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Orthogonal polynomials on the unit circle, q -Gamma weights, and discrete Painlevé equations.

Philippe Biane

ABSTRACT. We consider orthogonal polynomials on the unit circle with respect to a weight which is a quotient of q -gamma functions. We show that the Verblunsky coefficients of these polynomials satisfy discrete Painlevé equations, in a Lax form, which correspond to an $A_3^{(1)}$ surface in Sakai's classification.

1. Introduction

Motivated by number theoretic considerations, J.F. Burnol [4], [5], has raised the problem of realizing the Fourier transform as a scattering. More precisely, consider the Fourier transform restricted to even functions on \mathbf{R} which is given, after appropriate normalization, by

$$\mathcal{F}f(x) = 2 \int_0^\infty \cos(2\pi xy) f(y) dy.$$

One can rewrite this as an isometry of the space $L^2(]0, +\infty[, dx/x)$, by putting $f(x) = g(x)/\sqrt{x}$, to obtain

$$\tilde{\mathcal{F}}g(x) = 2 \int_0^\infty \sqrt{xy} \cos(2\pi xy) g(y) dy/y$$

The interaction between the additive Fourier transform and the Mellin transform is captured by the formula, where the Fourier transform acts on tempered distributions,

$$(1.1) \quad \tilde{\mathcal{F}}x^{+ik} = \pi^{-ik} \frac{\Gamma(1/4 - ik/2)}{\Gamma(1/4 + ik/2)} x^{-ik} \quad k \text{ real}$$

which yields the gamma factor coming from the place at infinity in the functional equation for Riemann zeta function. Gamma factors for other L -functions are obtained by similar considerations involving the radial Fourier transform in \mathbf{R}^n .

Interpreting x^{ik} as an incoming wave, and x^{-ik} as the corresponding outgoing wave (in logarithmic coordinates) on the space $]0, +\infty[$, it is a natural question to ask what is the scattering potential which realizes the map (1.1) (we shall recall

the basics of scattering theory in the next section). This was answered by Burnol [6] who showed that, more generally, the function

$$(1.2) \quad \frac{\Gamma(a - ik/2)\Gamma(b + ik/2)}{\Gamma(a + ik/2)\Gamma(b - ik/2)}$$

(a, b real, $a > b$), is the scattering phase shift for a Dirac equation on $]0, +\infty[$, where the potential is given in terms of Painlevé transcendents. In this paper we consider a discrete analogue of this problem. Discrete scattering problems have been introduced by Case and Kac [7] and studied further by Case and Geronimo [8]. In this last paper it is shown that, given a function f , analytic inside the unit disk, continuous on the unit circle, then $f(e^{i\theta})/\overline{f(e^{i\theta})}$ is the phase shift for a discrete scattering problem, and solving the inverse problem for this scattering problem is equivalent to finding the recursion relations for the orthogonal polynomials on the unit circle with respect to the measure $d\theta/|f(e^{i\theta})|^2$. Recall that, if ϕ_n is the sequence of these polynomials, and

$$\phi_n^*(z) = z^n \bar{\phi}_n(1/z)$$

then the recursion relation, due to Szegő, can be written, with

$$\Phi_n(z) = \begin{pmatrix} \phi_n(z) \\ \phi_n^*(z) \end{pmatrix}$$

as

$$(1.3) \quad \Phi_{n+1}(z) = B_n(z)\Phi_n(z)$$

where

$$(1.4) \quad B_n(z) = \begin{pmatrix} z & \alpha_{n+1} \\ z\bar{\alpha}_{n+1} & 1 \end{pmatrix}.$$

The α_n are the Verblunsky coefficients of the sequence ϕ_n , see e.g. [21], [13], or [19], [20]. Equation (1.3) is a discrete analogue of a Dirac system leading to scattering. Thus a natural problem, in view of Burnol's question, is to find the recursion coefficients for a weight w , expressed in terms of q -Gamma functions, of the form

$$w(z) = \frac{(az, q)_\infty (\bar{a}z^{-1}, q)_\infty}{(bz, q)_\infty (\bar{b}z^{-1}, q)_\infty}.$$

with, as usual,

$$(z, q)_\infty = \prod_{i=0}^{\infty} (1 - zq^i).$$

Indeed $w(e^{i\theta}) = 1/|f(e^{i\theta})|^2$ with $f(e^{i\theta}) = (be^{i\theta}, q)_\infty / (ae^{i\theta}, q)_\infty$, and taking a suitable limit of $f(e^{i\theta})/\overline{f(e^{i\theta})}$ for $q \rightarrow 1$ yields the phase shift (1.2).

The weight w satisfies a q -difference equation

$$w(qz) = \frac{V(z)}{W(z)} w(z)$$

where V and W are two polynomials of degree 2. It turns out that a method, which goes back to Laguerre, further used by Freud [10], allows, in this situation, to obtain nonlinear relations between these recursion coefficients. This kind of question has seen renewed interest recently, see e.g. [17], [11], [22]. We shall prove that

$$\Phi_n(qz) = \frac{1}{V(z)} A_n(z) \Phi_n(z)$$

where A_n is a matrix with polynomial entries, of degree 2, which can be computed from the coefficients of the orthogonal polynomials. Compatibility between this equation and (1.3) implies the relation

$$A_{n+1}(z)B_n(z) = B_n(qz)A_n(z).$$

This last equation can be interpreted as an isomonodromy deformation of the q difference operator associated with the matrix A_n . It is similar, although it is not equivalent to, the Lax form of the discrete Painlevé VI equation studied by Jimbo and Sakai in [15]. We shall study it in details and prove that the equation that we obtain in this way is a Lax form for a discrete Painlevé equation, corresponding to the $A_3^{(1)}$ surface, according to the classification of Sakai [18]. As we shall see, this equation looks like the one of [15], but is obtained from translation in a different direction in the affine Coxeter group $D_5^{(1)}$.

This paper is organized as follows. We start by recalling some basic material on one-dimensional scattering theory, following [9], for the continuous case, and [8] for the discrete case. We also recall some of the theory of orthogonal polynomials on the unit circle, following closely [11]. Laguerre's method is used in section 3 to derive nonlinear equations satisfied by the recursion coefficients of these orthogonal polynomials. Then, in section 4, we give an explicit formula for these nonlinear relations as a birational map in two variables, and show that they give rise to a regular isomorphism between two surfaces obtained by blowing up $\mathbf{P}^1 \times \mathbf{P}^1$ at eight points. In section 5 we recall Sakai's theory, and use it to identify our nonlinear relation as a discrete Painlevé equation. We determine its type in Sakai's classification, and find its expression as a certain translation in the affine Weyl group of type $D_5^{(1)}$. Finally in the last section we derive a differential system by taking a continuous limit of the discrete equations.

The results of this paper are based on numerous lengthy computations. In order not to bore the reader we have omitted the details of most of these computations, but given enough details so that they can be reproduced.

I would like to thank Jean-François Burnol for keeping me informed about his work which motivated me to undertake the study of these equations. I thank also Alexei Borodin and Philip Boalch for their useful comments at an early stage of this work.

2. One-dimensional scattering, continuous and discrete

2.1. Scattering for the Schrödinger equation. One starts with the equation

$$(2.1) \quad -\frac{d^2}{dx^2}\psi(x, s) + q(x)\psi(x, s) = s^2\psi(x, s) \quad s \in \mathbf{C}$$

on the half line $[0, +\infty[$. This equation has a unique solution, $\psi(x, s)$, satisfying the boundary condition

$$(2.2) \quad \psi(0, s) = 0, \quad \frac{d}{dx}\psi(0, s) = 1.$$

For a potential q decreasing sufficiently at infinity there are also two solutions, the Jost solutions $f(x, \pm s)$, determined by the conditions

$$(2.3) \quad f(x, \pm s)e^{\mp isx} \rightarrow_{x \rightarrow \infty} 1.$$

For real k , the Jost solutions $f(x, k)$ and $f(x, -k)$ correspond to incoming and outgoing waves and ψ is a linear combination

$$(2.4) \quad \psi(x, k) = \frac{1}{2ik} (f(x, k)\bar{M}(k) - f(x, -k)M(k))$$

such that M is the boundary value of the function $M(s) = f(0, s)$, analytic in the upper half-plane. It follows that, as $x \rightarrow \infty$,

$$\psi(x, k) = \frac{|M(k)|}{k} \sin(kx - \eta(k)) + o(1)$$

where $\eta(k) = \arg M(k)$ is the phase shift created by the potential q , with respect to the case $q = 0$. The problem of inverse scattering is, given the phase shift η , to reconstruct the potential q . For phase shifts satisfying a growth condition at infinity, there exists a unique solution without bound states, which can be recovered by the methods developed in the 50's, see e.g. [12] and [9].

2.2. Discrete scattering. We now follow the exposition of Case and Geronimo [8]. More information on orthogonal polynomials on the unit circle can be found e.g. in Szegő [21], Geronimus [13], or the comprehensive treatise by Simon [19], [20].

Let μ be a finite positive measure on the unit circle $U(1)$. Denote its moments

$$\int_0^{2\pi} e^{-ik\theta} d\mu(\theta) = c_k$$

and $\phi_n(z)$ the sequence of orthogonal polynomials obtained by the Gram-Schmidt procedure from $1, z, z^2, \dots$, normalized by

$$\phi_n(z) = z^n + \dots$$

This normalization will be more convenient for our purposes than the usual one, where the polynomials have L^2 -norm 1. Denote

$$\sigma_n = \int |\phi_n|^2 d\mu \quad \text{and} \quad \alpha_n = \phi_n(0).$$

The α_n are the Verblunsky coefficients of the system, cf [19]. They satisfy

$$\sigma_{n+1} = (1 - |\alpha_{n+1}|^2)\sigma_n$$

thus the sequence σ_n is determined by $\sigma_0 = \mu(U(1))$ and the sequence α_n . The orthogonal polynomials are subject to the recursion relation

$$(2.5) \quad \begin{cases} \phi_{n+1}(z) &= z\phi_n(z) + \alpha_{n+1}z^n\bar{\phi}_n(1/z) \\ \bar{\phi}_{n+1}(1/z) &= z^{-n}\phi_n(z) + \bar{\alpha}_{n+1}z^{-1}\bar{\phi}_n(1/z) \end{cases}$$

In matrix form, putting

$$\Psi(z, n) = \begin{pmatrix} \phi_n(z) \\ \bar{\phi}_n(1/z) \end{pmatrix}$$

and

$$D(n) = \begin{pmatrix} z & \alpha_{n+1}z^n \\ \bar{\alpha}_{n+1}z^{-n} & z^{-1} \end{pmatrix}$$

one can write

$$\Psi(z, n+1) = D(n)\Psi(z, n)$$

Assuming that the potential α satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ fast enough, one can construct the Jost solutions to equation (2.5)

$$\Psi_+(z, n) = \begin{pmatrix} \phi_+(z, n) \\ \hat{\phi}_+(z, n) \end{pmatrix} \quad \Psi_-(z, n) = \begin{pmatrix} \hat{\phi}_-(z, n) \\ \phi_-(z, n) \end{pmatrix}$$

satisfying the boundary conditions

$$\begin{aligned} \lim_n |\phi_+(z, n) - z^n| &= 0 & |z| > 1 \\ \lim_n |\phi_-(z, n) - z^{-n}| &= 0 & |z| < 1 \\ \lim_n |\hat{\phi}_+(z, n)| &= 0 & |z| > 1 \\ \lim_n |\hat{\phi}_-(z, n)| &= 0 & |z| < 1. \end{aligned}$$

The solutions Ψ_{\pm} are linearly independent, and

$$(2.6) \quad \Psi(z, n) = f_+(z)\Psi_+(z, n) + f_-(z)\Psi_-(z, n)$$

where

$$f_+(z) = \bar{f}_-(z) \quad \text{for } |z| = 1$$

Equation (2.6) is the discrete analogue of (2.4).

The function f_+ has analytic continuation inside the unit disc, it is related to the original measure μ on the unit circle by

$$d\mu(\theta) = \frac{d\theta}{|f_+(e^{i\theta})|^2}$$

The phase $\eta(\theta) = \arg(f_+(e^{i\theta}))$ is the phase shift induced by the potential α_n . It can be recovered from the function $\frac{1}{|f_+(e^{i\theta})|^2}$ on the unit circle.

One sees that reconstructing the phase shift from the potential α is equivalent to finding the orthogonality relation for the sequence of polynomials ϕ_n , solution to the recursion equation (2.5). Conversely, the inverse scattering problem is the easy problem of finding the recursion coefficients for the orthogonal polynomials, knowing their orthogonality measure. In fact the connection between scattering theory and orthogonal polynomials has been known for a long time and is at the heart of the Gelfand-Levitan method (see [12]).

2.3. Comparing the discrete and continuous cases. Given a solution A of the Riccati equation

$$q = A' + A^2$$

the equation (2.1), for $s = k$ real, is equivalent to the Dirac system

$$(2.7) \quad \begin{cases} dy/dx &= Ay + kz, \\ -dz/dx &= Az + ky. \end{cases}$$

The Dirac system (2.7) can then be put in the form

$$(2.8) \quad \begin{cases} dW/dx &= A\bar{W}e^{2ikx}, \\ d\bar{W}/dx &= AWe^{-2ikx}. \end{cases}$$

with

$$W = (y + iz)e^{ikx}$$

Similarly, putting

$$\omega_n(\theta) = e^{-in\theta}\phi_n(e^{i\theta})$$

the system (2.5) becomes

$$(2.9) \quad \begin{cases} \omega_{n+1} - \omega_n &= \alpha_{n+1} e^{-in\theta} \bar{\omega}_n, \\ \bar{\omega}_{n+1} - \bar{\omega}_n &= \bar{\alpha}_{n+1} e^{in\theta} \omega_n. \end{cases}$$

It is now clear that (2.9) is a discretized version of (2.8).

2.4. Recursion relations and the Caratheodory function. We recall a few properties of the orthogonal polynomials which will be needed in the sequel.

The function

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)$$

is called the Caratheodory function of μ , it is holomorphic in $\mathbf{C}^* \setminus U(1)$, and has the expansion

$$\begin{aligned} F(z) &= c_0 + 2 \sum_{k=1}^{\infty} c_k z^k \quad |z| < 1 \\ F(z) &= -c_0 - 2 \sum_{k=1}^{\infty} c_{-k} z^{-k} \quad |z| > 1 \end{aligned}$$

Let

$$\phi_n^*(z) = z^n \bar{\phi}_n(1/z)$$

and introduce the associated polynomials

$$\begin{aligned} \psi_n(z) &= \int \frac{e^{i\theta} + z}{e^{i\theta} - z} (\phi_n(e^{i\theta}) - \phi_n(z)) d\mu(\theta) \\ \psi_0(z) &= 1 \\ \psi_n^*(z) &= - \int \frac{e^{i\theta} + z}{e^{i\theta} - z} (z^n \overline{\phi_n(e^{i\theta})} - \phi_n^*(z)) d\mu(\theta) \\ \psi_0^*(z) &= 1 \end{aligned}$$

and the functions

$$\begin{aligned} \epsilon_n(z) &= \psi_n(z) + F(z) \phi_n(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \phi_n(e^{i\theta}) d\mu(\theta) \\ \epsilon_n^*(z) &= \psi_n^*(z) - F(z) \phi_n^*(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \overline{\phi_n(e^{i\theta})} z^n d\mu(\theta) \end{aligned}$$

then, for $n \geq 1$,

$$(2.10) \quad \epsilon_n(z) = 2\sigma_n z^n + O(z^{n+1}) \quad z \rightarrow 0$$

$$(2.11) \quad \epsilon_n(z) = 2\sigma_n \alpha_{n+1} z^{-1} + O(z^{-2}) \quad z \rightarrow \infty$$

$$(2.12) \quad \epsilon_n^*(z) = 2\sigma_n \bar{\alpha}_{n+1} z^{n+1} + O(z^{n+2}) \quad z \rightarrow 0$$

$$(2.13) \quad \epsilon_n^*(z) = 2\sigma_n + O(z^{-1}) \quad z \rightarrow \infty$$

In other words, the fractions $-\frac{\psi_n}{\phi_n}$ and $\frac{\psi_n^*}{\phi_n^*}$ are reduced continued fractions for F , near ∞ and 0 respectively. Furthermore,

$$\epsilon_{n+1}(z) = z\epsilon_n(z) - \alpha_{n+1}\epsilon_n^*(z)$$

$$\epsilon_{n+1}^*(z) = \epsilon_n^*(z) - \bar{\alpha}_{n+1}z\epsilon_n(z).$$

The recursion relations can be put in the form

$$(2.14) \quad Y_n(z) = \begin{pmatrix} \phi_n(z) & \frac{\epsilon_n(z)}{w(z)} \\ \phi_n^*(z) & -\frac{\epsilon_n^*(z)}{w(z)} \end{pmatrix}$$

$$(2.15) \quad Y_{n+1}(z) = B_n(z)Y_n(z).$$

with

$$(2.16) \quad B_n(z) = \begin{pmatrix} z & \alpha_{n+1} \\ z\bar{\alpha}_{n+1} & 1 \end{pmatrix}$$

There are also discrete Wronskian identities, derived from the recursion relations, namely

$$(2.17) \quad \begin{aligned} \phi_{n+1}(z)\psi_n(z) - \psi_{n+1}(z)\phi_n(z) &= \phi_{n+1}(z)\epsilon_n(z) - \epsilon_{n+1}(z)\phi_n(z) \\ &= 2\alpha_{n+1}\sigma_n z^n \end{aligned}$$

$$(2.18) \quad \begin{aligned} \phi_{n+1}^*(z)\psi_n^*(z) - \psi_{n+1}^*(z)\phi_n^*(z) &= \phi_{n+1}^*(z)\epsilon_n^*(z) - \epsilon_{n+1}^*(z)\phi_n^*(z) \\ &= 2\bar{\alpha}_{n+1}\sigma_n z^{n+1} \end{aligned}$$

$$(2.19) \quad \begin{aligned} \phi_n(z)\psi_n^*(z) + \psi_n(z)\phi_n^*(z) &= \phi_n(z)\epsilon_n^*(z) + \epsilon_n(z)\phi_n^*(z) \\ &= 2\sigma_n z^n. \end{aligned}$$

3. Nonlinear relations for the Verblunsky coefficients

3.1. Laguerre's method. We now adapt the method of Laguerre [16] and Freud [10] for finding nonlinear relations between the Verblunsky coefficients. The computations that follow are inspired by Forrester and Witte [11], who treat the case of difference equations of the form $w(z+1) = \rho(z)w(z)$. Another derivation for the q -recurrence relations can be found in the paper [14] by Ismail and Witte, however we shall need them in the form below, which is slightly different from the results of [14], so we give a detailed derivation here. Let us assume that the measure μ has the form

$$d\mu(\theta) = w(e^{i\theta})d\theta$$

with w holomorphic in \mathbf{C}^* , satisfying

$$(3.1) \quad w(qz) = \rho(z)w(z)$$

where $q \in]0, 1[$, and ρ is a rational function,

$$(3.2) \quad \rho(z) = \frac{V(z)}{W(z)}$$

for some polynomials V, W . The case of interest is when

$$(3.3) \quad w(e^{i\theta}) = \left| \frac{(ae^{i\theta}, q)_\infty}{(be^{i\theta}, q)_\infty} \right|^2 = \frac{(ae^{i\theta}, q)_\infty (\bar{a}e^{-i\theta}, q)_\infty}{(be^{i\theta}, q)_\infty (\bar{b}e^{-i\theta}, q)_\infty}$$

$q \in]0, 1[$, $a, b \in \mathbf{C}^*$, thus

$$w(z) = \frac{(az, q)_\infty (\bar{a}z^{-1}, q)_\infty}{(bz, q)_\infty (\bar{b}z^{-1}, q)_\infty}$$

satisfies

$$w(qz) = \frac{(qz - \bar{a})(1 - bz)}{(qz - \bar{b})(az - 1)} w(z),$$

and we may take

$$(3.4) \quad V(z) = (qz - \bar{a})(bz - 1)$$

$$(3.5) \quad W(z) = (qz - \bar{b})(1 - az).$$

For the moment we shall make the computations with general polynomials V and W , and come back to the specific values (3.4), (3.5) in the last section.

Equation (3.1) implies that the Caratheodory function satisfies

$$(3.6) \quad W(z)F(qz) = V(z)F(z) + U(z)$$

with U a polynomial of degree at most $D = \max(\deg W, \deg V)$. Indeed, take the partial fraction expansion

$$\rho(z) = \delta(z) + \sum_j \frac{\omega_j}{z - a_j}, \quad \delta \text{ polynomial of degree less than } (\deg V - \deg W)_+$$

with $W(z) = C \prod_j (z - a_j)$, then

$$\begin{aligned} F(qz) &= \int \frac{\zeta + qz}{\zeta - qz} w(\zeta) \frac{d\zeta}{i\zeta} \\ &= \int \frac{\zeta + z}{\zeta - z} w(q\zeta) \frac{d\zeta}{i\zeta} \quad \text{by a change of integration contour} \\ &= \int \frac{\zeta + z}{\zeta - z} \rho(\zeta) w(\zeta) \frac{d\zeta}{i\zeta} \\ &= \rho(z)F(z) + \int \frac{\zeta + z}{\zeta - z} (\rho(\zeta) - \rho(z)) w(\zeta) \frac{d\zeta}{i\zeta} \\ &= \rho(z)F(z) + \int \left[\tilde{\delta}(\zeta, z) + \sum_j \frac{\omega_j}{z - a_j} \frac{\zeta + z}{\zeta - a_j} w(\zeta) \right] \frac{d\zeta}{i\zeta} \\ &\quad \tilde{\delta}(\zeta, z) = (\zeta + z) \frac{\delta(\zeta) - \delta(z)}{\zeta - z} \\ &= (V(z)F(z) + U(z))/W(z) \end{aligned}$$

where U is a polynomial of degree at most D . We shall now use (3.6) and the characterization of the quotients ψ_n/ϕ_n and ψ_n^*/ϕ_n^* as reduced continued fraction expansions. Let us start from

$$\epsilon_n(z) = \psi_n(z) + F(z)\phi_n(z)$$

and

$$\epsilon_n(qz) = \psi_n(qz) + \left(\frac{V(z)}{W(z)} F(z) + \frac{U(z)}{W(z)} \right) \phi_n(qz) \quad \text{by (3.6)}$$

and eliminate $F(z)$ between these two equations, to get

$$\begin{aligned} W(z)\epsilon_n(qz)\phi_n(z) - V(z)\epsilon_n(z)\phi_n(qz) = \\ W(z)\psi_n(qz)\phi_n(z) - V(z)\psi_n(z)\phi_n(qz) + U(z)\phi_n(z)\phi_n(qz). \end{aligned}$$

The right hand side is a polynomial in z and, according to (2.10)-(2.13), the left hand side is $O(z^n)$ near 0 and $O(z^{D+n-1})$ near ∞ , therefore there exists a polynomial $\Theta_n(z)$ of degree less than $D - 1$, such that

$$(3.7) \quad 2\alpha_{n+1}\sigma_n z^n \Theta_n(z) = W(z)\epsilon_n(qz)\phi_n(z) - V(z)\epsilon_n(z)\phi_n(qz).$$

In case $\alpha_{n+1} = 0$, one has $\phi_{n+1}(z) = z\phi_n(z)$ and $\phi_{n+1}^*(z) = \phi_n^*(z)$, and the sought relation for $\phi_{n+1}(qz)$ can be obtained from that of $\phi_n(qz)$. From

$$2\alpha_{n+1}\sigma_n z^n \Theta_n(z) = W(z)\psi_n(qz)\phi_n(z) - V(z)\psi_n(z)\phi_n(qz) + U(z)\phi_n(z)\phi_n(qz)$$

we get, according to (2.17),

$$[\phi_{n+1}(z)\psi_n(z) - \psi_{n+1}(z)\phi_n(z)]\Theta_n(z) = W(z)\psi_n(qz)\phi_n(z) - V(z)\psi_n(z)\phi_n(qz) + U(z)\phi_n(z)\phi_n(qz)$$

or

$$\psi_n(z)(\phi_{n+1}(z)\Theta_n(z) + V(z)\phi_n(qz)) = \phi_n(z)(W(z)\psi_n(qz) + U(z)\phi_n(qz) + \psi_{n+1}\Theta_n(z)).$$

This expression is a polynomial, common multiple of ϕ_n and ψ_n which have no common zero (see (2.17)), so it can be put in the form

$$\Omega_n(z)\phi_n(z)\psi_n(z)$$

where Ω_n is a polynomial of degree at most D . Thus Ω_n satisfies

$$(3.8) \quad \Omega_n(z)\phi_n(z) = \phi_{n+1}(z)\Theta_n(z) + V(z)\phi_n(qz)$$

$$(3.9) \quad \Omega_n(z)\psi_n(z) = W(z)\psi_n(qz) + U(z)\phi_n(qz) + \psi_{n+1}(z)\Theta_n(z)$$

One obtains Ω_n by multiplying (3.9) by ϕ_{n+1} , subtracting (3.8) multiplied by ψ_{n+1} , and using (2.17), thus

$$(3.10) \quad 2\alpha_{n+1}\sigma_n z^n \Omega_n(z) = W(z)\epsilon_n(qz)\phi_{n+1}(z) - V(z)\epsilon_{n+1}(z)\phi_n(qz)$$

Equation (3.8) gives

$$(3.11) \quad V(z)\phi_n(qz) = \Omega_n(z)\phi_n(z) - \phi_{n+1}(z)\Theta_n(z)$$

or, using (2.5),

$$(3.12) \quad V(z)\phi_n(qz) = (\Omega_n(z) - z\Theta_n(z))\phi_n(z) - \alpha_{n+1}\Theta_n(z)\phi_n^*(z)$$

On multiplying (3.8) by $F(z)$ and adding (3.9)

$$\Omega_n(z)\epsilon_n(z) = V(z)\epsilon_n(qz)\frac{w(z)}{w(qz)} + \epsilon_{n+1}(z)\Theta_n(z)$$

therefore

$$\Omega_n(z)\frac{\epsilon_n(z)}{w(z)} = V(z)\frac{\epsilon_n(qz)}{w(qz)} + \frac{\epsilon_{n+1}(z)}{w(z)}\Theta_n(z).$$

Finally

$$(3.13) \quad V(z)\frac{\epsilon_n(qz)}{w(qz)} = \Omega_n(z)\frac{\epsilon_n(z)}{w(z)} - \frac{\epsilon_{n+1}(z)}{w(z)}\Theta_n(z)$$

and

$$(3.14) \quad V(z)\frac{\epsilon_n(qz)}{w(qz)} = (\Omega_n(z) - z\Theta_n(z))\frac{\epsilon_n(z)}{w(z)} + \alpha_{n+1}\Theta_n(z)\frac{\epsilon_n^*(z)}{w(z)}$$

Equations for ϕ_n^* and ϵ_n^* are obtained by similar arguments. One finds polynomials $\Theta_n^*(z)$ and $\Omega_n^*(z)$ with the same bounds on their degrees as $\Theta_n(z)$ and $\Omega_n(z)$, such that

$$(3.15) \quad V(z)\phi_n^*(qz) = (\Omega_n^*(z) - \Theta_n^*(z))\phi_n^*(z) - z\bar{\alpha}_{n+1}\Theta_n^*(z)\phi_n(z)$$

and

$$(3.16) \quad V(z)\frac{\epsilon_n^*(qz)}{w(qz)} = (\Omega_n^*(z) - \Theta_n^*(z))\frac{\epsilon_n^*(z)}{w(z)} + z\bar{\alpha}_{n+1}\Theta_n^*(z)\frac{\epsilon_n(z)}{w(z)}$$

PROPOSITION 3.1. *Let*

$$(3.17) \quad A_n(z) = \begin{pmatrix} \Omega_n(z) - z\Theta_n(z) & \alpha_{n+1}\Theta_n(z) \\ -z\bar{\alpha}_{n+1}\Theta_n^*(z) & \Omega_n^*(z) - \Theta_n^*(z) \end{pmatrix}.$$

then, with Y_n defined by (2.14), the relations (3.12), (3.14), (3.15), (3.16) can be put in matrix form as

$$(3.18) \quad Y_n(qz) = \frac{1}{V(z)} A_n(z) Y_n(z)$$

Compatibility relations between (2.15) and (3.18) imply that

$$(3.19) \quad A_{n+1}(z)B_n(z) = B_n(qz)A_n(z).$$

The coefficients of the polynomials $\Theta_n, \Theta_n^*, \Omega_n, \Omega_n^*$ in (3.17) can be computed from the coefficients of the polynomials ϕ_n . For example, with V and W given by (3.4) and (3.5) one finds, putting

$$\Theta_n(z) = \lambda z + \mu$$

$$(3.20) \quad 2\alpha_{n+1}z^n\Theta_n(z) = W(z)\epsilon_n(qz)\phi_n(z) - V(z)\epsilon_n(z)\phi_n(qz)$$

and comparing the two expressions near zero,

$$2\alpha_{n+1}z^n\mu = 2\alpha_n z^n(\bar{b}q^n - \bar{a})$$

Comparing at infinity gives λ , and finally

$$(3.21) \quad \Theta_n(z) = (a - bq^{n+1})z + (-\bar{a} + \bar{b}q^n)\frac{\alpha_n}{\alpha_{n+1}}.$$

Similar reasoning yield the other polynomials in terms of the Verblunsky coefficients. Using

$$\phi_n(z) = z^n + \beta_n z^{n-1} + \dots + \alpha_n$$

with

$$\beta_n = \sum_{j=1}^n \alpha_j \bar{\alpha}_{j-1}$$

One finds

$$(3.22) \quad \Omega_n(z) - z\Theta_n(z) = bq^{n+1}z^2 + tz + \bar{b}q^n$$

where t is a complicated expression in terms of α_n, α_{n+1} and β_n . Similarly

$$(3.23) \quad \Theta_n^*(z) = (aq - bq^{n+1})\frac{\bar{\alpha}_n}{\bar{\alpha}_{n+1}}z + bq^{n+1} - \bar{a}$$

$$(3.24) \quad \Omega_n^*(z) - \Theta_n^*(z) = aqz^2 + t^*z + \bar{a}$$

Equation (3.19) implies nonlinear recursion relations among these coefficients.

3.2. The isomonodromy deformation. Rather than do the tedious computations here, we shall, in the next section, start from the relation (3.19) and derive everything from its qualitative features. This will allow us to make the connection with Sakai's theory of discrete Painlevé equations. Before this we first make a few remarks. Equation (3.19) has the form

$$(3.25) \quad \tilde{A}(z) = B(qz)A(z)B(z)^{-1}$$

where

$$A(z) = A_d z^d + \dots + A_0, \quad \tilde{A}(z) = \tilde{A}_d z^d + \dots + \tilde{A}_0$$

and

$$B(z) = B \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

for some constant matrix B . Equation (3.25) is an isomonodromy transformation of a q -difference equation, as studied by Birkhoff [2], and recently by Borodin [3]. It is closely related to the equation appearing in [15]. Such isomonodromy transformations, for difference equations instead of q -difference equations, have been studied in depth by Arinkin and Borodin in [1], using geometric methods. These methods, involving the definition of “ q -connections” should also apply to our situation, although we do not use them here.

In our case, the matrices A_d, \tilde{A}_d are lower triangular matrices, and A_0, \tilde{A}_0 are upper triangular. Let κ_1, κ_2 be the eigenvalues of A_d , and θ_1, θ_2 those of A_0 , so that

$$A_d = \begin{pmatrix} \kappa_1 & 0 \\ * & \kappa_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \theta_1 & * \\ 0 & \theta_2 \end{pmatrix}$$

Consider the matrix

$$\hat{A}(z) = \begin{pmatrix} qz & 0 \\ 0 & 1 \end{pmatrix} A(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \hat{A}_2 z^2 + \hat{A}_1 z + \hat{A}_0$$

then

$$\hat{A}_d = \begin{pmatrix} q\kappa_1 & * \\ 0 & \kappa_2 \end{pmatrix}, \quad \hat{A}_0 = \begin{pmatrix} q\theta_1 & 0 \\ * & \theta_2 \end{pmatrix}, \quad \det \hat{A}(z) = q \det A(z)$$

We can now conjugate $\hat{A}(z)$ by a constant matrix B in order to get the matrix $\tilde{A}(z)$ with

$$\tilde{A}_d = \begin{pmatrix} q\kappa_1 & 0 \\ * & \kappa_2 \end{pmatrix}, \quad \tilde{A}_0 = \begin{pmatrix} q\theta_1 & * \\ 0 & \theta_2 \end{pmatrix}.$$

The matrix B is determined up to left multiplication by a diagonal invertible matrix. One can choose its coefficients to be expressed as rational functions of the coefficients of $A(z)$. Furthermore equation (3.25) can be used in the same way to deduce $A(z)$ from $\tilde{A}(z)$. It follows that equation (3.25) defines a birational transformation between two spaces of matrices with polynomial coefficients. We shall give an explicit formula for this transformation in the next section. The recursion equations for Verblunsky coefficients are given by iterating this transformation. In the next section we shall investigate in details the case where $d = 2$, which corresponds to the case of the Verblunsky coefficients for the q -Gamma weight (3.3).

4. A birational transformation

4.1. The space $X_{\kappa_1, \kappa_2, \theta_1, \theta_2, c_1, c_2, c_3, c_4}$. We now embark on the analysis of equation (3.25), in the case of degree 2 polynomial matrices. Consider a two-dimensional vector space H , and let

$$A(z) = A_0 + A_1 z + A_2 z^2$$

be an endomorphism of H which depends polynomially on the variable z , with A_0, A_1, A_2 , elements of $\text{End}(H)$.

DEFINITION 4.1. *We let*

$$\bar{X}_{\kappa_1, \kappa_2, \theta_1, \theta_2, c_1, c_2, c_3, c_4}$$

denote the space of all such $A(z)$ satisfying

$$\{\kappa_1, \kappa_2\} = \text{sp } A_2, \quad \{\theta_1, \theta_2\} = \text{sp } A_0$$

and

$$(4.1) \quad \det A(z) = \kappa_1 \kappa_2 (z - c_1)(z - c_2)(z - c_3)(z - c_4)$$

for some fixed complex constants $\kappa_1, \kappa_2, \theta_1, \theta_2, c_1, \dots, c_4$, satisfying the equation

$$(4.2) \quad \kappa_1 \kappa_2 c_1 c_2 c_3 c_4 = \theta_1 \theta_2$$

(which comes from considering $\det A(0)$). We let

$$X_{\kappa_1, \kappa_2, \theta_1, \theta_2, c_1, c_2, c_3, c_4}$$

denote its quotient by the action of $GL(H)$ by conjugation. These spaces are complex manifolds.

In the sequel c_1, \dots, c_4 will be fixed so we will omit them from the notation most of the time.

We consider the case of generic parameters, in particular $\kappa_1 \neq \kappa_2, \theta_1 \neq \theta_2$, and A_0, A_2 are semisimple. We denote E_1, E_2 the eigenspaces of A_2 , and F_1, F_2 those of A_0 . The set of $A(z)$ such that $E_2 \neq F_1$ is a dense open set Ω , invariant by conjugation, and for each element of this open set we can chose a basis (u, v) with $u \in F_1, v \in E_2$ so that, in this basis, A_0 and A_2 have matrices of the form

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ * & \kappa_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \theta_1 & * \\ 0 & \theta_2 \end{pmatrix}$$

As in the preceding section, let q be a nonzero complex number and consider the matrix

$$\hat{A}(z) = \begin{pmatrix} qz & 0 \\ 0 & 1 \end{pmatrix} A(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \hat{A}_2 z^2 + \hat{A}_1 z + \hat{A}_0$$

then

$$\hat{A}_2 = \begin{pmatrix} q\kappa_1 & * \\ 0 & \kappa_2 \end{pmatrix}, \quad \hat{A}_0 = \begin{pmatrix} q\theta_1 & 0 \\ * & \theta_2 \end{pmatrix}, \quad \det \hat{A}(z) = q \det A(z)$$

therefore the matrix $\hat{A}(z)$ belongs to the space $\bar{X}_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$. Thus $A(z) \rightarrow \hat{A}(z)$ defines a map from a dense open subset of $\bar{X}_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ to $\bar{X}_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$. Furthermore, as explained also at the end of the preceding section, this map can be inverted, thus defining a bijection between dense open subsets of $\bar{X}_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ and $\bar{X}_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$. Clearly the map passes to the quotients and induces a bijection between dense open subsets of $X_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ and $X_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$, which is a birational map.

DEFINITION 4.2. We call $\phi : X_{\kappa_1, \kappa_2, \theta_1, \theta_2} \rightarrow X_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$ the birational map constructed above.

In the next section, we shall introduce a parametrization of the spaces in order to give an explicit formula for this birational transformation.

4.2. Jimbo and Sakai parametrization. We will use the following parametrization of the space $X_{\kappa_1, \kappa_2, \theta_1, \theta_2}$, after Jimbo and Sakai [15]

$$(4.3) \quad A(z) = \begin{pmatrix} \kappa_1((z-y)(z-\alpha) + z_1) & z-y \\ z(\gamma z + \beta) & \kappa_2((z-y)(z-\delta) + z_2) \end{pmatrix}$$

Thus y is the unique root of the polynomial which is the 12 coefficient of A , and $\kappa_1 z_1, \kappa_2 z_2$ are the values of the 11, resp. 22 coefficient of $A(z)$ at $z = y$. One has

$$y\alpha + z_1 = \theta_1/\kappa_1, \quad y\alpha + z_2 = \theta_2/\kappa_2$$

The quantities y, z_1, z_2 satisfy

$$(4.4) \quad z_1 z_2 = (y - c_1)(y - c_2)(y - c_3)(y - c_4)$$

where the equality follows from considering the determinant of $A(y)$. Let us put

$$\xi = (y - c_1)(y - c_2)/\kappa_1 z_1 = \kappa_2 z_2/(y - c_3)(y - c_4).$$

The quantities β and γ are determined as rational functions of y, ξ using (4.1).

It follows that the variables (y, ξ) in $\mathbf{C}^* \times \mathbf{C}^*$ parametrize a dense open subset of $X_{\kappa_1, \kappa_2, \theta_1, \theta_2}$. Using this parametrization, the space $X_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ is identified with $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at the seven points

$$\begin{aligned} & (c_1, 0), \quad (c_2, 0), \quad (c_3, \infty), \quad (c_4, \infty) \\ & (0, c_1 c_2 / \theta_1), \quad (0, c_1 c_2 / \theta_2), \quad (\infty, 1/\kappa_1) \end{aligned}$$

minus the strict transforms of the four lines $y = 0, y = \infty, \xi = 0, \xi = \infty$, see Fig. 2 below.

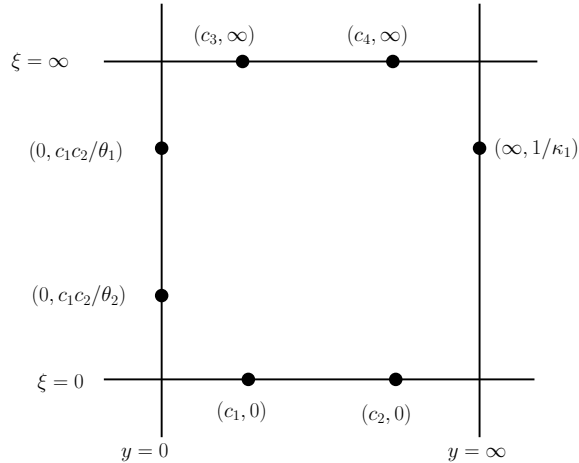


Fig. 2. The space X_{κ_1}

The curves corresponding to $F_2 = E_1$ and $F_2 = E_2$ are respectively the strict transforms of the points $(0, c_1 c_2 / \theta_2)$ and $(0, c_1 c_2 / \theta_1)$.

4.3. Explicit birational expression for ϕ . We now describe the map ϕ as a birational transformation, using the Jimbo-Sakai coordinates (4.3). Let $A(z)$ be as in (4.3) and consider the matrix

$$\begin{aligned}\hat{A}(z) &= \begin{pmatrix} qz & 0 \\ 0 & 1 \end{pmatrix} A(z) \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} q\kappa_1((z-y)(z-\alpha) + z_1) & qz(z-y) \\ (\gamma z + \beta) & \kappa_2((z-y)(z-\delta) + z_2) \end{pmatrix}\end{aligned}$$

Then $\hat{A}(z)$ belongs to the space $X_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$. In order to find its Jimbo-Sakai coordinates, we have to use the basis formed by the spaces \hat{E}_2 and \hat{F}_1 . It follows that

$$\tilde{A}(z) = B^{-1} \hat{A}(z) B$$

with

$$B = \begin{pmatrix} q\theta_1 - \theta_2 & -q \\ \beta & q\kappa_1 - \kappa_2 \end{pmatrix}$$

on the condition that the matrix B so defined is invertible. In order to go from \tilde{A} to A one proceeds in the opposite way. Start from

$$\tilde{A} = \begin{pmatrix} q\kappa_1((z-\tilde{y})(z-\tilde{\alpha}) + \tilde{z}_1) & z - \tilde{y} \\ \tilde{z}(\tilde{\gamma}z + \tilde{\beta}) & \kappa_2((z-\tilde{y})(z-\tilde{\delta}) + \tilde{z}_2) \end{pmatrix}$$

conjugate it by a constant matrix, so that its 21 coefficient has degree 1 and its 12 coefficient is $z \times$ (a polynomial with degree 1). For this use eigenvectors of \tilde{A}_2 and \tilde{A}_0 , thus $\hat{A}(z) = C^{-1} \tilde{A}(z) C$ with

$$C = \begin{pmatrix} q\kappa_1 - \kappa_2 & -\tilde{y} \\ q\tilde{\gamma} & q\theta_1 - \theta_2 \end{pmatrix}$$

We will not compute the inverse transformation.

Let us now compute an explicit formula for ϕ , using the coordinates (y, ξ) . We start from the matrix

$$A(z) = \begin{pmatrix} \kappa_1 z^2 + az + \theta_1 & z - y \\ z(\gamma z + \beta) & \kappa_2 z^2 + bz + \theta_2 \end{pmatrix}$$

with

$$(4.5) \quad a = -\kappa_1 y + \frac{(y - c_1)(y - c_2)}{y\xi} - \frac{\theta_1}{y}$$

$$(4.6) \quad b = -\kappa_2 y + \frac{\kappa_1 \kappa_2 \xi (y - c_3)(y - c_4)}{y} - \frac{\theta_2}{y}$$

and

$$(4.7) \quad \theta_1 b + \theta_2 a + \beta y = -\kappa_1 \kappa_2 \sigma_3$$

$$(4.8) \quad \kappa_1 b + \kappa_2 a - \gamma = -\kappa_1 \kappa_2 \sigma_1$$

where the σ_i are the elementary symmetric functions in the c_i . Then

$$(4.9) \quad \tilde{A}(z) = B(qz)A(z)B(z)^{-1} = \begin{pmatrix} q\kappa_1 z^2 + \tilde{a}z + q\theta_1 & w(z - \tilde{y}) \\ w^{-1}z(\tilde{\gamma}z + \tilde{\beta}) & \kappa_2 z^2 + \tilde{b}z + \theta_2 \end{pmatrix}$$

with

$$\begin{aligned} B(z) &= \begin{pmatrix} z(q\kappa_1 - \kappa_2) & q \\ -z\beta & q\theta_1 - \theta_2 \end{pmatrix} \\ B(z)^{-1} &= \Delta^{-1} \begin{pmatrix} z^{-1}(q\theta_1 - \theta_2) & -z^{-1}q \\ \beta & q\kappa_1 - \kappa_2 \end{pmatrix} \\ \Delta &= (q\kappa_1 - \kappa_2)(q\theta_1 - \theta_2) + q\beta \end{aligned}$$

A computation using (4.9) gives $w\tilde{y} = q$, and

$$w = R/\Delta, \quad \tilde{y} = \frac{q\Delta}{R}$$

with

$$\begin{aligned} \xi y^2 \Delta &= \xi y^2 (q\theta_1 - \theta_2)(q\kappa_1 - \kappa_2) + q\xi y^2 \beta \\ &= \xi y^2 (q\theta_1 - \theta_2)(q\kappa_1 - \kappa_2) \\ &\quad + qy\xi(-\kappa_1\kappa_2\sigma_3 - \theta_1 b - \theta_2 a) \quad \text{using (4.7)} \\ &= -q\kappa_1\kappa_2\theta_1\xi^2(y - c_3)(y - c_4) \\ &\quad + \xi((q^2\kappa_1\theta_1 + \kappa_2\theta_2)y^2 - q\kappa_1\kappa_2\sigma_3y + 2q\theta_1\theta_2) \\ &\quad - q\theta_2(y - c_1)(y - c_2) \quad \text{using (4.5), (4.6),} \end{aligned}$$

and

$$\begin{aligned} \xi y R &= \xi y((q\kappa_1 - \kappa_2)(-aq^2 + bq - q(q\kappa_1 - \kappa_2)y) - q^2\gamma) \\ &= \xi y(-q^3\kappa_1a - q\kappa_2b - q(q\kappa_1 - \kappa_2)^2y - q^2\kappa_1\kappa_2\sigma_1) \quad \text{by (4.8)} \\ &= q^3\kappa_1(\kappa_1\xi y^2 - (y - c_1)(y - c_2) + \theta_1\xi) \\ &\quad + q\kappa_2(\kappa_2\xi y^2 - \kappa_1\kappa_2\xi^2(y - c_3)(y - c_4) + \theta_2\xi) \\ &\quad - q(q\kappa_1 - \kappa_2)^2y^2\xi - q^2\kappa_1\kappa_2\sigma_1\xi y \quad \text{by (4.5), (4.6)} \\ &= -q\kappa_1\kappa_2^2\xi^2(y - c_3)(y - c_4) \\ &\quad + \xi(2q^2\kappa_1\kappa_2y^2 - q^2\kappa_1\kappa_2\sigma_1y + q(q^2\kappa_1\theta_1 + \kappa_2\theta_2)) \\ &\quad - q^3\kappa_1(y - c_1)(y - c_2). \end{aligned}$$

Therefore

$$(4.10) \quad \tilde{y} = \frac{S}{yT}$$

with

$$\begin{aligned} S &= -q\kappa_1\kappa_2\theta_1\xi^2(y - c_3)(y - c_4) \\ &\quad + \xi((q^2\kappa_1\theta_1 + \kappa_2\theta_2)y^2 - q\kappa_1\kappa_2\sigma_3y + 2q\theta_1\theta_2) \\ &\quad - q\theta_2(y - c_1)(y - c_2) \\ T &= -\kappa_1\kappa_2^2\xi^2(y - c_3)(y - c_4) \\ &\quad + \xi(2q\kappa_1\kappa_2y^2 - q\kappa_1\kappa_2\sigma_1y + (q^2\kappa_1\theta_1 + \kappa_2\theta_2)) \\ &\quad - q^2\kappa_1(y - c_1)(y - c_2). \end{aligned}$$

In order to find $\tilde{\xi}$ we proceed as follows. From the definition one has

$$\tilde{\xi} = \frac{(\tilde{y} - c_1)(\tilde{y} - c_2)}{q\kappa_1\tilde{y}^2 + \tilde{a}\tilde{y} + q\theta_1}$$

where \tilde{a} can be computed from (4.9). Plugging in the values of a, b, β, γ , and \tilde{y} given by (4.5), (4.6), (4.7) (4.8), (4.10) gives an unwieldy expression as a rational

fraction in the variables y, ξ . One can simplify this expression using the following observation. Computing the determinant $\det \tilde{A}(\tilde{y}) = q \det A(\tilde{y})$ from (4.9) gives the identity

$$(4.11) \quad q\kappa_1\kappa_2(\tilde{y} - c_1)(\tilde{y} - c_2)(\tilde{y} - c_3)(\tilde{y} - c_4) = (q\kappa_1\tilde{y}^2 + \tilde{a}\tilde{y} + q\theta_1)(\kappa_2\tilde{y}^2 + \tilde{b}\tilde{y} + \theta_2)$$

where $\tilde{y}, \tilde{a}, \tilde{b}$ are rational expressions in y, ξ . Of course this identity is compatible with the alternative expression

$$\tilde{\xi} = \frac{\kappa_2\tilde{y}^2 + \tilde{b}\tilde{y} + \theta_2}{q\kappa_1\kappa_2(\tilde{y} - c_3)(\tilde{y} - c_4)}.$$

Replacing $\tilde{y}, \tilde{a}, \tilde{b}$ in (4.11) by their values and reducing the denominators, we get a polynomial identity between the numerators. This leads us to expect that the numerator of each expression $\tilde{y} - c_i$ factorizes into a product of two terms, which divide respectively the numerators of $q\kappa_1\tilde{y}^2 + \tilde{a}\tilde{y} + q\theta_1$ and $\kappa_2\tilde{y}^2 + \tilde{b}\tilde{y} + \theta_2$. Since the degree of the numerator of $\tilde{y} - c_i$ in the variable ξ is two, we expect a factorization into two terms of degree 1 in ξ . Indeed a direct computation confirms that

$$\begin{aligned} S - c_1yT &= \kappa_1\kappa_2\xi^2(c_1y\kappa_2 - q\theta_1)(y - c_3)(y - c_4) \\ &\quad + \xi[-2qc_1\kappa_1\kappa_2y^3 + (q^2\kappa_1\theta_1 + \kappa_2\theta_2 + qc_1\kappa_1\kappa_2\sigma_1)y^2] \\ &\quad + \xi[(-q\kappa_1\kappa_2\sigma_3 + c_1(q^2\kappa_1\theta_1 + \kappa_2\theta_2))y + 2q\theta_1\theta_2] \\ &\quad + q(qc_1\kappa_1y - \theta_2)(y - c_1)(y - c_2) \\ &= (\xi(c_1y\kappa_2 - q\theta_1) - qc_1(y - c_2)) \times \\ &\quad \times (\kappa_1\kappa_2\xi(y - c_3)(y - c_4) - \frac{1}{c_1}(qc_1\kappa_1y - \theta_2)(y - c_1)) \end{aligned}$$

and similar expressions for the other c_i , e.g.

$$\begin{aligned} S - c_3yT &= \kappa_1\kappa_2\xi^2(c_3y\kappa_2 - q\theta_1)(y - c_3)(y - c_4) \\ &\quad + \xi[-2qc_3\kappa_1\kappa_2y^3 + (q^2\kappa_1\theta_1 + \kappa_2\theta_2 + qc_3\kappa_1\kappa_2\sigma_1)y^2] \\ &\quad + \xi[(-q\kappa_1\kappa_2\sigma_3 + c_3(q^2\kappa_1\theta_1 + \kappa_2\theta_2))y + 2q\theta_1\theta_2] \\ &\quad + q(qc_3\kappa_1y - \theta_2)(y - c_1)(y - c_2) \\ &= (\xi(y - c_4) - \frac{1}{\kappa_1\kappa_2c_3}(qc_3\kappa_1y - \theta_2)) \times \\ &\quad \times (\kappa_1\kappa_2\xi(y - c_3)(c_3y\kappa_2 - q\theta_1) - \kappa_1\kappa_2c_3(y - c_1)(y - c_2)) \end{aligned}$$

One checks then that one of these factors divides the numerator of $q\kappa_1\tilde{y}^2 + \tilde{a}\tilde{y} + q\theta_1$. This yields the simpler expression

$$(4.12) \quad \tilde{\xi} = \frac{c_1c_2}{\kappa_1\theta_1\xi} \frac{(\xi(y - \frac{q\theta_1}{c_1\kappa_2}) - \frac{q}{\kappa_2}(y - c_2))(\xi(y - \frac{q\theta_1}{c_2\kappa_2}) - \frac{q}{\kappa_2}(y - c_1))}{(\xi(y - c_4) - \frac{q}{\kappa_2}(y - \frac{\theta_2}{qc_3\kappa_1}))(\xi(y - c_3) - \frac{q}{\kappa_2}(y - \frac{\theta_2}{qc_4\kappa_1}))}.$$

PROPOSITION 4.3. *Formulas (4.10) and (4.12) express ϕ as a birational transformation $\phi : (y, \xi) \rightarrow (\tilde{y}, \tilde{\xi})$.*

4.4. Blowing up the spaces. The map ϕ , whose expression as a birational transformation we obtained in the preceding section, can be extended to a regular map between two surfaces, by blowing up the spaces $X_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ and $X_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$. For this we first extend the map to the set of endomorphisms such that $E_2 =$

F_1 . As noted before, this set corresponds to the blown up point with coordinates $(0, c_1 c_2 / \theta_2)$. Take a matrix in a neighbourhood of $E_2 = F_1$ in the form

$$\begin{pmatrix} \kappa_1 z^2 + \bar{a}z + \theta_2 & z - y \\ z(\bar{\gamma}z + \bar{\beta}) & \kappa_2 z^2 + \bar{b}z + \theta_1 \end{pmatrix}$$

with y small and $\xi - c_1 c_2 \kappa_1 / \theta_2 = \lambda y = (y - c_1)(y - c_2) / z_1$, or

$$\bar{a} = -\kappa_1 y + \frac{\kappa_1 y - \kappa_1(c_1 + c_2) - \lambda \theta_2}{\lambda y + c_1 c_2 \kappa_1 / \theta_2}$$

We can conjugate the matrix to put it in the form

$$\begin{pmatrix} \kappa_1 z^2 + az + \theta_1 & z - y \\ z(\gamma z + \beta) & \kappa_2 z^2 + bz + \theta_2 \end{pmatrix}$$

with

$$\begin{aligned} a &= \bar{a} + \frac{\theta_2 - \theta_1}{y} \\ b &= \bar{b} + \frac{\theta_1 - \theta_2}{y} \\ \gamma &= \sigma_1 + \kappa_1 \bar{b} + \kappa_2 \bar{a} + \frac{(\kappa_1 - \kappa_2)(\theta_1 - \theta_2)}{y} \\ \beta &= -\frac{\sigma_3 + \theta_1 \bar{b} + \theta_2 \bar{a}}{y} - \frac{(\theta_1 - \theta_2)^2}{y^2} \end{aligned}$$

then putting

$$\Delta = (q\theta_1 - \theta_2)(q\kappa_1 - \kappa_2) + q\beta$$

$$\tilde{y} = \frac{\Delta}{(q\kappa_1 - \kappa_2)(-qa - y(q\kappa_1 - \kappa_2)) + (-q\gamma + (q\kappa_1 - \kappa_2)b)}$$

we obtain

$$\tilde{a} = \frac{(q\kappa_1 - \kappa_2)(q\theta_1 - \theta_2)qa - qy\beta(q\kappa_1 - \kappa_2) + q\gamma(q\theta_1 - \theta_2) + q\beta b}{\Delta}$$

We keep only the main terms as $y \rightarrow 0$ to get

$$\begin{aligned} \Delta &\sim -q \frac{(\theta_1 - \theta_2)^2}{y^2} \\ \tilde{y} &\sim -q \frac{(\theta_1 - \theta_2)}{(q^2 \kappa_1 - \kappa_2)y} \\ \tilde{a} &\sim \frac{\theta_1 - \theta_2}{y} \end{aligned}$$

therefore

$$\kappa_1 \tilde{z}_1 \sim q\kappa_1 \tilde{y}^2 + \tilde{y} \tilde{a} \sim \kappa_2 \tilde{y} \frac{(\theta_1 - \theta_2)}{(q^2 \kappa_1 - \kappa_2)y} \sim \kappa_2 \tilde{y}^2 / q$$

or

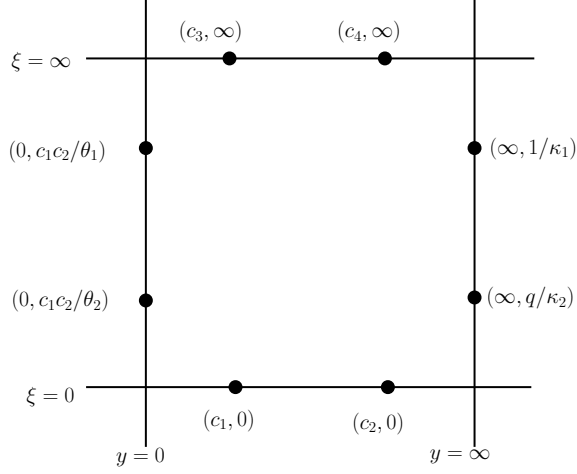
$$\tilde{\xi} = q / \kappa_2$$

Thus we see that the map can be extended if we blow up the final space at $(\infty, q / \kappa_2)$. A similar computation shows that the inverse map also extends to the curve $\tilde{E}_2 = \tilde{F}_1$ if we blow up the initial space at the point $(\infty, q / \kappa_2)$.

PROPOSITION 4.4. *Let $Y_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ denote $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at the eight points*

$$\begin{aligned} &(c_1, 0), \quad (c_2, 0), \quad (c_3, \infty), \quad (c_4, \infty) \\ &(0, c_1 c_2 / \theta_1), \quad (0, c_1 c_2 / \theta_2), \quad (\infty, 1 / \kappa_1), \quad (\infty, q / \kappa_2) \end{aligned}$$

minus the strict transforms of the four lines $y = 0, \infty, \xi = 0, \infty$. then the map ϕ extends to a regular isomorphism between the spaces $Y_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ and $Y_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$.

Fig. 3. The space $Y_{\kappa_1 \kappa_2, \theta_1 \theta_2}$

5. Discrete Painlevé equations

5.1. Sakai's theory. We recall here the basic facts about the relation between surface theory and discrete Painlevé equations, according to Sakai [18]. One starts with a complex surface X , obtained by blowing up $\mathbf{P}^2(\mathbf{C})$ at nine points x_1, \dots, x_9 , thus X comes with a morphism $\pi : X \rightarrow \mathbf{P}^2(\mathbf{C})$. Degenerate configurations, where some of the points x_i are infinitely close, are allowed. The Picard group of such a surface is a free abelian group of rank 10, generated by $\mathcal{E}_0 = \pi^{-1}(L)$, the inverse image of a generic line in $\mathbf{P}^2(\mathbf{C})$, and $\mathcal{E}_i = \pi^{-1}(x_i), i = 1, \dots, 9$. The intersection form on $\text{Pic}(X)$ satisfies

$$\mathcal{E}_0 \cdot \mathcal{E}_0 = 1, \quad \mathcal{E}_i \cdot \mathcal{E}_i = -1, \quad i = 1, \dots, 9, \quad \mathcal{E}_i \cdot \mathcal{E}_j = 0, \quad i \neq j$$

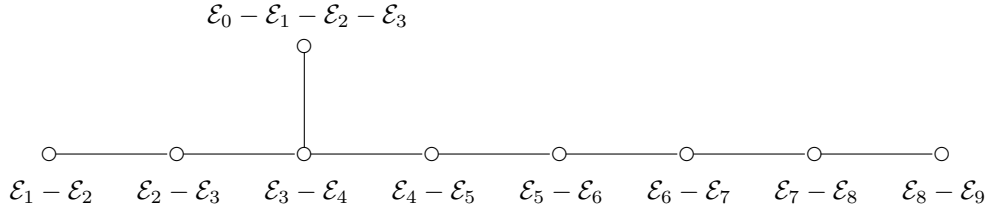
The canonical divisor class is

$$\mathcal{K}_X = -\delta = -3\mathcal{E}_0 + \mathcal{E}_1 + \dots + \mathcal{E}_9$$

and its orthogonal δ^\perp in $\text{Pic}(X)$ is a rank 9 lattice, generated by

$$\mathcal{E}_i - \mathcal{E}_{i+1}, i = 1, \dots, 8, \quad \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$$

which form the simple roots in an affine root system of type $E_8^{(1)}$ according to the Dynkin diagram

Fig. 1 Dynkin diagram of the affine root system $E_8^{(1)}$ inside $\text{Pic}(X)$

The surface X is called a generalized Halphen surface by Sakai, if the linear system $|\mathcal{K}_X|$ has a unique divisor, whose irreducible components are orthogonal to

\mathcal{K}_X . Let $\mathcal{D}_i, i \in I$ be these components, then they generate an affine subroot system R in $\delta^\perp \subset \text{Pic}(X)$, such that its orthogonal is also an affine subroot system $R^\perp \subset \delta^\perp$. These surfaces are classified according to their corresponding root systems. Let X and \tilde{X} be two such surfaces, corresponding to the same root system R , where the irreducible components of the canonical divisor are labelled $\mathcal{D}_i, i \in I$, resp. $\tilde{\mathcal{D}}_i, i \in I$, then Sakai's main result is that (up to some little twists, see [18], Theorem 25 for a precise statement), every isometry of the Picard groups $\phi : \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$ which preserves the components of the anticanonical divisor, $\phi((\mathcal{D}_i, i \in I)) = (\tilde{\mathcal{D}}_i, i \in I)$ comes from a unique isomorphism of surfaces $\phi : X \rightarrow \tilde{X}$. Such isometries are given by elements of the extended Weyl group of the dual root system R^\perp (i.e. the group generated by the Weyl group of R^\perp and the automorphisms of the Dynkin diagram). When viewed at the level of $\mathbf{P}^2(\mathbf{C})$, the isomorphisms $\phi : X \rightarrow \tilde{X}$ give rise to birational transformations $\mathbf{P}^2(\mathbf{C}) \rightarrow \mathbf{P}^2(\mathbf{C})$. The discrete Painlevé equations correspond to the translations in the affine Weyl group $W(R^\perp)$. On the other hand, transformations in the extended Weyl group of R give Bäcklund transformations. As an example, Sakai treats the case of the birational transformation

$$\rho : (f, g) \rightarrow (\bar{f}, \bar{g})$$

determined by

$$\begin{aligned} \bar{g} &= \frac{b_3 b_4 (f - b_5)(f - b_6)}{g (f - b_7)(f - b_8)} \\ \bar{f} &= \frac{b_7 b_8 (\bar{g} - q b_1)(\bar{g} - q b_2)}{f (\bar{g} - b_3)(\bar{g} - b_4)} \end{aligned}$$

where b_1, \dots, b_8 are some complex parameters and

$$q = \frac{b_3 b_4 b_5 b_6}{b_1 b_2 b_7 b_8}.$$

He shows that this transformation corresponds to an isomorphism between $A_3^{(1)}$ surfaces, which arises from a translation in an affine Weyl group $D_5^{(1)}$. Here $A_3^{(1)}$ and $D_5^{(1)}$ are two orthogonal affine root systems in $E_8^{(1)}$. Remark that this transformation is actually considered as a birational transformation of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$. Since $\mathbf{P}^2(\mathbf{C})$ blown up at 9 points is isomorphic to $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ blown up at 8 points the two pictures are equivalent, but it is sometimes easier to do formal reasoning on $\mathbf{P}^2(\mathbf{C})$ and to perform actual computations in $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.

As explained in [15], the transformation ρ can be considered as a step in a nonlinear difference equation

$$(f_{n+1}, g_{n+1}) = \rho(f_n, g_n)$$

and this difference equation leads, by taking a suitable continuous limit, to a rational differential system

$$\begin{aligned} df/dx &= p(x, f, g) \\ dg/dx &= q(x, f, g) \end{aligned}$$

which is equivalent to a Hamiltonian version of the Painlevé VI equation. The degenerescence scheme of discrete Painlevé equations corresponds to the inclusion of the root systems in $E_8^{(1)}$. The degenerescence to Painlevé differential equations is also discussed by Sakai.

5.2. According to the classification of Sakai, the spaces $Y_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ are examples of $A_3^{(1)}$ surfaces. We will now identify the map ϕ between $Y_{\kappa_1, \kappa_2, \theta_1, \theta_2}$ and $Y_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$ we have constructed as one of the discrete Painlevé maps. For this we will use intersection theory for surfaces obtained as blow-ups. Denote by $\mathcal{F}_i, i = 1, \dots, 8$ the divisors which are inverse images, respectively, of the points

$$(c_1, 0), (c_2, 0), (c_3, \infty), (c_4, \infty), \\ (0, c_1 c_2 / \theta_1), (0, c_1 c_2 / \theta_2), (\infty, 1/\kappa_1), (\infty, q/\kappa_2)$$

in the blow ups. Similarly we denote by $\tilde{\mathcal{F}}_i, i = 1, \dots, 8$ the ones in the space $Y_{q\kappa_1, \kappa_2, q\theta_1, \theta_2}$.

It will be convenient to identify, as in Sakai [18], the surfaces with \mathbf{P}^2 blown up at 9 points, called x_i . This identification is as follows. Consider $\mathbf{P}^1 \times \mathbf{P}^1$ with coordinates denoted by (f, g) , blown up at the eight points

$$(0, b_1), (0, b_2), (\infty, b_3), (\infty, b_4), \\ (b_5, 0), (b_6, 0), (b_7, \infty), (b_8, \infty)$$

We make the explicit identification

$$(5.1) \quad \begin{array}{ll} c_1 = b_1 & c_1 c_2 / \theta_1 = b_5 \\ c_2 = b_2 & c_1 c_2 / \theta_2 = b_6 \\ c_3 = b_3 & 1/\kappa_1 = b_7 \\ c_4 = b_4 & q/\kappa_2 = b_8 \end{array} \quad f = \xi \quad g = y$$

between our space and Sakai's. Consider also \mathbf{P}^2 blown up at the eight points

$$x_1 = (a_1 : 0 : 1), \quad x_2 = (a_2 : 0 : 1), \quad x_3 = (a_3 : 0 : 1), \quad x_4 = (0 : 1 : a_4), \\ x_5 = (0 : 1 : a_5), \quad x_6 = (1 : a_6 : 0), \quad x_7 = (1 : a_7 : 0), \quad x_8 = (0 : 1 : 0)$$

further blown up on $\pi^{-1}(0 : 1 : 0)$ at

$$x_9 = \left(\frac{x}{z}, \frac{z}{x} \right) = (0, a_8)$$

The two spaces are identified according to the transformation

$$(5.2) \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} ; f, g = \begin{pmatrix} 1/a_2 & 1/a_3 & 1/a_1 & a_8 \\ a_6 & a_7 & -1/a_4 a_1 & -1/a_5 a_1 \end{pmatrix} ; \frac{y}{x - a_1 z}, \frac{z}{x}$$

See pages 173 and 215 in [18]. The correspondance between divisors is

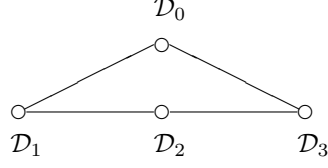
$$\mathcal{E}_2 = \mathcal{F}_1, \quad \mathcal{E}_3 = \mathcal{F}_2, \quad \mathcal{E}_4 = \mathcal{F}_5, \quad \mathcal{E}_5 = \mathcal{F}_6, \quad \mathcal{E}_6 = \mathcal{F}_7, \\ \mathcal{E}_7 = \mathcal{F}_8, \quad \mathcal{E}_9 = \mathcal{F}_4, \quad \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8 = \mathcal{F}_3$$

\mathcal{E}_1 is the line $y = a_3$, and $\mathcal{E}_8 - \mathcal{E}_9$ the line $\xi = \infty$.

The components of the anticanonical cycle are

$$(5.3) \quad \begin{array}{ll} \mathcal{D}_0 & = \mathcal{E}_8 - \mathcal{E}_9 \\ \mathcal{D}_2 & = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 \end{array} \quad \begin{array}{ll} \mathcal{D}_1 & = \mathcal{E}_0 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 \\ \mathcal{D}_3 & = \mathcal{E}_0 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8 \end{array}$$

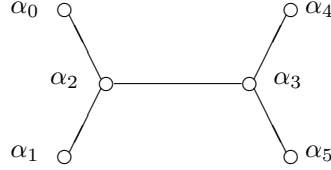
which form a root system of type $A_3^{(1)}$, with Dynkin diagram



The orthogonal is a root system of type $D_5^{(1)}$, with generators

$$(5.4) \quad \begin{aligned} \alpha_0 &= \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_8 - \mathcal{E}_9, & \alpha_1 &= \mathcal{E}_2 - \mathcal{E}_3, & \alpha_2 &= \mathcal{E}_1 - \mathcal{E}_2 \\ \alpha_3 &= \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_6, & \alpha_4 &= \mathcal{E}_6 - \mathcal{E}_7, & \alpha_5 &= \mathcal{E}_4 - \mathcal{E}_5 \end{aligned}$$

forming the Dynkin diagram



We denote by $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ the objects corresponding to the target space of the map ϕ . We saw that ϕ maps \mathcal{F}_6 to $\tilde{\mathcal{F}}_8$ and \mathcal{F}_7 to $\tilde{\mathcal{F}}_5$. We will compute the matrix of the map induced by ϕ between the Picard groups.

5.3. Let us start by computing the image of the curve $\mathcal{F}_1 = \mathcal{E}_2$. This curve consists of matrices of the form

$$A(z) = \begin{pmatrix} \kappa_1 z^2 + az + \theta_1 & w(z - c_1) \\ w^{-1}z(\gamma z + \beta) & \kappa_2(z - c_1)(z - \theta_2/c_1\kappa_2) \end{pmatrix}$$

where a is a local parameter along the curve and w an irrelevant constant which can be removed by conjugation with a diagonal matrix but is convenient for computations (this is the "gauge freedom" in [15]). The parameters γ and β are affine functions of a . Recall that

$$\Delta = (q\kappa_1 - \kappa_2)(q\theta_1 - \theta_2) + q\beta$$

then

$$\tilde{y} = \frac{\Delta}{qa(q\kappa_1 - \kappa_2) - (q\kappa_1 - \kappa_2)(-c_1 - \theta_1/a_1\kappa_1) + c_1(q\kappa_1 - \kappa_2)^2 + \gamma(q\kappa_1 - \kappa_2)}$$

The critical value to be investigated are $\tilde{y} = 0, c_i, \infty$. The case $\tilde{y} = 0$ corresponds to $\tilde{E}_1 = \tilde{F}_2$ so there is an intersection with $\tilde{\mathcal{F}}_6 = \mathcal{E}_5$. In the case $\tilde{y} = c_1$, the matrix $A(c_1)$ has the form $\begin{pmatrix} S & 0 \\ T & 0 \end{pmatrix}$, therefore

$$\tilde{A}(c_1) = \begin{pmatrix} (q\theta_1 - \theta_2)W & -qW \\ (q\theta_1 - \theta_2)Z & -qZ \end{pmatrix}$$

with $W = (q\kappa_1 - \kappa_2)qS + \beta T$, thus $\tilde{y} = c_1$ if $W = 0$, and then $\tilde{z}_1 = 0$, therefore the image does not intersect $\tilde{\mathcal{F}}_1 = \tilde{\mathcal{E}}_2$. In the case $y = c_i$, with $i = 1, 2, 3$, the matrix $A(c_i)$ has rank 1 so

$$A(c_i) = \begin{pmatrix} rG & G \\ rH & H \end{pmatrix}$$

for some constants H and G (independent of a), and

$$\tilde{A}(c_i) = \begin{pmatrix} ((q\theta_1 - \theta_2)r - \beta)((q\kappa_1 - \kappa_2)G - qH) & (qr + q\kappa_1 - \kappa_2)((q\kappa_1 - \kappa_2)G - qH) \\ (q\theta_1 - \theta_2)r - \beta)(\beta G + q\theta_1 - \theta_2) & (qr + q\kappa_1 - \kappa_2)(\beta G + q\theta_1 - \theta_2) \end{pmatrix}$$

The 12 coefficient vanishes only if $qr + q\kappa_1 - \kappa_2 = 0$, and then $\tilde{z}_2 = 0$, therefore the image intersects $\tilde{\mathcal{F}}_i$ if $i = 2$ but not $i = 3, 4$, and intersects $\tilde{\mathcal{E}}_1$. Finally for some value of a the equation $\tilde{y} = \infty$ holds and the image intersects $\tilde{\mathcal{F}}_7$. We conclude from these considerations that, in the Picard groups of the surfaces,

$$\phi(\mathcal{E}_2) = 2\tilde{\mathcal{E}}_0 - \tilde{\mathcal{E}}_1 - \tilde{\mathcal{E}}_3 - \tilde{\mathcal{E}}_5 - \tilde{\mathcal{E}}_6 - \tilde{\mathcal{E}}_8.$$

Similar tedious computations, which we omit, lead finally to the matrix induced by the map ϕ between the Picard groups, which is, from the basis $(\mathcal{E}_i)_{i=0,\dots,9}$ to $(\tilde{\mathcal{E}}_i)_{i=0,\dots,9}$ given by

$$\begin{pmatrix} 6 & 2 & 2 & 2 & 3 & 0 & 0 & 3 & 2 & 1 \\ -2 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 \\ -2 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 \\ -2 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & -1 & -1 & -1 & -2 & 0 & 0 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

One checks that this transformation preserves the components \mathcal{D}_i (cf (5.3))

$$\begin{aligned} \phi(\mathcal{D}_0) &= \tilde{\mathcal{D}}_2 & \phi(\mathcal{D}_2) &= \tilde{\mathcal{D}}_0 \\ \phi(\mathcal{D}_1) &= \tilde{\mathcal{D}}_3 & \phi(\mathcal{D}_3) &= \tilde{\mathcal{D}}_1 \end{aligned}$$

while its action on the orthogonal root system (5.4) is a translation in the affine Weyl group, namely

$$\begin{aligned} \phi(\alpha_0) &= \alpha_0, & \phi(\alpha_1) &= \alpha_1, & \phi(\alpha_2) &= \alpha_2, \\ \phi(\alpha_3) &= \alpha_3, & \phi(\alpha_4) &= \alpha_4 - \delta, & \phi(\alpha_5) &= \alpha_5 + \delta \end{aligned}$$

with

$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$$

This translation has the following reduced decomposition, where the w_i are the simple reflexions associated with the roots α_i , and σ is the automorphism of the Dynkin diagram such that $\sigma(\alpha_0) = \alpha_1, \sigma(\alpha_4) = \alpha_5$.

$$(5.5) \quad \phi = \sigma w_4 w_3 w_2 w_0 w_1 w_2 w_3 w_4$$

We summarize the results of this section in the following.

PROPOSITION 5.1. *The transformation ϕ induces, on the Picard group of the surfaces, a translation whose reduced decomposition is given by*

$$\phi = \sigma w_4 w_3 w_2 w_0 w_1 w_2 w_3 w_4$$

REMARK 5.2. The discrete Painlevé equation of [15] corresponds to the translation ψ given by

$$\begin{aligned} \psi(\alpha_i) &= \alpha_i, & i &= 0, 1, 4, 5 \\ \psi(\alpha_2) &= \alpha_2 - \delta, & \psi(\alpha_3) &= \alpha_3 + \delta \end{aligned}$$

The translations ϕ and ψ are not conjugate in the Weyl group.

5.4. As a check, we will obtain formulas (4.10), (4.12) using (5.5) and the explicit expressions for the birational transformations corresponding to the reflexions. After Sakai, we have the following formulas for the transformations corresponding to the roots

$$\begin{aligned} w_0 : \left(\begin{array}{cccc} a_8 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \end{array} ; x : y : z \right) \rightarrow \\ \left(\begin{array}{cccc} 1/a_1 & 1/a_8 & a_2 & a_3 \\ a_1 a_4 & a_5 a_1 & a_6 a_8 & a_7 a_8 \end{array} ; x(x - a_1 z) : y(x - z/a_8) : z(x - a_1 z) \right) \end{aligned}$$

w_1, w_4, w_5 are the transpositions of parameters

$$w_1 : (a_2, a_3); \quad w_4 : (a_4, a_5), \quad w_5 : (a_6, a_7)$$

Passing to $\mathbf{P}^1 \times \mathbf{P}^1$ with (5.2), w_2 and w_3 are

$$\begin{aligned} w_2 : \left(\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{array} ; f, g \right) \rightarrow \\ \left(\begin{array}{cccc} b_3 & b_2 & b_1 & b_4 \\ b_5 \frac{b_3}{b_1} & b_6 \frac{b_3}{b_1} & b_7 & b_8 \end{array} ; f \frac{g-b_3}{g-b_1}, g \right) \\ w_3 : \left(\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{array} ; f, g \right) \rightarrow \\ \left(\begin{array}{cccc} b_1 & b_2 & \frac{b_5}{b_7} b_3 & \frac{b_5}{b_7} b_4 \\ b_7 & b_6 & b_5 & b_8 \end{array} ; f, \frac{b_5}{b_7} g \frac{f-b_7}{f-b_5} \right) \end{aligned}$$

The automorphism σ is the inversion

$$\left(\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_8 & b_7 \end{array} ; f, g \right) \rightarrow \left(\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_8 & b_7 \end{array} ; b_7 b_8 / f, b_3 b_4 / g \right)$$

We can now compute ϕ induced by the translation (5.5). One gets

$$\begin{aligned} \bar{f} &= \frac{b_1 b_2 b_8}{b_5 f} \frac{(f(g - \frac{b_1 b_8}{b_5}) - b_8(g - b_1))(f(g - \frac{b_2 b_8}{b_5}) - b_8(g - b_2))}{(f(g - b_3) - b_8(g - \frac{b_3 b_5}{b_8}))(f(g - b_4) - b_8(g - \frac{b_4 b_5}{b_8}))} \\ \bar{g} &= \frac{b_5 b_7}{g} \frac{f - b_5}{f - b_8} \frac{f \frac{g \frac{f-b_8}{f-b_5} - b_3}{g \frac{f-b_8}{f-b_5} - b_1 \frac{b_8}{b_5}} \frac{g \frac{f-b_8}{f-b_5} - b_4}{g \frac{f-b_8}{f-b_5} - b_2 \frac{b_8}{b_5}} - \frac{b_3 b_4 b_5^2}{b_1 b_2 b_8}}{f \frac{g \frac{f-b_8}{f-b_5} - b_3}{g \frac{f-b_8}{f-b_5} - b_1 \frac{b_8}{b_5}} \frac{g \frac{f-b_8}{f-b_5} - b_4}{g \frac{f-b_8}{f-b_5} - b_2 \frac{b_8}{b_5}} - b_5} \end{aligned}$$

This gives (4.10), (4.12), using the correspondance (5.1) and after simplifying the expression for \bar{g} , noting that $f - b_5$ divides the denominator, while $f - b_8$ divide the numerator.

5.5. Finally the results of the preceding section allow one to derive relations for the Verblunsky coefficients. Indeed one has

$$\det A_n(z) = q^n V(z) W(z)$$

thus one can take $c_1 = \bar{a}/q$, $c_2 = 1/b$, $c_3 = \bar{b}/q$, $c_4 = 1/b$. From (3.21) one sees that the coordinate y corresponds to $\frac{\alpha_n(\bar{a}-\bar{b}q^n)}{\alpha_{n+1}(a-bq^{n+1})}$. According to equations (3.22), (3.24), one finds that $\kappa_1 = bq^{n+1}$, $\kappa_2 = aq$, $\theta_1 = \bar{b}q^n$, $\theta_2 = \bar{a}$. We conclude that these expressions satisfy a discrete Painlevé equation.

6. Recovering the differential equation

We shall now explain how to obtain a differential system as a limit of the discrete dynamical system given by (4.10), (4.12). For this we introduce a complex variable t , and consider the parameters $(t\kappa_1, \kappa_2, t\theta_1, \theta_2, c_1, c_2, c_3, c_4)$. Consider the map ϕ of Proposition (4.3) as a birational map $(y(tq), \xi(tq)) = \phi(y(t), \xi(t))$. Using the parametrisation

$$\begin{aligned} q &= 1 - \epsilon, \\ q\kappa_1 &= t(1 + \epsilon K_1), \quad \kappa_2 = 1 + \epsilon K_2 \\ q\theta_1 &= t(1 + \epsilon \Theta_1), \quad \theta_2 = 1 + \epsilon \Theta_2, \\ c_i &= 1 + \epsilon C_i, i = 1, \dots, 4 \\ y &= 1 + \epsilon u, \xi = v \end{aligned}$$

Letting ϵ go to zero gives a differential system in the variable t . We omit the details of the lengthy computation, and give the final result:

$$\begin{aligned} t(1-t) \frac{du}{dt} &= tv(u - C_3)(u - C_4) - t(t-1)v^{-1}(u - C_1)(u - C_2) \\ t(1-t) \frac{dv}{dt} &= v[2u + 2t(K_2 - \Theta_1) + C_1 + C_2 - K_1 - \Theta_1] + \\ &\quad 2u(t-1) + C_2 + C_1 - t(C_3 + C_4) + \\ &\quad v^{-1}[2ut - t(1 + K_2) + 2(1 + \Theta_2) - C_3 - C_4] \end{aligned}$$

Finally, we come back to the original problem about the Fourier transform and the scattering problem for phase shifts which are quotients of Gamma functions, like (1.2). First note that in the case of the weight w given by (3.3), one has, from (3.22), (3.24), and (3.4), (3.5),

$$\begin{aligned} \kappa_1 &= bq, \theta_1 = \bar{b}, \kappa_2 = aq, \theta_2 = \bar{a} \\ c_1 &= \bar{a}/q, c_2 = \bar{b}/q, c_3 = 1/b, c_4 = 1/a \end{aligned}$$

In order to recover (1.2) as a limit of the phase shift associated with the weight w we make the substitution $q \rightarrow 1 - \epsilon$, $a \rightarrow 1 + \epsilon a$, $b \rightarrow 1 + \epsilon b$ where a and b are real numbers, before letting $\epsilon \rightarrow 0$. The final system is then

$$\begin{aligned}
 t(1-t)\frac{du}{dt} &= tv(u+b)(u+a) - t(t-1)v^{-1}(u-a+1)(u-b+1) \\
 t(1-t)\frac{dv}{dt} &= v[2u+2t(a-b-1)+a-b+3] + \\
 &\quad 2u(t-1)+a+b+2+t(a+b)+ \\
 &\quad v^{-1}[(2u-a)t+2+3a+b]
 \end{aligned}$$

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