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► **To cite this version:**

Philippe Biane. Gog and Magog Triangles. Computation and Combinatorics in Dynamics, Stochastics and Control- The Abel Symposium, Rosendal, Norway, August 2016, In press. hal-01829708

HAL Id: hal-01829708

<https://hal.science/hal-01829708>

Submitted on 4 Jul 2018

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GOG AND MAGOG TRIANGLES

PHILIPPE BIANE

ABSTRACT. We survey the problem of finding an explicit bijection between Gog and Magog triangles, a combinatorial problem which has been open since the 80's. We give some of the ideas behind a recent approach to this question and also prove some properties of the distribution of inversions and coinversions in Gog triangles.

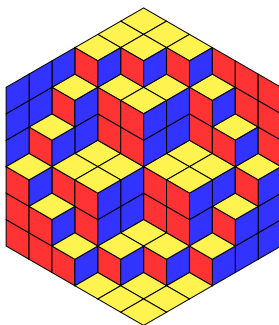
1. INTRODUCTION

An alternating sign matrix is a square matrix having coefficients in $\{-1, 0, 1\}$ so that, in each row and in each column, if one forgets the zeros, the 1 and -1 entries alternate and the sum is 1, e.g.

$$(1.1) \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

These matrices were introduced by Robbins and Rumsey [25] in 1986 as a generalization of permutation matrices, after they discovered some startling properties of the λ -determinant, a deformation of usual determinants of matrices.

Around the same time interest in the enumeration of plane partitions lead to the question of enumerating several symmetry classes of plane partitions. Among these classes the so-called totally symmetric self complementary plane partitions (TSSCPP in short), as the one below,



stood as the most symmetric ones. Empirical data showed that both the enumeration of ASM and of TSSCPP according to their size started with the numbers 1, 2, 7, 42, 429, 7436, 218348, 10850216 . . .

Date: October 17, 2017.

Key words and phrases. Gog and Magog triangles, Schützenberger Involution, alternating sign matrices, totally symmetric self complementary plane partitions.

which could be put into the closed form $A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. There was then two main problems: prove the enumeration formula and find a bijective proof of the coincidence, if it exists. As we explain below, the enumeration problems were finally solved around 1994/5, but the proofs gave no hint as to the existence of a natural bijection between these objects. It turns out that both classes of objects, the ASM and the TSSCPP, can be encoded as triangles of positive integers whose rows interlace, like the following (such triangles are usually called Gelfand-Tsetlin triangles in representation theory where they occur as labels for bases of irreducible representations)

$$\begin{array}{cccccc}
 1 & & 1 & & 3 & & 5 & & 5 \\
 & 1 & & 2 & & 4 & & 5 & \\
 & & 2 & & 2 & & 5 & & \\
 & & & 2 & & 3 & & & \\
 & & & & 3 & & & &
 \end{array}$$

Thus one can reformulate the problem as finding a bijection between two species of such Gelfand-Tsetlin triangles, which makes this problem completely elementary.

The purpose of this paper is to present a recent approach to this question which has lead to some progress towards the solution of the bijection problem. In particular we will introduce a new class of objects, the GOGAm triangles and present some new conjectures. We also study the distribution of inversions and coinversions in Gog triangles. In particular we determine the pairs (p, q) for which there exists Gog triangles of a given size with p inversions and q coinversions.

The paper is organized as follows. In the first part we present a short survey of plane partitions and their enumeration, in particular the TSSCPP and the Magog triangles. In the second part we present the other combinatorial objects in which we are interested, the alternating sign matrices. As we briefly explain, these objects arise in many places of algebra, representation theory and statistical physics. Much more information about these two parts can be found for example in the book by D. Bressoud [11]. Then we present the approach to the bijection problem. Finally we give some results on the joint enumeration of inversions and coinversions in Gog triangles.

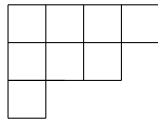
I thank both referees for their useful remarks and comments which lead to improvements in the presentation of this paper, in particular for correcting the statement of Proposition 5.3.

2. PLANE PARTITIONS

2.1. Partitions, tableaux and triangles. A partition of n is a nonincreasing sequence of nonnegative integers with sum n

$$n = \lambda_1 + \lambda_2 + \dots, \quad \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_i \geq 0.$$

This is a fundamental notion in mathematics which occurs in algebra, representation theory, combinatorics, number theory etc. See e.g. Andrews [1], Fulton [15], Macdonald [21], Ramanujan [24]. The usual way to depict a partition is by drawing superposed rows of squares with λ_i squares in row i from above:



$$8 = 4 + 3 + 1$$

It is easy to derive the following generating series for the set of all partitions, where $|\lambda| = \sum_i \lambda_i$

$$\sum_{\lambda} q^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)}$$

which is closely related to Dedekind's eta function.

A *semi-standard tableau* is obtained by putting positive integers in the boxes of a partition which are

- i) weakly increasing from left to right
 - ii) strictly increasing from top to bottom
- as below.

1	1	1	3
2	2	4	
4			

The shape of the tableau is the underlying partition λ .

The semi-standard tableaux themselves can be encoded by Gelfand-Tsetlin triangles.

Definition 2.1. A *Gelfand-Tsetlin triangle of size n* is a triangular array $X = (X_{i,j})_{n \geq i \geq j \geq 1}$ of nonnegative integers

$$\begin{array}{ccccccc}
 X_{n,1} & & X_{n,2} & & \dots & & X_{n,n-1} & & X_{n,n} \\
 & X_{n-1,1} & & X_{n-1,2} & & \dots & & X_{n-1,n-1} & \\
 & & \dots & & \dots & & \dots & & \\
 & & & X_{2,1} & & X_{2,2} & & & \\
 & & & & X_{1,1} & & & &
 \end{array}$$

such that

$$X_{i+1,j} \leq X_{i,j} \leq X_{i+1,j+1} \quad \text{for } n - 1 \geq i \geq j \geq 1.$$

Given a semi-standard tableau filled with numbers from 1 to n , one can construct a Gelfand-Tsetlin triangle of size n whose row k , as counted from below, consists of the partition, read backwards, formed by the boxes containing numbers from 1 to k in the semi-standard tableau. In the case of the semi-standard tableau above this gives

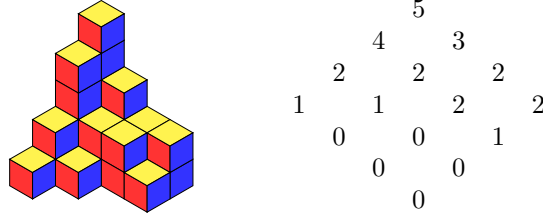
$$\begin{array}{cccc}
 0 & 1 & 3 & 4 \\
 & 0 & 2 & 4 \\
 & & 2 & 3 \\
 & & & 3
 \end{array}$$

Let $(x_i)_{i \geq 1}$ be a family of indeterminates. For a standard tableau t , let t_i be the number of i occurring in the tableau and $x^t = \prod_i x_i^{t_i}$. The generating function of semi-standard tableaux with shape λ , filled with numbers from 1 to n , is a Schur function

$$s_{\lambda}(x_1, \dots, x_n) = \sum_{t \text{ tableau of shape } \lambda} x^t.$$

These are symmetric functions, which occur as characters of irreducible representations of the group GL_n (see e.g. Macdonald [21]).

2.2. Plane Partitions. A plane partition is a stack of cubes in a corner. Putting the stack on a square basis and collecting the heights of the piles of cubes, one gets an array of integers:



Splitting the array into its left and right parts yields two Gelfand-Tsetlin triangles sharing the same upper row (which is the vertical diagonal of the square array):

$$\begin{array}{cccccc}
 0 & 0 & 2 & 5 & 0 & 0 & 2 & 5 \\
 & 0 & 1 & 4 & & 0 & 2 & 3 \\
 & & 0 & 2 & & & 1 & 2 \\
 & & & 1 & & & & 2
 \end{array}$$

From this one can infer that the generating series of plane partitions π according to their size (i.e. the number of cubes in the stack) is equal to

$$\sum_{\pi} q^{|\pi|} = \sum_{\lambda} s_{\lambda}(q, q^2, \dots, q^j, \dots)^2.$$

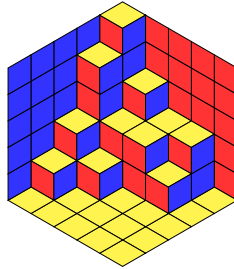
Using Cauchy's formula

$$(2.1) \quad \sum_{\lambda} s_{\lambda}(x_1, x_2, \dots, x_j, \dots) s_{\lambda}(y_1, y_2, \dots, y_j, \dots) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

one obtains Mac Mahon's formula which gives the generating function for plane partitions according to their size

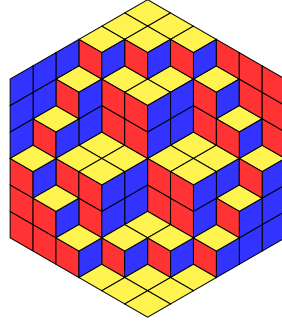
$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}.$$

Choosing a large cube that contains a plane partition, one can also encode it as a lozenge tiling of an hexagon.

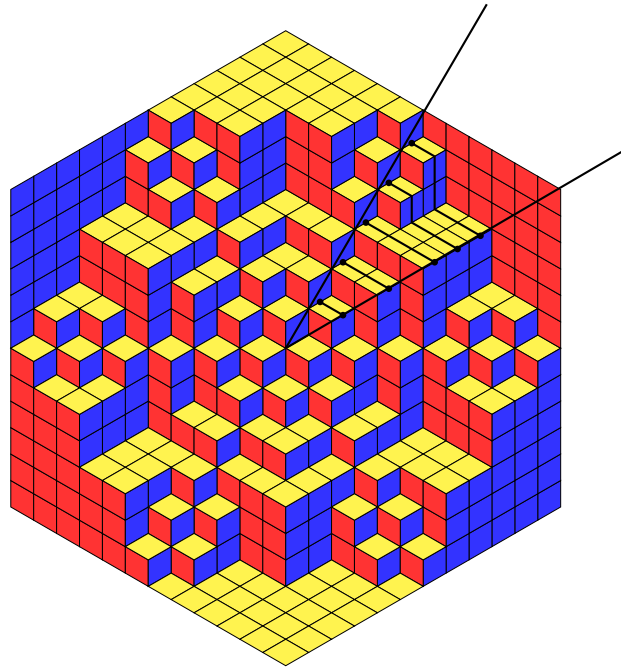


The symmetry group of the hexagon is a dihedral group. For each subgroup of this group, one can consider the class of plane partitions which are invariant under these symmetries of the hexagon. Various enumeration formulas have been derived for such symmetry classes. We will be interested in one of them.

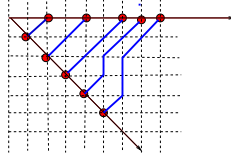
2.3. Totally Symmetric Self-Complementary Plane Partitions. A Totally Symmetric Self-Complementary Plane Partition (TSSCPP in short), of size n , is a plane partition, inside a cube of side $2n$, such that the lozenge tiling has all the dihedral symmetries of the hexagon, as in the picture below, where $n = 3$. Remarkably, a plane partition with all these symmetries can be superposed with its complement in the cube.



TSSCPPs can be encoded by non-intersecting paths as below, where the paths go through the yellow and blue tiles of a fundamental domain for the action of the dihedral group:



The paths can be drawn on a lattice using vertical steps to record blue tiles and diagonal steps for yellow tiles:



This correspondence has allowed J. Stembridge [27], building upon the Lindström-Gessel-Viennot technique, to express the number of these paths as a Pfaffian

$$t_n = pf(a_{ij})_{\epsilon_n \leq i < j \leq n-1}$$

$$a_{ij} = \sum_{r=2i-j+1}^{2j-i} \binom{i+j}{r}, \quad \epsilon_n = 0 \text{ if } n \text{ is even, } 1 \text{ if it is odd}$$

It is not an easy task however to evaluate this Pfaffian explicitly, but this was done by G. Andrews [2], who proved that

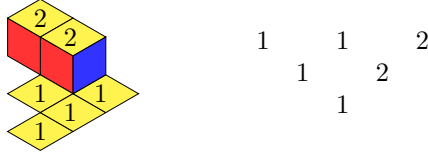
$$(2.2) \quad t_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

$t_n = 1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7436 \quad 218348 \quad \dots$

2.4. Magog triangles.

Definition 2.2. A Magog triangle of size n is a Gelfand-Tsetlin triangle of positive integers such that $X_{jj} \leq j$ for all $1 \leq j \leq n$.

Reading the heights of the cubes of a TSSCPP in a fundamental domain of the dihedral group gives a Magog triangle e.g. for the example above, with the heights starting at 1



This gives a bijection between Magog triangles of size n and TSSCPPs of size n . Thus the rather complicated objects which are TSSCPPs can be encoded by these triangles, satisfying a very simple condition.

3. ALTERNATING SIGN MATRICES

3.1. Jacobi-Desnanot identity and Dodgson algorithm. There are many polynomial identities relating the different minors of a matrix. One of them is the Jacobi-Desnanot identity which we now explain. For a square $n \times n$ matrix M let $M_{j_1 \dots j_r}^{i_1 \dots i_r}$ be the matrix obtained by deleting rows i_1, \dots, i_r and columns j_1, \dots, j_r . Then one has

$$\det(M) \det(M_{1n}^{1n}) = \det(M_1^1) \det(M_n^n) - \det(M_n^1) \det(M_1^n).$$

For a 2×2 matrix (the empty determinant is 1) this is just

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Using this identity Charles Dodgson (better known under the name of Lewis Carroll) devised an algorithm for computing the determinant of a matrix which uses only the computation of 2×2 determinants. For example, if you want to compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 4 & 6 & 0 \\ 2 & 1 & -3 & 1 \\ 3 & 2 & 1 & 5 \\ 3 & 2 & 2 & 0 \end{pmatrix}$$

start with two matrices, A and another matrix B , of size $(n-1) \times (n-1)$, with all its entries equal to one, then inside A insert a (red) matrix formed with the two by two minors of A divided by the corresponding entries of B ; inside B insert the (blue) values of A in the inner columns and rows

$$\begin{pmatrix} 1 & 4 & 6 & 0 \\ -7 & -18 & 6 & 6 \\ 2 & 1 & -3 & 1 \\ 1 & 7 & -16 & -16 \\ 3 & 2 & 1 & 5 \\ 0 & 2 & 2 & -10 \\ 3 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

then iterate with the new pair of matrices

$$A' = \begin{pmatrix} -7 & -18 & 6 \\ 1 & 7 & -16 \\ 0 & 2 & -10 \end{pmatrix} \quad B' = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$$

to get

$$\begin{pmatrix} -7 & -18 & 6 \\ -31 & -82 & -16 \\ 1 & 7 & -16 \\ 1 & -38 & -10 \\ 0 & 2 & -10 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 7 & 1 \end{pmatrix}$$

finally

$$\det(A) = \frac{\begin{vmatrix} -31 & -82 \\ 1 & -38 \end{vmatrix}}{7} = 180.$$

3.2. The λ -determinant. In 1983 David Robbins had the idea of replacing, in the above algorithm, every occurrence of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}_\lambda = ad + \lambda bc$, for some indeterminate λ . This defines the λ -determinant. The result is surprising, indeed although the algorithm implies taking a lot of quotients of rational fractions the result is always a Laurent polynomial in the coefficients of the matrix, namely one has, for a $d \times d$ matrix A ,

Theorem 3.1. (*D. Robbins, H. Rumsey [25]*)

$$(3.1) \quad \det_\lambda(A) = \sum_{M \in ASM(d)} (1 + \lambda)^{s(M)} \lambda^{i(M)} \prod_{ij} A_{ij}^{M_{ij}}$$

The sum is over the set of alternating sign matrices, defined at the beginning of the introduction while $i(M)$ is the number of inversions of M (to be defined later) and $s(M)$ is the number of -1 coefficients.

This is an example of the “Laurent phenomenon” which is at the heart of the deep theory of cluster algebras, see e.g. [14].

3.3. Alternating sign matrices. For the convenience of the reader we remind the definition of alternating sign matrices.

Definition 3.2. *An alternating sign matrix is a square matrix having coefficients in $\{-1, 0, 1\}$ so that, in each row and in each column, if one forgets the zeros, the 1 and -1 entries alternate and the sum is 1.*

Here is an example where we show an alternating sign matrix and the alternance of $+1$ and -1 in each row and column, once the zeros are removed:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} & & & & 1 \\ & & & 1 & -1 & 1 \\ & 1 & & -1 & 1 & \\ & & 1 & & & \\ 1 & & & & & \\ & & & & & 1 \end{pmatrix}$$

In particular, the alternating sign matrices without -1 are exactly the permutation matrices and (3.1) for $\lambda = -1$ gives the classical formula for the usual determinant.

It turns out that alternating sign matrices occur in a number of different contexts, in statistical physics, representation theory, or combinatorics. We shall give a few examples now.

3.3.1. The six-vertex model. An entry of an alternating sign matrix can take at most three values $\{-1, 0, 1\}$. For each entry with value zero consider the sum of entries lying respectively, on the right and on the left of this entry, then one of these sums is equal to 0 and the other is equal to 1. A similar property holds for the sum of entries lying above and the sum of entries lying below. It follows that one can divide the entries of the matrix into six groups, two corresponding to entries with the value 1 and -1 and four groups corresponding to the configurations of an entry with value 0. There are thus six possible configurations of each entry of an alternating sign matrix, listed below

$$(3.2) \quad \begin{array}{cccccc} & & & 0 & & 1 & & 0 & & 1 \\ & & & \mathbf{0} & 0 & 0 & \mathbf{0} & 1 & 0 & \mathbf{0} & 1 & 1 & \mathbf{0} & 0 \\ & & & 1 & & 0 & & & 1 & & 0 \end{array}$$

The configurations so obtained form an instance of a famous statistical physic model, known as the six-vertex model, which is one of the most studied models in statistical mechanics (see e.g. [4]). In order to study this model it is convenient to weigh the configurations as follows. Introduce indeterminates q , x_i and y_j , the indices ranging from 1 to n and corresponding to the rows and columns of the matrix. Endow each entry of an alternating sign matrix with a weight $w(i, j)$ (where i, j are the row and column of the entry) given by the following value, according to the configuration of the entry, as in (3.2)

$$(3.3) \quad \begin{array}{cccccc} x_i/y_j & y_j/x_i & [qx_i/y_j] & [qx_i/y_j] & [x_i/y_j] & [x_i/y_j] \end{array}$$

The convention used here is that $[a] = \frac{a-a^{-1}}{q-q^{-1}}$. One can then put on every alternating sign matrix the product of the weights of its entries. It turns out that the particular form of the

weights (3.3) allows one to use the Yang-Baxter equation to compute the partition function of this model, i.e. the sum over all alternating sign matrices of the weights, under the form of a remarkable determinant, the Izergin-Korepin determinant [16]

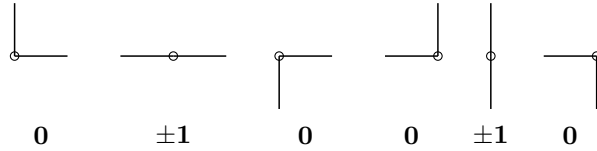
$$\sum_{ASM} \prod_{i,j} w(i,j) = \frac{\prod_i (x_i/y_i) \prod_{i,j} [x_i/y_j][qx_i/y_j]}{\prod_{i,j} [x_i/x_j][y_i/y_j]} \det \left[\frac{1}{[x_i/y_j][qx_i/y_j]} \right]$$

Using appropriate specializations of the variables x_i, y_j and the parameter q , G. Kuperberg [17] was able to deduce from this that the number of alternating sign matrices of size n is again, as in (2.2)

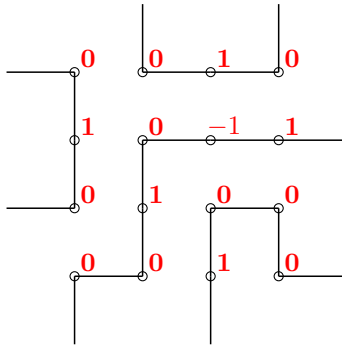
$$(3.4) \quad A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

This result had been obtained earlier by D. Zeilberger [29] in an indirect way, by showing that the alternating sign matrices of sign n are equinumerated with TSSCPPs of the same size and using (2.2). Recently a new proof of this result and related enumerations has been given by I. Fischer [13].

3.3.2. *Fully packed loops.* Another way to encode the six-vertex model is to replace each of the possible six configurations by one of the following



After fixing boundary conditions, there is a unique way to complete the diagram in a fully packed loop as in the picture below, which corresponds to the matrix (1.1).



Observe that in each configuration the $2n$ vertices on the boundary are related by noncrossing paths. This observation has led to the famous Razumov-Stroganov conjecture [26], relating alternating sign matrices and the $O(n)$ model, which has been solved recently [12].

3.3.3. *Alternating sign matrices, the Bruhat order and Gog triangles.* Recall that almost all invertible matrices can be factorized as $X = LU$ into a product of a lower and an upper triangular matrix (the LU-factorization) This can be refined into the Bruhat decomposition,

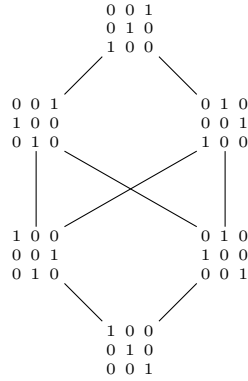
expressing the general linear group as a disjoint union of cells indexed by the symmetric group

$$GL_d = \cup_{w \in S_d} BwB$$

where B is the Borel subgroup of upper triangular matrices. For example, the matrices X having LU factorization are those such that $w_0 X \in Bw_0 B$ where w_0 is the permutation of $[1, d]$ such that $w_0(i) = d+1-i$. They form the cell of largest dimension. This decomposition induces an order relation (the Bruhat order) on the symmetric group by declaring for $\sigma, \tau \in S_n$:

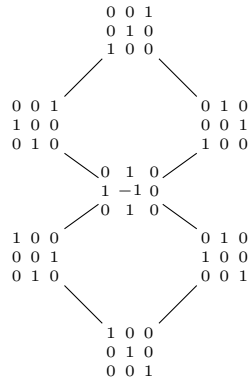
$$\sigma \leq \tau \quad \text{iff} \quad B\sigma B \subset \overline{B\tau B}$$

e.g. for S_3 we get the order relation with Hasse diagram



The Bruhat order on S_3

As we are going to explain, alternating sign matrices can be used to complete this order into a lattice order as in the following diagram



The lattice of 3×3 alternating sign matrices

For this we need to introduce a new species of Gelfand-Tsetlin triangles.

Definition 3.3. A Gog triangle of size n is a Gelfand-Tsetlin triangle such that

$$(i) \quad X_{i,j} < X_{i,j+1}, \quad j < i \leq n-1$$

in other words, such that its rows are strictly increasing, and such that

$$(ii) \quad X_{n,j} = j, \quad 1 \leq j \leq n.$$

There is a simple bijection between the sets of Gog triangles and of Alternating sign matrices of the same size, which goes as follows: If $(M_{ij})_{1 \leq i, j \leq n}$ is an ASM of size n , then the matrix $\tilde{M}_{ij} = \sum_{k=i}^n M_{kj}$ has exactly $i - 1$ entries 0 and $n - i + 1$ entries 1 on row i . Let $(X_{ij})_{j=1, \dots, i}$ be the columns (in increasing order) with a 1 entry of \tilde{M} on row $n - i + 1$. The triangle $X = (X_{ij})_{n \geq i \geq j \geq 1}$ is the Gog triangle corresponding to M .

For example, below are an alternating sign matrix of size 5 and its associated Gog triangle

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 3 & 4 & 5 \\ & & 1 & 4 & 5 \\ & & & 2 & 4 \\ & & & & 3 \end{matrix}$$

There is an order relation on Gog triangles obtained by entrywise comparison: for triangles X, Y of the same size, $X \leq Y$ if and only if each entry of X is smaller than the corresponding entry of Y . Clearly the Gog triangles of fixed size form a lattice for this order. It turns out that the restriction of this order relation to Gog triangles corresponding to permutations is exactly the reversed Bruhat order. The set of alternating sign matrices thus appears as the lattice completion of the set of permutations endowed with the Bruhat order, as first proved by Lascoux and Schützenberger [19].

4. THE GOG-MAGOG PROBLEM

4.1. The question. Since the sets of Gog and Magog triangles of size n have the same number of elements it is sensible to ask, in view of their very similar definitions, if there exists a natural bijection between these two sets. This problem is at the time of this writing still largely open. Observe that, although in our discussion we have encountered rather sophisticated mathematical objects, the actual definitions of the Gog and Magog triangles are completely elementary, one needs only to know what are the positive integers and how to compare two positive integers, it is not even necessary to know how to add or multiply them! Also many results have been obtained on the refined enumeration of Gog and Magog triangles according to different statistics and it has been observed that some of these refined enumerations coincide cf [6]. All these facts point towards the existence of a mathematical structure which would explain all these coincidences by showing that Gog and Magog triangles give two different ways of parametrizing the same mathematical objects, however for the moment the nature of this mathematical structure remains elusive.

Here are the seven Gog and Magog triangles of size 3. Already finding a “natural” bijection between them does not seem so obvious.

$$\begin{matrix} 1 & 1 & 1 & & 1 & 1 & 2 & & 1 & 1 & 3 & & 1 & 1 & 2 \\ & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 2 \\ & & 1 & & & & 1 & & & & 1 & & & & 1 \end{matrix}$$

$$\begin{matrix} & & 1 & 1 & 3 & & 1 & 2 & 2 & & 1 & 2 & 3 \\ & & & 1 & 2 & & & 1 & 2 & & & 1 & 2 \\ & & & & 1 & & & & 1 & & & & 1 \end{matrix}$$

Magog triangles of size 3

$$\begin{array}{cccc}
1 & 2 & 3 & \\
1 & 2 & & 3 \\
1 & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 3 \\
1 & 2 & 3 & \\
1 & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & \\
1 & 2 & 2 & 3 \\
2 & & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & \\
1 & 2 & 3 & \\
2 & & & \\
\end{array}$$

$$\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & \\
3 & & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & \\
2 & & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & \\
3 & & \\
\end{array}$$

Gog triangles of size 3

4.2. Gog and Magog trapezoids.

Definition 4.1. An (n, k) right (resp. left) Gog trapezoid (for $k \leq n$) is an array of positive integers formed from the k rightmost SW-NE diagonals (resp. leftmost NW-SE diagonals) of some Gog triangle of size n .

Below are two $(5, 2)$ Gog trapezoids.

$$\begin{array}{cc}
1 & 2 \\
1 & 3 \\
2 & 3 \\
2 & 4 \\
4 & \\
\end{array}
\quad
\begin{array}{cc}
4 & 5 \\
4 & 5 \\
3 & 4 \\
1 & 3 \\
2 & \\
\end{array}$$

left trapezoid right trapezoid

Analogously there is a notion of right and left Magog trapezoids. We will use only the right ones, of which below is a $(5, 2)$ example

$$\begin{array}{cc}
2 & 3 \\
1 & 3 \\
1 & 2 \\
1 & 2 \\
1 & \\
\end{array}$$

There is no known simple formula for enumerating Gog or Magog trapezoids of a fixed shape, however the following holds

Theorem 4.2 (Zeilberger [29]). *For all $k \leq n$, the (n, k) right Gog and Magog trapezoids are equienumerated.*

The proof of Zeilberger uses transformations of generating series for these objects and it does not seem possible to transform it into a bijective proof. Some conjectures on the enumeration of Gog and Magog trapezoids refined by some further statistics have been formulated by Krattenthaler [18]. A bijection between permutation matrices and a subset of Magog triangles has been proposed by J. Striker [28]. In the case of $(n, 2)$ right trapezoids a bijective proof incorporating a further statistic has been obtained in [9]. This proof is based on the Schützenberger involution, to be defined below, and uses the inversions of a Gog triangle. Bettinelli [8] found another, simpler bijection which however does not seem to preserve any of the statistics considered by Krattenthaler.

4.3. An approach to the bijection problem. In this section we will describe an approach to the bijection problem which has lead to some recent progress. For this approach we need to introduce some statistics on Gog and Magog triangles.

For a Gog triangle X we define

$$\beta_{Gog}(X) = X_{1,1}$$

For a Magog triangle of size n we let

$$\beta_{Magog}(X) = \sum_{i=1}^n X_{n,i} - \sum_{i=1}^{n-1} X_{n-1,i}$$

Remark that if we identify a Gog triangle with an alternating sign matrix, then the statistics $\beta_{Gog}(X)$ corresponds to the position of the 1 in the bottom line. Some recent results on the joint enumeration of this and other similar statistics can be found in [3], [6]. In particular, it is known that the number of Gog triangles of size n with $\beta_{Gog}(X) = k$ is equal to the number of Magog triangles of size n with $\beta_{Magog}(X) = k$.

Consider now the bottom triangle

$$\begin{array}{cc} a & b \\ & c \end{array}$$

made of the two lowest rows of some Gog triangle of size $n \geq 2$. Thus a, b, c are integers satisfying the inequalities

$$1 \leq a \leq c \leq b \leq n; \quad a < b.$$

Consider now a triangle

$$\begin{array}{cc} a' & b' \\ & c' \end{array}$$

extracted from the two rightmost NW-SE diagonals of Magog triangle of size n , such as this one:

$$\begin{array}{cccccc} 1 & & 1 & & 1 & & a' & & b' \\ & & 1 & & 1 & & 1 & & c' \\ & & & & 1 & & 1 & & 1 \\ & & & & & & 1 & & 1 \\ & & & & & & & & 1 \end{array}$$

These triangles are characterized by the inequalities

$$a' \leq c' \leq b' \leq n; \quad c' \leq n - 1.$$

It is now easy to find a bijection between these two sets of triangles, mapping the statistics β_{Gog} to β_{Magog} i.e. c to $a' + b' - c'$, as follows:

start from a triangle extracted from a Gog triangle $\begin{array}{cc} a & b \\ & c \end{array}$ with $1 \leq a \leq c \leq b \leq n; \quad a < b$ and then

- if $a < c$ map $\begin{array}{cc} a & b \\ & c \end{array}$ to $\begin{array}{cc} a & b \\ & a+b-c \end{array}$
- if $a = c$ map $\begin{array}{cc} a & b \\ a & \end{array}$ to $\begin{array}{cc} a & b-1 \\ & b-1 \end{array}$.

We leave to the reader the task of verifying that this is a bijection. Observe that it can be obtained in two steps: first we make the transformation

$$\begin{array}{ccc} a & b & \\ c & & \end{array} \rightarrow \begin{array}{ccc} a & b & \\ c & & \end{array} \quad \text{if } a < c$$

$$\begin{array}{ccc} a & b & \\ a & & \end{array} \rightarrow \begin{array}{ccc} a & b-1 & \\ a & & \end{array} \quad \text{if } a = c$$

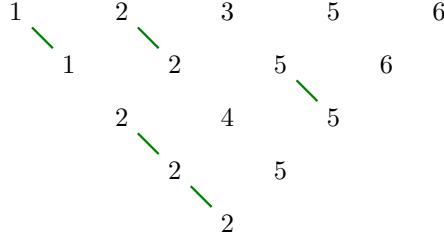
then a symmetry

$$\begin{array}{ccc} a & b & \\ c & & \end{array} \rightarrow \begin{array}{ccc} a & & b \\ a+b-c & & \end{array}$$

This idea was generalized in [9]. The first step leads to considering inversions of Gog triangles while the second leads to the Schützenberger involution. We shall explain these two ideas now.

4.3.1. *Inversions.* An inversion in a Gog triangle X is a pair (i, j) such that $X_{i,j} = X_{i+1,j}$.

For example the triangle below has 5 inversions.



4.3.2. *Schützenberger involution.* The Schützenberger involution is a fundamental tool in the theory of Young tableaux, which has a nice geometric interpretation [20]. Its simplest description uses the RSK transformation, which is a bijection between the set of two-dimensional arrays of nonnegative integers, $(M_{ij})_{i,j \geq 1}$ and pairs (S, T) of semi-standard Young tableaux of the same shape λ . This bijection allows to give a bijective proof of Cauchy's identity (2.1). Using the bijection between semi-standard tableaux and Gelfand-Tsetlin triangles the Schützenberger involution can be transported to Gelfand-Tsetlin triangles. The following description of this involution has been given by Berenstein and Kirillov [7].

First define involutions \mathfrak{s}_k , for $k \leq n-1$, acting on the set of Gelfand-Tsetlin triangles of size n . If $X = (x_{i,j})_{n \geq i \geq j \geq 1}$ is such a triangle the action of \mathfrak{s}_k on X is given by $\mathfrak{s}_k X = (\tilde{X}_{i,j})_{n \geq i \geq j \geq 1}$ with

$$\begin{aligned} \tilde{X}_{i,j} &= X_{i,j}, & \text{if } i \neq k \\ \tilde{X}_{k,j} &= \max(X_{k+1,j}, X_{k-1,j-1}) + \min(X_{k+1,j+1}, X_{k-1,j}) - X_{i,j} \end{aligned}$$

It is understood that $\max(a, b) = \max(b, a) = a$ and $\min(a, b) = \min(b, a) = a$ if the entry b of the triangle is not defined. The geometric meaning of the transformation of an entry is the following: on row k , any entry $X_{k,j}$ is surrounded by four (or less if it is on the boundary) numbers, increasing from left to right.

$$\begin{array}{ccccc} & X_{k+1,j} & & X_{k+1,j+1} & \\ & & X_{k,j} & & \\ X_{k-1,j-1} & & & & X_{k-1,j} \end{array}$$

These four numbers determine a smallest interval containing $X_{k,j}$, namely

$$[\max(X_{k+1,j}, X_{k-1,j-1}), \min(X_{k+1,j+1}, X_{k-1,j})]$$

and the transformation maps $X_{k,j}$ to its mirror image with respect to the center of this interval.

Define $\omega_j = \mathfrak{s}_j \mathfrak{s}_{j-1} \dots \mathfrak{s}_2 \mathfrak{s}_1$.

Theorem 4.3 (Berenstein and Kirillov [7]). *The Schützenberger involution, acting on Gelfand-Tsetlin triangles of size n , is given by the formula*

$$S = \omega_1 \omega_2 \dots \omega_{n-1}$$

Using inversions and the Schützenberger involution a bijection between $(n, 2)$ Gog and Magog trapezoids was given in [9].

4.3.3. *GOGAm triangles and trapezoids.*

Definition 4.4. *A GOGAm triangle of size n is a Gelfand-Tsetlin triangle which is the image by the Schützenberger involution of a Magog triangle (of size n).*

Remark 4.5. *The name GOGAm is obtained from Magog by reading backwards and changing the case as a reminder of the description of the Schützenberger involution on words (cf [15]).*

It is shown in [9] that the GOGAm triangles of size n are the Gelfand-Tsetlin triangles $X = (X_{i,j})_{n \geq i \geq j \geq 1}$ such that $X_{nn} \leq n$ and, for all $1 \leq k \leq n-1$, and all $n = j_0 > j_1 > j_2 \dots > j_{n-k} \geq 1$, one has

$$(4.1) \quad \left(\sum_{i=0}^{n-k-1} X_{j_i+i, j_i} - X_{j_{i+1}+i, j_{i+1}} \right) + X_{j_{n-k}+n-k, j_{n-k}} \leq k$$

The problem of finding an explicit bijection between Gog and Magog triangles can therefore be reduced to that of finding an explicit bijection between Gog and GOGAm triangles. Again one can define right or left GOGAm trapezoids.

Conjecture 4.6. *For all $k \leq n$, the (n, k) left Gog and GOGAm trapezoids are equienumerated.*

In [10] it was shown that the ideas of [9] could be used to provide a simple bijection between $(n, 2)$ Gog and GOGAm left trapezoids. These bijections suggested some further conjectures which we describe in the next section.

4.3.4. *Pentagons.*

Definition 4.7. *For (n, k, l, m) , with $n \geq k, l, m$, an (n, k, l, m) Gog (resp. GOGAm) pentagon is an array of positive integers $X = (x_{i,j})_{n \geq i \geq j \geq 1; k \geq j; j+l \geq i+1}$ formed from the intersection of the k leftmost NW-SE diagonals, the l rightmost SW-NE diagonals and the m bottom lines of a Gog (resp. GOGAm) triangle of size n .*

$$\begin{array}{ccccc} & & 3 & & 5 \\ & & & & & & 6 \\ & & 2 & & 5 & & \\ & & & & & & & 5 \\ & & 2 & & 4 & & 5 \\ & & & & 2 & & 5 \\ & & & & & & & 2 \\ & & & & 2 & & & & & 5 \end{array}$$

A (6, 4, 3, 5) Gog pentagon

Remark that if $m \geq k + l - 1$ then the pentagon is a rectangle, whereas if $m \leq k, l$ then it is a Gelfand-Tsetlin triangle of size m .

Conjecture 4.8. *For any n, k, l, m the (n, k, l, m) Gog and GOGAm pentagons are equinumerated.*

This conjecture can even be refined into

Conjecture 4.9. *For each n, k, l the (n, k) left Gog and GOGAm trapezoids with bottom entry $X_{1,1} = l$ are equinumerated.*

Some numerical evidence for these conjectures has been given in [10].

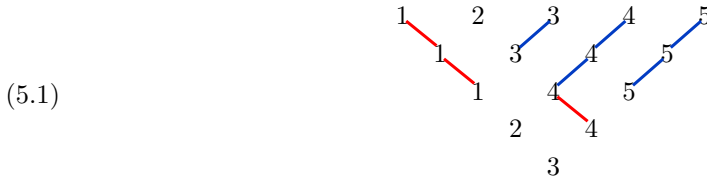
5. ON THE DISTRIBUTION OF INVERSIONS AND COINVERSIONS

5.1. We recall the definition of inversions and introduce the dual notion of coinversion.

Definition 5.1. *An inversion in a Gog triangle X is a pair (i, j) such that $X_{i,j} = X_{i+1,j}$.*

A coinversion is a pair (i, j) such that $X_{i,j} = X_{i+1,j+1}$.

For example, the Gog triangle in (5.1) contains three inversions, (2, 2), (3, 1), (4, 1) and five coinversions, (3, 2), (3, 3), (4, 2), (4, 3), (4, 4).



We denote by $\nu(X)$ (resp. $\mu(X)$) the number of inversions (resp. coinversions) of a Gog triangle X . Since a pair (i, j) cannot be an inversion and a coinversion at the same time in a Gog triangle and the top row does not contain any inversion or coinversion, one has

$$\nu(X) + \mu(X) \leq \frac{n(n-1)}{2}$$

Actually one can easily see that $\frac{n(n-1)}{2} - \nu(X) - \mu(X)$ is the number of -1 's in the alternating sign matrix associated to the Gog triangle X . Also inversions and coinversions correspond to different types of vertices in the six vertex model, see e.g. [5].

Let us denote by $Z(n, x, y)$ the generating function of Gog triangles of size n according to ν and μ .

$$(5.2) \quad Z(n, x, y) = \sum_{X \in \text{Gog}_n} x^{\nu(X)} y^{\mu(X)}.$$

where the sum is over the set Gog_n of Gog triangles of size n .

The following formula has been proved in [5], using properties of the six vertex model.

Proposition 5.2.

$$(5.3) \quad Z(n, x, y) = \det_{0 \leq i, j \leq n-1} \left(-y^i \delta_{i, j+1} + \sum_{k=0}^{\min(i, j+1)} \binom{i-1}{i-k} \binom{j+1}{k} x^k \right).$$

For example, for Gog triangles of size 3, we have

$$(5.4) \quad Z(3, x, y) = \det \begin{pmatrix} 1 & 1 & 1 \\ -y+x & 2x & 3x \\ x & -y^2+2x+x^2 & 3x+3x^2 \end{pmatrix} = x^3 + xy + y^3 + 2x^2y + 2xy^2.$$

which matches part (a) of Table 1.

It is however not so easy to use this formula in order to prove results on the distribution of inversion and coinversions.

5.2. Table 1 below shows the joint distribution of μ and ν , for $n = 3$ and $n = 4$.

					0	1	2	3	4	5	6
				0							1
				1				1	2	3	
				2				6	5		
				3		1	6	6			
				4		2	5				
				5		3					
				6	1						
0	0	1	2	3							
1				1							
2			1	2							
3	1										

TABLE 1. The number of Gog triangles of size 3 (a) and 4 (b) with k inversions (horizontal values) and l coinversions (vertical values).

We remark that the numbers on the antidiagonal are the Mahonian numbers counting permutations according to the number of their inversions.

Let us denote by $A_{n,k}$ the set of pairs of nonnegative integers (i, j) such that

$$i \geq \frac{k(k+1)}{2}, \quad j \geq \frac{(n-k-1)(n-k)}{2}, \quad i+j \leq \frac{n(n-1)}{2}$$

and let

$$A_n = \cup_{k=0}^{n-1} A_{n,k}.$$

We will give a simple combinatorial proof of the following.

Theorem 5.3. *There exists a Gog triangle of size n , with i inversions and j coinversions, if and only if (i, j) belongs to the set A_n . If $i = \frac{k(k+1)}{2}$ and $j = \frac{(n-k-1)(n-k)}{2}$ for some $k \in [0, n-1]$ then this triangle is unique, furthermore its bottom value is $n - k$.*

Remark 5.4. *Note, for future reference, that if (l, m) belongs to the set A_n and if $l < \frac{p(p+1)}{2}$ then $m \geq \frac{(n-p)(n-p+1)}{2}$.*

5.3. Proof of Theorem 5.3.

5.3.1. Existence. First we show that there exists a triangle of size n with $\frac{k(k+1)}{2}$ inversions and $\frac{(n-k-1)(n-k)}{2}$ coinversions. Indeed the triangle is defined by

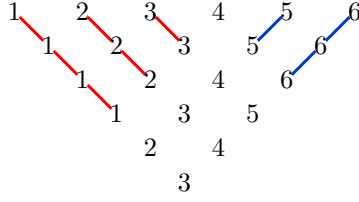
$$(5.5) \quad X_{ij} = j \quad \text{for } j \leq i - n + k$$

$$(5.6) \quad X_{ij} = n + j - i \quad \text{for } j \geq k + 1$$

$$(5.7) \quad X_{ij} = n - k + 2j - i - 1 \quad \text{for } i - n + k + 1 \leq j \leq k$$

The bottom entry of this triangle is $n - k$, as expected.

We give an example below: for $n = 6$ and $k = 3$, the triangle has 6 inversions and 3 coinversions:

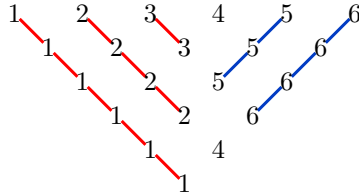


Observe that the entries which are neither inversions nor coinversions form a rectangle of size $k \times (n - k - 1)$ at the bottom of the triangle.

The ASM corresponding to such a triangle has a diamond shape:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Starting from this triangle, it is not difficult, for a pair of integers (l, m) such that $l \geq \frac{k(k+1)}{2}$, $m \geq \frac{(n-k-1)(n-k)}{2}$, and $l + m \leq \frac{n(n-1)}{2}$, to construct (at least) one triangle with l inversions and m coinversions, for example one can add inversions by decreasing some entries, starting from the westmost corner of the rectangle, and add coinversions by increasing entries, starting from the eastmost corner. Here is an example with $n = 6$, $l = 9$, $m = 5$, details of the general case are left to the reader.



5.3.2. Standardization of Gog triangles. In order to prove the only if part of the Theorem, as well as the uniqueness statement, we now introduce two *standardization* operations. These operations build a Gog triangle of size $n - 1$ from a Gog triangle of size n .

5.3.3. Left standardization. Let X be a Gog triangle of size n then its $(n - 1)^{th}$ row (counted from bottom to top) has the form $1, 2, \dots, k, k + 2, \dots, n$ for some $k \in [1, n]$. For $j \leq k$, let m_j be the smallest integer such that $X_{n,j} = X_{m_j,j} = j$.

The left standardization of X is the triangle LX of size $n - 1$ obtained as follows:

$$(5.8) \quad LX_{i,j} = X_{i,j} = j \quad \text{for } j \leq k \quad \text{and } n - 1 \geq i \geq m_j.$$

$$(5.9) \quad LX_{i,j} = X_{i,j} - 1 \quad \text{for other values of } i, j.$$

5.3.4. Right standardization. Let X be a Gog triangle of size n with $(n-1)^{th}$ row of the form $1, 2, \dots, k, k+2, \dots, n$, and for $j \geq k+1$ let $p_j \geq 1$ be the largest integer such that $X_{n,j} = X_{n-p_j, j+1-p_j} = j+1$.

The right standardization of X is the triangle RX of size $n-1$ obtained as follows:

$$(5.10) \quad RX_{n-l, j+1-l} = X_{n-l, j+1-l} - 1 = j \quad \text{for } j \geq k+1 \quad \text{and } 1 \leq l \leq p_j.$$

$$(5.11) \quad RX_{i,j} = X_{i,j} \quad \text{for other values of } i, j.$$

Below are a Gog triangle of size 6, together with its left and right standardizations:

$$\begin{array}{cccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
 & & & 1 & 2 & 3 & 5 & 6 \\
 & & & & 1 & 3 & 5 & 6 \\
 & & & & & 1 & 4 & 6 \\
 & & & & & & 3 & 5 \\
 & & & & & & & 4 \\
 & & & & & & & X \\
 \\
 & & 1 & 2 & 3 & 4 & 5 & & & & 1 & 2 & 3 & 4 & 5 \\
 & & & 1 & 2 & 4 & 5 & & & & & 1 & 3 & 4 & 5 \\
 & & & & 1 & 3 & 5 & & & & & & 1 & 4 & 5 \\
 & & & & & 2 & 4 & & & & & & & 3 & 5 \\
 & & & & & & 3 & & & & & & & & 4 \\
 & & & & & & & LX & & & & & & & RX
 \end{array}$$

5.3.5.

Proposition 5.5. ¹ Let X be a Gog triangle of size n , with l inversions and m coinversions with $(n-1)^{th}$ row $1, 2, 3, \dots, k, k+2, \dots, n$.

Then LX is a Gog triangle of size $n-1$ with at most $m-n+k+1$ coinversions and at most l inversions.

Similarly, RX is a Gog triangle of size $n-1$ with at most $l-k$ inversions and at most m coinversions.

Proof. We prove the proposition only for the left standardization. The case of right standardization can be proven in an analogous way, or by noticing that the mapping $X \mapsto (n+1 - X_{i, i+1-j})_{i,j}$ is an involution, which exchanges inversions and coinversions.

Observe that there are exactly k inversions and $n-k-1$ coinversions on row $n-1$ of X . It follows that X has $l-k$ inversions and $m-n+k+1$ coinversions on rows $1, \dots, n-2$.

We first prove that LX is a Gog triangle. For any i, j we have to prove that

$$LX_{i,j} \geq LX_{i-1, j-1}, \quad LX_{i,j} \geq LX_{i+1, j}, \quad LX_{i,j} > LX_{i, j-1}.$$

Since X is a Gog triangle one has $X_{i,j} \geq X_{i-1, j-1}$, therefore the first inequality may fail only if $LX_{i,j} = X_{i,j} - 1$ and $LX_{i-1, j-1} = X_{i-1, j-1}$. If this is the case then $X_{i,j} > j$ and

¹In the first version of this paper the statement of this proposition was incorrect. I would like to thank the referees for pointing out the mistake.

$X_{i-1,j-1} = j - 1$, therefore $LX_{i,j} > LX_{i-1,j-1}$. This shows also that LX cannot have more coinversions than X on its first $n - 2$ rows, therefore the number of coinversions of LX is at most $m - n + k + 1$. A similar reasoning yields the other two inequalities, moreover the number of inversions of LX can increase at most by k with respect to that of X in its first $n - 2$ rows, more precisely by at most one in each of the k leftmost NW-SE diagonals. It follows that LX has at most l inversions. \square

5.3.6. We can now finish the proof of Theorem 5.3 by induction on n . For $n = 3$ or 4 , the claim follows by inspection of Table 1. Let X be a Gog triangle of size n , with l inversions and m coinversions. We have to prove that (l, m) belongs to some $A_{n,r}$. We have seen that LX is a Gog triangle of size $n - 1$ with at most $m - n + k + 1$ coinversions and at most l inversions, whereas RX is a Gog triangle of size $n - 1$ with at most $l - k$ inversions at most m coinversions. By induction hypothesis there exists some p such that

$$(5.12) \quad l \geq \frac{p(p+1)}{2}, \quad m - n + k + 1 \geq \frac{(n-p-2)(n-p-1)}{2}$$

and there exists q such that

$$(5.13) \quad l - k \geq \frac{q(q+1)}{2}, \quad m \geq \frac{(n-q-2)(n-q-1)}{2}.$$

If $p > q$, then (5.12) implies $l \geq \frac{(q+1)(q+2)}{2}$ and since $m \geq \frac{(n-q-2)(n-q-1)}{2}$ by (5.13) one has $(l, m) \in A_{n,q+1}$.

Similarly if $q > p$ then (5.13) implies $l \geq \frac{(p+1)(p+2)}{2}$ and (5.12) implies $m \geq \frac{(n-p-2)(n-p-1)}{2}$ so that $(l, m) \in A_{n,p+1}$.

If now $p = q$ then either $k > p$ and then $l - k \geq \frac{q(q+1)}{2}$ implies $l \geq \frac{(p+1)(p+2)}{2}$ and $(l, m) \in A_{n,p+1}$, or $k \leq p$ then $m - n + k + 1 \geq \frac{(n-p-2)(n-p-1)}{2}$ implies $m \geq \frac{(n-p-1)(n-p)}{2}$ and $(l, m) \in A_{n,p}$.

Suppose now that $l = \frac{p(p+1)}{2}$, $m = \frac{(n-p-1)(n-p)}{2}$. We wish to prove that there exists a unique Gog triangle with these numbers of inversions and coinversions. Let X be such a triangle and consider RX , which has at most $l - k$ inversions. If $k > p$ then $l - k < \frac{(p-1)p}{2}$, therefore, by Remark 5.4, RX has at least $\frac{(n-p)(n-p+1)}{2}$ coinversions, which contradicts the fact that RX has at most $m = \frac{(n-p-1)(n-p)}{2}$ coinversions; it follows that $k \leq p$. A similar reasoning with LX shows that in fact $k = p$, and RX has at most $\frac{(p-1)p}{2}$ inversions, and at most $m = \frac{(n-p-1)(n-p)}{2}$ coinversions. By induction hypothesis RX is the unique Gog triangle with $l = \frac{(p-1)p}{2}$ inversions, and $\frac{(n-p-1)(n-p)}{2}$ coinversions. The triangle X is completely determined by RX and k and we have $k = p$ therefore X is unique. Comparing with the formula (5.5)-(5.7) for this RX , we check that X is the unique triangle of size n with $l = \frac{p(p+1)}{2}$, $m = \frac{(n-p-1)(n-p)}{2}$. \square

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