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# Minimal graphs for 2-factor extension

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## Abstract

Let  $G = (V, E)$  be a simple loopless finite undirected graph. We say that  $G$  is *(2-factor) expandable* if for any non-edge  $uv$ ,  $G + uv$  has a 2-factor  $F$  that contains  $uv$ . We are interested in the following: Given a positive integer  $n = |V|$ , what is the minimum cardinality of  $E$  such that there exists  $G = (V, E)$  which is 2-factor expandable? This minimum number is denoted by  $Exp_2(n)$ . We give an explicit formula for  $Exp_2(n)$  and provide 2-factor expandable graphs of minimum size  $Exp_2(n)$ .

**Keywords:** 2-factor,

## 1 Introduction

Matching extension has been studied for a long time by many authors, see for instance [1], [7] and [9]. In [4] a different kind of matching extension has been explored: the goal was to characterize graphs  $G = (V, E)$  on  $n$  vertices with a minimum number of edges such that for every pair  $x, y$  of non adjacent vertices, there exists an (almost) perfect matching in  $G_{xy} = (V, E \cup xy)$ . This work was motivated by a reliability problem in which edges subject to breakdowns could be reinforced so that to any pair  $xy$  of non adjacent vertices, one could associate reinforced edges to obtain an (almost) perfect matching  $M \cup xy$ .

Here we intend to concentrate on the problem in which (almost) perfect matchings (1-factors) are replaced by 2-factors. The precise formulation will be given below. In [8] a different 2-factor extension problem has been studied.

We will consider a simple finite graph  $G = (V, E)$  with  $n$  vertices and  $m$  edges. A pair  $u, v$  of vertices is a *non-edge* if  $uv \notin E$ . For any subset  $X \subseteq V$  the subgraph induced by  $X$  is denoted by  $G[X]$ . We write  $G - X = G[V \setminus X]$  and  $G - v$  for  $G - \{v\}$ .  $N(v)$  is the set of neighbors of a vertex  $v$ ;  $\delta(v) = |N(v)|$  is the degree of  $v$ ; a *p-vertex* is a vertex of degree  $p$ ; if  $\delta(v) = n - 1$  then  $v$  is *universal*. The closed

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neighborhood of  $v$  is  $N[v] = N(v) \cup \{v\}$ .  $\delta(G)$  (resp.  $\Delta(G)$ ) is the minimum (resp. maximum) degree of the vertices in  $G$ . An induced path with  $p$  edges is called a  $p$ -path. By  $d(u, v)$  we denote the *distance* between  $u$  and  $v$ , i.e., the length of a shortest path (number of edges) between  $u$  and  $v$  in  $G$ .

A subset  $F \subseteq E$  is a *2-factor* if every vertex  $v$  has exactly two edges in  $F$  which are incident in  $v$ . Equivalently  $F$  is a collection of vertex-disjoint cycles covering all vertices.  $C_k$  (resp.  $K_k$ ) is the cycle (resp. complete graph) on  $k$  vertices.  $K_4$  with an edge missing is called a *diamond*.

For all definitions related to graphs, see [3].

We intend to determine, for any integer  $n \geq 3$ , a graph  $G = (V, E)$  with  $n$  vertices and a minimum number of edges such that for every pair  $x, y$  of non adjacent vertices of  $G$  it is always possible to include the non-edge  $xy$  into a 2-factor, i.e., the graph  $G_{xy} = (V, E \cup \{xy\})$  has a 2-factor  $F, F \supset xy$ . In such a case we shall say that  $xy$  has been *extended* (to a 2-factor).

**Definition 1.1** *A graph  $G = (V, E)$  is 2-factor expandable (or shortly expandable) if every non-edge  $xy$  can be extended.*

**Definition 1.2** *An expandable graph  $G = (V, E)$  with  $|V| = n$  and with a minimum number of edges is a minimum expandable graph ( $meg(n)$ ). The size  $|E|$  of its edge set is denoted by  $Exp_2(n)$ .*

Our problem may be related to the idea of edge-criticality. This concept has been considered in various contexts (see for instance [2], [3], [5]). We may call a graph *2-factor edge-critical* if it contains no 2-factor but for any non-edge  $uv$  there is a 2-factor in  $G_{uv}$  (which necessarily uses  $uv$ ). Our  $meg(n)$  are different : they may contain a 2-factor and they have a minimum number of edges.

We state our main result which will be proved in the following sections:

**Proposition 1.1** *The minimum size of a 2-factor expandable graph is:*

- $Exp_2(3) = 2, Exp_2(4) = 4, Exp_2(5) = 6, Exp_2(6) = 9, Exp_2(7) = 10, Exp_2(8) = 11, Exp_2(9) = 12;$
- $Exp_2(n) = \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil, n \geq 10.$

The paper is organized as follows. In Section 2 some elementary properties of expandable graphs will be stated for later use. Section 3 will be dedicated to the presentation of  $meg(n)$  for  $3 \leq n \leq 9$ . In Section 4 a lowerbound for  $Exp_2(n)$  will be established for  $n \geq 10$ , while it will be shown in Section 5 that it is best possible. Variations of the construction for  $n = 8p$  will be presented in Section 5.3 to handle the case  $n \not\equiv 0 \pmod{8}, n \geq 14$ . Finally constructions will be given for  $10 \leq n \leq 13$  in Section 5.4. Some conclusions and suggestions for further research are presented in Section 6.

## 2 Properties of expandable graphs

We shall state some basic properties of minimal expandable graphs which will be used to compute  $Exp_2(n)$ .

**Fact 2.1** *If  $G = (V, E)$  is not expandable, any partial graph  $G' = (V, E')$ , with  $E' \subset E$ , is not expandable.*

**Property 2.1** *If  $G = (V, E)$  is expandable then  $G$  is connected.*

**Proof:** If  $u$  and  $v$  are in two distinct components then clearly  $uv$  cannot be extended.  $\square$

**Property 2.2** *If  $n \geq 5$  and  $\delta(G) = 1$  then  $Exp_2(n) \geq \frac{3}{2}n$ .*

**Proof:** Let  $u$  be a 1-vertex of  $G$ . If  $G$  is expandable then  $G - u$  induces a clique. Since  $n \geq 5$  we have  $m \geq \frac{3}{2}n$ .  $\square$

**Property 2.3** *Let  $G = (V, E)$  be expandable,  $n \geq 5$ . If  $v \in V$  is universal then  $m \geq \frac{3}{2}(n - 1)$ .*

**Proof:** From Property 2.2 if there is a universal vertex then  $\sum_{u \in V} \delta(u) \geq n - 1 + 2(n - 1) = 3(n - 1)$ .  $\square$

**Property 2.4** *Let  $G = (V, E)$  be expandable. Let  $v$  be a 2-vertex of  $G$  with  $N(v) = \{a, b\}$  and  $ab \notin E$ . If  $c \in N(a) \cap N(b)$ ,  $c \neq v$ , then  $\delta(c) \geq 4$ .*

**Proof:** Consider any extension of  $ab$ : the triangle  $(a, b, v)$  is in the 2-factor. Since  $c$  is necessarily covered by another cycle, we have  $\delta(c) \geq 4$ .  $\square$

**Property 2.5** *Let  $G = (V, E)$  be expandable and  $u, v$  be two 2-vertices. If  $d(u, v) = 4$  with a 4-path  $uu'wv'$  from  $u$  to  $v$  then  $\delta(w) > 3$ .*

**Proof:**  $d(u, v) = 4$  implies that  $u'v' \notin E$ . Now if  $\delta(w) \leq 3$  the non-edge  $u'v'$  cannot be extended.  $\square$

**Property 2.6** *Let  $n \geq 7$ . If  $u, v$  are two 2-vertices and  $N(u) \cap N(v) \neq \emptyset$  then  $Exp_2(n) \geq \frac{3}{2}n$ .*

**Proof:** Let  $G$  be an expandable graph with  $m$  edges. If  $\delta(G) = 1$ , from Property 2.2 we have  $m \geq \frac{3}{2}n$ . Now let  $\delta(G) \geq 2$ . If  $w \in N(u) \cap N(v)$  then  $w$  is universal. If there are at most two 2-vertices we have  $\sum_{u \in V} \delta(u) \geq (n - 1) + 4 + 3(n - 3) = 4n - 6$  and  $m \geq \frac{3}{2}n$  for  $n \geq 7$ . If  $a, b, c$  are three 2-vertices then any non-edge  $xy$  with  $x, y \neq a, b, c$  cannot be extended so  $\delta(v) \geq n - 4$  for all  $v \neq a, b, c$ . We have  $\sum_{u \in V} \delta(u) \geq (n - 1) + 6 + (n - 4)(n - 4) = n^2 - 7n + 21$  and  $m \geq \frac{3}{2}n$  for  $n \geq 7$ .  $\square$

### 3 Meg(n) for $3 \leq n \leq 9$

We will compute  $Exp_2(n)$  for small values of  $n$ .

- $Exp_2(3) = 2$ : Trivially  $P_3$  the path on three vertices (Figure 1 left) is a  $meg(3)$ .

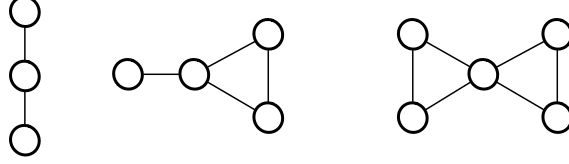


Figure 1:  $P_3$ , the paw, the butterfly.

- $Exp_2(4) = 4$ : The paw (see Figure 1 center) is expandable. If  $G = (V, E)$  is a  $meg(4)$  with  $|E| < 4$  then from Property 2.1  $G$  is a tree. So  $G$  is either  $P_4$  or the claw. None of those is expandable.
- $Exp_2(5) = 6$ : The butterfly (see Figure 1 right) is expandable. Let  $G = (V, E)$  be a  $meg(5)$  with  $|E| \leq 5$ . Then  $|E| < \frac{3}{2}n$  and from Property 2.2,  $\delta(G) \geq 2$ , so  $G$  is the cycle  $C_5$  on five vertices which is not expandable.
- $Exp_2(6) = 9$ : The graph  $G_6$  (see Figure 2 left) is expandable. From Fact 2.1 it is sufficient to consider a  $meg(6)$ ,  $G = (V, E)$ , with  $|E| = 8$ . If there is a 1-vertex  $v$  then  $G - v$  is  $K_5$  and  $|E| = 11$ , a contradiction. So  $\delta(G) \geq 2$ . Let

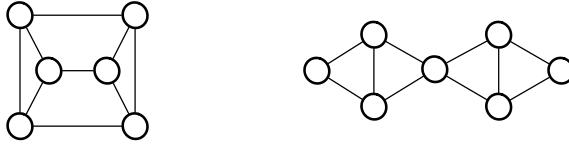


Figure 2:  $G_6$  and  $G_7$  two minimal expandable graphs with 6 and 7 vertices.

$n_2$  be the number of 2-vertices. We have  $2 \leq n_2 \leq 4$ .

Let  $n_2 = 2$ . We have  $\delta(v_i) = 2, 1 \leq i \leq 2$  and  $\delta(v_i) = 3, 3 \leq i \leq 6$ . If  $v_i \in N(v_1) \cap N(v_2)$  then  $v_i$  is universal which is impossible. Thus w.l.o.g.  $N(v_1) = \{v_3, v_4\}, N(v_2) = \{v_5, v_6\}$ . From Property 2.4 if  $v_3v_4 \notin E$  then  $\delta(v_5) > 3$ , a contradiction. So w.l.o.g.  $v_3v_4, v_5v_6, v_3v_5, v_4v_6 \in E$ . But  $v_3v_6 \notin E$  cannot be extended.

Let  $n_2 = 3$ . W.l.o.g.  $\delta(v_1) = \delta(v_2) = \delta(v_3) = 2, \delta(v_4) = \delta(v_5) = 3, \delta(v_6) = 4$ . We have  $|N(v_6) \cap \{v_1, v_2, v_3\}| \geq 2$ , so  $v_6$  is universal a contradiction.

Let  $n_2 = 4$ . W.l.o.g.  $\delta(v_1) = \delta(v_2) = \delta(v_3) = \delta(v_4) = 2$ . If  $\delta(v_5) = \delta(v_6) = 4$  then  $|N(v_6) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$ , so  $v_6$  is universal a contradiction. So we have  $d(v_5) = 3, d(v_6) = 5$  but  $|N(v_5) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$  and  $v_5$  is universal a contradiction.

- $Exp_2(7) = 10$ : The graph  $G_7$  (see Figure 2 right) is expandable. Let  $G = (V, E)$  be a  $meg(7)$  with  $|E| < 10$ . Then  $|E| < \frac{3}{2}n$ : from Property 2.2 we have  $\delta(G) \geq 2$  and from Property 2.6 two 2-vertices have no common neighbor. Each 2-vertex has its proper neighbor of degree at least three, thus there are exactly three 2-vertices, says  $v_1, v_2, v_3$ . Thus there are exactly four 3-vertices  $v_4, v_5, v_6, v_7$  and  $|E| = 9$ . Using Property 2.6 again, w.l.o.g.  $v_1v_2, v_1v_4, v_2v_5, v_3v_6, v_3v_7 \in E$ . If  $v_6v_7 \notin E$  then  $v_4 \in N(v_6) \cap N(v_7)$  and from Property 2.4  $\delta(v_4) > 3$ , a contradiction. So w.l.o.g.  $v_4v_6, v_5v_7, v_4v_5, v_6v_7 \in E$  but  $v_4v_7$  cannot be extended.
- $Exp_2(8) = 11$ :

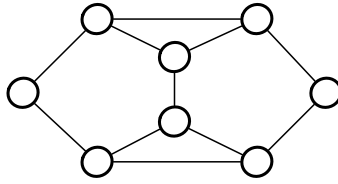


Figure 3: The graph  $G_8$  is a  $meg(8)$ .

One can check that  $G_8$  (see Figure 3) is expandable. Let  $G = (V, E)$  be a  $meg(8)$  with  $|E| < 11$ . From Properties 2.2 and 2.6  $\delta(G) \geq 2$  and each 2-vertex has its proper neighbor of degree at least three. It follows that there are at most four 2-vertices. Since  $|E| \leq 10$  there are exactly four 2-vertices, says  $v_1, v_2, v_3, v_4$ , and four 3-vertices  $v_5, v_6, v_7, v_8$ . W.l.o.g.  $v_1v_2, v_3v_4, v_1v_5, v_2v_6, v_3v_7, v_4v_8 \in E$ . From Property 2.4  $v_5v_6 \notin E$ . So  $v_5v_7, v_5v_8, v_6v_7, v_6v_8 \in E$ , but  $v_5v_6$  cannot be extended.

- $Exp_2(9) = 12$ : One can check that  $G_9$  (see Figure 4) is expandable.

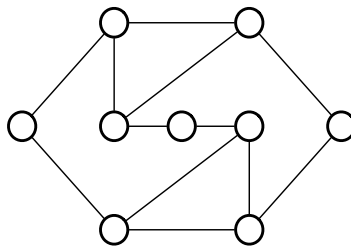


Figure 4: The graph  $G_9$  is a  $meg(9)$ .

Let  $G = (V, E), |E| < 12$ , be a  $meg(9)$ . Properties 2.2 and 2.6 imply that  $\delta(G) \geq 2$  and each 2-vertex has its proper neighbor of degree at least three. Then there are at most four 2-vertices. It follows that  $\sum_{v \in V} \delta(v) \geq 23$  and  $|E| \geq 12$ . So  $Exp_2(9) = 12$ , a contradiction.

## 4 A lower bound for $Exp_2(n)$ , $n \geq 10$

We will concentrate in showing a lower bound of  $Exp_2(n)$  for  $n \geq 10$ . It will be shown in the next section that it is best possible.

**Lemma 4.1** *If  $G = (V, E)$  is a meg( $n$ ),  $n \geq 10$ , then  $m \geq \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil$ .*

**Proof:** From Property 2.1  $G$  is connected. If there is a 1-vertex then from Property 2.2,  $m \geq \frac{3}{2}n$ . If two 2-vertices have a common neighbor then from Property 2.6,  $m \geq \frac{3}{2}n$ .

So from now on we examine the case where  $\delta(G) \geq 2$  and for any two 2-vertices  $u, v$  we have  $N_G(u) \cap N_G(v) = \emptyset$ .

Let  $V_0, V_1, V_2$  be a partition of  $V$  with  $V_0 = \{v \in V : \delta(v) = 2\}$ ,  $V_1 = \{v \in V, \delta(v) \geq 3, \exists u \in V_0, uv \in E\}$ ,  $V_2 = V - (V_0 \cup V_1)$ .

Each connected component of  $G[V_0]$  is isomorphic either to  $K_1$  or to  $K_2$  (in this second case its two vertices are denoted by  $v$  and  $\bar{v}$ ). Every  $v \in V_1$  is such that:  $|N(v) \cap V_0| = 1$ .

If  $|V_0| \leq 1$  then  $m \geq \lceil \frac{3n-1}{2} \rceil$ . So from now on we have  $|V_0| \geq 2$  and  $V_1 \neq \emptyset$ .

Let  $V_0^1, V_0^2$  be the partition of  $V_0$  such that  $v \in V_0^1$  if and only if  $N(v) \cap V_0 = \emptyset$ ,  $V_0^2 = V_0 \setminus V_0^1$ . Define  $X_0 = \{v \in V_0^1 : N(v) = \{x, y\}, xy \in E\}$  and  $Y_0 = V_0^1 \setminus X_0 = \{v \in V_0^1 : N(v) = \{x, y\}, xy \notin E\}$ . For  $v \in V_0^2$  let  $N(v) = \{\bar{v}, x\}$  with  $\bar{v} \in V_0^2, x \in V_1$ ,  $N(\bar{v}) = \{v, y\}$  with  $v \in V_0^2, y \in V_1$ ; let  $Z_0 = \{v, \bar{v} \mid xy \in E\}$  and  $W_0 = V_0^2 \setminus Z_0$ .

For such  $v, \bar{v} \in Z_0$ , from Property 2.4 we have  $\delta(x), \delta(y) \geq 4$ .

We will use a discharging procedure: a weight  $w(v)$  is assigned to every vertex  $v \in V$ . At each step of the process the  $w(v)$  are changed to  $w'(v)$  in such a way that  $\sum_{v \in V} w(v) = \sum_{v \in V} w'(v)$ . At the beginning we take  $w(v) = \delta(v)$  for every vertex  $v$ . When the procedure will be completed we will have  $\sum_{v \in V} w'(v) = 2m \geq 3n - \lfloor \frac{n}{4} \rfloor$ .

The procedure consists of two steps. In the first one vertices  $r$  with  $\delta(r) \geq 4$  and possibly some 3-vertices  $u$  with a 3-path  $usr$  such that  $s$  is a 2-vertex transfer part of their charge to all 2-vertices  $v$  with  $d(v, r) \leq 2$ . In the second step 3-vertices  $u$  transfer part of their charge to all 2-vertices  $v$  with  $d(v, u) \leq 2$ .

Moreover the weight of any vertex  $v \in V_0$  will be modified exactly once; initially the vertex  $v$  is *active*; after increasing its charge,  $v$  will be considered as *neutral* for all further discharging operations.

We proceed as follows:

- As long as there is  $r$  with  $\delta(r) \geq 4$  such that there exists an active  $v$  with  $d(v, r) = 1$ : we take  $w'(v) = w(v) + \frac{3}{4} = \frac{11}{4}$  and  $w'(r) = w(r) - \frac{3}{4} \geq \frac{11}{4}$ . Recall that no two vertices of  $V_0$  share a same  $r$ .

From now on, each active vertex has all its neighbors of degree 2 or 3.

- For each  $r$  with  $\delta(r) \geq 4$  such that there exists an active  $v$  with  $d(v, r) = 2$ : note that  $v \notin Z_0$  otherwise  $\delta(x) \geq 4$ , a contradiction. We now consider the neighborhood of  $r$ .

–  $\delta(s) \geq 3, \forall s \in N(r)$ : From our assumption there are at most  $\delta(r)$  vertices  $u_i \in V_0$  such that  $d(u_i, r) = 2$ . Let  $u_1, u_2, \dots, u_k$  with  $v = u_1$  be the vertices which are active among those. Note that in the case where  $u_i \in V_0^1$  there is either one 2-path or two disjoint 2-paths between  $r$  and  $u_i$ . If there is one path from  $r$  to  $u_i$  let  $x_i$  be its internal vertex; if there are two paths, let  $x_i, y_i$  be their internal vertices. For  $u_i \in V_0^2$  there is exactly one 2-path from  $u_i$  to  $r$ , let  $x_i$  denotes its internal vertex. Let  $\lambda$  be the number of vertices  $x_i$  and  $y_i$ , i.e., the number of 2-paths between  $r$  and the vertices  $u_i, i = 1, \dots, k$ . We take  $w'(r) = w(r) - \frac{\lambda}{4}, w'(x_i) = w(x_i) - \frac{1}{4}, w'(y_i) = w(y_i) - \frac{1}{4}, 1 \leq i \leq k$ . In the case where  $u_i \in V_0^1$  and there is exactly one 2-path from  $u_i$  to  $r$  we take  $w'(y_i) = w(y_i) - \frac{1}{4}$ . Note that since  $\lambda \leq \delta(r)$  we have  $w'(r) \geq 3$  and, from Property 2.6,  $x_i$  and  $y_i$  are discharged at most once so we have  $w'(x_i) \geq \frac{11}{4}$  and  $w'(y_i) \geq \frac{11}{4}$ . In the case where  $u_i \in V_0^2$ , let  $z_i$  be a neighbor of  $x_i$  distinct from  $r$  and  $u_i$ , we take  $w'(z_i) = w(z_i) - \frac{1}{4}$ ; note that  $z_i \notin V_1$  else, let  $s \in V_0 \cap N(z_i), sz_i x_i u_i \bar{u}_i$  is a 4-path and from Property 2.5  $\delta(x_i) \geq 4$ , a contradiction; if there is another active vertex  $u_j \in V_0^2$  with a 2-path  $u_j x_j z_i$ , there is a 4-path between  $u_i$  and  $u_j$  so  $\delta(z_i) \geq 4$ ; since  $z_i \notin V_1$ ,  $z_i$  transfers at most  $\frac{1}{4}\delta(z_i)$ ; hence, when the procedure ends, we have  $w'(z_i) \geq \frac{11}{4}$ . Finally, we set  $w'(u_i) = w(u_i) + 1 \geq \frac{11}{4}$  if there are two 2-paths from  $u_i$  to  $r$ , or  $w'(u_i) = w(u_i) + \frac{3}{4} = \frac{11}{4}$  else, for  $1 \leq i \leq k$ .

–  $\exists s \in N(r) \cap V_0$  ( $s$  is neutral):

From our assumption we have  $(N(r) - \{s\}) \cap V_0 = \emptyset$ . If  $v \in W_0$  then there is a 4-path from  $\bar{v}$  to  $s$ ; by Property 2.5 the neighbor  $x$  of  $v$  has degree at least 4, a contradiction. Hence  $v \in V_0^1$ .

Consider the case where there are two 2-paths  $vxr, vyr$ : since  $s$  is a 2-vertex, to extend  $vr$  any 2-factor must contain  $vr, rs$  and needs that at least one of  $x, y$  has a degree at least 4, a contradiction.

Thus there is exactly one 2-path  $vxr$ . Let  $u_1, \dots, u_k$  with  $v = u_1$  be the active vertices such that  $d(u_i, r) = 2$ . As for  $u_1$ , we have  $u_2, \dots, u_k \in V_0^1$  with exactly one 2-path  $u_i x_i r$ ; let  $y_i \neq x_i$  denote the second neighbor of  $u_i, 1 \leq i \leq k$ . There are three cases to consider.

- \*  $s \in V_0^1$ : we have  $k \leq \delta(r) - 1$ . Let  $y_s \neq r$  be the second neighbor of  $s$ . We suppose first that  $s$  has been charged by  $r$  in the first step of the procedure. Notice that either  $\delta(y_s) = 3$  and  $y_s$  was not used before, or  $\delta(y_s) \geq 4$  and its charge is at least  $\delta(y_s) - \frac{1}{4}(\delta(y_s) - 1) \geq 3$ . We take  $w'(r) = w(r) - \frac{(k-1)}{4}, w'(y_s) = w(y_s) - \frac{1}{4}$ . Second assume that  $s$  has been charged by  $y_s$ , so the charge of  $r$  has not been modified: we take  $w'(r) = w(r) - \frac{k}{4}$ . Then in both cases  $w'(x_i) = w(x_i) - \frac{1}{4}, w'(y_i) = w(y_i) - \frac{1}{4}, w'(u_i) = w(u_i) + \frac{3}{4}, 1 \leq i \leq k$ . Notice that  $x_i$  and  $y_i$  have not been discharged before because  $u_i$  is the only 2-vertex in their neighborhood. Hence all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$ .



\*  $s \in Z_0$ : let  $N(\bar{s}) = \{s, \bar{x}\}$  (recall that  $\bar{x}r \in E$  and  $\delta(\bar{x}) \geq 4$ ; so  $\bar{s}$  is neutral). Since  $\delta(x_1) = \delta(y_1) = 3$  we have  $x_1, y_1 \neq \bar{x}$ . From our assumption it does not exist  $v' \in V_0 \cap N(\bar{x}), v' \neq \bar{s}$ . So  $k \leq \delta(r) - 2$ . We take  $w'(r) = w(r) - \frac{k}{4}$ ,  $w'(x_i) = w(x_i) - \frac{1}{4}$ ,  $w'(y_i) = w(y_i) - \frac{1}{4}$ ,  $1 \leq i \leq k$ , and  $w'(u_i) = w(u_i) + \frac{3}{4}$ ,  $1 \leq i \leq k$ . Hence all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$ .

\*  $s \in W_0$ : let  $N(\bar{s}) = \{s, \bar{x}\}$  (recall that  $y_1r \notin E$ ). If  $u_1 \in X_0$  then  $ry_1$  cannot be extended, indeed  $rx_1$  and  $rs$  must be in the 2-factor. So  $u_1 \in Y_0$ .

Let  $N(x_1) = \{u_1, r, z\}$ . Let  $z' \in N(z), z' \neq x_1$ ; assume  $z' \in V_0$ . If  $rz \notin E$  then  $srx_1zz'$  is a 4-path and from Property 2.5  $\delta(x_1) \geq 4$ , a contradiction. Now we consider the case  $rz \in E$ :

First  $\delta(r) = 4$ . Assume that  $u_1, u_2, u_3$  with  $u_1 = v, u_2 = z', u_3 \neq \bar{s}$  and  $d(r, u_3) = 2$  are three active 2-vertices. Let  $rx_3u_3$  be the 2-path from  $r$  to  $u_3$ . Since  $u_3$  is active  $x_3$  is a 3-vertex. At least one of  $x_3y_1, x_3y_2$  is a non-edge. W.l.o.g.  $y_1x_3 \notin E$ . This non edge cannot be extended: If a 2-factor  $F, y_1x_3 \in F$ , exists then  $x_3u_3 \in F$ , so  $rx_3 \notin F$ ,  $sr, x_1u_1, zz' \in F$ . W.l.o.g  $rz \in F, rx_1 \notin F$ , so  $x_1z \notin F$ , a contradiction.  $u_3$  cannot be active. We do as follows:  $w'(r) = w(r) - \frac{1}{2} = (4 - \frac{3}{4} - \frac{1}{2}) = \frac{11}{4}$ ,  $w'(x_1) = w(x_1) - \frac{1}{4}$ ,  $w'(y_1) = w(y_1) - \frac{1}{4}$ ,  $w'(x_2 = z) = w(x_2) - \frac{1}{4}$ ,  $w'(y_2) = w(y_2) - \frac{1}{4}$ , and  $w'(u_1) = w(u_1) + \frac{3}{4}$ ,  $w'(u_2 = z') = w(u_2) + \frac{3}{4}$ .

Second  $\delta(r) \geq 5$ . Let  $k$  be the number of active 2-vertices distinct from  $\bar{s}$  at distance 2 of  $r$ . We do as follows:  $w'(r) = w(r) - \frac{k}{4}$ ,  $w'(x_i) = w(x_i) - \frac{1}{4}$ ,  $w'(y_i) = w(y_i) - \frac{1}{4}$ ,  $w'(u_i) = w(u_i) + \frac{3}{4}$ ,  $1 \leq i \leq k$ .

Now any neighbor of  $z$  has a degree at least 3. In the case  $\delta(z) = 3$  there does not exist a 2-vertex  $w, w \neq u_1, d(z, w) = 2$  else there is a 4-path between  $u_1$  and  $w$  and  $z$  has degree greater than 3, a contradiction. So  $z$  has not been used before in the discharging procedure. We do as follows:  $w'(r) = w(r) - \frac{(k-1)}{4}$ ,  $w'(z) = w(z) - \frac{1}{4}$ ,  $w'(x_i) = w(x_i) - \frac{1}{4}$ ,  $w'(y_i) = w(y_i) - \frac{1}{4}$ ,  $w'(u_i) = w(u_i) + \frac{3}{4}$ ,  $1 \leq i \leq k$ . Since  $k \leq \delta(r) - 1$  all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$ .

In the case  $\delta(z) > 3$ , since all the neighbors of  $z$  have a degree at least 3, the case has been treated before and  $v$  is neutral: a contradiction.

From now on, any active  $v \in V_0$  is such that  $\forall r$  with  $\delta(r) \geq 4$ ,  $d(v, r) \geq 3$ . It follows that  $v \notin Z_0$ .

- For each active  $v \in V_0$  such that  $d(v, u) \geq 4$  for any  $u \in V_0 \setminus N[v]$

If  $d(v, u) = 4$ , there is a 4-path  $vxxz'u$ , then from Property 2.5,  $\delta(z) \geq 4$ , a contradiction. So  $d(v, u) \geq 5$  for any  $u \in V_0 \setminus N[v]$ . We have two cases:

- $v \in V_0^1$ . Remark that from the discharging procedure the charge of any 3-vertex at distance at most two of  $v$  has not been decreased before since  $d(v, v') > 3$  for any 2-vertex  $v'$ . Let  $N(v) = \{x, y\}$  and  $z \in N(x) \cup N(y)$ ,  $z \neq v, x, y$ . We take  $w'(v) = w(v) + \frac{3}{4}$ ,  $w'(x) = w(x) - \frac{1}{4}$ ,  $w'(y) = w(y) - \frac{1}{4}$  and  $w'(z) = w(z) - \frac{1}{4}$ . Hence  $w'(v) = \frac{11}{4}$  and all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$ .

–  $v \in W_0$ . Let  $N(x) = \{v, z_1, z_2\}$  and  $N(y) = \{\bar{v}, z_3, z_4\}$ . If  $\{z_1, z_2\} \cap \{z_3, z_4\} \neq \emptyset$  we can suppose that  $z_1 = z_3$ . Then  $xy$  cannot be extended since  $\delta(z_1) = 3$ .

Now  $\{z_1, z_2\} \cap \{z_3, z_4\} = \emptyset$ . We take  $w'(v) = w(v) + \frac{3}{4}$ ,  $w'(x) = w(x) - \frac{1}{4}$  and  $w'(z_i) = w(z_i) - \frac{1}{4}$ ,  $1 \leq i \leq 2$ . In the case where  $\bar{v}$  is active we also take  $w'(\bar{v}) = w(\bar{v}) + \frac{3}{4}$ ,  $w'(y) = w(y) - \frac{1}{4}$  and  $w'(z_i) = w(z_i) - \frac{1}{4}$ ,  $3 \leq i \leq 4$ . Hence  $w'(v) = \frac{11}{4}$ ,  $w'(\bar{v}) \geq \frac{11}{4}$  and all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$  since for any 2-vertex  $s$ ,  $s \neq v, \bar{v}$ , we have  $d(s, z_i) \geq 3$ ,  $i = 1, \dots, 4$ .

Notice that until now, the only 3-vertices which have been discharged are at distance 1 or 2 of a 2-vertex.

- For each remaining active vertex  $v \in V_0$   $v$  is such that there is  $u \in V_0$  with  $d(v, u) \leq 3$ .

From Property 2.6, the case  $d(v, u) = 2$  cannot occur. In the case where  $d(v, u) = 1$  we have  $v \in W_0$  so  $u = \bar{v}$ . There is no other vertex  $u'$  in  $V_0$  with  $d(v, u') = 3$ , else either  $d(\bar{v}, u') = 2$ , which is forbidden, or there is a 4-path  $\bar{v}vxx'u'$  and  $\delta(x) \geq 4$  from Property 2.5, a contradiction. So for any  $u' \in V_0$ ,  $u' \neq v, \bar{v}$ ,  $d(v, u') \geq 4$ ,  $d(\bar{v}, u') \geq 4$ : this case has been handled before.

From now on each active  $v \in V_0$  is such that  $v \in V_0^1$  and there is  $u \in V_0$  with  $d(v, u) = 3$ . With the same arguments as for  $v$ ,  $u \in V_0^1$ . Let  $N(v) = \{x, y\}$ ,  $N(u) = \{x', y'\}$  and  $yx' \in E$  (recall that  $\delta(x) = \delta(y) = \delta(x') = \delta(y') = 3$ ).

We have  $xx' \notin E$  (resp.  $yy' \notin E$ ) else  $vx'$  (resp.  $yu$ ) cannot be extended. Moreover  $xy \notin E$  (resp.  $x'y' \notin E$ ) else  $xx'$  (resp.  $yy'$ ) cannot be extended. So  $v, u \in Y_0$  and  $xx', yy' \notin E$ .

If there is  $z \in N(x) \cap N(y)$ ,  $z \neq v$ , then  $\delta(z) \geq 4$  otherwise  $xy$  cannot be extended: a contradiction since  $d(v, z) = 2$ . With a similar argument for  $u$ , we have:

**Fact 4.1**  $N(x) \cap N(y) = \{v\}$  and  $N(x') \cap N(y') = \{u\}$ .

Let  $N(y) = \{v, x', z\}$  and  $N(x) = \{v, x'', y''\}$ . From Fact 4.1  $xz, yx'', yy'' \notin E$  and  $z \neq y', x'', y''$ . Assume there is a 2-vertex  $w$  with  $N(w) = \{z, w'\}$ : if  $x'z \notin E$  then there is a 4-path  $wzyx'u$  and  $\delta(y) \geq 4$ , a contradiction; if  $x'z \in E$  then, either one of the three edges  $xy', xw', w'y'$  is not in  $E$  and this edge cannot be extended, or  $G$  is not connected: indeed  $n \geq 10$ ,  $\delta(x) = \delta(w') = \delta(y') = 3$  and  $G'$  the subgraph induced by the nine vertices  $u, v, w, x, y, z, w', x', y'$  is isomorphic to  $G_9$  (see Figure 4) and not connected to the rest of  $G$ , a contradiction. Thus there is no 2-vertex at distance 1 of  $z$ .

If there is a 2-vertex  $w'$ ,  $w' \neq v, u$ , with  $d(w', z) = 2$  then there is a 4-path  $w'wzyv$  and  $\delta(z) \geq 4$ , a contradiction. Thus there is no 2-vertex  $w$ ,  $w \neq u, v$ , with  $d(w, z) \leq 2$ , so  $z$  has not been discharged before.

Remember that any 3-vertex with an active vertex in the neighborhood has not been discharged. Thus  $w(x) = w(y) = w(z) = 3$ . We take  $w'(v) = w(v) + \frac{3}{4}$ ,  $w'(x) = w(x) - \frac{1}{4}$ ,  $w'(y) = w(y) - \frac{1}{4}$  and  $w'(z) = w(z) - \frac{1}{4}$ . This discharging is made for all the active vertices.

For each active vertex  $v$  we have considered its neighborhood  $\{x, y\}$  plus a vertex  $z$  at distance 2 of  $v$  which has no other 2-vertex at distance 1 or 2. Thus the weight of each vertex  $x, y, z$  associated to an active vertex  $v$  is decreased exactly once. Hence all vertices  $q$  involved here satisfy  $w'(q) \geq \frac{11}{4}$ .

The discharging procedure is done for all active vertices and  $\forall v \in V$ ,  $w'(v) \geq \frac{11}{4}$ . So  $2m = \sum_{i=1}^n w(v) = \sum_{i=1}^n w'(v) \geq \frac{11}{4}n$ . Thus  $m \geq \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil$ .  $\square$

## 5 Meg(n) for $n \geq 10$

### 5.1 A basic module

To build the minimum expandable graphs we define their components. Figure 5 gives the component  $\mathcal{H}$ .

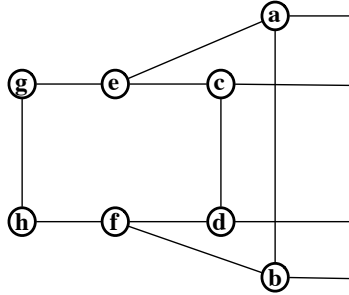


Figure 5: The subgraph  $\mathcal{H}$  with two 2-vertices and six 3-vertices.

The graph  $\mathcal{H}(2)$  is as follows (see Figure 6):  $\mathcal{H}(2)$  contains 2 copies  $\mathcal{H}_1, \mathcal{H}_2$  of  $\mathcal{H}$ . The vertices of  $\mathcal{H}_i$  are denoted by  $a_i, b_i, \dots, h_i, 1 \leq i \leq 2$ . The edges between  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are  $a_1b_2, b_1a_2, c_1c_2, d_1d_2$ . Notice that since  $\mathcal{H}_2$  is a copy of  $\mathcal{H}_1$ , there are symmetries in  $\mathcal{H}(2)$ . In each module,  $a_i, c_i, e_i, g_i$  is symmetric to  $b_i, d_i, f_i, h_i$  and  $a_i$  (resp.  $b_i$ ) and  $c_i$  (resp.  $d_i$ ) play identical roles. As a consequence, if there is a 2-factor containing, for instance, the non-edge  $a_1f_1$  then, by *symmetry*, there is also a 2-factor containing  $c_1f_1$  as well as a 2-factor containing  $b_1e_1$  or  $d_1e_1, a_2f_2, b_2e_2, d_2e_2, c_2f_2$ . Also,  $a_1c_2$  plays the same role as  $a_1d_2$ .

The graph  $\mathcal{H}(p), p \geq 3$ , is built from  $\mathcal{H}(p-1)$  as follows (see Fig. 7): add one copy  $\mathcal{H}_p$  of  $\mathcal{H}$  to  $\mathcal{H}(p-1)$ . The vertices of  $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$  are denoted by  $a_i, b_i, \dots, h_i, 1 \leq i \leq p$ . Remove the two edges  $b_1a_{p-1}$  and  $d_1d_{p-1}$  of  $\mathcal{H}(p-1)$ . Add the four edges  $b_1a_p, a_{p-1}b_p, d_1d_p, c_p d_{p-1}$ . The modules  $\mathcal{H}_i$  are arranged around a cycle and numbered clockwise from 1 to  $p$ . Notice that there is a symmetry on each side of  $\mathcal{H}_1$  between  $\mathcal{H}_i$  and  $\mathcal{H}_{p-i+2}$ , for  $i = 2, \dots, \lfloor p/2 \rfloor + 1$ . As for  $\mathcal{H}(2)$ ,  $a_i$  (resp.  $b_i$ ) and

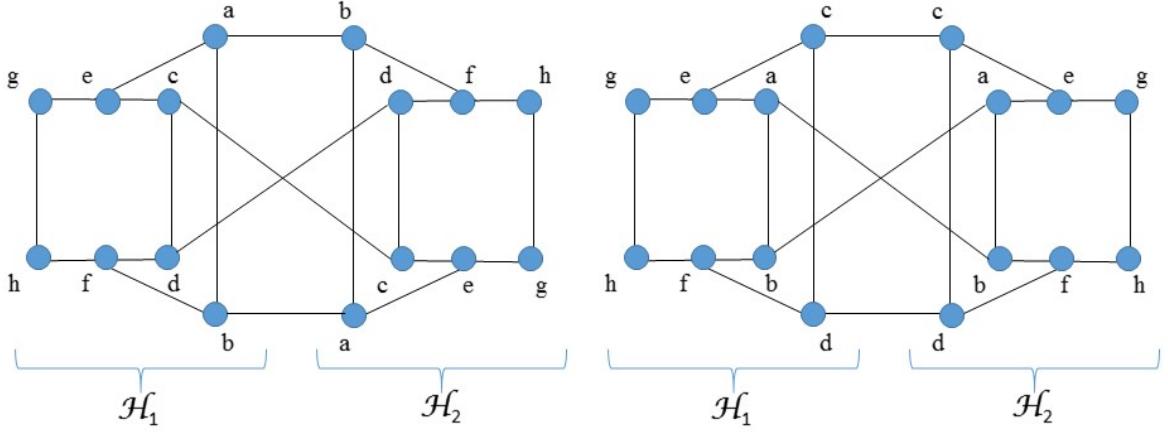


Figure 6: Two representations of  $\mathcal{H}(2)$ . (The index vertices are omitted).

$c_i$  (resp.  $d_i$ ) play identical roles: this can be seen by looking at Figure 7. If there is a 2-factor containing, for instance,  $a_1f_2$  then, there is also a 2-factor containing  $b_1f_5$  as well as a 2-factor containing  $c_1f_2$  or  $d_1f_5$ . Notice also that  $a_p b_1$  and  $d_p d_1$ , the two edges linking  $\mathcal{H}_p$  to  $\mathcal{H}_1$  play the same role.

In the following, we shall shorten many proofs by referring to all these properties as *symmetries*.

**Remark 5.1** *The graph  $\mathcal{H}_p$  has a 2-factor: for instance, take the cycle  $(a_1, b_2, a_2, \dots, a_p, b_1, a_1)$  and the  $p$  cycles  $(c_i, e_i, g_i, h_i, f_i, d_i, c_i)$ ,  $i = 1, \dots, p$ .*

**Remark 5.2** *The subgraph induced by  $\mathcal{V}_i \cup \mathcal{V}_{i+1}$ ,  $1 < i < p$ , is hamiltonian. A hamiltonian cycle is  $(a_i, b_i, f_i, h_i, g_i, e_i, c_i, d_i, c_{i+1}, d_{i+1}, f_{i+1}, h_{i+1}, g_{i+1}, e_{i+1}, a_{i+1}, b_{i+1}, a_i)$ .*

## 5.2 Meg(n) for $n = 8p$ , $p \geq 2$

We use a recurrence to prove that  $\mathcal{H}(p)$  is a  $meg(8p)$ .

**Property 5.1**  $\mathcal{H}(2)$  is a  $meg(16)$ .

**Proof:**  $\mathcal{H}(2)$  contains  $m = 22 = \lceil \frac{1}{2}(3 \times 16 - \lfloor \frac{16}{4} \rfloor) \rceil$  edges.

We show that  $\mathcal{H}(2)$  is expandable. Let  $xy \notin E$ . We give, first a chain  $(x, \dots, y)$ , and then, possibly, a set of cycles that provide a 2-factor of  $\mathcal{H}(2)$ .

- $xy = a_1c_1$ :  $(a_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1)$ ; by symmetry  $b_1d_1, b_2d_2, a_2c_2$  can be extended;
- $xy = a_1d_1$ :  $(d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, a_1), (d_2, c_2, e_2, g_2, h_2, f_2, d_2)$ ; by symmetry  $b_1c_1, b_2c_2, a_2d_2$  can be extended;

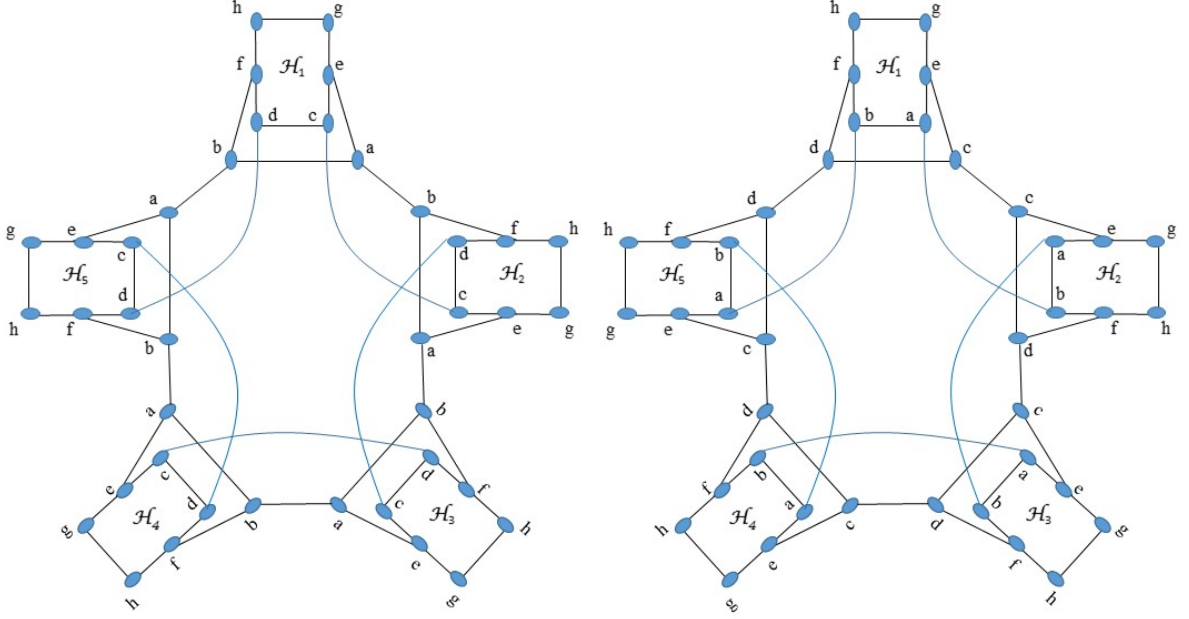


Figure 7: Two representation of  $\mathcal{H}(5)$ . (The vertex indices are omitted).

- $xy = a_1f_1$ :  $(f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$ ; by symmetry  $d_1e_1, c_1f_1, b_1e_1, c_2f_2, d_2e_2, a_2f_2, b_2e_2$  can be extended;
- $xy = a_1g_1$ :  $(g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1, e_1, a_1)$ ; by symmetry  $c_1g_1, b_1h_1, d_1h_1, b_2h_2, d_2h_2, a_2g_2, c_2g_2$  can be extended;
- $xy = a_1a_2$ :  $(a_2, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$ ; by symmetry  $b_1b_2, c_1d_2, d_1c_2$  can be extended.
- $xy = a_1c_2$ :  $(c_2, d_2, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$ ; by symmetry  $b_1d_2, b_1c_2, a_1d_2, b_2c_1, d_1b_2, a_2c_1, a_2d_1$  can be extended.
- $xy = d_1g_1$ :  $(g_1, h_1, f_1, b_1, a_2, b_2, a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, d_1)$ ; by symmetry  $b_1g_1, c_1h_1, a_1h_1, c_2h_2, a_2h_2, d_2g_2, b_2g_2$  can be extended;
- $xy = e_1a_2$ :  $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, a_2), (c_1, d_1, d_2, c_2, c_1)$ ; by symmetry  $b_2f_1, b_1f_2, a_1e_2, d_2e_1, c_2f_1, d_1e_2, c_1f_2$  can be extended;
- $xy = e_1b_2$ :  $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_1, a_1, b_2)$ ; by symmetry  $a_2f_1, a_1f_2, b_1e_2, c_2e_1, d_2f_1, c_1e_2, d_1f_2$  can be extended;
- $xy = e_1f_1$ :  $(e_1, g_1, h_1, f_1), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1), (c_1, d_1, d_2, c_2, c_1)$ ; by symmetry  $e_2f_2$  can be extended;
- $xy = e_1d_2$ :  $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$ ; by symmetry  $a_2e_1, c_2f_1, b_2f_1, c_1f_2, d_1e_2, a_1e_2, b_1f_2$  can be extended;
- $xy = e_1f_2$ :  $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, a_2, e_2, g_2, h_2, f_2), (c_1, d_1, d_2, c_2, c_1)$ ; by symmetry  $e_1e_2, f_1f_2, e_2f_1$  can be extended;

- $xy = e_1h_2$ :  $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, b_2, a_1, b_1, a_2, e_2, g_2, h_2)$ ; by symmetry  $e_1g_2, f_1h_2, f_1g_2, e_2g_1, f_2h_1, f_2g_1, e_2h_1$  can be extended;
- $xy = f_1g_1$ :  $(g_1, h_1, f_1), (b_1, a_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1)$ ; by symmetry  $e_1h_1, f_2g_2, e_2h_2$  can be extended;
- $xy = g_1b_2$ :  $(g_1, h_1, f_1, d_1, d_2, c_2, c_1, e_1, a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2)$ ; by symmetry  $c_2g_1, a_2h_1, d_2h_1, a_1h_2, d_1h_2, b_1g_2, c_1g_2$  can be extended;
- $xy = g_1d_2$ :  $(g_1, h_1, f_1, d_1, d_2), (a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$ ; by symmetry  $a_2g_1, c_2h_1, b_2h_1, c_1h_2, b_1h_2, d_1g_2, a_1g_2$  can be extended;
- $xy = g_1h_2$ :  $(g_1, h_1, f_1, d_1, d_2, f_2, b_2, a_2, b_1, a_1, e_1, c_1, c_2, e_2, g_2, h_2)$ ; by symmetry  $g_1g_2, h_1h_2, h_1g_2$  can be extended;

□

**Property 5.2**  $\mathcal{H}(3)$  is a meg(24).

**Proof:**  $\mathcal{H}(3)$  contains  $m = 33 = \lceil \frac{1}{2}(3 \times 24 - \lfloor \frac{24}{4} \rfloor) \rceil$  edges.

We show that  $\mathcal{H}(3)$  is expandable. Let  $xy \notin E$ .

Case 1.  $x, y \notin \mathcal{H}_3$ . Looking at the 2-factors given for  $\mathcal{H}(2)$ , we observe that they all contain at least one of the two edges  $b_1a_2, d_1d_2$  and then we can build a 2-factor containing  $xy$  in  $\mathcal{H}(3)$  from a 2-factor  $F$  containing  $xy$  in  $\mathcal{H}(2)$ . If  $b_1a_2 \in F$  and  $d_1d_2 \notin F$ , we substitute  $b_1a_2$  for  $b_1a_3b_3a_2$  and we add the cycle  $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$  to obtain a 2-factor in  $\mathcal{H}(3)$ . The case  $b_1a_2 \notin F$  and  $d_1d_2 \in F$  is symmetric. If  $b_1a_2 \in F$  and  $d_1d_2 \in F$ , we substitute  $b_1a_2$  for  $b_1a_3b_3a_2$  and  $d_1d_2$  for  $d_1d_3f_3h_3g_3e_3c_3d_2$  to obtain a 2-factor in  $\mathcal{H}(3)$ .

Case 2.  $x, y \notin \mathcal{H}_2$ : by symmetry this is equivalent to Case 1.

Case 3.  $x \in \mathcal{H}_2, y \in \mathcal{H}_3$ . For each  $xy$ , we give, first a chain  $(x, \dots, y)$ , and then, possibly, a set of cycles that provide a 2-factor of  $\mathcal{H}(3)$ .

- $xy = a_2a_3$ :  $(a_2, b_3, f_3, h_3, g_3, e_3, a_3); (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$ ; by symmetry  $b_2b_3, d_2d_3, c_2c_3$  can be extended;
- $xy = a_2c_3$ :  $(a_2, b_3, a_3, e_3, g_3, h_3, f_3, d_3, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3)$ ; by symmetry  $d_2b_3$  can be extended;
- $xy = a_2d_3$ :  $(a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3), (c_1, d_1, f_1, h_1, g_1, e_1, c_1)$ ; by symmetry  $b_2c_3, c_2b_3, d_2a_3$  can be extended;
- $xy = a_2e_3$ :  $(a_2, b_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3), (c_1, d_1, f_1, h_1, g_1, e_1, c_1)$ ; by symmetry  $f_2b_3, d_2f_3, e_2c_3$  can be extended;
- $xy = a_2f_3$ :  $(a_2, b_3, a_3, e_3, g_3, h_3, f_3), (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$ ; by symmetry  $e_2b_3, f_2c_3, d_2e_3$  can be extended;
- $xy = a_2g_3$ :  $(a_2, b_3, a_3, e_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$ ; by symmetry  $h_2b_3, g_2c_3, d_2h_3$  can be extended;

- $xy = a_2h_3$ :  $(a_2, b_3, f_3, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$ ,  $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$ ; by symmetry  $g_2b_3, h_2c_3, d_2g_3$  can be extended;
- $xy = b_2a_3$ :  $(b_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3)$ ,  $(c_1, d_1, d_3, c_3, d_2, c_2, c_1)$ ,  $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$ ; by symmetry  $c_2d_3$  can be extended;
- $xy = b_2d_3$ :  $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, e_3, g_3, h_3, f_3, d_3)$ ; by symmetry  $c_2a_3$  can be extended;
- $xy = b_2e_3$ :  $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$ ; by symmetry  $f_2a_3, e_2d_3, c_2f_3$  can be extended;
- $xy = b_2f_3$ :  $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$ ; by symmetry  $e_2a_3, f_2d_3, c_2e_3$  can be extended;
- $xy = b_2g_3$ :  $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, e_3, c_3, d_3, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3)$ ; by symmetry  $h_2a_3, g_2d_3, c_2h_3$  can be extended;
- $xy = b_2h_3$ :  $(b_2, a_2, b_3, f_3, d_3, c_3, d_2, f_2, h_2, g_2, e_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$ ; by symmetry  $g_2a_3, h_2d_3, c_2g_3$  can be extended;
- $xy = e_2e_3$ :  $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$ ; by symmetry  $f_2f_3$  can be extended;
- $xy = e_2f_3$ :  $(e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$ ;  $(a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, b_2, a_1)$ ; by symmetry  $f_2e_3$  can be extended;
- $xy = e_2g_3$ :  $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, e_3, c_3, d_2, c_2, c_1, d_1, d_3, f_3, h_3, g_3)$ ;  $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$ ; by symmetry  $h_2f_3, f_2h_3, g_2e_3$  can be extended;
- $xy = e_2h_3$ :  $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, d_3, c_3, d_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$ ; by symmetry  $g_2f_3, f_2g_3, h_2e_3$  can be extended;
- $xy = g_2g_3$ :  $(g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$ ,  $(a_2, e_2, c_2, d_2, c_3, e_3, a_3, b_3, a_2)$ ; by symmetry  $h_2h_3$  can be extended;
- $xy = g_2h_3$ :  $(g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$ ,  $(a_2, e_2, c_2, d_2, c_3, d_3, f_3, b_3, a_2)$ ,  $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$ ; by symmetry  $h_2g_3$  can be extended;

□

**Remark 5.3** Let  $xy \notin \mathcal{H}(p)$ ,  $p \geq 3$ . If  $x, y \neq h_1$  then any 2-factor containing  $xy$  contains  $a_p b_1$  or  $d_p d_1$ .

**Proof:** Let  $F$  be a 2-factor containing  $xy$  and assume that none of  $a_p b_1$  and  $d_p d_1$  are in  $F$ . To cover  $b_1$  and  $d_1$ ,  $F$  must contain the edges  $b_1 a_1, b_1 f_1, d_1 c_1$  and  $d_1 f_1$ . Clearly these 4 edges cannot be completed in a 2-factor if  $x$  and  $y \neq h_1$ . □

**Property 5.3**  $\mathcal{H}(p)$  is a meg( $8p$ ),  $p \geq 4$ .

**Proof:**  $\mathcal{H}(p)$  contains  $m = 11p = \lceil \frac{1}{2}(3 \times 8p - \lfloor \frac{8p}{4} \rfloor) \rceil$  edges.

The proof is by induction. From Properties 5.1 and 5.2,  $\mathcal{H}(2)$  and  $\mathcal{H}(3)$  are expandable. So for  $p \geq 3$  we assume that  $\mathcal{H}(p)$  and  $\mathcal{H}(p-1)$  are expandable.

We now prove that for any  $p \geq 3$ ,  $\mathcal{H}(p+1)$  is expandable.

Let  $xy \notin \mathcal{H}(p+1)$ . We examine several cases.

Case 1.  $x, y \notin \mathcal{V}_1 \cup \mathcal{V}_{p+1}$ . Let  $F$  be a 2-factor containing  $xy$  in  $\mathcal{H}(p)$ . From Remark 5.3,  $a_p b_1 \in F$  or  $d_p d_1 \in F$ . If  $F$  contains  $a_p b_1$  but not  $d_p d_1$  then  $a_p b_1$  is replaced by  $a_p b_{p+1} a_{p+1} b_1$ ; adding the cycle  $(c_{p+1}, d_{p+1}, f_{p+1}, h_{p+1}, g_{p+1}, e_{p+1}, c_{p+1})$  we have a 2-factor in  $\mathcal{H}(p+1)$ . By symmetry, the case where  $d_p d_1 \in F$  and  $a_p b_1 \notin F$  is equivalent. If both  $a_p b_1$  and  $d_p d_1$  are in  $F$ , then  $a_p b_1$  is substituted for  $a_p b_{p+1} a_{p+1} b_1$  and  $d_p d_1$  for  $d_p c_{p+1} e_{p+1} g_{p+1} h_{p+1} f_{p+1} d_{p+1}, d_1$  to obtain a 2-factor in  $\mathcal{H}(p+1)$ .

Case 2.  $x \in \mathcal{V}_1$ .

Case 2.1.  $y \notin \mathcal{V}_p \cup \mathcal{V}_{p+1}$ . From our assumption  $xy$  can be extended in  $\mathcal{H}(p-1)$ . If the 2-factor in  $\mathcal{H}(p-1)$  contains  $a_{p-1} b_1$  but not  $d_{p-1} d_1$  then  $a_{p-1} b_1$  is substituted for  $a_{p-1} b_p a_p b_{p+1} a_{p+1} b_1$ ; adding the two cycles  $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$ ,  $p \leq j \leq p+1$ , we have a 2-factor in  $\mathcal{H}(p+1)$ . By symmetry, the case where the 2-factor in  $\mathcal{H}(p-1)$  contains  $d_{p-1} d_1$  but not  $a_{p-1} b_1$  is equivalent. If both  $a_{p-1} b_1$  and  $d_{p-1} d_1$  are in the 2-factor of  $\mathcal{H}(p-1)$ , then  $a_{p-1} b_1$  is substituted for  $a_{p-1} b_p a_p b_{p+1} a_{p+1} b_1$  and  $d_{p-1} d_1$  for  $d_{p-1} c_p e_p g_p h_p f_p d_p c_{p+1} e_{p+1} g_{p+1} h_{p+1} f_{p+1} d_{p+1}, d_1$  to obtain a 2-factor in  $\mathcal{H}(p+1)$ . Finally, if  $a_{p-1} b_1, d_{p-1} d_1 \notin F$ , we add to  $F$  the hamiltonian cycle on  $\mathcal{V}_{p+1} \cup \mathcal{V}_p$  (see Remark 5.2) to obtain a 2-factor in  $\mathcal{H}(p+1)$ .

Case 2.2.  $y \in \mathcal{V}_p \cup \mathcal{V}_{p+1}$ . By symmetry,  $y \in \mathcal{V}_{p+1}$  is equivalent to  $y \in \mathcal{V}_2$  and, if  $p \geq 4$ ,  $y \in \mathcal{V}_p$  is equivalent to  $y \in \mathcal{V}_3$  seen just before.

It remains the case  $p = 3$ ,  $y \in \mathcal{V}_3$ : we remove from  $\mathcal{H}(4)$  the modules  $\mathcal{H}_2$  and  $\mathcal{H}_4$  and we obtain  $\mathcal{H}'(2)$  isomorphic to  $\mathcal{H}(2)$ . Looking at the 2-factors obtained in  $\mathcal{H}(2)$  for  $xy \notin E$ ,  $x \in \mathcal{V}_1, y \in \mathcal{V}_2$ , we see that they all contain exactly three edges among  $a_1 b_2, b_1 a_2, c_1 c_2, d_1 d_2$ . Thus in  $\mathcal{H}'(2)$ , any 2-factor  $F$  containing  $xy \notin E$ ,  $x \in \mathcal{V}_1, y \in \mathcal{V}_3$  contains also exactly three edges among  $a_1 b_3, b_1 a_3, c_1 c_3, d_1 d_3$  or, equivalently, they all miss exactly one edge among  $a_1 b_3, b_1 a_3, c_1 c_3, d_1 d_3$ . To obtain a 2-factor in  $\mathcal{H}(4)$ , we make the following substitutions: if  $a_1 b_3 \notin F$ , we replace  $c_1 c_3$  by  $c_1 c_2 d_2 c_3$  and we add the cycle  $(c_2, d_2, f_2, h_2, g_2, e_2, c_2)$ ; if  $b_1 a_3 \notin F$ , we replace  $d_1 d_3$  by  $d_1 d_4 c_4 d_3$  and we add the cycle  $(c_4, d_4, f_4, h_4, g_4, e_4, c_4)$ ; if  $c_1 c_3 \notin F$ , we replace  $a_1 b_3$  by  $a_1 b_2 a_2 b_3$  and we add the cycle  $(c_2, d_2, f_2, h_2, g_2, e_2, c_2)$ ; if  $d_1 d_3 \notin F$ , we replace  $b_1 a_3$  by  $b_1 a_4 b_4 a_3$  and we add the cycle  $(c_4, d_4, f_4, h_4, g_4, e_4, c_4)$ .

Case 3.  $x \in \mathcal{V}_{p+1}$ .

Case 3.1.  $y \in \mathcal{V}_i, i \neq 2$ . If  $y \in \mathcal{V}_1$ , by symmetry, this equivalent to the case  $x \in \mathcal{V}_1, y \in \mathcal{V}_{p+1}$  treated in Case 2.

If  $y \in \mathcal{V}_i, i = 3, \dots, p-1$ , these cases are equivalent to cases  $x \in \mathcal{V}_2, y \in \mathcal{V}_{p-i+3}$  which were treated in Case 1.

If  $y \in \mathcal{V}_p$ , this is equivalent to  $x \in \mathcal{V}_2, y \in \mathcal{V}_3$  treated in Case 1.



Case 3.2  $y \in \mathcal{V}_2$ ,  $p \geq 4$ . There are two adjacent modules  $\mathcal{H}_i$  and  $\mathcal{H}_{i+1}$  with  $i \neq 2$  and  $i+1 \neq p+1$ . We remove these modules and add the edges  $a_{i-1}b_{i+2}$  and  $d_{i-1}c_{i+2}$ . We obtain a graph  $\mathcal{H}'(p-1)$  isomorphic to  $\mathcal{H}(p-1)$  which has a 2-factor  $F$  containing  $xy$  by assumption. If  $a_{i-1}b_{i+2} \in F$  but  $d_{i-1}c_{i+2} \notin F$  then  $a_{i-1}b_{i+2}$  is substituted for  $a_{i-1}b_i a_i b_{i+1} a_{i+1} b_{i+2}$ ; adding the two cycles  $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$ ,  $i \leq j \leq i+1$ , we have a 2-factor in  $\mathcal{H}(p+1)$ . By symmetry, the case where  $d_{i-1}c_{i+2} \in F$  and  $a_{i-1}b_{i+2} \notin F$  is equivalent. If both  $a_{i-1}b_{i+2}$  and  $d_{i-1}c_{i+2}$  are in the 2-factor of  $\mathcal{H}'(p-1)$ , then  $a_{i-1}b_{i+2}$  is substituted for  $a_{i-1}b_i a_i b_{i+1} a_{i+1} b_{i+2}$  and  $d_{i-1}c_{i+2}$  for  $d_{i-1}c_i, e_i g_i h_i f_i d_i c_{i+1} e_{i+1} g_{i+1} h_{i+1} f_{i+1} d_{i+1} c_{i+2}$  to obtain a 2-factor in  $\mathcal{H}(p+1)$ . Finally, if neither  $a_{i-1}b_{i+2}$  nor  $d_{i-1}c_{i+2}$  are in the 2-factor of  $\mathcal{H}'(p-1)$ , then adding the hamiltonian cycle covering  $\mathcal{V}_i \cup \mathcal{V}_{i+1}$  (see Remark 5.2) we obtain a 2-factor in  $\mathcal{H}(p+1)$ .

Case 3.3  $y \in \mathcal{V}_2$ ,  $p = 3$ . This is the last case:  $x \in \mathcal{V}_4$  and  $y \in \mathcal{V}_2$ . We remove the module  $\mathcal{H}_3$  from  $\mathcal{H}(4)$  and add the edges  $a_2b_4$  and  $d_2c_4$ : we get a graph  $\mathcal{H}'(3)$  isomorphic to  $\mathcal{H}(3)$  in which  $xy$  can be extended. Let  $F$  be the 2-factor of  $\mathcal{H}'(3)$  containing  $xy$ .

Looking at the 2-factors given for  $\mathcal{H}(3)$  to complete an edge  $xy \notin \mathcal{H}(3)$  with  $x \in \mathcal{V}_2$  and  $y \in \mathcal{V}_3$ , we see that they all contain  $a_2b_3$  or  $d_2c_3$ . Equivalently,  $F$  contains  $a_2b_4$  or  $d_2c_4$ .

If  $F$  contains  $a_2b_4$  but not  $d_2c_4$  then  $a_2b_4$  is substituted for  $a_2b_3 a_3 b_4$ ; adding the cycle  $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$  we have a 2-factor in  $\mathcal{H}(4)$ . By symmetry, the case where  $\mathcal{H}'(3)$  contains  $d_2c_4$  but not  $a_2b_4$  is equivalent. If both  $a_2b_4$  and  $d_2c_4$  are in  $F$ , then  $a_2b_4$  is substituted for  $a_2b_3 a_3 b_4$  and  $d_2c_4$  for  $d_2c_3 e_3 g_3 h_3 f_3 d_3 c_4$  to obtain a 2-factor in  $\mathcal{H}'(4)$ .  $\square$

### 5.3 Meg(n) for $n \neq 0 \pmod{8}$ , $n \geq 14$

Now, we give minimal expandable graphs when  $n \geq 10$  is not a multiple of 8. The graphs  $\mathcal{H}^{-1}(p)$ ,  $\mathcal{H}^{-2}(p)$ ,  $p \geq 2$ , are obtained from  $\mathcal{H}(p)$  by contracting one edge  $g_i h_i$ , respectively two edges  $g_i h_i, g_j h_j$ ,  $i \neq j$ . The vertex resulting of the contraction of  $gh$  is denoted by  $gh$ .

**Property 5.4**  $\mathcal{H}^{-1}(p)$  is a  $meg(8p-1)$ ,  $p \geq 2$  and  $\mathcal{H}^{-2}(p)$  is a  $meg(8p-2)$ ,  $p \geq 2$ .

**Proof:**  $\mathcal{H}^{-1}(p)$  contains  $m = 11p - 1 = \lceil \frac{1}{2}(3 \times (8p - 1) - \lfloor \frac{8p-1}{4} \rfloor) \rceil$  edges.  $\mathcal{H}^{-2}(p)$  contains  $m = 11p - 2 = \lceil \frac{1}{2}(3 \times (8p - 2) - \lfloor \frac{8p-2}{4} \rfloor) \rceil$  edges.

From Properties 5.1 and 5.3 we have  $\mathcal{H}(p)$  is a  $meg(8p)$ ,  $p \geq 2$ . The 2-factor of  $\mathcal{H}^{-1}(p)$ , resp.  $\mathcal{H}^{-2}(p)$ , corresponds to the 2-factor of  $\mathcal{H}(p)$  where the subsequence of two consecutive vertices  $g, h$  in  $\mathcal{H}(p)$  is replaced by the contracted vertex  $gh$  in  $\mathcal{H}^{-1}(p)$ , resp.  $\mathcal{H}^{-2}(p)$ .  $\square$

The graph  $\mathcal{H}^{+1}(p)$ ,  $p \geq 2$ , is obtained from  $\mathcal{H}(p)$  by adding a 2-vertex  $i$  and the two edges  $ig, ih$  to  $\mathcal{H}(p)$ .

**Property 5.5**  $\mathcal{H}^{+1}(p)$  is a  $meg(8p+1)$ ,  $p \geq 2$ .

**Proof:**  $\mathcal{H}^{+1}(p)$  contains  $m = 11p + 2 = \lceil \frac{1}{2}(3 \times (8p + 1) - \lfloor \frac{8p+1}{4} \rfloor) \rceil$  edges.

Observe that all 2-factors of  $\mathcal{H}(p)$  contain  $gh$ . Let  $xy \notin E$ . If  $x, y \neq i_1$  then the 2-factor of  $\mathcal{H}^{+1}(p)$  corresponds to the 2-factor of  $\mathcal{H}(p)$  where  $g_1h_1$  in  $\mathcal{H}(p)$  is replaced by  $g_1i_1h_1$  in  $\mathcal{H}^{+1}(p)$ . If  $x = i_1$  then  $gy \notin E$  or  $hy \notin E$ . W.l.o.g. assume that  $gy \notin E$ . Then the 2-factor of  $\mathcal{H}^{+1}(p)$  corresponds to the 2-factor of  $\mathcal{H}(p)$  where  $h_1g_1$  in  $\mathcal{H}(p)$  is replaced by  $h_1i_1g_1$  in  $\mathcal{H}^{+1}(p)$ .  $\square$

The graph  $\mathcal{H}^{+2}(p)$  is obtained from  $\mathcal{H}(p)$  as follows: the two edges  $a_1e_1, c_1e_1$  are subdivided into the paths  $(a_1, i, e_1), (c_1, j, e_1)$ , respectively; the edge  $ij$  is added.

**Property 5.6**  $\mathcal{H}^{+2}(p)$  is a meg( $8p + 2$ ),  $p \geq 2$ .

**Proof:**  $\mathcal{H}^{+2}(p)$  contains  $m = 11p + 3 = \lceil \frac{1}{2}(3 \times (8p + 2) - \lfloor \frac{8p+2}{4} \rfloor) \rceil$  edges.

We show that  $\mathcal{H}^{+2}(p)$  is expandable. Let  $xy \notin E$ . In the case where  $x, y \notin \{a_1, c_1, e_1\}$  we proceed as follows: we know that  $xy$  can be extended in  $\mathcal{H}(p)$ ; the corresponding 2-factor contains either the subsequence  $a_1e_1c_1$ , or  $a_1e_1g_1$  or  $c_1e_1g_1$ . In the first case the subsequence is substituted for  $a_1ie_1jc_1$ , in the second for  $a_1ije_1g_1$ , in the third for  $c_1jie_1g_1$ . So  $xy$  can be extended in  $\mathcal{H}^{+2}(p)$ .

Now let  $x = i$  (the case where  $x = j$  is similar). If  $y \neq c_1, g_1$ , we know that  $e_1y$  can be extended in  $\mathcal{H}(p)$ ; the corresponding 2-factor contains  $e_1g_1$ ; we replace it by  $ije_1g_1$ . If  $y = c_1$  (resp.  $y = g_1$ ): in  $\mathcal{H}(p)$ , the 2-factor corresponding to the non-edge  $g_1c_1$  (resp.  $g_1a_1$ ) contains  $c_1e_1a_1$ ; we replace it by  $c_1je_1ia_1$ .

If  $y = g_1$ : in  $\mathcal{H}(p)$ , the 2-factor corresponding to the non-edge  $g_1a_1$  contains  $c_1e_1a_1$ ; we replace it by  $c_1je_1ia_1$ .

So  $iy$  can be extended in  $\mathcal{H}^{+2}(p)$ .

Now let  $x = e_1$ . We know that  $g_1y$  can be extended in  $\mathcal{H}(p)$ ; the corresponding 2-factor contains both  $g_1h_1$  and  $a_1e_1c_1$  (or  $c_1e_1a_1$ , but from symmetry this case is the same). We substitute the first subsequence for  $e_1g_1h_1$  and the second for  $a_1ijc_1$  and  $e_1y$  can be extended in  $\mathcal{H}^{+2}(p)$ .  $\square$

The graph  $\mathcal{H}^{+3}(p)$ ,  $p \geq 2$ , is obtained from  $\mathcal{H}(p)$  by adding a triangle  $(i, j, k)$  and the two edges  $ia_1, ja_1$  to  $\mathcal{H}(p)$ .

**Property 5.7**  $\mathcal{H}^{+3}(p)$  is a meg( $8p + 3$ ),  $p \geq 2$ .

**Proof:**  $\mathcal{H}^{+3}(p)$  contains  $m = 11p + 5 = \lceil \frac{1}{2}(3 \times (8p + 3) - \lfloor \frac{8p+3}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ . If  $x, y \notin \{i, j, k\}$  then the 2-factor of  $\mathcal{H}^{+3}(p)$  corresponds to the 2-factor of  $\mathcal{H}(p)$  plus  $(i, j, k)$ .

Let  $x = i$  (the case  $x = j$  is the same): note that  $y \notin \{i, j, k, a_1\}$ . Suppose first that  $y$  is not a neighbor of  $a_1$  in  $\mathcal{H}(p)$ :  $ya_1$  can be extended to a 2-factor and substituting  $ya_1$  for  $yikja_1$ , ( $i = x$ ), we have a 2-factor for  $\mathcal{H}^{+3}(p)$ . Now let  $y = e_1$ . We know that  $e_1f_1$  can be extended in  $\mathcal{H}(p)$ . The corresponding 2-factor contains  $(e_1g_1h_1f_1e_1)$  and a cycle  $C$  with the edge  $a_1b_1$ . Substituting  $a_1b_1$  for  $a_1jkie_1g_1h_1f_1b_1$  we have a 2-factor of  $\mathcal{H}^{+3}(p)$ . Let  $y \in \{b_1, b_2\}$ ;  $e_1f_1$  can be extended in  $\mathcal{H}(p)$ , and the corresponding 2-factor contains  $(e_1, g_1, h_1, f_1, e_1)$ , a cycle  $C$  with the edges  $a_1b_1$  and  $a_1b_2$ , and a cycle  $C'$  with the edge  $c_1d_1$ . So replacing  $a_1y$  by  $a_1jkiiy$  and  $c_1d_1$  by  $c_1e_1g_1h_1f_1d_1$  we have a 2-factor of  $\mathcal{H}^{+3}(p)$ .

For  $x = k$ , if  $y \neq a_1$  we do as above by replacing the sequence  $ikj$  by  $kij$ . If  $y = a_1$ , we do as follows: we know that  $g_1a_1$  can be extended in  $\mathcal{H}(p)$ ; the corresponding 2-factor contains  $e_1a_1g_1$ . Thus substituting  $e_1a_1g_1$  for  $e_1g_1$  and adding  $(a_1, i, j, k, a_1)$  we obtain a 2-factor of  $\mathcal{H}^{+3}(p)$ .  $\square$

The graph  $\mathcal{H}^{+4}(p), p \geq 2$ , is obtained from  $\mathcal{H}(p)$  by adding a diamond  $(i, j, k, l), ij \notin E$  and the edge  $ia_1$  to  $\mathcal{H}(p)$ .

**Property 5.8**  $\mathcal{H}^{+4}(p)$  is a  $meg(8p + 4), p \geq 2$ .

**Proof:**  $\mathcal{H}^{+4}(p)$  contains  $m = 11p + 6 = \lceil \frac{1}{2}(3 \times (8p + 4) - \lfloor \frac{8p+4}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ . If  $x, y \notin \{i, j, k, l\}$  then to obtain a 2-factor of  $\mathcal{H}^{+4}(p)$  we add the cycle  $(i, k, j, l, i)$  to the 2-factor of  $\mathcal{H}(p)$  obtained for the extension of  $xy$ .

For  $xy = ij$  we do as follows: we take a 2-factor covering  $\mathcal{H}(p)$  (see Remark 5.1) and the cycle  $(i, j, l, k, i)$  to obtain a 2-factor of  $\mathcal{H}^{+4}(p)$ .

Let  $x = i$ : we have  $y \notin \{i, j, k, l, a_1\}$ . First we suppose that  $y$  is not a neighbor of  $a_1$ . We know that  $ya_1$  can be extended in  $\mathcal{H}(p)$ , so in the corresponding 2-factor there is a cycle with the sequence  $ya_1$ . Substituting  $ya_1$  for  $ya_1a_1$  and adding the cycle  $(j, k, l, j)$  we have a 2-factor for  $\mathcal{H}^{+4}(p)$ . Now let  $y = e_1$ : We know that  $e_1f_1$  can be extended in  $\mathcal{H}(p)$ . The corresponding 2-factor contains  $(e_1, g_1, h_1, f_1, e_1)$  and a cycle  $C$  with the edge  $a_1b_1$ . Substituting the sequence  $a_1b_1$  for  $a_1ie_1g_1h_1f_1b_1$  and adding  $(j, k, l, j)$  we have a 2-factor of  $\mathcal{H}^{+4}(p)$ . When  $y \in \{b_1, b_2\}$  we do as follows:  $e_1f_1$  can be extended in  $\mathcal{H}(p)$ , the corresponding 2-factor contains  $(e_1, g_1, h_1, f_1, e_1)$ , a cycle  $C$  with  $a_1y$ , and a cycle  $C'$  with  $c_1d_1$ . So substituting the sequence  $a_1y$  for  $a_1iy$ , the sequence  $c_1d_1$  for  $c_1e_1g_1h_1f_1d_1$  and adding  $(j, k, l, j)$  we have a 2-factor of  $\mathcal{H}^{+4}(p)$ .

Let  $x = k$  (resp.  $x = j$ ): We proceed as for  $x = i$  but instead of  $a_1i$  we take  $a_1iljk$  (resp.  $a_1ilkj$ ).  $\square$

The graph  $\mathcal{H}^{+5}(p)$ , is obtained from  $\mathcal{H}^{+2}(p)$  by subdividing the edge  $a_1b_1$  into the path  $a_1lkb_1$ , adding a 2-vertex  $r$  together with two edges  $rl, rk$ . See Figure 8.

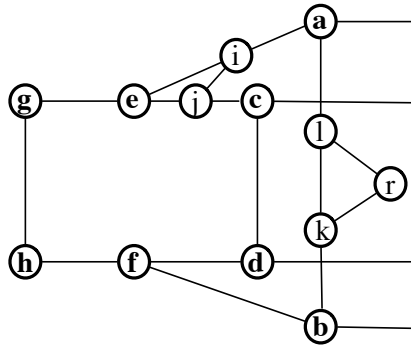


Figure 8: The graph  $\mathcal{H}^{+5}(p)$  is a  $meg(11p + 7)$ .

**Property 5.9**  $\mathcal{H}^{+5}(p)$  is a  $meg(8p + 5)$ .

**Proof:**  $\mathcal{H}^{+5}(p)$  contains  $m = 11p + 7 = \lceil \frac{1}{2}(3 \times (8p + 5) - \lfloor \frac{8p+5}{4} \rfloor) \rceil$  edges.

From Property 5.6 we know that  $\mathcal{H}^{+2}(p)$  is expandable.

Let  $xy \notin E$ . If  $xy$  is also a non-edge of  $\mathcal{H}^{+2}(p)$  and the corresponding 2-factor contains  $a_1b_1$  then we substitute the sequence  $a_1b_1$  for  $a_1lrkb_1$ ; else if  $a_1b_1$  is not in the 2-factor, so we add the cycle  $(k, l, r, k)$  to the 2-factor. In both cases we obtain a 2-factor for  $\mathcal{H}^{+5}(p)$ .

Now let  $xy = a_1b_1$ :  $(a_1, b_2, a_2, b_3, a_3, \dots, b_p, a_p, b_1), (c_1, j, i, e_1, g_1, h_1, f_1, d_1, c_1), (c_i, e_i, g_i, h_i, f_i, d_i, c_i), 2 \leq i \leq p$ , and  $(k, l, r, k)$  is a 2-factor.

Let  $x = k$  (the case where  $x = l$  is symmetric). If  $y \neq a_1$ :  $a_1y$  can be extended in  $\mathcal{H}^{+2}(p)$ ; the corresponding 2-factor contains the path  $a_1 \cdots y$ ; we substitute it for  $krla_1 \cdots y$  and we have 2-factor for  $\mathcal{H}^{+5}(p)$ . If  $y = a_1$ :  $(k, r, l, a_1), (c_1, d_1, d_p, c_p, d_{p-1}, c_{p-1}, \dots, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_4, f_4, h_4, g_4, e_4, a_4, \dots, b_p, f_p, h_p, g_p, e_p, a_p, b_1, f_1, h_1, g_1, e_1, i, j, c_1)$  is a 2-factor.

Let  $x = r$ . Remark that all 2-factors containing the non-edge  $ky$  contain also  $kr$ ; to extend  $ry$  we replace  $kr$  by  $rk$  in these 2-factors.  $\square$

## 5.4 Meg(n) for $10 \leq n \leq 13$

The constructions given in the previous sections do not cover the cases  $10 \leq n \leq 13$ . We give constructions for all the situations.

We start from the graph  $G_{10}$  shown in Figure 9. Note that  $G_{10}$  and  $G_{10} - a$  are hamiltonian.

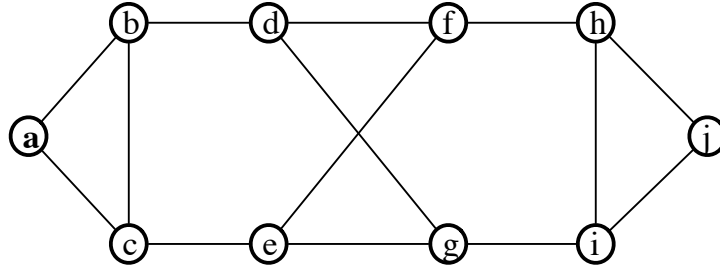


Figure 9: The graph  $G_{10}$  is a  $meg(10)$ .

**Property 5.10**  $G_{10}$  is a  $meg(10)$ .

**Proof:**  $G_{10}$  contains  $m = 14 = \lceil \frac{1}{2}(3 \times 10 - \lfloor \frac{10}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ .

- $xy = ad$ :  $(a, b, c, e, f, h, j, i, g, d)$  yields a 2-factor; by symmetry  $ae, jf, jg$  can be extended;
- $xy = af$ :  $(a, b, c, e, g, d, f), (h, i, j, h)$  yields a 2-factor; by symmetry  $ag, jd, je$  can be extended;

- $xy = ah$ :  $(a, b, c, e, f, d, g, i, j, h)$  yields a 2-factor; by symmetry  $ai, jb, jc$  can be extended;
- $xy = aj$ :  $(a, b, c, e, f, d, g, i, h, j)$  yields a 2-factor;
- $xy = be$ :  $(b, a, c, e), (d, g, i, j, h, f, d)$  yields a 2-factor; by symmetry  $cd, hg, if$  can be extended;
- $xy = bf$ :  $(b, a, c, e, g, d, f), (h, i, j, h)$  yields a 2-factor; by symmetry  $cg, hd, ie$  can be extended;
- $xy = bg$ :  $(b, a, c, e, f, d, g), (h, i, j, h)$  yields a 2-factor; by symmetry  $cf, he, id$  can be extended;
- $xy = bh$ :  $(b, a, c, e, f, d, g, i, j, h)$  yields a 2-factor; by symmetry  $ci$  can be extended;
- $xy = bi$ :  $(b, a, c, e, g, d, f, h, j, i)$  yields a 2-factor; by symmetry  $ch$  can be extended;
- $xy = de$ :  $(d, f, h, j, i, g, e), (a, b, c, a)$  yields a 2-factor; by symmetry  $fg$  can be extended.

□

The graph  $G_{11}$  shown in Figure 10 is obtained from  $G_{10}$  by the addition of the vertex  $k$ , the removal of  $dg, ef$  and the addition of  $dk, ek, fk, gk$  to  $E$ .

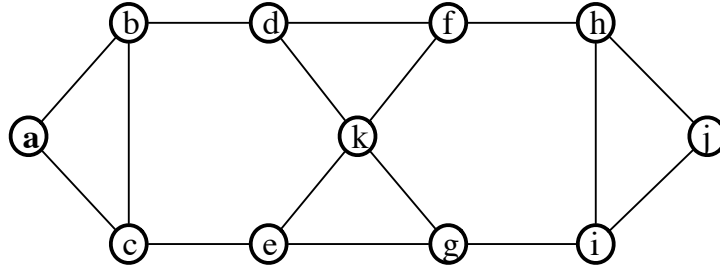


Figure 10: The graph  $G_{11}$  is a  $meg(11)$ .

**Property 5.11**  $G_{11}$  is a  $meg(11)$ .

**Proof:**  $G_{11}$  contains  $m = 16 = \lceil \frac{1}{2}(3 \times 11 - \lfloor \frac{11}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ . If  $xy \notin \{dg, ef, kz \mid z \in \{a, \dots, j\}\}$  from Property 5.10  $xy$  can be extended to a 2-factor  $F$  in  $G_{10}$ . Since at least one of  $dg, ef, df, eg$  is an edge of  $F$ ,  $F$  can easily be modified to a 2-factor  $F'$  of  $G_{11}$  by replacing the edge by the 3-path passing through  $k$ .

- $xy = dg$ :  $(d, f, k, e, g), (a, b, c, a), (h, i, j, h)$  is a 2-factor; by symmetry  $ef$  can be extended;

- $xy = ak$ :  $(a, b, c, e, g, i, j, h, f, d, k)$  is a 2-factor; by symmetry  $jk$  can be extended;
- $xy = bk$ :  $(b, a, c, e, g, i, j, h, f, d, k)$  is a 2-factor; by symmetry  $ck, hk, ik$  can be extended.

□

The graph  $G_{12}$  shown in Figure 11 is obtained from  $G_{10}$  by the addition of the vertices  $k, l$ , the removal of  $dg, ef$  and the addition of  $dk, ek, fl, gl, kl$  to  $E$ .

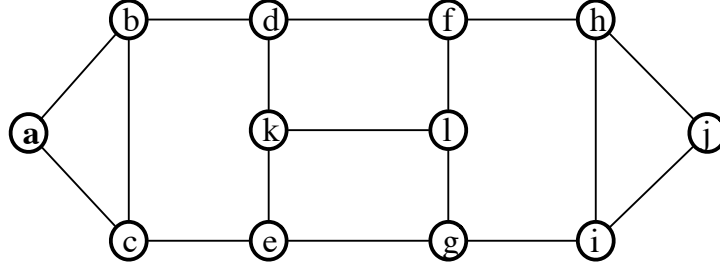


Figure 11: The graph  $G_{12}$  is a  $meg(12)$ .

**Property 5.12**  $G_{12}$  is a  $meg(12)$ .

**Proof:**  $G_{12}$  contains  $m = 17 = \lceil \frac{1}{2}(3 \times 12 - \lfloor \frac{12}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ . If  $xy \notin \{dg, ef, kz, lz \mid z \in \{a, \dots, j\}\}$  from Property 5.10  $xy$  can be extended to a 2-factor  $F$  in  $G_{10}$ . Since at least one of  $dg, ef, df, eg$  is an edge of  $F$ ,  $F$  can easily be modified to a 2-factor  $F'$  of  $G_{12}$  by replacing the edge by the 4-path passing through  $k$  and  $l$ .

- $xy = dg$ :  $(d, b, a, c, e, k, l, f, h, j, i, g)$  is a 2-factor; by symmetry  $ef$  can be extended;
- $xy = kf$ :  $(k, l, g, e, c, a, b, d, f), (h, i, j, h)$  is a 2-factor; by symmetry  $kg, dl, el$  can be extended;
- $xy = ka$ :  $(k, d, f, l, g, e, c, b, a), (h, i, j, h)$  is a 2-factor; by symmetry  $lj$  can be extended;
- $xy = kb$ :  $(k, d, f, l, g, e, c, a, b), (h, i, j, h)$  is a 2-factor; by symmetry  $kc, lh, li$  can be extended;
- $xy = kj$ :  $(k, l, g, e, c, a, b, d, f, h, i, j)$  is a 2-factor; by symmetry  $la$  can be extended;
- $xy = ki$ :  $(k, l, g, e, c, a, b, d, f, h, j, i)$  is a 2-factor; by symmetry  $kh, lb, lc$  can be extended.

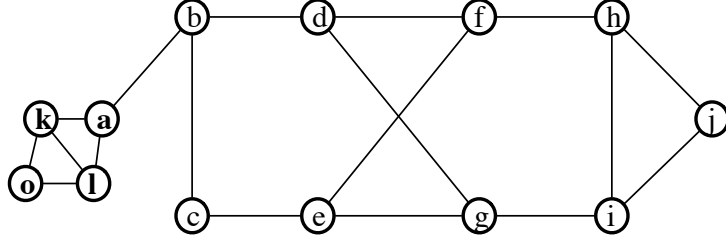


Figure 12: The graph  $G_{13}$  is a  $meg(13)$ .

□

The graph  $G_{13}$  shown in Figure 12 is obtained from  $G_{10}$  by the addition of the vertices  $k, l, o$ , the removal of  $ac$ .  $\{a, k, l, o\}$  induce the diamond with  $ao \notin E$ .

**Property 5.13**  $G_{13}$  is a  $meg(13)$ .

**Proof:**  $G_{13}$  contains  $m = 18 = \lceil \frac{1}{2}(3 \times 13 - \lfloor \frac{13}{4} \rfloor) \rceil$  edges.

Let  $xy \notin E$ . If  $x, y \notin \{a, k, l, o\}$  then we take  $F$  the corresponding 2-factor of  $G_{10}$ .  $F$  has  $bac$  as subsequence. Substituting  $bac$  for  $bc$  and adding  $(a, k, o, l, a)$  we have a 2-factor for  $G_{13}$ . If  $xy = ao$  then  $(a, k, l, o)$  and an hamiltonian cycle of  $G_{10} - a$  is a 2-factor for  $G_{13}$ .

Let  $x = a$ . If  $y \in \{d, f, h, j\}$  we know from Property 5.10 that there exists a 2-factor  $F$  for  $G_{10}$  which does not contain  $ac$ . So adding  $(k, l, o, k)$  we have a 2-factor for  $G_{13}$ .

- $y = c$ :  $(a, b, d, f, h, j, i, g, e, c), (k, l, o, k)$  is a 2-factor;
- $y = e$ :  $(a, b, c, e), (d, g, i, j, h, f, d), (k, l, o, k)$  is a 2-factor;
- $y = g$ :  $(a, b, c, e, f, d, g), (h, i, j, h), (k, l, o, k)$  is a 2-factor;
- $y = i$ :  $(a, b, c, e, g, d, f, h, j, i), (k, l, o, k)$  is a 2-factor.

Now when  $x \in \{k, l, o\}$  we use the 2-factors we obtained above for  $x = a$ : the first sequence begins with  $kola$  or  $loka$  or  $okla$  instead of  $a$ . □

## 6 Conclusion

We have determined the values of  $Exp_2(n)$  for all values of  $n$ . It could be interesting to characterize the  $meg(n)$ , for instance by a (finite ?) collection of forbidden induced subgraphs. Furthermore one could consider a generalization of the basic problem: the edges of a complete graph  $K_n$  are coloured in blue and in red. We want to color a minimal set of red edges in white so that any red edge  $uv$  can be extended to a 2-factor using only white edges. In our case, we had only white and red edges. Furthermore instead of just one red edge we could require that any appropriate

subset of  $d$  red edges could be extended to a 2-factor by adding only white edges. More generally we could concentrate on  $k$ -factors,  $k \geq 3$ , or even other structured set of edges.

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