_0$. For such $n \geq n_0$, let us define $y_n \in \mathcal{V}$ to be the antecedent in \mathcal{V} of x_n by g . Since g is a \mathcal{C}^k -diffeomorphism from \mathcal{V} to \mathcal{U} we get that y_n converges to y . Let $\pi : \mathcal{D} \rightarrow Q$ denote the quotient projection. Since π is continuous we get that $\pi(y_n)$ converges to $\pi(y)$. Finally, noting that $\pi(y_n) = \hat{g}^{-1}(x_n)$ and $\pi(y) = \hat{g}^{-1}(x)$, we obtain the sought convergence. In conclusion we have that \hat{g} is a bi-continuous isomorphism of groups between (Q, \star) and $(\mathbb{R}^2, +)$. In particular \hat{g} is a homeomorphism between Q and \mathbb{R}^2 so the latter two are homeomorphic. Now, let $\gamma : [0, 1] \rightarrow \mathcal{D}$ be a continuous path such that $\gamma(0) = y_0$ and $\gamma(1) \in \ker g \setminus \{y_0\}$. Clearly, $\pi \circ \gamma : [0, 1] \rightarrow Q$ is a continuous loop that cannot be continuously reduced to a single point. Therefore the topological group of Q is not trivial, so Q is not homeomorphic to \mathbb{R}^2 (whose topological group is trivial). This is a contradiction, therefore g is injective.

Combining what is proved above, we obtain that g is bijective from \mathcal{D} to \mathbb{R}^2 and \mathcal{C}^k . Recall that $Jg(y) = M \cdot (J_2f(y, y_0))^{-1}$ is invertible for any $y \in \mathcal{D}$. Therefore we can conclude that g is a \mathcal{C}^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 . Since g is also a group homomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$, it is even a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$, which concludes the proof. \square

Lemma 4.4. *Let $\mathcal{D} \subset \mathbb{R}^2$ be an open simply connected domain, and let $f : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ define a **non-commutative** \mathcal{C}^k -Lie group structure (\mathcal{D}, \star) on \mathcal{D} , for some $k \geq 2$.*

Let $Y_0 := [V(e_1), V(e_2)]_{y_0}$ which can be computed explicitly using (4.45), and let $M := \begin{pmatrix} \pi_2(Y_0) & -\pi_1(Y_0) \\ \pi_1(Y_0) & \pi_2(Y_0) \end{pmatrix}$. Define $g_1 : \mathcal{D} \rightarrow \mathbb{R}$ via

$$g_1(y) = \int_a^b \pi_1 \left(M \cdot [J_2f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) \right) ds, \quad (4.50)$$

where $y_0 \in \mathcal{D}$ is the neutral element of (\mathcal{D}, \star) , $J_2f(z, \cdot)$ denotes the Jacobian matrix of the function $f(z, \cdot) : \mathcal{D} \rightarrow \mathcal{D}$, and where $\gamma : [a, b] \rightarrow \mathcal{D}$ is any path, locally \mathcal{C}^1 , with $\gamma(a) = y_0$ and $\gamma(b) = y$. Define $g_2 : \mathcal{D} \rightarrow \mathbb{R}$ via

$$g_2(y) = \int_a^b e^{g_1(\gamma(s))} \pi_2 \left(M \cdot [J_2f(\gamma(s), y_0)]^{-1} \cdot \gamma'(s) \right) ds, \quad (4.51)$$

where $\gamma : [a, b] \rightarrow \mathcal{D}$ is any path, locally \mathcal{C}^1 , with $\gamma(a) = y_0$ and $\gamma(b) = y$. Then g_1 and g_2 are well-defined, $g := (g_1, g_2)$ is a \mathcal{C}^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 , and a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) .

Proof. If g is the sought isomorphism, then, reasoning as in the beginning of the proof of Lemma 4.2, we see that the Jacobian matrix of g has to be $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & e^{g_1(y)} \end{pmatrix} \cdot Q \cdot (J_2 f(y, y_0))^{-1}$ where Q is an invertible 2×2 matrix. This explains why the definition of g in the statement of the lemma is a natural candidate for the isomorphism. Here again, thanks to Lemma 4.1 applied with $(u, v) = (y, y_0)$, we can see that $J_2 f(y, y_0)$ is invertible for any $y \in \mathcal{D}$ and that $y \mapsto (J_2 f(y, y_0))^{-1}$ is of class \mathcal{C}^{k-1} . To prove that g is well-defined and \mathcal{C}^k , we thus only need to prove the existence of a primitive function, g_1 , for $y \mapsto \pi_1(M \cdot (J_2 f(y, y_0))^{-1} \cdot z)$, and then to prove the existence of a primitive function, g_2 , for $y \mapsto e^{g_1(y)} \pi_2(M \cdot (J_2 f(y, y_0))^{-1} \cdot z)$. It will appear that the 2×2 matrix M has been chosen in such a way so that it is possible to apply Poincaré's Lemma to these differential forms.

For any $y \in \mathcal{D}$, let us denote by d_y^1 the linear form that sends $z \in \mathbb{R}^2$ to $\pi_1(M \cdot (J_2 f(y, y_0))^{-1} \cdot z) \in \mathbb{R}^2$. We wish to apply Poincaré's Lemma to the differential form $y \mapsto d_y^1$. Let us check that it is closed. For $y \in \mathcal{D}$ we have

$$\begin{aligned} & \frac{d}{dy_2} [d_y^1(e_1)] - \frac{d}{dy_1} [d_y^1(e_2)] = \pi_1 \left(M \cdot \left[\frac{d}{dy_2} (J_2 f(y, y_0))^{-1} \cdot e_1 - \frac{d}{dy_1} (J_2 f(y, y_0))^{-1} \cdot e_2 \right] \right) \\ &= -\pi_1 \left(M \cdot \left[(J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1} \cdot e_1] \right. \right. \\ & \quad \left. \left. - (J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1} \cdot e_2] \right] \right) \\ &= \pi_1 \left(M \cdot (J_2 f(y, y_0))^{-1} \cdot \left[J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1} \cdot e_2] - J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1} \cdot e_1] \right] \right) \\ &= \pi_1 \left(M \cdot (J_2 f(y, y_0))^{-1} \cdot \left[V((J_2 f(y, y_0))^{-1} \cdot e_1), V((J_2 f(y, y_0))^{-1} \cdot e_2) \right]_y \right) \\ &= \pi_1 \left(M \cdot (J_2 f(y, y_0))^{-1} \cdot J_2 f(y, y_0) \cdot \left[V((J_2 f(y, y_0))^{-1} \cdot e_1), V((J_2 f(y, y_0))^{-1} \cdot e_2) \right]_{y_0} \right) \\ &= \pi_1 \left(M \cdot \left[V((J_2 f(y, y_0))^{-1} \cdot e_1), V((J_2 f(y, y_0))^{-1} \cdot e_2) \right]_{y_0} \right) \\ &= \det((J_2 f(y, y_0))^{-1}) \times \pi_1(M \cdot Y_0) = 0. \end{aligned}$$

In the above we have recognized the expression of the commutator in (4.45), used (4.43), (4.46), and the fact, that can easily be seen from the definition of M , that $\pi_1(M \cdot Y_0) = 0$. This proves that the differential form d_y^1 is closed. Then, since \mathcal{D} is simply connected, Poincaré's Lemma ensures the existence of a function $g_1 : \mathcal{D} \rightarrow \mathbb{R}$ such that $g_1(y_0) = 0$ and for any $y \in \mathcal{D}$, $dg_1(y)$, the differential of g_1 at y , is $d_y^1 = \pi_1(M \cdot (J_2 f(y, y_0))^{-1} \cdot z)$. Therefore g_1 is \mathcal{C}^k , and for any $y \in \mathcal{D}$ and any locally \mathcal{C}^1 path $\gamma : [a, b] \rightarrow \mathcal{D}$ with $\gamma(a) = y_0, \gamma(b) = y$, the derivative of the function $g_1 \circ \gamma : [a, b] \rightarrow \mathbb{R}$ at $s \in [a, b]$ is clearly $\pi_1(M \cdot (J_2 f(\gamma(s), y_0))^{-1} \cdot \gamma'(s))$, so (4.50) follows.

Now that g_1 is defined, we can define d_y^2 for any $y \in \mathcal{D}$ as the linear form that sends $z \in \mathbb{R}^2$ to $e^{g_1(y)} \pi_2(M \cdot (J_2 f(y, y_0))^{-1} \cdot z) \in \mathbb{R}^2$. Here again, we wish to apply Poincaré's Lemma to the differential form $y \mapsto d_y^2$ so we first check that it is closed. For $y \in \mathcal{D}$ we have

$$\begin{aligned} & \frac{d}{dy_2} [d_y^2(e_1)] - \frac{d}{dy_1} [d_y^2(e_2)] \\ &= \pi_2 \left(M \cdot \left(\frac{d}{dy_2} \left[e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \right] \cdot e_1 - \frac{d}{dy_1} \left[e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \cdot e_2 \right] \right) \right) \\ &= \pi_2 \left(M \cdot \left(-e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1} \cdot e_1] + \left(\frac{d}{dy_2} g_1(y) \right) \cdot e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \cdot e_1 \right. \right. \\ & \quad \left. \left. + e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \cdot J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1} \cdot e_2] - \left(\frac{d}{dy_1} g_1(y) \right) \cdot e^{g_1(y)} \cdot (J_2 f(y, y_0))^{-1} \cdot e_2 \right) \right) \end{aligned}$$

$$\begin{aligned}
&= e^{g_1(y)} \times \pi_2 \left(M. (J_2 f(y, y_0))^{-1} \cdot \left(-J_1 J_2 f(y, y_0) [e_2, (J_2 f(y, y_0))^{-1}.e_1] + \left(\frac{d}{dy_2} g_1(y) \right) .e_1 \right. \right. \\
&\quad \left. \left. + J_1 J_2 f(y, y_0) [e_1, (J_2 f(y, y_0))^{-1}.e_2] - \left(\frac{d}{dy_1} g_1(y) \right) .e_2 \right) \right) \\
&= e^{g_1(y)} \times \pi_2 \left(M. (J_2 f(y, y_0))^{-1} \cdot \left([V((J_2 f(y, y_0))^{-1}.e_1), V((J_2 f(y, y_0))^{-1}.e_2)]_y \right. \right. \\
&\quad \left. \left. + \left(\frac{d}{dy_2} g_1(y) \right) .e_1 - \left(\frac{d}{dy_1} g_1(y) \right) .e_2 \right) \right) \\
&= e^{g_1(y)} \times \pi_2 \left(M. (J_2 f(y, y_0))^{-1} \cdot \left(J_2 f(y, y_0) \cdot [V((J_2 f(y, y_0))^{-1}.e_1), V((J_2 f(y, y_0))^{-1}.e_2)]_{y_0} \right. \right. \\
&\quad \left. \left. + \left(\frac{d}{dy_2} g_1(y) \right) .e_1 - \left(\frac{d}{dy_1} g_1(y) \right) .e_2 \right) \right) \\
&= e^{g_1(y)} \times \pi_2 \left(M. [V((J_2 f(y, y_0))^{-1}.e_1), V((J_2 f(y, y_0))^{-1}.e_2)]_{y_0} \right) \\
&\quad + e^{g_1(y)} \left(\left(\frac{d}{dy_2} g_1(y) \right) \times \pi_2 (M. (J_2 f(y, y_0))^{-1}.e_1) - \left(\frac{d}{dy_1} g_1(y) \right) \times \pi_2 (M. (J_2 f(y, y_0))^{-1}.e_2) \right). \tag{4.52}
\end{aligned}$$

In the above we have recognized the expression of the commutator in (4.45) and used (4.43). Then, using (4.46), and the definition of M :

$$\begin{aligned}
&\pi_2 \left(M. [V((J_2 f(y, y_0))^{-1}.e_1), V((J_2 f(y, y_0))^{-1}.e_2)]_{y_0} \right) \\
&= \det((J_2 f(y, y_0))^{-1}) \times \pi_2(M.Y_0) \\
&= \det((J_2 f(y, y_0))^{-1}) \times (\pi_1(Y_0)^2 + \pi_2(Y_0)^2). \tag{4.53}
\end{aligned}$$

On the other hand, recall that the differential of g_1 at y , is $\pi_1(M.(J_2 f(y, y_0))^{-1})$. In particular we have $\frac{d}{dy_2} g_1(y) = (M.(J_2 f(y, y_0))^{-1})_{1,2}$ and $\frac{d}{dy_1} g_1(y) = (M.(J_2 f(y, y_0))^{-1})_{1,1}$. Also, we have obviously $\pi_2(M.(J_2 f(y, y_0))^{-1}.e_1) = (M.(J_2 f(y, y_0))^{-1})_{2,1}$ and $\pi_2(M.(J_2 f(y, y_0))^{-1}.e_2) = (M.(J_2 f(y, y_0))^{-1})_{2,2}$. We thus get

$$\begin{aligned}
&\left(\frac{d}{dy_2} g_1(y) \right) \times \pi_2(M.(J_2 f(y, y_0))^{-1}.e_1) - \left(\frac{d}{dy_1} g_1(y) \right) \times \pi_2(M.(J_2 f(y, y_0))^{-1}.e_2) \\
&= (M.(J_2 f(y, y_0))^{-1})_{1,2} \times (M.(J_2 f(y, y_0))^{-1})_{2,1} - (M.(J_2 f(y, y_0))^{-1})_{1,1} \times (M.(J_2 f(y, y_0))^{-1})_{2,2} \\
&= -\det(M.(J_2 f(y, y_0))^{-1}) = -\det((J_2 f(y, y_0))^{-1}) \times \det(M) \\
&= -\det((J_2 f(y, y_0))^{-1}) \times (\pi_1(Y_0)^2 + \pi_2(Y_0)^2). \tag{4.54}
\end{aligned}$$

Putting (4.53) and (4.54) into (4.52) we obtain $\frac{d}{dy_2} [d_y^2(e_1)] - \frac{d}{dy_1} [d_y^2(e_2)] = 0$, that is, the differential form $y \mapsto d_y^2$ is closed. Then, since \mathcal{D} is simply connected, Poincaré's Lemma ensures the existence of a function $g_2 : \mathcal{D} \rightarrow \mathbb{R}$ such that $g_2(y_0) = 0$ and for any $y \in \mathcal{D}$, $dg_2(y)$, the differential of g_2 at y , is $d_y^2 = e^{g_1(y)} \pi_2(M.(J_2 f(y, y_0))^{-1})$. Therefore g_2 is \mathcal{C}^k , and for any $y \in \mathcal{D}$ and any locally \mathcal{C}^1 path $\gamma : [a, b] \rightarrow \mathcal{D}$ with $\gamma(a) = y_0, \gamma(b) = y$, the derivative of the function $g_2 \circ \gamma : [a, b] \rightarrow \mathbb{R}^2$ at $s \in [a, b]$ is clearly $e^{g_1(\gamma(s))} \pi_2(M.(J_2 f(\gamma(s), y_0))^{-1}.\gamma'(s))$, so (4.51) follows.

We have seen that g_1 and g_2 are well-defined and \mathcal{C}^k , so $g := (g_1, g_2)$ is also well-defined and \mathcal{C}^k . We now justify that g is a group homomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) . Let $y, x \in \mathcal{D}$ and choose a locally \mathcal{C}^1 path $\gamma : [0, 2] \rightarrow \mathcal{D}$ with $\gamma(0) = y_0, \gamma(1) = y, \gamma(2) = f(y, x)$. Using two

times (4.50), we have

$$\begin{aligned}
 g_1(f(y, x)) &= \int_0^2 \pi_1 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds \\
 &= \int_0^1 \pi_1 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds + \int_1^2 \pi_1 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds \\
 &= g_1(y) + \int_1^2 \pi_1 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds.
 \end{aligned} \tag{4.55}$$

Now, let h_y denote the inverse function of $f(y, \cdot)$ ($h_y := f(y^{-1}, \cdot)$) and let us consider the locally \mathcal{C}^1 path $\tilde{\gamma} : [1, 2] \rightarrow \mathcal{D}$ defined by $\forall s \in [1, 2], \tilde{\gamma}(s) := h_y(\gamma(s))$. In particular we have $\tilde{\gamma}(1) = y_0, \tilde{\gamma}(2) = x$ and

$$\forall s \in [1, 2], \gamma(s) = f(y, \tilde{\gamma}(s)), \gamma'(s) = J_2f(y, \tilde{\gamma}(s)).\tilde{\gamma}'(s). \tag{4.56}$$

As a consequence, the second term in the right hand side of (4.55) equals

$$\int_1^2 \pi_1 (M.[J_2f(f(y, \tilde{\gamma}(s)), y_0)]^{-1}.J_2f(y, \tilde{\gamma}(s)).\tilde{\gamma}'(s)) ds = \int_1^2 \pi_1 (M.[J_2f(\tilde{\gamma}(s), y_0)]^{-1}.\tilde{\gamma}'(s)) ds = g_1(x),$$

where we have used (4.39) with $u = y, v = \tilde{\gamma}(s)$ and (4.50). Putting into (4.55) we obtain

$$g_1(f(y, x)) = g_1(y) + g_1(x). \tag{4.57}$$

Then, using two times (4.51), we have

$$\begin{aligned}
 g_2(f(y, x)) &= \int_0^2 e^{g_1(\gamma(s))} \pi_2 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds \\
 &= \int_0^1 e^{g_1(\gamma(s))} \pi_2 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds \\
 &\quad + \int_1^2 e^{g_1(\gamma(s))} \pi_2 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds \\
 &= g_2(y) + \int_1^2 e^{g_1(\gamma(s))} \pi_2 (M.[J_2f(\gamma(s), y_0)]^{-1}.\gamma'(s)) ds.
 \end{aligned} \tag{4.58}$$

Recall the definition of the locally \mathcal{C}^1 path $\tilde{\gamma}$ above, and that it satisfies (4.56). Note that for all $s \in [1, 2]$ we have $g_1(\gamma(s)) = g_1(f(y, \tilde{\gamma}(s))) = g_1(y) + g_1(\tilde{\gamma}(s))$ thanks to (4.57). The second term in the right hand side of (4.58) can thus be expressed in term of $\tilde{\gamma}$. It equals

$$\begin{aligned}
 &e^{g_1(y)} \int_1^2 e^{g_1(\tilde{\gamma}(s))} \pi_2 (M.[J_2f(f(y, \tilde{\gamma}(s)), y_0)]^{-1}.J_2f(y, \tilde{\gamma}(s)).\tilde{\gamma}'(s)) ds \\
 &= e^{g_1(y)} \int_1^2 e^{g_1(\tilde{\gamma}(s))} \pi_2 (M.[J_2f(\tilde{\gamma}(s), y_0)]^{-1}.\tilde{\gamma}'(s)) ds = e^{g_1(y)} g_2(x),
 \end{aligned}$$

where we have used (4.39) with $u = y, v = \tilde{\gamma}(s)$ and (4.51). Putting into (4.58) we obtain

$$g_2(f(y, x)) = g_2(y) + e^{g_1(y)} g_2(x). \tag{4.59}$$

The combination of (4.57) and (4.59) yields $g(f(y, x)) = g(y)Tg(x)$ so g is indeed a group homomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) .

We now justify that g is onto \mathbb{R}^2 . Since $e^{g(y_0)} = 1$, we have $Jg(y_0) = M.(J_2f(y_0, y_0)^{-1}) = M$ which is invertible so, by the *local inversion theorem*, $g(\mathcal{D})$ contains a neighborhood in \mathbb{R}^2 of

$g(y_0) = (0, 0)$. Let us justify that any neighborhood \mathcal{V} of $(0, 0)$ generates the group (\mathbb{R}^2, T) . For $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ let us define

$$z_n := \begin{pmatrix} y_1/n \\ y_2 / \left(\sum_{k=0}^{n-1} e^{ky_1/n} \right) \end{pmatrix}.$$

Then, we see that for any $n \geq 1$, y is equal to the T -product of n times the element z_n : $y = z_n T \dots T z_n$. Moreover, we can see that $z_n \in \mathcal{V}$ when n is large enough. Therefore \mathcal{V} generates the group (\mathbb{R}^2, T) . Then, since $g(\mathcal{D})$ is a subgroup of (\mathbb{R}^2, T) that contains a neighborhood of 0, we conclude that $g(\mathcal{D}) = \mathbb{R}^2$.

To prove that g is injective, we can readily repeat the argument that justified the injectiveness in the proof of Lemma 4.3.

Combining what is proved above, we obtain that g is bijective from \mathcal{D} to \mathbb{R}^2 and \mathcal{C}^k . Recall that g has been defined in such a way that for all $y \in \mathcal{D}$, $Jg(y)$, the Jacobian of g at y , equals $\begin{pmatrix} 1 & 0 \\ 0 & e^{g_1(y)} \end{pmatrix} \cdot M \cdot (J_2 f(y, y_0))^{-1}$ where M is invertible and $J_2 f(y, y_0)$ is invertible for any $y \in \mathcal{D}$, as mentioned in the beginning of the proof. Therefore, for all $y \in \mathcal{D}$ the Jacobian of g at y is invertible so we can conclude that g is a \mathcal{C}^k -diffeomorphism from \mathcal{D} to \mathbb{R}^2 . Since g is also a group homomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) , it is even a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) , which concludes the proof. □

We will also need the following simple lemma about the group (\mathbb{R}^2, T) .

Lemma 4.5. *Let $m : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous group homomorphism from (\mathbb{R}^2, T) to $(\mathbb{R}, +)$. Then there exists $\beta \in \mathbb{R}$ such that*

$$\forall z = (z_1, z_2) \in \mathbb{R}^2, \quad m(z) = \beta z_1.$$

A consequence of the above lemma is that, if \tilde{m} is a continuous group homomorphism from (\mathbb{R}^2, T) to (\mathbb{R}_+^*, \times) , then there exists $\beta \in \mathbb{R}$ such that $\forall z = (z_1, z_2) \in \mathbb{R}^2$, $\tilde{m}(z) = e^{\beta z_1}$.

Proof. of Lemma 4.5

The proof is very basic. Note that $(\lambda \mapsto (\lambda, 0))$ and $(\lambda \mapsto (0, \lambda))$ are continuous group homomorphisms from $(\mathbb{R}, +)$ to (\mathbb{R}^2, T) . Therefore, if we define $m_1, m_2 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$m_1(\lambda) := m(\lambda, 0) \quad \text{and} \quad m_2(\lambda) := m(0, \lambda),$$

then m_1 and m_2 are continuous group homomorphisms from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$, so there exist β and β' such that $\forall \lambda \in \mathbb{R}, m_1(\lambda) = \beta \lambda, m_2(\lambda) = \beta' \lambda$. Then, note that for any $z = (z_1, z_2) \in \mathbb{R}^2$,

$$\begin{aligned} m(z) &= m \left(\begin{pmatrix} 0 \\ z_2 \end{pmatrix} T \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right) = m(0, z_2) + m(z_1, 0) \\ &= m_2(z_2) + m_1(z_1) = \beta z_1 + \beta' z_2. \end{aligned} \tag{4.60}$$

(4.60) yields $m(1, 1) = \beta + \beta'$. On the other hand we have

$$\begin{aligned} m(1, 1) &= m \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = m(0, 1) + m(1, 0) \\ &= m(1, 0) + m(0, 1) = m \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= m(1, e^1) = \beta + \beta' e^1, \end{aligned}$$

where we have used (4.60) for the last equality. Putting together the above and $m(1, 1) = \beta + \beta'$ we deduce that $\beta' = 0$ and, substituting 0 to β' in (4.60), we obtain the asserted result. \square

5. LÉVY PROCESSES ON (\mathbb{R}^2, T)

The aim of this section is to identify Lévy processes on the group (\mathbb{R}^2, T) , and in particular to express these Lévy processes in term of exponential functionals of classical Lévy processes on $(\mathbb{R}^2, +)$. Recall that the starting point of a Lévy process on a group is always the neutral element of the group. Our result is the following.

Proposition 5.1. *Let Y be a Lévy process on (\mathbb{R}^2, T) , then there exists a Lévy process (ξ, η) on $(\mathbb{R}^2, +)$ such that*

$$\forall t \geq 0, Y(t) = \left(\int_0^t e^{\xi(s-)} d\eta(s) \right). \quad (5.61)$$

Let us mention that the general form of generators of Lévy processes on the group (\mathbb{R}^2, T) can be obtained by an application of Hunt's theorem (see Theorem 5.3.3 in [3]) to the Lie group (\mathbb{R}^2, T) . It is possible to prove Proposition 5.1 by 1) computing the generator of the right-hand-side of (5.61) thanks to Ito's Lemma for Lévy processes, 2) expliciting Hunt's theorem in the case of the Lie group (\mathbb{R}^2, T) , and 3) identifying the expressions of the two generators. However, we have chosen to proceed in a more intuitive way that does not require an application of Hunt's theorem and relies on simple computations on stochastic integrals. If Y is a Lévy process on (\mathbb{R}^2, T) , the idea is to define

$$\forall t \geq 0, \xi(t) := \pi_1(Y(t)), \quad \eta(t) := \int_0^t e^{-\pi_1(Y(s-))} d\pi_2(Y(s)),$$

and then to justify that (ξ, η) is a Lévy process on $(\mathbb{R}^2, +)$ and that (5.61) holds. This argument is very simple modulo the fact that we can write stochastic integrals with respect to the real-valued process $\pi_2(Y)$. Since we do not know *a priori* what this process looks like we first justify the fact that these integrals are well-defined. We proceed in several steps. We first study the procedure of removing the big jumps for a Lévy process on the group (\mathbb{R}^2, T) . This procedure is classical but we provide all the details for the sake of clarity.

For any $M > 0$ let

$$A_M := \{x \in \mathbb{R}^2, |x|_\infty \geq M\}, \quad (5.62)$$

where $|(x_1, x_2)|_\infty := |x_1| \vee |x_2|$. For Y a Lévy process on (\mathbb{R}^2, T) and $t \geq 0$, the jump of Y at t is $\Delta Y(t) := (Y(t-))^{-1}TY(t)$ (note that $\Delta Y(t) = (0, 0)$ if Y is continuous at t). Let $R^M[Y]$ be the process obtained by removing the jumps of Y that are in A_M . Let us describe briefly the construction of $R^M[Y]$. If Y has no jump in A_M then we put $R^M[Y] := Y$. If Y does have jumps in A_M then, since Y is càd-làg, there are almost surely finitely many jumps of Y in A_M on finite time intervals. Let $0 < t_1 < t_2 < \dots$ be the discrete sequence of times associated to the jumps in A_M , we then put $R^M[Y](t) := R^M[Y](t_i-)T(Y(t_i))^{-1}TY(t)$ for $t \in [t_i, t_{i+1}[$, where by convention $t_0 := 0$ and $R^M[Y](0-) := (0, 0)$. It will be convenient to work with $R^M[Y]$, which has only bounded jumps, and to notice that $R^M[Y]$ coincides with Y until the instant of the first jump of Y in A_M . We have the following lemma about $R^M[Y]$:

Lemma 5.2. *Let Y be a Lévy process on (\mathbb{R}^2, T) and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration associated with Y . For any $M > 0$, $R^M[Y]$ is a Lévy process on (\mathbb{R}^2, T) adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$. For any $t \geq 0$ we have*

$$\sup_{s \in [0, t]} \mathbb{E} \left[e^{2|\pi_1(R^M[Y](s-))|} \right] < +\infty, \quad (5.63)$$

and

$$\sup_{s \in [0, t]} \mathbb{E} [|\pi_2(R^M[Y](s))|^2] < +\infty. \quad (5.64)$$

Proof. Let Y be a Lévy process on (\mathbb{R}^2, T) and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration associated with Y . The fact that $R^M[Y]$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$ is plain from the fact that $(R^M[Y](s), 0 \leq s \leq t)$ has been constructed as a function of $(Y(s), 0 \leq s \leq t)$. We first justify that $R^M[Y]$ is a Lévy process on (\mathbb{R}^2, T) . If Y has a.s. no jumps in A_M then $Y = R^M[Y]$ so the latter is clearly a Lévy process on (\mathbb{R}^2, T) . We thus assume that Y admits jumps in A_M .

By construction, $R^M[Y]$ is a càd-làg process starting from $(0, 0)$. We now justify that $R^M[Y]$ has stationary and independent increments. Let us fix $t \geq 0$ and define $S_t[Y](\cdot) := (Y(t))^{-1}TY(t + \cdot)$. Then $S_t[Y]$ is independent from \mathcal{F}_t and has the same law as Y . For any $s \geq 0$ we have

$$\begin{aligned} \Delta Y(t + s) &= (Y(t + s-))^{-1}TY(t + s) = ((Y(t))^{-1}TY(t + s-))^{-1}T(Y(t))^{-1}TY(t + s) \\ &= (S_t[Y](s-))^{-1}TS_t[Y](s) = \Delta S_t[Y](s). \end{aligned}$$

As a consequence, $S_t[Y]$ has a jump at s if and only if Y has a jump at $t + s$. Let $0 < s_1 < s_2 < \dots$ be the sequence of the jumps of $S_t[Y]$ in A_M (then $t + s_1 < t + s_2 < \dots$ is the sequence of the jumps of Y in A_M after time t). If Y has jumps in A_M on $[0, t]$, then let u be the instant of the last jump of Y in A_M . If Y has no jump in $[0, t]$, then let $u := 0$. Let us prove by induction on i that

$$R^M[S_t[Y]](s) = S_t[R^M[Y]](s), \quad \forall s \in [s_i, s_{i+1}[), \quad (5.65)$$

where by convention $s_0 := 0$. For $s \in [0, s_1[$ we have by the definitions of R^M , s_1 , and S_t :

$$\begin{aligned} R^M[S_t[Y]](s) &= S_t[Y](s) = (Y(t))^{-1}TY(t + s) \\ &= (R^M[Y](u-)T(Y(u))^{-1}TY(t))^{-1}TR^M[Y](u-)T(Y(u))^{-1}TY(t + s) \\ &= (R^M[Y](t))^{-1}TR^M[Y](t + s) = S_t[R^M[Y]](s). \end{aligned}$$

Therefore, (5.65) holds for $i = 0$. Let us assume that it holds for some $i \geq 0$ and prove it for $i + 1$. By the induction hypothesis we have that $R^M[S_t[Y]](s_{i+1}-) = S_t[R^M[Y]](s_{i+1}-)$. Then, for any $s \in [s_{i+1}, s_{i+2}[$, using the definition of R^M , the previous relation and the definition of S_t we have

$$\begin{aligned} R^M[S_t[Y]](s) &= R^M[S_t[Y]](s_{i+1}-)T(S_t[Y](s_{i+1}))^{-1}TS_t[Y](s) \\ &= S_t[R^M[Y]](s_{i+1}-)T(S_t[Y](s_{i+1}))^{-1}TS_t[Y](s) \\ &= (R^M[Y](t))^{-1}TR^M[Y](t + s_{i+1}-)T((Y(t))^{-1}TY(t + s_{i+1}))^{-1}T(Y(t))^{-1}TY(t + s) \\ &= (R^M[Y](t))^{-1}TR^M[Y](t + s_{i+1}-)T(Y(t + s_{i+1}))^{-1}TY(t + s) \\ &= (R^M[Y](t))^{-1}TR^M[Y](t + s) = S_t[R^M[Y]](s). \end{aligned}$$

Then the induction is completed and we conclude that we have a.s. $R^M[S_t[Y]] = S_t[R^M[Y]]$. This means that $S_t[R^M[Y]]$ is expressed as a deterministic function, R^M , of the process $S_t[Y]$ that is independent from \mathcal{F}_t and has the same law as Y . As a consequence $S_t[R^M[Y]]$ is independent from \mathcal{F}_t and has the same law as $R^M[Y]$. This proves the independence and stationarity of the increments of $R^M[Y]$ which is therefore a Lévy process.

We now justify the claim about moments. For this we follow the proof of Theorem 2.4.7 in [2], just working on (\mathbb{R}^2, T) instead of $(\mathbb{R}^d, +)$. Let $T_0 := 0$ and recursively for $n \geq 0$,

$T_{n+1} := \inf\{t > T_n, |(R^M[Y](T_n))^{-1}TR^M[Y](t)|_\infty > M\}$. By definition of T_n we have a.s. that

$$\forall s \in [T_n, T_{n+1}[, |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](s))| \leq M, \quad (5.66)$$

$$|\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](s))| \leq M. \quad (5.67)$$

Then, $|\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}))|$ equals a.s.

$$\begin{aligned} & |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-)T\Delta R^M[Y](T_{n+1}))| \\ &= |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-)) + \pi_1(\Delta R^M[Y](T_{n+1}))| \\ &\leq |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-))| + |\pi_1(\Delta R^M[Y](T_{n+1}))|. \end{aligned}$$

According to (5.66) the first term is a.s. less than M and, since $R^M[Y]$ has no jump in A_M , the second term is a.s. less than M . We thus get $|\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}))| \leq 2M$ a.s. and combining with (5.66) we get that a.s.

$$\sup_{s \in [T_n, T_{n+1}]} |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](s))| \leq 2M. \quad (5.68)$$

Then, $|\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}))|$ a.s. equals

$$\begin{aligned} & |\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-)T\Delta R^M[Y](T_{n+1}))| \\ &= |\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-)) + e^{\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-))}\pi_2(\Delta R^M[Y](T_{n+1}))| \\ &\leq |\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-))| + e^{\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}-))}|\pi_2(\Delta R^M[Y](T_{n+1}))|. \end{aligned}$$

According to (5.67) the first term is a.s. less than M , and according to (5.66) the first factor in the second term is a.s. less than e^M . Since $R^M[Y]$ has no jump in A_M , the second factor in the second term is a.s. less than M . We thus get that $|\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](T_{n+1}))| \leq (1 + e^M)M \leq 2Me^M$ a.s. and, combining with (5.67) we get that a.s.

$$\sup_{s \in [T_n, T_{n+1}]} |\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](s))| \leq 2Me^M. \quad (5.69)$$

Let us prove by induction that a.s.

$$\forall s \in [0, T_n], |\pi_1(R^M[Y](s))| \leq 2nM, |\pi_2(R^M[Y](s))| \leq 2nMe^{2nM}. \quad (5.70)$$

(5.70) is clearly true for $n = 0$. Let us assume that it holds for some $n \geq 0$ and prove it for $n + 1$. If the supremum of $|\pi_1(R^M[Y](s))|$ on the interval $[0, T_{n+1}]$ is attained in $[0, T_n[$, then the induction hypothesis implies that $\sup_{s \in [0, T_{n+1}]} |\pi_1(R^M[Y](s))| \leq 2nM \leq 2(n + 1)M$. If the supremum of $|\pi_1(R^M[Y](s))|$ on the interval $[0, T_{n+1}]$ is attained in $[T_n, T_{n+1}]$ then we have

$$\begin{aligned} \sup_{s \in [0, T_{n+1}]} |\pi_1(R^M[Y](s))| &= \sup_{s \in [T_n, T_{n+1}]} |\pi_1(R^M[Y](T_n)T(R^M[Y](T_n))^{-1}TR^M[Y](s))| \\ &= \sup_{s \in [T_n, T_{n+1}]} |\pi_1(R^M[Y](T_n)) + \pi_1((R^M[Y](T_n))^{-1}TR^M[Y](s))| \\ &\leq |\pi_1(R^M[Y](T_n))| + \sup_{s \in [T_n, T_{n+1}]} |\pi_1((R^M[Y](T_n))^{-1}TR^M[Y](s))|. \end{aligned}$$

By the induction hypothesis the first term is a.s. less than $2nM$ and by (5.68) the second term is a.s. less than $2M$. In any case we thus have $\sup_{s \in [0, T_{n+1}]} |\pi_1(R^M[Y](s))| \leq 2(n + 1)M$. Then, if the supremum of $|\pi_2(R^M[Y](s))|$ on the interval $[0, T_{n+1}]$ is attained in $[0, T_n[$, then the induction hypothesis implies that $\sup_{s \in [0, T_{n+1}]} |\pi_2(R^M[Y](s))| \leq 2nMe^{2nM} \leq 2(n + 1)Me^{2(n+1)M}$. If the

supremum of $|\pi_2(R^M[Y](s))|$ on the interval $[0, T_{n+1}]$ is attained in $[T_n, T_{n+1}]$ then we have

$$\begin{aligned} \sup_{s \in [0, T_{n+1}]} |\pi_2(R^M[Y](s))| &= \sup_{s \in [T_n, T_{n+1}]} |\pi_2(R^M[Y](T_n)T(R^M[Y](T_n))^{-1}TR^M[Y](s))| \\ &= \sup_{s \in [T_n, T_{n+1}]} |\pi_2(R^M[Y](T_n)) + e^{\pi_1(R^M[Y](T_n))}\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](s))| \\ &\leq |\pi_2(R^M[Y](T_n))| + e^{\pi_1(R^M[Y](T_n))} \sup_{s \in [T_n, T_{n+1}]} |\pi_2((R^M[Y](T_n))^{-1}TR^M[Y](s))|. \end{aligned}$$

By the induction hypothesis the first term is a.s. less than $2nMe^{2nM} \leq 2nMe^{2(n+1)M}$ and the first factor in the second term is a.s. less than e^{2nM} . By (5.69) the second factor of the second term is a.s. less than $2Me^M \leq 2Me^{2M}$. In any case we thus have $\sup_{s \in [0, T_{n+1}]} |\pi_2(R^M[Y](s))| \leq 2(n+1)Me^{2(n+1)M}$. In conclusion, (5.70) holds for $n+1$ so the induction is completed.

Then, since $R^M[Y]$ is a Lévy process, the sequence $(T_{n+1} - T_n)_{n \geq 0}$ is *iid* so in particular $\mathbb{E}[e^{-\lambda T_n}] = \mathbb{E}[e^{-\lambda(T_n - T_{n-1})} \dots e^{-\lambda T_1}] = (\mathbb{E}[e^{-\lambda T_1}])^n$. Then for any $t \geq 0, z > 0$,

$$\begin{aligned} \mathbb{P}(|\pi_1(R^M[Y](t))| > z) &\leq \mathbb{P}(|\pi_1(R^M[Y](t))| > 2M \lfloor z/2M \rfloor) \\ &\leq \mathbb{P}(T_{\lfloor z/2M \rfloor} < t) \leq e^{\lambda t} \left(\mathbb{E} \left[e^{-\lambda T_1} \right] \right)^{\lfloor z/2M \rfloor}, \end{aligned}$$

where we have used (5.70) (which shows that $|\pi_1(R^M[Y](t))| > 2nM \Rightarrow T_n < t$) and Chernoff's inequality. Let us choose $\lambda_{M,1}$ large enough so that $(\mathbb{E}[e^{-\lambda_{M,1} T_1}])^{1/2M} < e^{-2}$. For such a choice of $\lambda_{M,1}$ we get

$$\begin{aligned} \forall t \geq 0, \mathbb{E} \left[e^{2|\pi_1(R^M[Y](t))|} \right] &= 1 + 2 \int_0^{+\infty} e^{2z} \mathbb{P}(|\pi_1(R^M[Y](t))| > z) dz \\ &\leq 1 + 2e^{\lambda_{M,1} t} \int_0^{+\infty} e^{2z} \left(\mathbb{E} \left[e^{-\lambda_{M,1} T_1} \right] \right)^{\lfloor z/2M \rfloor} dz \\ &\leq 1 + e^{\lambda_{M,1} t} C_1, \end{aligned}$$

where C_1 is some positive constant. We thus get

$$\sup_{s \in [0, t]} \mathbb{E} \left[e^{2|\pi_1(R^M[Y](s))|} \right] < +\infty.$$

Taking left-limits and using Fatou's Lemma we obtain (5.63).

Let z_M be such that $e^{3M \lfloor \log(z)/3M \rfloor} > 2M \lfloor \log(z)/3M \rfloor e^{2M \lfloor \log(z)/3M \rfloor}$ for any $z > z_M$. Then for any $t \geq 0$ and $z > z_M$,

$$\begin{aligned} \mathbb{P}(|\pi_2(R^M[Y](t))| > z) &\leq \mathbb{P}(|\pi_2(R^M[Y](t))| > e^{3M \lfloor \log(z)/3M \rfloor}) \\ &\leq \mathbb{P}(|\pi_2(R^M[Y](t))| > 2M \lfloor \log(z)/3M \rfloor e^{2M \lfloor \log(z)/3M \rfloor}) \\ &\leq \mathbb{P}(T_{\lfloor \log(z)/3M \rfloor} < t) \leq e^{\lambda_{M,1} t} \left(\mathbb{E} \left[e^{-\lambda_{M,1} T_1} \right] \right)^{\lfloor \log(z)/3M \rfloor}, \end{aligned}$$

where we have used (5.70) (which shows that $|\pi_2(R^M[Y](t))| > 2nMe^{2nM} \Rightarrow T_n < t$) and Chernoff's inequality. Let us choose $\lambda_{M,2}$ large enough so that $(\mathbb{E}[e^{-\lambda_{M,2} T_1}])^{1/3M} < e^{-3}$. For

such a choice of $\lambda_{M,2}$ we have $(\mathbb{E}[e^{-\lambda_{M,2}T_1}])^{\lfloor \log(z)/3M \rfloor} < (\mathbb{E}[e^{-\lambda_{M,2}T_1}])^{-1}z^{-3}$. We thus get:

$$\begin{aligned} \forall t \geq 0, \mathbb{E} [|\pi_2(R^M[Y](t))|^2] &= 2 \int_0^{+\infty} z \mathbb{P} (|\pi_2(R^M[Y](t))| > z) dz \\ &\leq 2e^{\lambda_{M,2}t} \int_0^{+\infty} z \left(\mathbb{E} \left[e^{-\lambda_{M,2}T_1} \right] \right)^{\lfloor \log(z)/3M \rfloor} dz \\ &\leq e^{\lambda_{M,2}t} C_2, \end{aligned}$$

where C_2 is some positive constant. This yields (5.64). \square

We now study a procedure of re-centering for the second coordinate of $R^M[Y]$, in order to obtain a martingale.

Lemma 5.3. *Let Y be a Lévy process on (\mathbb{R}^2, T) and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration associated with Y . For any $M > 0$, there is a unique $\alpha_M \in \mathbb{R}$ such that the process*

$$W^M[Y] := \left(\pi_2(R^M[Y](t)) - \alpha_M \int_0^t e^{\pi_1(R^M[Y](u-))} du, t \geq 0 \right) \quad (5.71)$$

is a càd-làg martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, locally bounded in L^2 (that is, for any $t > 0$, $\sup_{u \in [0,t]} \mathbb{E}[(W^M[Y](s))^2] < +\infty$).

Proof. Let Y be a Lévy process on (\mathbb{R}^2, T) and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration associated with Y . The process $(\int_0^t e^{\pi_1(R^M[Y](u-))} du, t \geq 0)$ is well-defined and a.s. continuous, because $R^M[Y]$ is a.s. càd-làg as a Lévy process, according to Lemma 5.2. Note from Lemma 5.2 that the expectations $\mathbb{E}[\pi_2(R^M[Y](1))]$ and $\mathbb{E}[\int_0^1 e^{\pi_1(R^M[Y](u-))} du]$ are finite, and notice that the second is positive. Let us put

$$\alpha_M := \mathbb{E} [\pi_2(R^M[Y](1))] / \mathbb{E} \left[\int_0^1 e^{\pi_1(R^M[Y](u-))} du \right], \quad (5.72)$$

and define $W^M[Y]$ with this α_M , as in (5.71). Then, $W^M[Y]$ is well-defined and a.s. càd-làg.

We now justify the local boundedness in L^2 . Let us fix $t > 0$ and $s \in [0, t]$. We have

$$\begin{aligned} \mathbb{E} [(W^M[Y](s))^2] &\leq 2\mathbb{E} [(\pi_2(R^M[Y](s)))^2] + 2\alpha_M^2 \mathbb{E} \left[\left(\int_0^s e^{\pi_1(R^M[Y](u-))} du \right)^2 \right] \\ &\leq 2 \sup_{u \in [0,t]} \mathbb{E} [|\pi_2(R^M[Y](u))|^2] + 2\alpha_M^2 t^2 \mathbb{E} \left[\left(\frac{1}{t} \int_0^t e^{\pi_1(R^M[Y](u-))} du \right)^2 \right] \\ &\leq 2 \sup_{u \in [0,t]} \mathbb{E} [|\pi_2(R^M[Y](u))|^2] + 2\alpha_M^2 t \mathbb{E} \left[\int_0^t e^{2\pi_1(R^M[Y](u-))} du \right] \\ &\leq 2 \sup_{u \in [0,t]} \mathbb{E} [|\pi_2(R^M[Y](u))|^2] + 2\alpha_M^2 t \int_0^t \mathbb{E} \left[e^{2\pi_1(R^M[Y](u-))} \right] du \\ &\leq 2 \sup_{u \in [0,t]} \mathbb{E} [|\pi_2(R^M[Y](u))|^2] + 2\alpha_M^2 t^2 \sup_{u \in [0,t]} \mathbb{E} \left[e^{2|\pi_1(R^M[Y](u-))|} \right] \\ &< +\infty, \end{aligned} \quad (5.73)$$

where we have used Jensen's inequality, Fubini's theorem, and Lemma 5.2 for the finiteness of the two supremums. Since the final upper bound is independent of $s \in [0, t]$, we obtain that $W^M[Y]$ is indeed locally bounded in L^2 .

We now justify that $W^M[Y]$ is a martingale. Recall from Lemma 5.2 that $R^M[Y]$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$. From this it is not difficult to see that $W^M[Y]$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$. Now, for any $h > 0$ let us put

$$\alpha_M(h) := \mathbb{E} [\pi_2(R^M[Y](h))] / \mathbb{E} \left[\int_0^h e^{\pi_1(R^M[Y](u-))} du \right],$$

and notice that α_M defined in (5.72) coincides with $\alpha_M(1)$. By definition, $\alpha_M(h)$ is the unique real number that satisfies

$$\mathbb{E} \left[\pi_2(R^M[Y](h)) - \alpha_M(h) \int_0^h e^{\pi_1(R^M[Y](u-))} du \right] = 0.$$

Then recall that, since $R^M[Y]$ is a Lévy process on (\mathbb{R}^2, T) adapted to $(\mathcal{F}_t)_{t \geq 0}$, for any $h > 0$ and $u > 0$ we have $R^M[Y](h+u) = R^M[Y](h)TS_h[R^M[Y]](u)$ where $S_h[R^M[Y]]$ is equal in law to $R^M[Y]$ and is independent from \mathcal{F}_h . Then $\mathbb{E}[\pi_2(R^M[Y](2h)) - \alpha_M(h) \int_0^{2h} e^{\pi_1(R^M[Y](u-))} du]$ equals

$$\begin{aligned} & \mathbb{E} \left[\pi_2(R^M[Y](h)TS_h[R^M[Y]](h)) - \alpha_M(h) \int_0^h e^{\pi_1(R^M[Y](u-))} du \right. \\ & \left. - \alpha_M(h) \int_0^h e^{\pi_1(R^M[Y](h)TS_h[R^M[Y]](u-))} du \right] \\ &= \mathbb{E} \left[\pi_2(R^M[Y](h)) - \alpha_M(h) \int_0^h e^{\pi_1(R^M[Y](u-))} du \right] \\ &+ \mathbb{E} \left[e^{\pi_1(R^M[Y](h))} \left(\pi_2(S_h[R^M[Y]](h)) - \alpha_M(h) \int_0^h e^{\pi_1(S_h[R^M[Y]](u-))} du \right) \right] \\ &= 0 + \mathbb{E} \left[e^{\pi_1(R^M[Y](h))} \mathbb{E} \left[\pi_2(S_h[R^M[Y]](h)) - \alpha_M(h) \int_0^h e^{\pi_1(S_h[R^M[Y]](u-))} du \middle| \mathcal{F}_h \right] \right] \\ &= \mathbb{E} \left[e^{\pi_1(R^M[Y](h))} \times \mathbb{E} \left[\pi_2(R^M[Y](h)) - \alpha_M(h) \int_0^h e^{\pi_1(R^M[Y](u-))} du \right] \right] \\ &= \mathbb{E} \left[e^{\pi_1(R^M[Y](h))} \times 0 \right] = 0. \end{aligned} \tag{5.74}$$

In the above we have used two times the definition of $\alpha_M(h)$ and one time the fact that $S_h[R^M[Y]]$ is equal in law to $R^M[Y]$ and is independent from \mathcal{F}_h . By unicity we obtain $\alpha_M(2h) = \alpha_M(h)$ and the previous reasoning can be used to prove by induction that we have actually $\alpha_M(nh) = \alpha_M(h)$, for all $n \in \mathbb{N}$. In particular, the function $\alpha_M(\cdot)$ is constant on the set of positive dyadic numbers. Then, reasoning as in (5.73), we see that for any $t > 0$ the families $(\pi_2(R^M[Y](s)), 0 \leq s \leq t)$ and $(\int_0^s e^{\pi_1(R^M[Y](u-))} du, 0 \leq s \leq t)$ are bounded in L^2 and therefore uniformly integrable. Since $R^M[Y]$ is a Lévy process, these two family are also stochastically continuous with respect to $s \in [0, t]$. We deduce that these family are continuous in L^1 with respect to $s \in [0, t]$. This proves the continuity of the function $\alpha_M(\cdot)$. Combining with the constantness on the set of positive dyadic numbers, we obtain that this function is constant equal to α_M . As a consequence, for any $h > 0$ we have

$$\mathbb{E} \left[\pi_2(R^M[Y](h)) - \alpha_M \int_0^h e^{\pi_1(R^M[Y](u-))} du \right] = 0.$$

Using this and reasoning as in (5.74) we can now prove that for any $t, s > 0$

$$\mathbb{E} [W^M[Y](t+s) | \mathcal{F}_t] = W^M[Y](t).$$

This concludes the proof of the fact that $W^M[Y]$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Conversely, the fact that $W^M[Y]$ is a martingale implies that $\mathbb{E}[W^M[Y](1)] = \mathbb{E}[W^M[Y](0)] = \mathbb{E}[0] = 0$. This can hold only for α_M being as in (5.72). This proves unicity for the choice of α_M for which $W^M[Y]$ is a martingale. \square

We are now ready to prove Proposition 5.1.

Proof. of Proposition 5.1

Let Y be a Lévy process on (\mathbb{R}^2, T) and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous filtration associated with Y . We want to define

$$\forall t \geq 0, \xi(t) := \pi_1(Y(t)), \quad \eta(t) := \int_0^t e^{-\pi_1(Y(s-))} d\pi_2(Y(s)). \quad (5.75)$$

We first need to justify that the stochastic integral in (5.75) is well-defined. Let us fix $M > 0$, and let $W^M[Y]$ denote the process defined in (5.71). According to Lemma 5.3, $W^M[Y]$ is a càd-làg martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, locally bounded in L^2 . Also, $(\int_0^t e^{\pi_1(R^M[Y](u-))} du, t \geq 0)$ is continuous, adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$ and has locally bounded variation. Then, since by definition of $W^M[Y]$ in (5.71) we have $\pi_2(R^M[Y]) = W^M[Y] + \alpha_M \int_0^\cdot e^{\pi_1(R^M[Y](u-))} du$, we deduce that the process $\pi_2(R^M[Y])$ (that is càd-làg and adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$) satisfies the definition of a decomposable process, just before Theorem II.3.9 in [13]. By this theorem, $\pi_2(R^M[Y])$ is a semimartingale, in the meaning of [13], for the filtration $(\mathcal{F}_t)_{t \geq 0}$. In the remainder, the concept of semimartingale has to be always understood in the meaning of [13].

Let $T_M := \inf\{t \geq 0, \Delta Y(t) \in A_M\}$, where A_M is defined in (5.62). T_M is a stopping time for the filtration $(\mathcal{F}_t)_{t \geq 0}$ and T_M converges a.s. to $+\infty$ when M goes to $+\infty$ (otherwise, with positive probability, a finite time interval could contain infinitely many big jumps of Y , which would not be compatible with Y being a.s. càd-làg). For Z a real process starting at 0 and adapted to $(\mathcal{F}_t)_{t \geq 0}$, we define Z stopped at T_M- by $Z^{T_M-}(t) := Z(t)\mathbb{1}_{0 \leq t < T_M} + Z(T_M-)\mathbb{1}_{t \geq T_M}$. For any $M > 0$ we have clearly $(\pi_2(R^M[Y]))^{T_M-} = (\pi_2(Y))^{T_M-}$. Moreover $\pi_2(Y)$ is càd-làg and adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$. By Theorem II.3.6 in [13], we can conclude that $\pi_2(Y)$ is a semimartingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$.

The process $(e^{-\pi_1(Y(t-))}, t \geq 0)$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$ and left continuous with right limits. Section II.4 in [13] ensures that the stochastic integral of such a process, with respect to a semimartingale for $(\mathcal{F}_t)_{t \geq 0}$, is well-defined (as a limit of stochastic integrals of simple predictable processes with respect to the semimartingale in question), a.s. càd-làg, and adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$.

We deduce that the process (ξ, η) defined in (5.75) is indeed well-defined, a.s. càd-làg, and adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$. We now justify that (ξ, η) is a Lévy process on $(\mathbb{R}^2, +)$. We only need to justify the independence and stationarity of the increments. Let us fix $t \geq 0$ and define $\tilde{Y}(\cdot) := (Y(t))^{-1}TY(t + \cdot)$. Then \tilde{Y} is independent from \mathcal{F}_t and has the same law as Y . Since $Y(t + \cdot) = Y(t)T\tilde{Y}(\cdot)$, it is not difficult to see that we have a.s.

$$\forall s \geq 0, \tilde{Y}(s) = \begin{pmatrix} \pi_1(Y(t+s)) - \pi_1(Y(t)) \\ e^{-\pi_1(Y(t))} [\pi_2(Y(t+s)) - \pi_2(Y(t))] \end{pmatrix}.$$

Let $(\tilde{\xi}, \tilde{\eta})$ be constructed from \tilde{Y} just as (ξ, η) is constructed from Y in (5.75). Then $(\tilde{\xi}, \tilde{\eta})$ is independent from \mathcal{F}_t and has the same law as (ξ, η) . Moreover, we have a.s. that for any $s \geq 0$,

$$\begin{aligned} \begin{pmatrix} \tilde{\xi}(s) \\ \tilde{\eta}(s) \end{pmatrix} &= \begin{pmatrix} \pi_1(\tilde{Y}(s)) \\ \int_0^s e^{-\pi_1(\tilde{Y}(u-))} d\pi_2(\tilde{Y}(u)) \end{pmatrix} = \begin{pmatrix} \pi_1(Y(t+s)) - \pi_1(Y(t)) \\ \int_0^s e^{-\pi_1(Y(t+u-)) + \pi_1(Y(t))} e^{-\pi_1(Y(t))} d\pi_2(Y(t+u)) \end{pmatrix} \\ &= \begin{pmatrix} \pi_1(Y(t+s)) - \pi_1(Y(t)) \\ \int_0^s e^{-\pi_1(Y(t+u-))} d\pi_2(Y(t+u)) \end{pmatrix} = \begin{pmatrix} \pi_1(Y(t+s)) - \pi_1(Y(t)) \\ \int_t^{t+s} e^{-\pi_1(Y(u-))} d\pi_2(Y(u)) \end{pmatrix} \\ &= \begin{pmatrix} \xi(t+s) \\ \eta(t+s) \end{pmatrix} - \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}. \end{aligned}$$

This proves that (ξ, η) has stationary increments. Moreover, recall that $((\xi(s), \eta(s)), 0 \leq s \leq t)$ is measurable with respect to \mathcal{F}_t and that $(\tilde{\xi}, \tilde{\eta})$ is independent from \mathcal{F}_t , so the independence of increments for (ξ, η) follows. (ξ, η) is thus indeed a Lévy process on $(\mathbb{R}^2, +)$.

It now only remains to justify that the expression (5.61) is satisfied. Since η is defined via (5.75), Theorem II.5.19 in [13] ensures that η is a semimartingale for the filtration $(\mathcal{F}_t)_{t \geq 0}$ (alternatively, this follows from the fact that η is a Lévy process, adapted to $(\mathcal{F}_t)_{t \geq 0}$, and the corollary of Theorem II.3.9 in [13]). The process $(e^{\pi_1(Y(t-))}, t \geq 0)$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$ and left continuous with right limits. Therefore, by Section II.4 in [13], this process can be integrated with respect to η , and the stochastic integral in the sens of [13] coincides with the classical stochastic integral with respect to the Lévy process η (see for example Section 4 in [2] for the definition of stochastic integrals with respect to real Lévy processes). Indeed, both integrals are limits of stochastic integrals, with respect to η , of simple predictable processes approaching the integrand. By the associativity property in Theorem II.5.19 of [13], we have $\int_0^t e^{\pi_1(Y(s-))} d\eta(s) = \int_0^t e^{\pi_1(Y(s-))} e^{-\pi_1(Y(s-))} d\pi_2(Y(s))$. Then we get that, a.s. for any $t \geq 0$,

$$\begin{aligned} \begin{pmatrix} \xi(t) \\ \int_0^t e^{\xi(s-)} d\eta(s) \end{pmatrix} &= \begin{pmatrix} \pi_1(Y(t)) \\ \int_0^t e^{\pi_1(Y(s-))} d\eta(s) \end{pmatrix} = \begin{pmatrix} \pi_1(Y(t)) \\ \int_0^t e^{\pi_1(Y(s-))} e^{-\pi_1(Y(s-))} d\pi_2(Y(s)) \end{pmatrix} \\ &= \begin{pmatrix} \pi_1(Y(t)) \\ \int_0^t 1 d\pi_2(Y(s)) \end{pmatrix} = \begin{pmatrix} \pi_1(Y(t)) \\ \pi_2(Y(t)) \end{pmatrix} = Y(t). \end{aligned}$$

Therefore, (5.61) holds a.s. for our definition of (ξ, η) . □

6. PROOF OF THE MAIN THEOREMS

6.1. Proof of Theorems 1.16 and 1.5. We first prove Theorem 1.16. Let I be an open interval of \mathbb{R} , and X be a \mathcal{C}^k -gssMp on I , for some $k \geq 1$. $((f_y, c_y), y \in E)$ is a family of \mathcal{C}^k good invariance components associated with X , relatively to some reference point $y_0 \in I$. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = I$. Lemma 7.1 then guaranties that Assumption 1 is satisfied. According to Lemma 7.2, $Sym(X_{y_0})$ is discrete.

We can thus apply Proposition 3.1. Let us define an interne composition law \star on I by $y \star x := f_y(x)$. According to Proposition 3.1, (I, \star) is a \mathcal{C}^k -Lie group (for the natural differential structure on I , arising from the fact that it is an open subset of \mathbb{R}) with neutral element y_0 , and $(y \mapsto c_y)$ is a \mathcal{C}^k -Lie group homomorphism from (I, \star) to (\mathbb{R}_+^*, \times) . According to Proposition 3.3, there is a Lévy process L , on (I, \star) , such that if we set $\forall t \in [0, +\infty]$, $\varphi(t) := \int_0^t 1/c_{L(s)} ds$, then $\zeta(X_{y_0}) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty) = \zeta(X_{y_0}), \quad X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (6.76)$$

Let us define $g : I \longrightarrow \mathbb{R}$ as in (1.18). Then, according to Lemma 4.2, g is well-defined, is a \mathcal{C}^k diffeomorphism from I to \mathbb{R} , and even a \mathcal{C}^k -Lie group isomorphism from (I, \star) to $(\mathbb{R}, +)$. Let us define the real valued process ξ as $\xi(t) := g(L(t))$. Since g is a continuous group homomorphism from (I, \star) to $(\mathbb{R}, +)$ we have clearly that ξ is a real Lévy process. Then, we have $\forall t \in [0, +\infty]$, $\varphi(t) = \int_0^t 1/c_{g^{-1}(\xi(s))} ds$ and (6.76) can be re-written as

$$\forall 0 \leq t < \varphi(+\infty) = \zeta(X_{y_0}), \quad X_{y_0}(t) = g^{-1}(\xi(\varphi^{-1}(t))).$$

Then, since g^{-1} is a continuous group homomorphism from $(\mathbb{R}, +)$ to (I, \star) , and $(y \longmapsto c_y)$ is a continuous group homomorphism from (I, \star) to (\mathbb{R}_+^*, \times) , we get that $(x \longmapsto c_{g^{-1}(x)})$ is a continuous group homomorphism from $(\mathbb{R}, +)$ to (\mathbb{R}_+^*, \times) . Therefore, there exists $\alpha \in \mathbb{R}$ such that $c_{g^{-1}(x)} = e^{-\alpha x}$, $\forall x \in \mathbb{R}$. Clearly, this α is equal to $-\log(c_{g^{-1}(1)})$ and " $\varphi(t) = \int_0^t 1/c_{g^{-1}(\xi(s))} ds$ " can be re-written as $\varphi(t) = \int_0^t e^{\alpha \xi(s)} ds$, which terminates the proof of the asserted representation for X .

Let us now justify Remark 1.17. Clearly we only need to justify that for all $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$. In the above proof, an application of Proposition 3.3 yielded the existence of a process L satisfying the relation (6.76), with $\forall t \in [0, +\infty]$, $\varphi(t) = \int_0^t 1/c_{L(s)} ds$. According to Remark 3.4, this implies that we have $0 \leq t < \zeta(X_{y_0})$, $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$, which is the claim.

We now justify Theorem 1.5. Under the assumptions of the theorem, Lemma 7.1 guaranties that Assumption 1 is satisfied. Then, Lemma 7.2 applies so the conditions are satisfied to apply Proposition 1.15. We can thus produce a family $((f_y, c_y), y \in I)$ of \mathcal{C}^k good invariance components associated with X , relatively to the reference point y_0 . Then Theorem 1.16 applies and yields the direct part of the theorem. The reciprocal is straightforward to verify, following the procedure from the proof of Proposition 1.11.

6.2. Proof of Theorems 1.19 and 1.6. We first prove Theorem 1.19. Let \mathcal{D} be an open simply connected domain of \mathbb{R}^2 , and X be a \mathcal{C}^k -gssMp on \mathcal{D} , for some $k \geq 2$. $((f_y, c_y), y \in E)$ is a family of \mathcal{C}^k good invariance components associated with X , relatively to some reference point $y_0 \in \mathcal{D}$. We assume that either Assumption 1 or Assumption 2 is satisfied for $E = \mathcal{D}$ and that $Sym(X_{y_0})$ is discrete.

Let us define an interne composition law \star on \mathcal{D} by $y \star x := f_y(x)$. According to Proposition 3.1, (\mathcal{D}, \star) is a \mathcal{C}^k -Lie group (for the natural differential structure on \mathcal{D} , arising from the fact that it is an open subset of \mathbb{R}^2) with neutral element y_0 , and $(y \longmapsto c_y)$ is a \mathcal{C}^k -Lie group homomorphism from (\mathcal{D}, \star) to (\mathbb{R}_+^*, \times) . According to Proposition 3.3, there is a Lévy process L , on (\mathcal{D}, \star) , such that if we set $\forall t \in [0, +\infty]$, $\varphi(t) := \int_0^t 1/c_{L(s)} ds$, then $\zeta(X_{y_0}) = \varphi(+\infty)$ and

$$\forall 0 \leq t < \varphi(+\infty) = \zeta(X_{y_0}), \quad X_{y_0}(t) = L(\varphi^{-1}(t)). \quad (6.77)$$

We now distinguish two cases:

1) If $((f_y, c_y), y \in \mathcal{D})$ are *commutative invariance components*, then the Lie group (\mathcal{D}, \star) is commutative. Let us define $g : \mathcal{D} \longrightarrow \mathbb{R}^2$ as in (1.23). Since the matrix M , defined a little before (1.23), is always invertible, Lemma 4.3 ensures that g is well-defined, is a \mathcal{C}^k diffeomorphism from \mathcal{D} to \mathbb{R}^2 , and even a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to $(\mathbb{R}^2, +)$. Let us define the \mathbb{R}^2 valued process (ξ, η) as $(\xi(t), \eta(t)) := g(L(t))$, where L is the Lévy process on (\mathcal{D}, \star) that appears in (6.77). Since g is a continuous group homomorphism we have that (ξ, η) is a Lévy

process on $(\mathbb{R}^2, +)$. Also, replacing L by $g^{-1}(\xi, \eta)$ we obtain

$$\forall t \in [0, +\infty], \varphi(t) = \int_0^t \frac{1}{c_{L(s)}} ds = \int_0^t \frac{1}{c_{g^{-1}(\xi(s), \eta(s))}} ds, \quad (6.78)$$

and (6.77) can be re-written as (1.26) (but with $\varphi(\cdot)$ as in (6.78)). It thus only remains to prove that $\varphi(t) = \int_0^t e^{\alpha \xi(s)} ds$ where α is as defined in the statement of the theorem, that is, $\alpha = -\log(c_{g^{-1}(1,0)})$.

Note that $(z \mapsto c_{g^{-1}(z)})$ is a \mathcal{C}^k -Lie group homomorphism from $(\mathbb{R}^2, +)$ to (\mathbb{R}_+^*, \times) . Therefore, there are $\beta_1, \beta_2 \in \mathbb{R}$ such that $\forall z = (z_1, z_2) \in \mathbb{R}^2, c_{g^{-1}(z)} = e^{\beta_1 z_1 + \beta_2 z_2}$. Differentiating at $(0, 0)$ with respect to respectively z_1 and z_2 we get

$$\beta_1 = {}^t Y_0 \cdot Jg^{-1}(0, 0) \cdot e_1 = {}^t Y_0 \cdot (Jg(y_0))^{-1} \cdot e_1 = {}^t Y_0 \cdot M^{-1} \cdot e_1 \quad \text{and} \quad \beta_2 = {}^t Y_0 \cdot M^{-1} \cdot e_2. \quad (6.79)$$

In the above expression we have used that, as defined in the statement of the theorem, Y_0 is the gradient at y_0 of the application $(y \mapsto c_y)$, and the fact that $Jg(y_0) = M$ (see Lemma 4.3).

If $Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then we have clearly from (6.79) that $\beta_2 = 0$. If $Y_0 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then recall that $M = \begin{pmatrix} \pi_1(Y_0) & \pi_2(Y_0) \\ -\pi_2(Y_0) & \pi_1(Y_0) \end{pmatrix}$ so $M^{-1} = \frac{1}{\|Y_0\|^2} \begin{pmatrix} \pi_1(Y_0) & -\pi_2(Y_0) \\ \pi_2(Y_0) & \pi_1(Y_0) \end{pmatrix}$, where $\|\cdot\|$ denotes the Euclidian norm of vectors of \mathbb{R}^2 . Combining with (6.79) we get $\beta_2 = 0$. Therefore, in any case, we have $\forall z = (z_1, z_2) \in \mathbb{R}^2, c_{g^{-1}(z)} = e^{\beta_1 z_1}$ and, evaluating at $z = (1, 0)$, $\beta_1 = \log(c_{g^{-1}(1,0)}) = -\alpha$. Putting the obtained expression of $c_{g^{-1}(\cdot)}$ into (6.78) we get $\varphi(t) = \int_0^t e^{\alpha \xi(s)} ds$ as required, which terminates the proof.

2) If $((f_y, c_y), y \in \mathcal{D})$ are not *commutative invariance components*, then the Lie group (\mathcal{D}, \star) is non-commutative. Let us define $g : \mathcal{D} \rightarrow \mathbb{R}^2$ as in Lemma 4.4. Then, according to that lemma, g is well-defined, is a \mathcal{C}^k diffeomorphism from \mathcal{D} to \mathbb{R}^2 , and even a \mathcal{C}^k -Lie group isomorphism from (\mathcal{D}, \star) to (\mathbb{R}^2, T) . Let us define the \mathbb{R}^2 valued process Y as $Y(t) := g(L(t))$, where L is the Lévy process on (\mathcal{D}, \star) that appears in (6.77). Since g is a continuous group homomorphism we have that Y is a Lévy process on (\mathbb{R}^2, T) . According to Proposition 5.1, there is a Lévy process (ξ, η) on $(\mathbb{R}^2, +)$ such that $Y(t) = (\xi(t), \int_0^t e^{\xi(s^-)} d\eta(s))$, $\forall t \geq 0$. Therefore, we can replace $L(\cdot)$ by $g^{-1}(\xi(\cdot), \int_0^\cdot e^{\xi(s^-)} d\eta(s))$. We obtain

$$\forall t \in [0, +\infty], \varphi(t) = \int_0^t \frac{1}{c_{L(s)}} ds = \int_0^t \frac{1}{c_{g^{-1}(\xi(s), \int_0^s e^{\xi(u^-)} d\eta(u))}} ds, \quad (6.80)$$

and (6.77) can be re-written as (1.29) (but with $\varphi(\cdot)$ as in (6.80)). It thus only remains to prove that $\varphi(t) = \int_0^t e^{\alpha \xi(s)} ds$ where α is as defined in the statement of the theorem, that is, $\alpha = -\log(c_{g^{-1}(1,0)})$.

$(z \mapsto c_{g^{-1}(z)})$ is a \mathcal{C}^k -Lie group homomorphism from (\mathbb{R}^2, T) to (\mathbb{R}_+^*, \times) . Therefore, according to Lemma 4.5, there is $\beta \in \mathbb{R}$ such that $\forall z = (z_1, z_2) \in \mathbb{R}^2, c_{g^{-1}(z)} = e^{\beta z_1}$ and, evaluating at $z = (1, 0)$, $\beta = \log(c_{g^{-1}(1,0)}) = -\alpha$. Putting the obtained expression of $c_{g^{-1}(\cdot)}$ into (6.80) we get $\varphi(t) = \int_0^t e^{\alpha \xi(s)} ds$ as required, which terminates the proof.

Here, we still have the alternative expression $\varphi^{-1}(t) = \int_0^t c_{X_{y_0}(u)} du$ for $0 \leq t < \zeta(X_{y_0})$. The justification is just as in the proof of Theorem 1.16.

We now justify Theorem 1.6. Under the assumptions of the theorem, Proposition 1.15 applies. We can thus produce a family $((f_y, c_y), y \in I)$ of \mathcal{C}^k good invariance components associated with X , relatively to the reference point y_0 . Then Theorem 1.19 applies and yields the direct

part of the theorem. Indeed, if the good invariance components are commutative then Theorem 1.19 shows that (1.9), (1.10) and (1.11) are true for $\psi = g^{-1}$, $\beta = 0$ (and g , (ξ, η) , α are given in the first point of Theorem 1.19). If the good invariance components are not commutative then Theorem 1.19 shows that (1.9), (1.10) and (1.11) are true for $\psi = g^{-1}$, $\beta = 1$ (and g , (ξ, η) , α are given in the second point of Theorem 1.19). The reciprocal is straightforward to verify, following the procedure from the proof of Proposition 1.11.

7. TECHINICAL RESULTS

We now state and prove Lemmas 7.1 and 7.2, about the assumptions that are discussed in the end of Subsection 1.1 and used all along the paper.

Lemma 7.1. *Let E be a connected locally compact separable metric space, and let X be a gssMp on E with invariance components $((f_y, c_y), y \in E)$ relatively to some reference point $y_0 \in E$. Then, Assumption 2 implies Assumption 1.*

Proof. Let us assume that Assumption 2 holds. Let y be such that $y \in \widehat{Supp}(X_y)$. Clearly we have $Supp(X_y) = f_y(Supp(X_{y_0}))$ so, since f_y is an homeomorphism that maps y_0 to y , we have $y_0 \in \widehat{Supp}(X_{y_0})$. Similarly, $Supp(X_z) = f_z(Supp(X_{y_0}))$ with f_z an homeomorphism that maps y_0 to z , so $z \in \widehat{Supp}(X_z)$ for all $z \in E$. In other words, for all $z \in E$, $Supp(X_z)$ is a neighborhood of z .

We now justify that $z \in Supp(X_{y_0}) \Rightarrow Supp(X_z) \subset Supp(X_{y_0})$. Let $z \in Supp(X_{y_0})$, $v \in Supp(X_z)$ and $\epsilon > 0$, we need to prove the existence of $r \geq 0$ such that $\mathbb{P}(X_{y_0}(r) \in B(v, \epsilon)) > 0$. By definition of $Supp(X_z)$, there is $t \geq 0$ such that $\mathbb{P}(X_z(t) \in B(v, \epsilon)) > 0$. By self-similarity, $\mathbb{P}(f_z(X_{y_0}(c_z t)) \in B(v, \epsilon)) > 0$. Since $(y, x) \mapsto f_y(x)$ and $y \mapsto c_y$ are continuous, and X_{y_0} is stochastically continuous, we have that $f_w(X_{y_0}(c_w t))$ converges in distribution to $f_z(X_{y_0}(c_z t))$ as w goes to z . Therefore, there is a neighborhood \mathcal{U} of z such that

$$\forall w \in \mathcal{U}, \mathbb{P}(X_w(t) \in B(v, \epsilon)) = \mathbb{P}(f_w(X_{y_0}(c_w t)) \in B(v, \epsilon)) > 0. \quad (7.81)$$

Then, since \mathcal{U} is a neighborhood of $z \in Supp(X_{y_0})$, there is $s \geq 0$ such that

$$\mathbb{P}(X_{y_0}(s) \in \mathcal{U}) > 0. \quad (7.82)$$

The combination of (7.82), (7.81), and the Markov property at time s yields that $\mathbb{P}(X_{y_0}(t+s) \in B(v, \epsilon)) > 0$, proving the claim.

We have established that for each $z \in Supp(X_{y_0})$, $Supp(X_{y_0})$ contains $Supp(X_z)$ which is a neighborhood of z . Therefore $Supp(X_{y_0})$ is open. Since it also close (see (1.4)) and nonempty (it contains y_0 , as previously justified), and since E is connected, we conclude that $Supp(X_{y_0}) = E$, that is, Assumption 1. □

Lemma 7.2. *Let $I \subset \mathbb{R}$ be an open interval, $y_0 \in I$, X_{y_0} be a Markovian process on I that satisfies Assumption 1 with $E = I$, and let $Sym(X_{y_0})$ be defined as in (1.5). Then $Sym(X_{y_0})$ is discrete.*

Proof. Let h be an increasing element of $Sym(X_{y_0})$, and let us fix $x \in I$. Because of Assumption 1 and of the definition of the support (see (1.4)), we have that for any $\epsilon > 0$, we can find $t_1, t_2 \geq 0$ such that $\mathbb{P}(X_{y_0}(t_1) \in]x, x + \epsilon]) > 0$ and $\mathbb{P}(X_{y_0}(t_2) \in]x - \epsilon, x]) > 0$. Therefore we have

$$\mathbb{P}(X_{y_0}(t_1) > x) > \mathbb{P}(X_{y_0}(t_1) > x + \epsilon) \quad \text{and} \quad \mathbb{P}(X_{y_0}(t_2) > x) < \mathbb{P}(X_{y_0}(t_2) > x - \epsilon). \quad (7.83)$$

$h^{-1} \in \text{Sym}(X_{y_0})$ and is also increasing so we have

$$\begin{aligned}\mathbb{P}(X_{y_0}(t_1) > x) &= \mathbb{P}(h^{-1}(X_{y_0}(t_1)) > x) = \mathbb{P}(X_{y_0}(t_1) > h(x)) \\ \mathbb{P}(X_{y_0}(t_2) > x) &= \mathbb{P}(h^{-1}(X_{y_0}(t_2)) > x) = \mathbb{P}(X_{y_0}(t_2) > h(x)).\end{aligned}$$

Combining with (7.83) we obtain $\mathbb{P}(X_{y_0}(t_1) > h(x)) > \mathbb{P}(X_{y_0}(t_1) > x + \epsilon)$ and $\mathbb{P}(X_{y_0}(t_2) > h(x)) < \mathbb{P}(X_{y_0}(t_2) > x - \epsilon)$ which implies $h(x) \in]x - \epsilon, x + \epsilon[$. Since this is true for any $\epsilon > 0$ we get $h(x) = x$, and since $x \in I$ is arbitrary we conclude that $h = id_I$.

Since any $h \in \text{Sym}(X_{y_0})$ is an homeomorphisms of I , it is either increasing or decreasing. Let h be a decreasing element of $\text{Sym}(X_{y_0})$, if such an element exists. If \tilde{h} is any other decreasing element of $\text{Sym}(X_{y_0})$, then $\tilde{h}^{-1} \circ h$ is an increasing element of $\text{Sym}(X_{y_0})$ so, by the previous part, it equals id_I , so $\tilde{h} = h$. In conclusion $\text{Sym}(X_{y_0})$ contains at most two elements so it is discrete.

□

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