



HAL
open science

Minimal graphs for matching extension

Marie-Christine Costa, Dominique de Werra, Christophe Picouleau

► **To cite this version:**

Marie-Christine Costa, Dominique de Werra, Christophe Picouleau. Minimal graphs for matching extension. *Discrete Applied Mathematics*, 2018, 234, pp.47-55. 10.1016/j.dam.2015.11.007. hal-01829546

HAL Id: hal-01829546

<https://hal.science/hal-01829546>

Submitted on 4 Jul 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Minimal graphs for 2-factor extension

M.-C. Costa ^{*} D. de Werra [†] C. Picouleau [‡]

May 7, 2018

Abstract

Let $G = (V, E)$ be a simple loopless finite undirected graph. We say that G is *(2-factor) expandable* if for any non-edge uv , $G + uv$ has a 2-factor F that contains uv . We are interested in the following: Given a positive integer $n = |V|$, what is the minimum cardinality of E such that there exists $G = (V, E)$ which is 2-factor expandable? This minimum number is denoted by $Exp_2(n)$. We give an explicit formula for $Exp_2(n)$ and provide 2-factor expandable graphs of minimum size $Exp_2(n)$.

Keywords: 2-factor,

1 Introduction

Matching extension has been studied for a long time by many authors, see for instance [1], [7] and [9]. In [4] a different kind of matching extension has been explored: the goal was to characterize graphs $G = (V, E)$ on n vertices with a minimum number of edges such that for every pair x, y of non adjacent vertices, there exists an (almost) perfect matching in $G_{xy} = (V, E \cup xy)$. This work was motivated by a reliability problem in which edges subject to breakdowns could be reinforced so that to any pair xy of non adjacent vertices, one could associate reinforced edges to obtain an (almost) perfect matching $M \cup xy$.

Here we intend to concentrate on the problem in which (almost) perfect matchings (1-factors) are replaced by 2-factors. The precise formulation will be given below. In [8] a different 2-factor extension problem has been studied.

We will consider a simple finite graph $G = (V, E)$ with n vertices and m edges. A pair u, v of vertices is a *non-edge* if $uv \notin E$. For any subset $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. We write $G - X = G[V \setminus X]$ and $G - v$ for $G - \{v\}$. $N(v)$ is the set of neighbors of a vertex v ; $\delta(v) = |N(v)|$ is the degree of v ; a *p-vertex* is a vertex of degree p ; if $\delta(v) = n - 1$ then v is *universal*. The closed

^{*}Ecole Nationale Supérieure des Techniques Avancées Paris-Tech and CEDRIC laboratory, Paris (France). Email: marie-christine.costa@ensta-paristech.fr

[‡]Conservatoire National des Arts et Métiers, CEDRIC laboratory, Paris (France). Email: christophe.picouleau@cnam.fr

[†]Ecole Polytechnique Fédérale de Lausanne, Lausanne (Switzerland). Email: dominique.dewerra@epfl.ch

neighborhood of v is $N[v] = N(v) \cup \{v\}$. $\delta(G)$ (resp. $\Delta(G)$) is the minimum (resp. maximum) degree of the vertices in G . An induced path with p edges is called a p -path. By $d(u, v)$ we denote the *distance* between u and v , i.e., the length of a shortest path (number of edges) between u and v in G .

A subset $F \subseteq E$ is a *2-factor* if every vertex v has exactly two edges in F which are incident in v . Equivalently F is a collection of vertex-disjoint cycles covering all vertices. C_k (resp. K_k) is the cycle (resp. complete graph) on k vertices. K_4 with an edge missing is called a *diamond*.

For all definitions related to graphs, see [3].

We intend to determine, for any integer $n \geq 3$, a graph $G = (V, E)$ with n vertices and a minimum number of edges such that for every pair x, y of non adjacent vertices of G it is always possible to include the non-edge xy into a 2-factor, i.e., the graph $G_{xy} = (V, E \cup \{xy\})$ has a 2-factor $F, F \supset xy$. In such a case we shall say that xy has been *extended* (to a 2-factor).

Definition 1.1 *A graph $G = (V, E)$ is 2-factor expandable (or shortly expandable) if every non-edge xy can be extended.*

Definition 1.2 *An expandable graph $G = (V, E)$ with $|V| = n$ and with a minimum number of edges is a minimum expandable graph ($meg(n)$). The size $|E|$ of its edge set is denoted by $Exp_2(n)$.*

Our problem may be related to the idea of edge-criticality. This concept has been considered in various contexts (see for instance [2], [3], [5]). We may call a graph *2-factor edge-critical* if it contains no 2-factor but for any non-edge uv there is a 2-factor in G_{uv} (which necessarily uses uv). Our $meg(n)$ are different : they may contain a 2-factor and they have a minimum number of edges.

We state our main result which will be proved in the following sections:

Proposition 1.1 *The minimum size of a 2-factor expandable graph is:*

- $Exp_2(3) = 2, Exp_2(4) = 4, Exp_2(5) = 6, Exp_2(6) = 9, Exp_2(7) = 10, Exp_2(8) = 11, Exp_2(9) = 12;$
- $Exp_2(n) = \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil, n \geq 10.$

The paper is organized as follows. In Section 2 some elementary properties of expandable graphs will be stated for later use. Section 3 will be dedicated to the presentation of $meg(n)$ for $3 \leq n \leq 9$. In Section 4 a lowerbound for $Exp_2(n)$ will be established for $n \geq 10$, while it will be shown in Section 5 that it is best possible. Variations of the construction for $n = 8p$ will be presented in Section 5.3 to handle the case $n \not\equiv 0 \pmod{8}, n \geq 14$. Finally constructions will be given for $10 \leq n \leq 13$ in Section 5.4. Some conclusions and suggestions for further research are presented in Section 6.

2 Properties of expandable graphs

We shall state some basic properties of minimal expandable graphs which will be used to compute $Exp_2(n)$.

Fact 2.1 *If $G = (V, E)$ is not expandable, any partial graph $G' = (V, E')$, with $E' \subset E$, is not expandable.*

Property 2.1 *If $G = (V, E)$ is expandable then G is connected.*

Proof: If u and v are in two distinct components then clearly uv cannot be extended. \square

Property 2.2 *If $n \geq 5$ and $\delta(G) = 1$ then $Exp_2(n) \geq \frac{3}{2}n$.*

Proof: Let u be a 1-vertex of G . If G is expandable then $G - u$ induces a clique. Since $n \geq 5$ we have $m \geq \frac{3}{2}n$. \square

Property 2.3 *Let $G = (V, E)$ be expandable, $n \geq 5$. If $v \in V$ is universal then $m \geq \frac{3}{2}(n - 1)$.*

Proof: From Property 2.2 if there is a universal vertex then $\sum_{u \in V} \delta(u) \geq n - 1 + 2(n - 1) = 3(n - 1)$. \square

Property 2.4 *Let $G = (V, E)$ be expandable. Let v be a 2-vertex of G with $N(v) = \{a, b\}$ and $ab \notin E$. If $c \in N(a) \cap N(b)$, $c \neq v$, then $\delta(c) \geq 4$.*

Proof: Consider any extension of ab : the triangle (a, b, v) is in the 2-factor. Since c is necessarily covered by another cycle, we have $\delta(c) \geq 4$. \square

Property 2.5 *Let $G = (V, E)$ be expandable and u, v be two 2-vertices. If $d(u, v) = 4$ with a 4-path $uu'wv'$ from u to v then $\delta(w) > 3$.*

Proof: $d(u, v) = 4$ implies that $u'v' \notin E$. Now if $\delta(w) \leq 3$ the non-edge $u'v'$ cannot be extended. \square

Property 2.6 *Let $n \geq 7$. If u, v are two 2-vertices and $N(u) \cap N(v) \neq \emptyset$ then $Exp_2(n) \geq \frac{3}{2}n$.*

Proof: Let G be an expandable graph with m edges. If $\delta(G) = 1$, from Property 2.2 we have $m \geq \frac{3}{2}n$. Now let $\delta(G) \geq 2$. If $w \in N(u) \cap N(v)$ then w is universal. If there are at most two 2-vertices we have $\sum_{u \in V} \delta(u) \geq (n - 1) + 4 + 3(n - 3) = 4n - 6$ and $m \geq \frac{3}{2}n$ for $n \geq 7$. If a, b, c are three 2-vertices then any non-edge xy with $x, y \neq a, b, c$ cannot be extended so $\delta(v) \geq n - 4$ for all $v \neq a, b, c$. We have $\sum_{u \in V} \delta(u) \geq (n - 1) + 6 + (n - 4)(n - 4) = n^2 - 7n + 21$ and $m \geq \frac{3}{2}n$ for $n \geq 7$. \square

3 Meg(n) for $3 \leq n \leq 9$

We will compute $Exp_2(n)$ for small values of n .

- $Exp_2(3) = 2$: Trivially P_3 the path on three vertices (Figure 1 left) is a $meg(3)$.

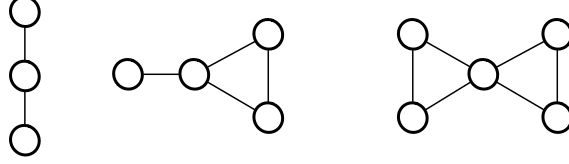


Figure 1: P_3 , the paw, the butterfly.

- $Exp_2(4) = 4$: The paw (see Figure 1 center) is expandable. If $G = (V, E)$ is a $meg(4)$ with $|E| < 4$ then from Property 2.1 G is a tree. So G is either P_4 or the claw. None of those is expandable.
- $Exp_2(5) = 6$: The butterfly (see Figure 1 right) is expandable. Let $G = (V, E)$ be a $meg(5)$ with $|E| \leq 5$. Then $|E| < \frac{3}{2}n$ and from Property 2.2, $\delta(G) \geq 2$, so G is the cycle C_5 on five vertices which is not expandable.
- $Exp_2(6) = 9$: The graph G_6 (see Figure 2 left) is expandable. From Fact 2.1 it is sufficient to consider a $meg(6)$, $G = (V, E)$, with $|E| = 8$. If there is a 1-vertex v then $G - v$ is K_5 and $|E| = 11$, a contradiction. So $\delta(G) \geq 2$. Let

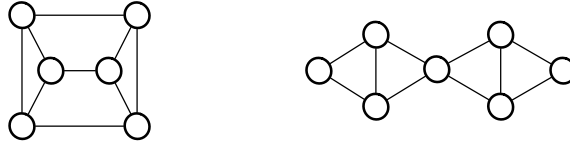


Figure 2: G_6 and G_7 two minimal expandable graphs with 6 and 7 vertices.

n_2 be the number of 2-vertices. We have $2 \leq n_2 \leq 4$.

Let $n_2 = 2$. We have $\delta(v_i) = 2, 1 \leq i \leq 2$ and $\delta(v_i) = 3, 3 \leq i \leq 6$. If $v_i \in N(v_1) \cap N(v_2)$ then v_i is universal which is impossible. Thus w.l.o.g. $N(v_1) = \{v_3, v_4\}, N(v_2) = \{v_5, v_6\}$. From Property 2.4 if $v_3v_4 \notin E$ then $\delta(v_5) > 3$, a contradiction. So w.l.o.g. $v_3v_4, v_5v_6, v_3v_5, v_4v_6 \in E$. But $v_3v_6 \notin E$ cannot be extended.

Let $n_2 = 3$. W.l.o.g. $\delta(v_1) = \delta(v_2) = \delta(v_3) = 2, \delta(v_4) = \delta(v_5) = 3, \delta(v_6) = 4$. We have $|N(v_6) \cap \{v_1, v_2, v_3\}| \geq 2$, so v_6 is universal a contradiction.

Let $n_2 = 4$. W.l.o.g. $\delta(v_1) = \delta(v_2) = \delta(v_3) = \delta(v_4) = 2$. If $\delta(v_5) = \delta(v_6) = 4$ then $|N(v_6) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$, so v_6 is universal a contradiction. So we have $d(v_5) = 3, d(v_6) = 5$ but $|N(v_5) \cap \{v_1, v_2, v_3, v_4\}| \geq 2$ and v_5 is universal a contradiction.

- $Exp_2(7) = 10$: The graph G_7 (see Figure 2 right) is expandable. Let $G = (V, E)$ be a $meg(7)$ with $|E| < 10$. Then $|E| < \frac{3}{2}n$: from Property 2.2 we have $\delta(G) \geq 2$ and from Property 2.6 two 2-vertices have no common neighbor. Each 2-vertex has its proper neighbor of degree at least three, thus there are exactly three 2-vertices, says v_1, v_2, v_3 . Thus there are exactly four 3-vertices v_4, v_5, v_6, v_7 and $|E| = 9$. Using Property 2.6 again, w.l.o.g. $v_1v_2, v_1v_4, v_2v_5, v_3v_6, v_3v_7 \in E$. If $v_6v_7 \notin E$ then $v_4 \in N(v_6) \cap N(v_7)$ and from Property 2.4 $\delta(v_4) > 3$, a contradiction. So w.l.o.g. $v_4v_6, v_5v_7, v_4v_5, v_6v_7 \in E$ but v_4v_7 cannot be extended.
- $Exp_2(8) = 11$:

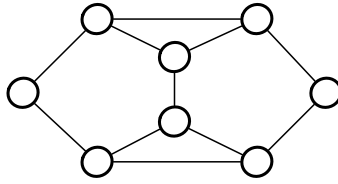


Figure 3: The graph G_8 is a $meg(8)$.

One can check that G_8 (see Figure 3) is expandable. Let $G = (V, E)$ be a $meg(8)$ with $|E| < 11$. From Properties 2.2 and 2.6 $\delta(G) \geq 2$ and each 2-vertex has its proper neighbor of degree at least three. It follows that there are at most four 2-vertices. Since $|E| \leq 10$ there are exactly four 2-vertices, says v_1, v_2, v_3, v_4 , and four 3-vertices v_5, v_6, v_7, v_8 . W.l.o.g. $v_1v_2, v_3v_4, v_1v_5, v_2v_6, v_3v_7, v_4v_8 \in E$. From Property 2.4 $v_5v_6 \notin E$. So $v_5v_7, v_5v_8, v_6v_7, v_6v_8 \in E$, but v_5v_6 cannot be extended.

- $Exp_2(9) = 12$: One can check that G_9 (see Figure 4) is expandable.

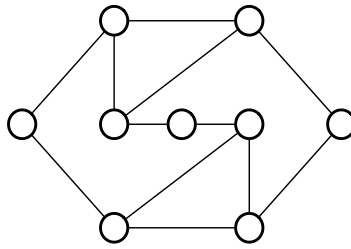


Figure 4: The graph G_9 is a $meg(9)$.

Let $G = (V, E), |E| < 12$, be a $meg(9)$. Properties 2.2 and 2.6 imply that $\delta(G) \geq 2$ and each 2-vertex has its proper neighbor of degree at least three. Then there are at most four 2-vertices. It follows that $\sum_{v \in V} \delta(v) \geq 23$ and $|E| \geq 12$. So $Exp_2(9) = 12$, a contradiction.

4 A lower bound for $Exp_2(n)$, $n \geq 10$

We will concentrate in showing a lower bound of $Exp_2(n)$ for $n \geq 10$. It will be shown in the next section that it is best possible.

Lemma 4.1 *If $G = (V, E)$ is a meg(n), $n \geq 10$, then $m \geq \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil$.*

Proof: From Property 2.1 G is connected. If there is a 1-vertex then from Property 2.2, $m \geq \frac{3}{2}n$. If two 2-vertices have a common neighbor then from Property 2.6, $m \geq \frac{3}{2}n$.

So from now on we examine the case where $\delta(G) \geq 2$ and for any two 2-vertices u, v we have $N_G(u) \cap N_G(v) = \emptyset$.

Let V_0, V_1, V_2 be a partition of V with $V_0 = \{v \in V : \delta(v) = 2\}$, $V_1 = \{v \in V, \delta(v) \geq 3, \exists u \in V_0, uv \in E\}$, $V_2 = V - (V_0 \cup V_1)$.

Each connected component of $G[V_0]$ is isomorphic either to K_1 or to K_2 (in this second case its two vertices are denoted by v and \bar{v}). Every $v \in V_1$ is such that: $|N(v) \cap V_0| = 1$.

If $|V_0| \leq 1$ then $m \geq \lceil \frac{3n-1}{2} \rceil$. So from now on we have $|V_0| \geq 2$ and $V_1 \neq \emptyset$.

Let V_0^1, V_0^2 be the partition of V_0 such that $v \in V_0^1$ if and only if $N(v) \cap V_0 = \emptyset$, $V_0^2 = V_0 \setminus V_0^1$. Define $X_0 = \{v \in V_0^1 : N(v) = \{x, y\}, xy \in E\}$ and $Y_0 = V_0^1 \setminus X_0 = \{v \in V_0^1 : N(v) = \{x, y\}, xy \notin E\}$. For $v \in V_0^2$ let $N(v) = \{\bar{v}, x\}$ with $\bar{v} \in V_0^2, x \in V_1$, $N(\bar{v}) = \{v, y\}$ with $v \in V_0^2, y \in V_1$; let $Z_0 = \{v, \bar{v} \mid xy \in E\}$ and $W_0 = V_0^2 \setminus Z_0$.

For such $v, \bar{v} \in Z_0$, from Property 2.4 we have $\delta(x), \delta(y) \geq 4$.

We will use a discharging procedure: a weight $w(v)$ is assigned to every vertex $v \in V$. At each step of the process the $w(v)$ are changed to $w'(v)$ in such a way that $\sum_{v \in V} w(v) = \sum_{v \in V} w'(v)$. At the beginning we take $w(v) = \delta(v)$ for every vertex v . When the procedure will be completed we will have $\sum_{v \in V} w'(v) = 2m \geq 3n - \lfloor \frac{n}{4} \rfloor$.

The procedure consists of two steps. In the first one vertices r with $\delta(r) \geq 4$ and possibly some 3-vertices u with a 3-path usr such that s is a 2-vertex transfer part of their charge to all 2-vertices v with $d(v, r) \leq 2$. In the second step 3-vertices u transfer part of their charge to all 2-vertices v with $d(v, u) \leq 2$.

Moreover the weight of any vertex $v \in V_0$ will be modified exactly once; initially the vertex v is *active*; after increasing its charge, v will be considered as *neutral* for all further discharging operations.

We proceed as follows:

- As long as there is r with $\delta(r) \geq 4$ such that there exists an active v with $d(v, r) = 1$: we take $w'(v) = w(v) + \frac{3}{4} = \frac{11}{4}$ and $w'(r) = w(r) - \frac{3}{4} \geq \frac{11}{4}$. Recall that no two vertices of V_0 share a same r .

From now on, each active vertex has all its neighbors of degree 2 or 3.

- For each r with $\delta(r) \geq 4$ such that there exists an active v with $d(v, r) = 2$: note that $v \notin Z_0$ otherwise $\delta(x) \geq 4$, a contradiction. We now consider the neighborhood of r .

– $\delta(s) \geq 3, \forall s \in N(r)$: From our assumption there are at most $\delta(r)$ vertices $u_i \in V_0$ such that $d(u_i, r) = 2$. Let u_1, u_2, \dots, u_k with $v = u_1$ be the vertices which are active among those. Note that in the case where $u_i \in V_0^1$ there is either one 2-path or two disjoint 2-paths between r and u_i . If there is one path from r to u_i let x_i be its internal vertex; if there are two paths, let x_i, y_i be their internal vertices. For $u_i \in V_0^2$ there is exactly one 2-path from u_i to r , let x_i denotes its internal vertex. Let λ be the number of vertices x_i and y_i , i.e., the number of 2-paths between r and the vertices $u_i, i = 1, \dots, k$. We take $w'(r) = w(r) - \frac{\lambda}{4}, w'(x_i) = w(x_i) - \frac{1}{4}, w'(y_i) = w(y_i) - \frac{1}{4}, 1 \leq i \leq k$. In the case where $u_i \in V_0^1$ and there is exactly one 2-path from u_i to r we take $w'(y_i) = w(y_i) - \frac{1}{4}$. Note that since $\lambda \leq \delta(r)$ we have $w'(r) \geq 3$ and, from Property 2.6, x_i and y_i are discharged at most once so we have $w'(x_i) \geq \frac{11}{4}$ and $w'(y_i) \geq \frac{11}{4}$. In the case where $u_i \in V_0^2$, let z_i be a neighbor of x_i distinct from r and u_i , we take $w'(z_i) = w(z_i) - \frac{1}{4}$; note that $z_i \notin V_1$ else, let $s \in V_0 \cap N(z_i), sz_i x_i u_i \bar{u}_i$ is a 4-path and from Property 2.5 $\delta(x_i) \geq 4$, a contradiction; if there is another active vertex $u_j \in V_0^2$ with a 2-path $u_j x_j z_i$, there is a 4-path between u_i and u_j so $\delta(z_i) \geq 4$; since $z_i \notin V_1$, z_i transfers at most $\frac{1}{4}\delta(z_i)$; hence, when the procedure ends, we have $w'(z_i) \geq \frac{11}{4}$. Finally, we set $w'(u_i) = w(u_i) + 1 \geq \frac{11}{4}$ if there are two 2-paths from u_i to r , or $w'(u_i) = w(u_i) + \frac{3}{4} = \frac{11}{4}$ else, for $1 \leq i \leq k$.

– $\exists s \in N(r) \cap V_0$ (s is neutral):

From our assumption we have $(N(r) - \{s\}) \cap V_0 = \emptyset$. If $v \in W_0$ then there is a 4-path from \bar{v} to s ; by Property 2.5 the neighbor x of v has degree at least 4, a contradiction. Hence $v \in V_0^1$.

Consider the case where there are two 2-paths vxr, vyr : since s is a 2-vertex, to extend vr any 2-factor must contain vr, rs and needs that at least one of x, y has a degree at least 4, a contradiction.

Thus there is exactly one 2-path vxr . Let u_1, \dots, u_k with $v = u_1$ be the active vertices such that $d(u_i, r) = 2$. As for u_1 , we have $u_2, \dots, u_k \in V_0^1$ with exactly one 2-path $u_i x_i r$; let $y_i \neq x_i$ denote the second neighbor of $u_i, 1 \leq i \leq k$. There are three cases to consider.

- * $s \in V_0^1$: we have $k \leq \delta(r) - 1$. Let $y_s \neq r$ be the second neighbor of s . We suppose first that s has been charged by r in the first step of the procedure. Notice that either $\delta(y_s) = 3$ and y_s was not used before, or $\delta(y_s) \geq 4$ and its charge is at least $\delta(y_s) - \frac{1}{4}(\delta(y_s) - 1) \geq 3$. We take $w'(r) = w(r) - \frac{(k-1)}{4}, w'(y_s) = w(y_s) - \frac{1}{4}$. Second assume that s has been charged by y_s , so the charge of r has not been modified: we take $w'(r) = w(r) - \frac{k}{4}$. Then in both cases $w'(x_i) = w(x_i) - \frac{1}{4}, w'(y_i) = w(y_i) - \frac{1}{4}, w'(u_i) = w(u_i) + \frac{3}{4}, 1 \leq i \leq k$. Notice that x_i and y_i have not been discharged before because u_i is the only 2-vertex in their neighborhood. Hence all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$.

* $s \in Z_0$: let $N(\bar{s}) = \{s, \bar{x}\}$ (recall that $\bar{x}r \in E$ and $\delta(\bar{x}) \geq 4$; so \bar{s} is neutral). Since $\delta(x_1) = \delta(y_1) = 3$ we have $x_1, y_1 \neq \bar{x}$. From our assumption it does not exist $v' \in V_0 \cap N(\bar{x}), v' \neq \bar{s}$. So $k \leq \delta(r) - 2$. We take $w'(r) = w(r) - \frac{k}{4}$, $w'(x_i) = w(x_i) - \frac{1}{4}$, $w'(y_i) = w(y_i) - \frac{1}{4}$, $1 \leq i \leq k$, and $w'(u_i) = w(u_i) + \frac{3}{4}$, $1 \leq i \leq k$. Hence all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$.

* $s \in W_0$: let $N(\bar{s}) = \{s, \bar{x}\}$ (recall that $y_1r \notin E$). If $u_1 \in X_0$ then ry_1 cannot be extended, indeed rx_1 and rs must be in the 2-factor. So $u_1 \in Y_0$.

Let $N(x_1) = \{u_1, r, z\}$. Let $z' \in N(z), z' \neq x_1$; assume $z' \in V_0$. If $rz \notin E$ then srx_1zz' is a 4-path and from Property 2.5 $\delta(x_1) \geq 4$, a contradiction. Now we consider the case $rz \in E$:

First $\delta(r) = 4$. Assume that u_1, u_2, u_3 with $u_1 = v, u_2 = z', u_3 \neq \bar{s}$ and $d(r, u_3) = 2$ are three active 2-vertices. Let rx_3u_3 be the 2-path from r to u_3 . Since u_3 is active x_3 is a 3-vertex. At least one of x_3y_1, x_3y_2 is a non-edge. W.l.o.g. $y_1x_3 \notin E$. This non edge cannot be extended: If a 2-factor $F, y_1x_3 \in F$, exists then $x_3u_3 \in F$, so $rx_3 \notin F$, $sr, x_1u_1, zz' \in F$. W.l.o.g. $rz \in F, rx_1 \notin F$, so $x_1z \notin F$, a contradiction. u_3 cannot be active. We do as follows: $w'(r) = w(r) - \frac{1}{2} = (4 - \frac{3}{4} - \frac{1}{2}) = \frac{11}{4}$, $w'(x_1) = w(x_1) - \frac{1}{4}$, $w'(y_1) = w(y_1) - \frac{1}{4}$, $w'(x_2 = z) = w(x_2) - \frac{1}{4}$, $w'(y_2) = w(y_2) - \frac{1}{4}$, and $w'(u_1) = w(u_1) + \frac{3}{4}$, $w'(u_2 = z') = w(u_2) + \frac{3}{4}$.

Second $\delta(r) \geq 5$. Let k be the number of active 2-vertices distinct from \bar{s} at distance 2 of r . We do as follows: $w'(r) = w(r) - \frac{k}{4}$, $w'(x_i) = w(x_i) - \frac{1}{4}$, $w'(y_i) = w(y_i) - \frac{1}{4}$, $w'(u_i) = w(u_i) + \frac{3}{4}$, $1 \leq i \leq k$.

Now any neighbor of z has a degree at least 3. In the case $\delta(z) = 3$ there does not exist a 2-vertex $w, w \neq u_1, d(z, w) = 2$ else there is a 4-path between u_1 and w and z has degree greater than 3, a contradiction. So z has not been used before in the discharging procedure. We do as follows: $w'(r) = w(r) - \frac{(k-1)}{4}$, $w'(z) = w(z) - \frac{1}{4}$, $w'(x_i) = w(x_i) - \frac{1}{4}$, $w'(y_i) = w(y_i) - \frac{1}{4}$, $w'(u_i) = w(u_i) + \frac{3}{4}$, $1 \leq i \leq k$. Since $k \leq \delta(r) - 1$ all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$.

In the case $\delta(z) > 3$, since all the neighbors of z have a degree at least 3, the case has been treated before and v is neutral: a contradiction.

From now on, any active $v \in V_0$ is such that $\forall r$ with $\delta(r) \geq 4$, $d(v, r) \geq 3$. It follows that $v \notin Z_0$.

- For each active $v \in V_0$ such that $d(v, u) \geq 4$ for any $u \in V_0 \setminus N[v]$

If $d(v, u) = 4$, there is a 4-path $vxx'z'u$, then from Property 2.5, $\delta(z) \geq 4$, a contradiction. So $d(v, u) \geq 5$ for any $u \in V_0 \setminus N[v]$. We have two cases:

- $v \in V_0^1$. Remark that from the discharging procedure the charge of any 3-vertex at distance at most two of v has not been decreased before since $d(v, v') > 3$ for any 2-vertex v' . Let $N(v) = \{x, y\}$ and $z \in N(x) \cup N(y)$, $z \neq v, x, y$. We take $w'(v) = w(v) + \frac{3}{4}$, $w'(x) = w(x) - \frac{1}{4}$, $w'(y) = w(y) - \frac{1}{4}$ and $w'(z) = w(z) - \frac{1}{4}$. Hence $w'(v) = \frac{11}{4}$ and all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$.

– $v \in W_0$. Let $N(x) = \{v, z_1, z_2\}$ and $N(y) = \{\bar{v}, z_3, z_4\}$. If $\{z_1, z_2\} \cap \{z_3, z_4\} \neq \emptyset$ we can suppose that $z_1 = z_3$. Then xy cannot be extended since $\delta(z_1) = 3$.

Now $\{z_1, z_2\} \cap \{z_3, z_4\} = \emptyset$. We take $w'(v) = w(v) + \frac{3}{4}$, $w'(x) = w(x) - \frac{1}{4}$ and $w'(z_i) = w(z_i) - \frac{1}{4}$, $1 \leq i \leq 2$. In the case where \bar{v} is active we also take $w'(\bar{v}) = w(\bar{v}) + \frac{3}{4}$, $w'(y) = w(y) - \frac{1}{4}$ and $w'(z_i) = w(z_i) - \frac{1}{4}$, $3 \leq i \leq 4$. Hence $w'(v) = \frac{11}{4}$, $w'(\bar{v}) \geq \frac{11}{4}$ and all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$ since for any 2-vertex s , $s \neq v, \bar{v}$, we have $d(s, z_i) \geq 3$, $i = 1, \dots, 4$.

Notice that until now, the only 3-vertices which have been discharged are at distance 1 or 2 of a 2-vertex.

- For each remaining active vertex $v \in V_0$ v is such that there is $u \in V_0$ with $d(v, u) \leq 3$.

From Property 2.6, the case $d(v, u) = 2$ cannot occur. In the case where $d(v, u) = 1$ we have $v \in W_0$ so $u = \bar{v}$. There is no other vertex u' in V_0 with $d(v, u') = 3$, else either $d(\bar{v}, u') = 2$, which is forbidden, or there is a 4-path $\bar{v}vxx'u'$ and $\delta(x) \geq 4$ from Property 2.5, a contradiction. So for any $u' \in V_0$, $u' \neq v, \bar{v}$, $d(v, u') \geq 4$, $d(\bar{v}, u') \geq 4$: this case has been handled before.

From now on each active $v \in V_0$ is such that $v \in V_0^1$ and there is $u \in V_0$ with $d(v, u) = 3$. With the same arguments as for v , $u \in V_0^1$. Let $N(v) = \{x, y\}$, $N(u) = \{x', y'\}$ and $yx' \in E$ (recall that $\delta(x) = \delta(y) = \delta(x') = \delta(y') = 3$).

We have $xx' \notin E$ (resp. $yy' \notin E$) else vx' (resp. yu) cannot be extended. Moreover $xy \notin E$ (resp. $x'y' \notin E$) else xx' (resp. yy') cannot be extended. So $v, u \in Y_0$ and $xx', yy' \notin E$.

If there is $z \in N(x) \cap N(y)$, $z \neq v$, then $\delta(z) \geq 4$ otherwise xy cannot be extended: a contradiction since $d(v, z) = 2$. With a similar argument for u , we have:

Fact 4.1 $N(x) \cap N(y) = \{v\}$ and $N(x') \cap N(y') = \{u\}$.

Let $N(y) = \{v, x', z\}$ and $N(x) = \{v, x'', y''\}$. From Fact 4.1 $xz, yx'', yy'' \notin E$ and $z \neq y', x'', y''$. Assume there is a 2-vertex w with $N(w) = \{z, w'\}$: if $x'z \notin E$ then there is a 4-path $wzyx'u$ and $\delta(y) \geq 4$, a contradiction; if $x'z \in E$ then, either one of the three edges $xy', xw', w'y'$ is not in E and this edge cannot be extended, or G is not connected: indeed $n \geq 10$, $\delta(x) = \delta(w') = \delta(y') = 3$ and G' the subgraph induced by the nine vertices $u, v, w, x, y, z, w', x', y'$ is isomorphic to G_9 (see Figure 4) and not connected to the rest of G , a contradiction. Thus there is no 2-vertex at distance 1 of z .

If there is a 2-vertex w' , $w' \neq v, u$, with $d(w', z) = 2$ then there is a 4-path $w'wzyv$ and $\delta(z) \geq 4$, a contradiction. Thus there is no 2-vertex w , $w \neq u, v$, with $d(w, z) \leq 2$, so z has not been discharged before.

Remember that any 3-vertex with an active vertex in the neighborhood has not been discharged. Thus $w(x) = w(y) = w(z) = 3$. We take $w'(v) = w(v) + \frac{3}{4}$, $w'(x) = w(x) - \frac{1}{4}$, $w'(y) = w(y) - \frac{1}{4}$ and $w'(z) = w(z) - \frac{1}{4}$. This discharging is made for all the active vertices.

For each active vertex v we have considered its neighborhood $\{x, y\}$ plus a vertex z at distance 2 of v which has no other 2-vertex at distance 1 or 2. Thus the weight of each vertex x, y, z associated to an active vertex v is decreased exactly once. Hence all vertices q involved here satisfy $w'(q) \geq \frac{11}{4}$.

The discharging procedure is done for all active vertices and $\forall v \in V$, $w'(v) \geq \frac{11}{4}$. So $2m = \sum_{i=1}^n w(v) = \sum_{i=1}^n w'(v) \geq \frac{11}{4}n$. Thus $m \geq \lceil \frac{1}{2}(3n - \lfloor \frac{n}{4} \rfloor) \rceil$. \square

5 Meg(n) for $n \geq 10$

5.1 A basic module

To build the minimum expandable graphs we define their components. Figure 5 gives the component \mathcal{H} .

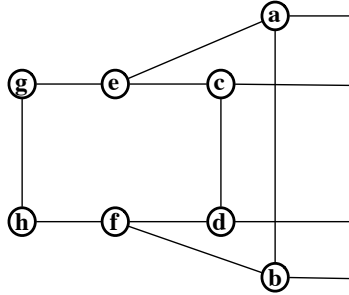


Figure 5: The subgraph \mathcal{H} with two 2-vertices and six 3-vertices.

The graph $\mathcal{H}(2)$ is as follows (see Figure 6): $\mathcal{H}(2)$ contains 2 copies $\mathcal{H}_1, \mathcal{H}_2$ of \mathcal{H} . The vertices of \mathcal{H}_i are denoted by $a_i, b_i, \dots, h_i, 1 \leq i \leq 2$. The edges between \mathcal{H}_1 and \mathcal{H}_2 are $a_1b_2, b_1a_2, c_1c_2, d_1d_2$. Notice that since \mathcal{H}_2 is a copy of \mathcal{H}_1 , there are symmetries in $\mathcal{H}(2)$. In each module, a_i, c_i, e_i, g_i is symmetric to b_i, d_i, f_i, h_i and a_i (resp. b_i) and c_i (resp. d_i) play identical roles. As a consequence, if there is a 2-factor containing, for instance, the non-edge a_1f_1 then, by *symmetry*, there is also a 2-factor containing c_1f_1 as well as a 2-factor containing b_1e_1 or $d_1e_1, a_2f_2, b_2e_2, d_2e_2, c_2f_2$. Also, a_1c_2 plays the same role as a_1d_2 .

The graph $\mathcal{H}(p), p \geq 3$, is built from $\mathcal{H}(p-1)$ as follows (see Fig. 7): add one copy \mathcal{H}_p of \mathcal{H} to $\mathcal{H}(p-1)$. The vertices of $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$ are denoted by $a_i, b_i, \dots, h_i, 1 \leq i \leq p$. Remove the two edges b_1a_{p-1} and d_1d_{p-1} of $\mathcal{H}(p-1)$. Add the four edges $b_1a_p, a_{p-1}b_p, d_1d_p, c_pd_{p-1}$. The modules \mathcal{H}_i are arranged around a cycle and numbered clockwise from 1 to p . Notice that there is a symmetry on each side of \mathcal{H}_1 between \mathcal{H}_i and \mathcal{H}_{p-i+2} , for $i = 2, \dots, \lfloor p/2 \rfloor + 1$. As for $\mathcal{H}(2)$, a_i (resp. b_i) and

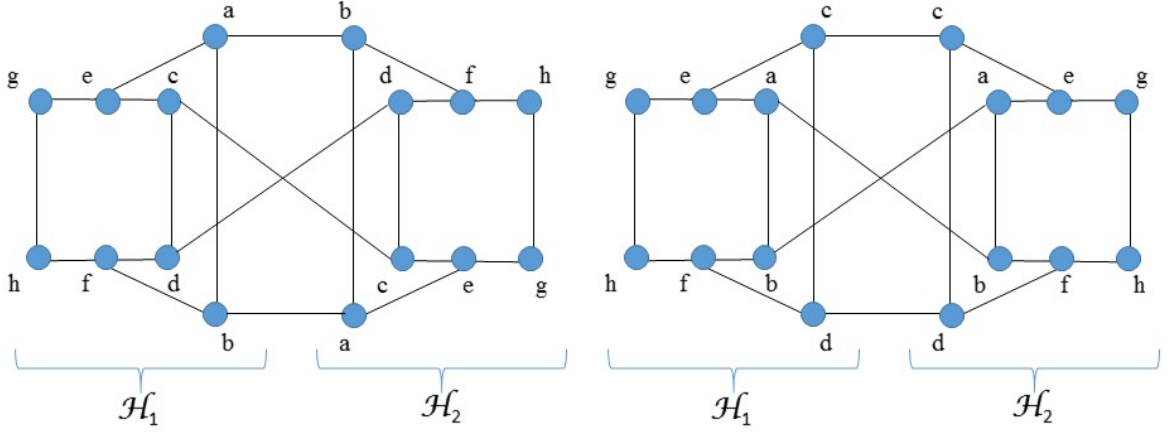


Figure 6: Two representations of $\mathcal{H}(2)$. (The index vertices are omitted).

c_i (resp. d_i) play identical roles: this can be seen by looking at Figure 7. If there is a 2-factor containing, for instance, a_1f_2 then, there is also a 2-factor containing b_1f_5 as well as a 2-factor containing c_1f_2 or d_1f_5 . Notice also that $a_p b_1$ and $d_p d_1$, the two edges linking \mathcal{H}_p to \mathcal{H}_1 play the same role.

In the following, we shall shorten many proofs by referring to all these properties as *symmetries*.

Remark 5.1 *The graph \mathcal{H}_p has a 2-factor: for instance, take the cycle $(a_1, b_2, a_2, \dots, a_p, b_1, a_1)$ and the p cycles $(c_i, e_i, g_i, h_i, f_i, d_i, c_i)$, $i = 1, \dots, p$.*

Remark 5.2 *The subgraph induced by $\mathcal{V}_i \cup \mathcal{V}_{i+1}$, $1 < i < p$, is hamiltonian. A hamiltonian cycle is $(a_i, b_i, f_i, h_i, g_i, e_i, c_i, d_i, c_{i+1}, d_{i+1}, f_{i+1}, h_{i+1}, g_{i+1}, e_{i+1}, a_{i+1}, b_{i+1}, a_i)$.*

5.2 Meg(n) for $n = 8p$, $p \geq 2$

We use a recurrence to prove that $\mathcal{H}(p)$ is a $meg(8p)$.

Property 5.1 $\mathcal{H}(2)$ is a $meg(16)$.

Proof: $\mathcal{H}(2)$ contains $m = 22 = \lceil \frac{1}{2}(3 \times 16 - \lfloor \frac{16}{4} \rfloor) \rceil$ edges.

We show that $\mathcal{H}(2)$ is expandable. Let $xy \notin E$. We give, first a chain (x, \dots, y) , and then, possibly, a set of cycles that provide a 2-factor of $\mathcal{H}(2)$.

- $xy = a_1c_1$: $(a_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1)$; by symmetry b_1d_1, b_2d_2, a_2c_2 can be extended;
- $xy = a_1d_1$: $(d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, b_2, a_1), (d_2, c_2, e_2, g_2, h_2, f_2, d_2)$; by symmetry b_1c_1, b_2c_2, a_2d_2 can be extended;

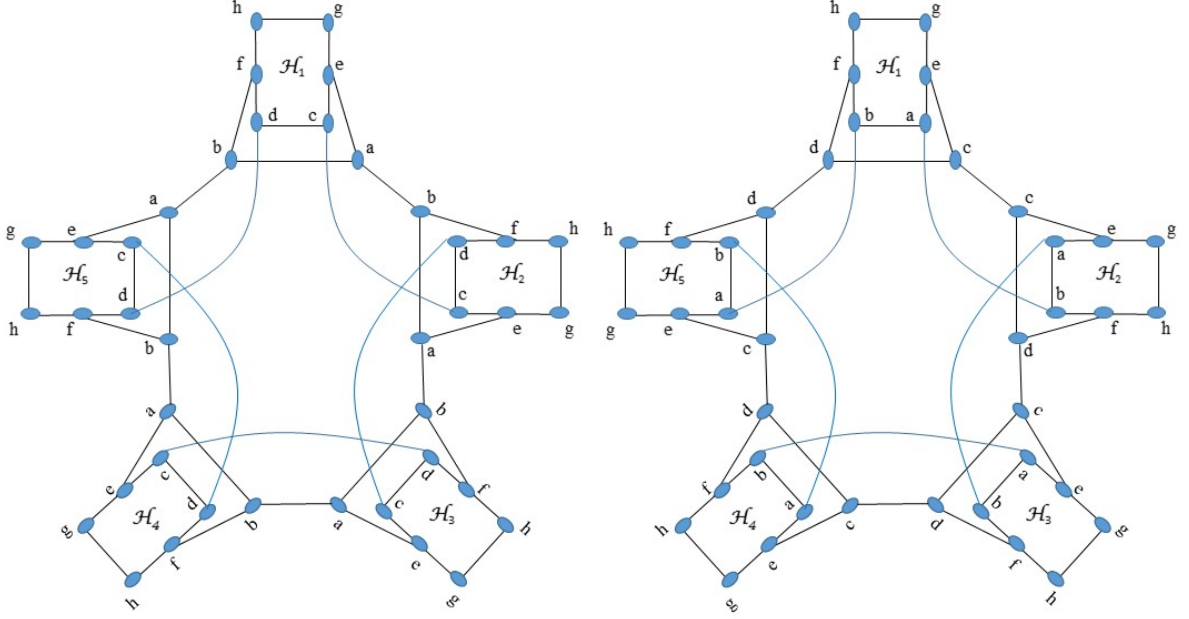


Figure 7: Two representation of $\mathcal{H}(5)$. (The vertex indices are omitted).

- $xy = a_1f_1$: $(f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$; by symmetry $d_1e_1, c_1f_1, b_1e_1, c_2f_2, d_2e_2, a_2f_2, b_2e_2$ can be extended;
- $xy = a_1g_1$: $(g_1, h_1, f_1, b_1, a_2, b_2, f_2, h_2, g_2, e_2, c_2, d_2, d_1, c_1, e_1, a_1)$; by symmetry $c_1g_1, b_1h_1, d_1h_1, b_2h_2, d_2h_2, a_2g_2, c_2g_2$ can be extended;
- $xy = a_1a_2$: $(a_2, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry b_1b_2, c_1d_2, d_1c_2 can be extended.
- $xy = a_1c_2$: $(c_2, d_2, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry $b_1d_2, b_1c_2, a_1d_2, b_2c_1, d_1b_2, a_2c_1, a_2d_1$ can be extended.
- $xy = d_1g_1$: $(g_1, h_1, f_1, b_1, a_2, b_2, a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, d_1)$; by symmetry $b_1g_1, c_1h_1, a_1h_1, c_2h_2, a_2h_2, d_2g_2, b_2g_2$ can be extended;
- $xy = e_1a_2$: $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, a_2), (c_1, d_1, d_2, c_2, c_1)$; by symmetry $b_2f_1, b_1f_2, a_1e_2, d_2e_1, c_2f_1, d_1e_2, c_1f_2$ can be extended;
- $xy = e_1b_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_1, a_1, b_2)$; by symmetry $a_2f_1, a_1f_2, b_1e_2, c_2e_1, d_2f_1, c_1e_2, d_1f_2$ can be extended;
- $xy = e_1f_1$: $(e_1, g_1, h_1, f_1), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1), (c_1, d_1, d_2, c_2, c_1)$; by symmetry e_2f_2 can be extended;
- $xy = e_1d_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2), (a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry $a_2e_1, c_2f_1, b_2f_1, c_1f_2, d_1e_2, a_1e_2, b_1f_2$ can be extended;
- $xy = e_1f_2$: $(e_1, g_1, h_1, f_1, b_1, a_1, b_2, a_2, e_2, g_2, h_2, f_2), (c_1, d_1, d_2, c_2, c_1)$; by symmetry e_1e_2, f_1f_2, e_2f_1 can be extended;

- $xy = e_1h_2$: $(e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, f_2, b_2, a_1, b_1, a_2, e_2, g_2, h_2)$; by symmetry $e_1g_2, f_1h_2, f_1g_2, e_2g_1, f_2h_1, f_2g_1, e_2h_1$ can be extended;
- $xy = f_1g_1$: $(g_1, h_1, f_1), (b_1, a_1, e_1, c_1, d_1, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1)$; by symmetry e_1h_1, f_2g_2, e_2h_2 can be extended;
- $xy = g_1b_2$: $(g_1, h_1, f_1, d_1, d_2, c_2, c_1, e_1, a_1, b_1, a_2, e_2, g_2, h_2, f_2, b_2)$; by symmetry $c_2g_1, a_2h_1, d_2h_1, a_1h_2, d_1h_2, b_1g_2, c_1g_2$ can be extended;
- $xy = g_1d_2$: $(g_1, h_1, f_1, d_1, d_2), (a_1, e_1, c_1, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_1, a_1)$; by symmetry $a_2g_1, c_2h_1, b_2h_1, c_1h_2, b_1h_2, d_1g_2, a_1g_2$ can be extended;
- $xy = g_1h_2$: $(g_1, h_1, f_1, d_1, d_2, f_2, b_2, a_2, b_1, a_1, e_1, c_1, c_2, e_2, g_2, h_2)$; by symmetry g_1g_2, h_1h_2, h_1g_2 can be extended;

□

Property 5.2 $\mathcal{H}(3)$ is a meg(24).

Proof: $\mathcal{H}(3)$ contains $m = 33 = \lceil \frac{1}{2}(3 \times 24 - \lfloor \frac{24}{4} \rfloor) \rceil$ edges.

We show that $\mathcal{H}(3)$ is expandable. Let $xy \notin E$.

Case 1. $x, y \notin \mathcal{H}_3$. Looking at the 2-factors given for $\mathcal{H}(2)$, we observe that they all contain at least one of the two edges b_1a_2, d_1d_2 and then we can build a 2-factor containing xy in $\mathcal{H}(3)$ from a 2-factor F containing xy in $\mathcal{H}(2)$. If $b_1a_2 \in F$ and $d_1d_2 \notin F$, we substitute b_1a_2 for $b_1a_3b_3a_2$ and we add the cycle $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$ to obtain a 2-factor in $\mathcal{H}(3)$. The case $b_1a_2 \notin F$ and $d_1d_2 \in F$ is symmetric. If $b_1a_2 \in F$ and $d_1d_2 \in F$, we substitute b_1a_2 for $b_1a_3b_3a_2$ and d_1d_2 for $d_1d_3f_3h_3g_3e_3c_3d_2$ to obtain a 2-factor in $\mathcal{H}(3)$.

Case 2. $x, y \notin \mathcal{H}_2$: by symmetry this is equivalent to Case 1.

Case 3. $x \in \mathcal{H}_2, y \in \mathcal{H}_3$. For each xy , we give, first a chain (x, \dots, y) , and then, possibly, a set of cycles that provide a 2-factor of $\mathcal{H}(3)$.

- $xy = a_2a_3$: $(a_2, b_3, f_3, h_3, g_3, e_3, a_3); (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry b_2b_3, d_2d_3, c_2c_3 can be extended;
- $xy = a_2c_3$: $(a_2, b_3, a_3, e_3, g_3, h_3, f_3, d_3, d_1, c_1, e_1, g_1, h_1, f_1, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3)$; by symmetry d_2b_3 can be extended;
- $xy = a_2d_3$: $(a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3), (c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry b_2c_3, c_2b_3, d_2a_3 can be extended;
- $xy = a_2e_3$: $(a_2, b_3, a_3, b_1, a_1, b_2, f_2, h_2, g_2, e_2, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3), (c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry f_2b_3, d_2f_3, e_2c_3 can be extended;
- $xy = a_2f_3$: $(a_2, b_3, a_3, e_3, g_3, h_3, f_3), (a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1)$; by symmetry e_2b_3, f_2c_3, d_2e_3 can be extended;
- $xy = a_2g_3$: $(a_2, b_3, a_3, e_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$; by symmetry h_2b_3, g_2c_3, d_2h_3 can be extended;

- $xy = a_2h_3$: $(a_2, b_3, f_3, d_3, c_3, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry g_2b_3, h_2c_3, d_2g_3 can be extended;
- $xy = b_2a_3$: $(b_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3)$, $(c_1, d_1, d_3, c_3, d_2, c_2, c_1)$, $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$; by symmetry c_2d_3 can be extended;
- $xy = b_2d_3$: $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, e_3, g_3, h_3, f_3, d_3)$; by symmetry c_2a_3 can be extended;
- $xy = b_2e_3$: $(b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, e_2, g_2, h_2, f_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$; by symmetry f_2a_3, e_2d_3, c_2f_3 can be extended;
- $xy = b_2f_3$: $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$; by symmetry e_2a_3, f_2d_3, c_2e_3 can be extended;
- $xy = b_2g_3$: $(b_2, a_1, e_1, g_1, h_1, f_1, b_1, a_3, e_3, c_3, d_3, d_1, c_1, c_2, d_2, f_2, h_2, g_2, e_2, a_2, b_3, f_3, h_3, g_3)$; by symmetry h_2a_3, g_2d_3, c_2h_3 can be extended;
- $xy = b_2h_3$: $(b_2, a_2, b_3, f_3, d_3, c_3, d_2, f_2, h_2, g_2, e_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$; by symmetry g_2a_3, h_2d_3, c_2g_3 can be extended;
- $xy = e_2e_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, b_1, a_1, e_1, g_1, h_1, f_1, d_1, c_1, c_2, d_2, c_3, d_3, f_3, h_3, g_3, e_3)$; by symmetry f_2f_3 can be extended;
- $xy = e_2f_3$: $(e_2, g_2, h_2, f_2, d_2, c_2, c_1, d_1, d_3, c_3, e_3, g_3, h_3, f_3)$; $(a_1, e_1, g_1, h_1, f_1, b_1, a_3, b_3, a_2, b_2, a_1)$; by symmetry f_2e_3 can be extended;
- $xy = e_2g_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, a_3, e_3, c_3, d_2, c_2, c_1, d_1, d_3, f_3, h_3, g_3)$; $(a_1, b_1, f_1, h_1, g_1, e_1, a_1)$; by symmetry h_2f_3, f_2h_3, g_2e_3 can be extended;
- $xy = e_2h_3$: $(e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, d_3, c_3, d_2, c_2, c_1, d_1, f_1, h_1, g_1, e_1, a_1, b_1, a_3, e_3, g_3, h_3)$; by symmetry g_2f_3, f_2g_3, h_2e_3 can be extended;
- $xy = g_2g_3$: $(g_2, h_2, f_2, b_2, a_1, b_1, f_1, h_1, g_1, e_1, c_1, d_1, d_3, f_3, h_3, g_3)$, $(a_2, e_2, c_2, d_2, c_3, e_3, a_3, b_3, a_2)$; by symmetry h_2h_3 can be extended;
- $xy = g_2h_3$: $(g_2, h_2, f_2, b_2, a_1, b_1, a_3, e_3, g_3, h_3)$, $(a_2, e_2, c_2, d_2, c_3, d_3, f_3, b_3, a_2)$, $(c_1, d_1, f_1, h_1, g_1, e_1, c_1)$; by symmetry h_2g_3 can be extended;

□

Remark 5.3 *Let $xy \notin \mathcal{H}(p)$, $p \geq 3$. If $x, y \neq h_1$ then any 2-factor containing xy contains $a_p b_1$ or $d_p d_1$.*

Proof: Let F be a 2-factor containing xy and assume that none of $a_p b_1$ and $d_p d_1$ are in F . To cover b_1 and d_1 , F must contain the edges $b_1 a_1, b_1 f_1, d_1 c_1$ and $d_1 f_1$. Clearly these 4 edges cannot be completed in a 2-factor if x and $y \neq h_1$. □

Property 5.3 $\mathcal{H}(p)$ is a meg($8p$), $p \geq 4$.

Proof: $\mathcal{H}(p)$ contains $m = 11p = \lceil \frac{1}{2}(3 \times 8p - \lfloor \frac{8p}{4} \rfloor) \rceil$ edges.

The proof is by induction. From Properties 5.1 and 5.2, $\mathcal{H}(2)$ and $\mathcal{H}(3)$ are expandable. So for $p \geq 3$ we assume that $\mathcal{H}(p)$ and $\mathcal{H}(p-1)$ are expandable.

We now prove that for any $p \geq 3$, $\mathcal{H}(p+1)$ is expandable.

Let $xy \notin \mathcal{H}(p+1)$. We examine several cases.

Case 1. $x, y \notin \mathcal{V}_1 \cup \mathcal{V}_{p+1}$. Let F be a 2-factor containing xy in $\mathcal{H}(p)$. From Remark 5.3, $a_p b_1 \in F$ or $d_p d_1 \in F$. If F contains $a_p b_1$ but not $d_p d_1$ then $a_p b_1$ is replaced by $a_p b_{p+1} a_{p+1} b_1$; adding the cycle $(c_{p+1}, d_{p+1}, f_{p+1}, h_{p+1}, g_{p+1}, e_{p+1}, c_{p+1})$ we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where $d_p d_1 \in F$ and $a_p b_1 \notin F$ is equivalent. If both $a_p b_1$ and $d_p d_1$ are in F , then $a_p b_1$ is substituted for $a_p b_{p+1} a_{p+1} b_1$ and $d_p d_1$ for $d_p c_{p+1} e_{p+1} g_{p+1} h_{p+1} f_{p+1} d_{p+1}, d_1$ to obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 2. $x \in \mathcal{V}_1$.

Case 2.1. $y \notin \mathcal{V}_p \cup \mathcal{V}_{p+1}$. From our assumption xy can be extended in $\mathcal{H}(p-1)$. If the 2-factor in $\mathcal{H}(p-1)$ contains $a_{p-1} b_1$ but not $d_{p-1} d_1$ then $a_{p-1} b_1$ is substituted for $a_{p-1} b_p a_p b_{p+1} a_{p+1} b_1$; adding the two cycles $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$, $p \leq j \leq p+1$, we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where the 2-factor in $\mathcal{H}(p-1)$ contains $d_{p-1} d_1$ but not $a_{p-1} b_1$ is equivalent. If both $a_{p-1} b_1$ and $d_{p-1} d_1$ are in the 2-factor of $\mathcal{H}(p-1)$, then $a_{p-1} b_1$ is substituted for $a_{p-1} b_p a_p b_{p+1} a_{p+1} b_1$ and $d_{p-1} d_1$ for $d_{p-1} c_p e_p g_p h_p f_p d_p c_{p+1} e_{p+1} g_{p+1} h_{p+1} f_{p+1} d_{p+1} d_1$ to obtain a 2-factor in $\mathcal{H}(p+1)$. Finally, if $a_{p-1} b_1, d_{p-1} d_1 \notin F$, we add to F the hamiltonian cycle on $\mathcal{V}_{p+1} \cup \mathcal{V}_p$ (see Remark 5.2) to obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 2.2. $y \in \mathcal{V}_p \cup \mathcal{V}_{p+1}$. By symmetry, $y \in \mathcal{V}_{p+1}$ is equivalent to $y \in \mathcal{V}_2$ and, if $p \geq 4$, $y \in \mathcal{V}_p$ is equivalent to $y \in \mathcal{V}_3$ seen just before.

It remains the case $p = 3$, $y \in \mathcal{V}_3$: we remove from $\mathcal{H}(4)$ the modules \mathcal{H}_2 and \mathcal{H}_4 and we obtain $\mathcal{H}'(2)$ isomorphic to $\mathcal{H}(2)$. Looking at the 2-factors obtained in $\mathcal{H}(2)$ for $xy \notin E$, $x \in \mathcal{V}_1, y \in \mathcal{V}_2$, we see that they all contain exactly three edges among $a_1 b_2, b_1 a_2, c_1 c_2, d_1 d_2$. Thus in $\mathcal{H}'(2)$, any 2-factor F containing $xy \notin E$, $x \in \mathcal{V}_1, y \in \mathcal{V}_3$ contains also exactly three edges among $a_1 b_3, b_1 a_3, c_1 c_3, d_1 d_3$ or, equivalently, they all miss exactly one edge among $a_1 b_3, b_1 a_3, c_1 c_3, d_1 d_3$. To obtain a 2-factor in $\mathcal{H}(4)$, we make the following substitutions: if $a_1 b_3 \notin F$, we replace $c_1 c_3$ by $c_1 c_2 d_2 c_3$ and we add the cycle $(c_2, d_2, f_2, h_2, g_2, e_2, c_2)$; if $b_1 a_3 \notin F$, we replace $d_1 d_3$ by $d_1 d_4 c_4 d_3$ and we add the cycle $(c_4, d_4, f_4, h_4, g_4, e_4, c_4)$; if $c_1 c_3 \notin F$, we replace $a_1 b_3$ by $a_1 b_2 a_2 b_3$ and we add the cycle $(c_2, d_2, f_2, h_2, g_2, e_2, c_2)$; if $d_1 d_3 \notin F$, we replace $b_1 a_3$ by $b_1 a_4 b_4 a_3$ and we add the cycle $(c_4, d_4, f_4, h_4, g_4, e_4, c_4)$.

Case 3. $x \in \mathcal{V}_{p+1}$.

Case 3.1. $y \in \mathcal{V}_i, i \neq 2$. If $y \in \mathcal{V}_1$, by symmetry, this equivalent to the case $x \in \mathcal{V}_1, y \in \mathcal{V}_{p+1}$ treated in Case 2.

If $y \in \mathcal{V}_i, i = 3, \dots, p-1$, these cases are equivalent to cases $x \in \mathcal{V}_2, y \in \mathcal{V}_{p-i+3}$ which were treated in Case 1.

If $y \in \mathcal{V}_p$, this is equivalent to $x \in \mathcal{V}_2, y \in \mathcal{V}_3$ treated in Case 1.

Case 3.2 $y \in \mathcal{V}_2$, $p \geq 4$. There are two adjacent modules \mathcal{H}_i and \mathcal{H}_{i+1} with $i \neq 2$ and $i+1 \neq p+1$. We remove these modules and add the edges $a_{i-1}b_{i+2}$ and $d_{i-1}c_{i+2}$. We obtain a graph $\mathcal{H}'(p-1)$ isomorphic to $\mathcal{H}(p-1)$ which has a 2-factor F containing xy by assumption. If $a_{i-1}b_{i+2} \in F$ but $d_{i-1}c_{i+2} \notin F$ then $a_{i-1}b_{i+2}$ is substituted for $a_{i-1}b_i a_i b_{i+1} a_{i+1} b_{i+2}$; adding the two cycles $(c_j, d_j, f_j, h_j, g_j, e_j, c_j)$, $i \leq j \leq i+1$, we have a 2-factor in $\mathcal{H}(p+1)$. By symmetry, the case where $d_{i-1}c_{i+2} \in F$ and $a_{i-1}b_{i+2} \notin F$ is equivalent. If both $a_{i-1}b_{i+2}$ and $d_{i-1}c_{i+2}$ are in the 2-factor of $\mathcal{H}'(p-1)$, then $a_{i-1}b_{i+2}$ is substituted for $a_{i-1}b_i a_i b_{i+1} a_{i+1} b_{i+2}$ and $d_{i-1}c_{i+2}$ for $d_{i-1}c_i, e_i g_i h_i f_i d_i c_{i+1} e_{i+1} g_{i+1} h_{i+1} f_{i+1} d_{i+1} c_{i+2}$ to obtain a 2-factor in $\mathcal{H}(p+1)$. Finally, if neither $a_{i-1}b_{i+2}$ nor $d_{i-1}c_{i+2}$ are in the 2-factor of $\mathcal{H}'(p-1)$, then adding the hamiltonian cycle covering $\mathcal{V}_i \cup \mathcal{V}_{i+1}$ (see Remark 5.2) we obtain a 2-factor in $\mathcal{H}(p+1)$.

Case 3.3 $y \in \mathcal{V}_2$, $p = 3$. This is the last case: $x \in \mathcal{V}_4$ and $y \in \mathcal{V}_2$. We remove the module \mathcal{H}_3 from $\mathcal{H}(4)$ and add the edges a_2b_4 and d_2c_4 : we get a graph $\mathcal{H}'(3)$ isomorphic to $\mathcal{H}(3)$ in which xy can be extended. Let F be the 2-factor of $\mathcal{H}'(3)$ containing xy .

Looking at the 2-factors given for $\mathcal{H}(3)$ to complete an edge $xy \notin \mathcal{H}(3)$ with $x \in \mathcal{V}_2$ and $y \in \mathcal{V}_3$, we see that they all contain a_2b_3 or d_2c_3 . Equivalently, F contains a_2b_4 or d_2c_4 .

If F contains a_2b_4 but not d_2c_4 then a_2b_4 is substituted for $a_2b_3a_3b_4$; adding the cycle $(c_3, d_3, f_3, h_3, g_3, e_3, c_3)$ we have a 2-factor in $\mathcal{H}(4)$. By symmetry, the case where $\mathcal{H}'(3)$ contains d_2c_4 but not a_2b_4 is equivalent. If both a_2b_4 and d_2c_4 are in F , then a_2b_4 is substituted for $a_2b_3a_3b_4$ and d_2c_4 for $d_2c_3e_3g_3h_3f_3d_3c_4$ to obtain a 2-factor in $\mathcal{H}'(4)$. \square

5.3 Meg(n) for $n \neq 0 \pmod{8}$, $n \geq 14$

Now, we give minimal expandable graphs when $n \geq 10$ is not a multiple of 8. The graphs $\mathcal{H}^{-1}(p)$, $\mathcal{H}^{-2}(p)$, $p \geq 2$, are obtained from $\mathcal{H}(p)$ by contracting one edge g_ih_i , respectively two edges g_ih_i, g_jh_j , $i \neq j$. The vertex resulting of the contraction of gh is denoted by gh .

Property 5.4 $\mathcal{H}^{-1}(p)$ is a $meg(8p-1)$, $p \geq 2$ and $\mathcal{H}^{-2}(p)$ is a $meg(8p-2)$, $p \geq 2$.

Proof: $\mathcal{H}^{-1}(p)$ contains $m = 11p - 1 = \lceil \frac{1}{2}(3 \times (8p - 1) - \lfloor \frac{8p-1}{4} \rfloor) \rceil$ edges. $\mathcal{H}^{-2}(p)$ contains $m = 11p - 2 = \lceil \frac{1}{2}(3 \times (8p - 2) - \lfloor \frac{8p-2}{4} \rfloor) \rceil$ edges.

From Properties 5.1 and 5.3 we have $\mathcal{H}(p)$ is a $meg(8p)$, $p \geq 2$. The 2-factor of $\mathcal{H}^{-1}(p)$, resp. $\mathcal{H}^{-2}(p)$, corresponds to the 2-factor of $\mathcal{H}(p)$ where the subsequence of two consecutive vertices g, h in $\mathcal{H}(p)$ is replaced by the contracted vertex gh in $\mathcal{H}^{-1}(p)$, resp. $\mathcal{H}^{-2}(p)$. \square

The graph $\mathcal{H}^{+1}(p)$, $p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a 2-vertex i and the two edges ig, ih to $\mathcal{H}(p)$.

Property 5.5 $\mathcal{H}^{+1}(p)$ is a $meg(8p+1)$, $p \geq 2$.

Proof: $\mathcal{H}^{+1}(p)$ contains $m = 11p + 2 = \lceil \frac{1}{2}(3 \times (8p + 1) - \lfloor \frac{8p+1}{4} \rfloor) \rceil$ edges.

Observe that all 2-factors of $\mathcal{H}(p)$ contain gh . Let $xy \notin E$. If $x, y \neq i_1$ then the 2-factor of $\mathcal{H}^{+1}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ where g_1h_1 in $\mathcal{H}(p)$ is replaced by $g_1i_1h_1$ in $\mathcal{H}^{+1}(p)$. If $x = i_1$ then $gy \notin E$ or $hy \notin E$. W.l.o.g. assume that $gy \notin E$. Then the 2-factor of $\mathcal{H}^{+1}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ where h_1g_1 in $\mathcal{H}(p)$ is replaced by $h_1i_1g_1$ in $\mathcal{H}^{+1}(p)$. \square

The graph $\mathcal{H}^{+2}(p)$ is obtained from $\mathcal{H}(p)$ as follows: the two edges a_1e_1, c_1e_1 are subdivided into the paths $(a_1, i, e_1), (c_1, j, e_1)$, respectively; the edge ij is added.

Property 5.6 $\mathcal{H}^{+2}(p)$ is a $meg(8p + 2), p \geq 2$.

Proof: $\mathcal{H}^{+2}(p)$ contains $m = 11p + 3 = \lceil \frac{1}{2}(3 \times (8p + 2) - \lfloor \frac{8p+2}{4} \rfloor) \rceil$ edges.

We show that $\mathcal{H}^{+2}(p)$ is expandable. Let $xy \notin E$. In the case where $x, y \notin \{a_1, c_1, e_1\}$ we proceed as follows: we know that xy can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains either the subsequence $a_1e_1c_1$, or $a_1e_1g_1$ or $c_1e_1g_1$. In the first case the subsequence is substituted for $a_1ie_1jc_1$, in the second for $a_1ije_1g_1$, in the third for $c_1jie_1g_1$. So xy can be extended in $\mathcal{H}^{+2}(p)$.

Now let $x = i$ (the case where $x = j$ is similar). If $y \neq c_1, g_1$, we know that e_1y can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains e_1g_1 ; we replace it by ije_1g_1 . If $y = c_1$ (resp. $y = g_1$): in $\mathcal{H}(p)$, the 2-factor corresponding to the non-edge g_1c_1 (resp. g_1a_1) contains $c_1e_1a_1$; we replace it by $c_1je_1ia_1$.

If $y = g_1$: in $\mathcal{H}(p)$, the 2-factor corresponding to the non-edge g_1a_1 contains $c_1e_1a_1$; we replace it by $c_1je_1ia_1$.

So iy can be extended in $\mathcal{H}^{+2}(p)$.

Now let $x = e_1$. We know that g_1y can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains both g_1h_1 and $a_1e_1c_1$ (or $c_1e_1a_1$, but from symmetry this case is the same). We substitute the first subsequence for $e_1g_1h_1$ and the second for a_1ijc_1 and e_1y can be extended in $\mathcal{H}^{+2}(p)$. \square

The graph $\mathcal{H}^{+3}(p), p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a triangle (i, j, k) and the two edges ia_1, ja_1 to $\mathcal{H}(p)$.

Property 5.7 $\mathcal{H}^{+3}(p)$ is a $meg(8p + 3), p \geq 2$.

Proof: $\mathcal{H}^{+3}(p)$ contains $m = 11p + 5 = \lceil \frac{1}{2}(3 \times (8p + 3) - \lfloor \frac{8p+3}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{i, j, k\}$ then the 2-factor of $\mathcal{H}^{+3}(p)$ corresponds to the 2-factor of $\mathcal{H}(p)$ plus (i, j, k) .

Let $x = i$ (the case $x = j$ is the same): note that $y \notin \{i, j, k, a_1\}$. Suppose first that y is not a neighbor of a_1 in $\mathcal{H}(p)$: ya_1 can be extended to a 2-factor and substituting ya_1 for $yikja_1, (i = x)$, we have a 2-factor for $\mathcal{H}^{+3}(p)$. Now let $y = e_1$. We know that e_1f_1 can be extended in $\mathcal{H}(p)$. The corresponding 2-factor contains $(e_1g_1h_1f_1e_1)$ and a cycle C with the edge a_1b_1 . Substituting a_1b_1 for $a_1jkie_1g_1h_1f_1b_1$ we have a 2-factor of $\mathcal{H}^{+3}(p)$. Let $y \in \{b_1, b_2\}$; e_1f_1 can be extended in $\mathcal{H}(p)$, and the corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$, a cycle C with the edges a_1b_1 and a_1b_2 , and a cycle C' with the edge c_1d_1 . So replacing a_1y by a_1jkiiy and c_1d_1 by $c_1e_1g_1h_1f_1d_1$ we have a 2-factor of $\mathcal{H}^{+3}(p)$.

For $x = k$, if $y \neq a_1$ we do as above by replacing the sequence ikj by kij . If $y = a_1$, we do as follows: we know that g_1a_1 can be extended in $\mathcal{H}(p)$; the corresponding 2-factor contains $e_1a_1g_1$. Thus substituting $e_1a_1g_1$ for e_1g_1 and adding (a_1, i, j, k, a_1) we obtain a 2-factor of $\mathcal{H}^{+3}(p)$. \square

The graph $\mathcal{H}^{+4}(p), p \geq 2$, is obtained from $\mathcal{H}(p)$ by adding a diamond $(i, j, k, l), ij \notin E$ and the edge ia_1 to $\mathcal{H}(p)$.

Property 5.8 $\mathcal{H}^{+4}(p)$ is a $meg(8p + 4), p \geq 2$.

Proof: $\mathcal{H}^{+4}(p)$ contains $m = 11p + 6 = \lceil \frac{1}{2}(3 \times (8p + 4) - \lfloor \frac{8p+4}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{i, j, k, l\}$ then to obtain a 2-factor of $\mathcal{H}^{+4}(p)$ we add the cycle (i, k, j, l, i) to the 2-factor of $\mathcal{H}(p)$ obtained for the extension of xy .

For $xy = ij$ we do as follows: we take a 2-factor covering $\mathcal{H}(p)$ (see Remark 5.1) and the cycle (i, j, l, k, i) to obtain a 2-factor of $\mathcal{H}^{+4}(p)$.

Let $x = i$: we have $y \notin \{i, j, k, l, a_1\}$. First we suppose that y is not a neighbor of a_1 . We know that ya_1 can be extended in $\mathcal{H}(p)$, so in the corresponding 2-factor there is a cycle with the sequence ya_1 . Substituting ya_1 for ya_1a_1 and adding the cycle (j, k, l, j) we have a 2-factor for $\mathcal{H}^{+4}(p)$. Now let $y = e_1$: We know that e_1f_1 can be extended in $\mathcal{H}(p)$. The corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$ and a cycle C with the edge a_1b_1 . Substituting the sequence a_1b_1 for $a_1ie_1g_1h_1f_1b_1$ and adding (j, k, l, j) we have a 2-factor of $\mathcal{H}^{+4}(p)$. When $y \in \{b_1, b_2\}$ we do as follows: e_1f_1 can be extended in $\mathcal{H}(p)$, the corresponding 2-factor contains $(e_1, g_1, h_1, f_1, e_1)$, a cycle C with a_1y , and a cycle C' with c_1d_1 . So substituting the sequence a_1y for a_1iy , the sequence c_1d_1 for $c_1e_1g_1h_1f_1d_1$ and adding (j, k, l, j) we have a 2-factor of $\mathcal{H}^{+4}(p)$.

Let $x = k$ (resp. $x = j$): We proceed as for $x = i$ but instead of a_1i we take a_1iljk (resp. a_1ilkj). \square

The graph $\mathcal{H}^{+5}(p)$, is obtained from $\mathcal{H}^{+2}(p)$ by subdividing the edge a_1b_1 into the path a_1lkb_1 , adding a 2-vertex r together with two edges rl, rk . See Figure 8.

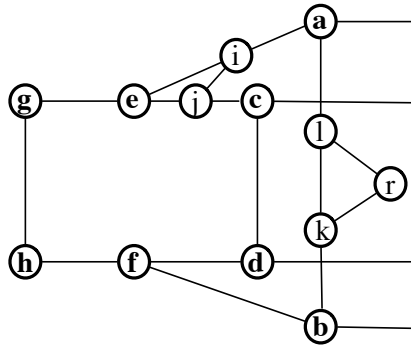


Figure 8: The graph $\mathcal{H}^{+5}(p)$ is a $meg(11p + 7)$.

Property 5.9 $\mathcal{H}^{+5}(p)$ is a $meg(8p + 5)$.

Proof: $\mathcal{H}^{+5}(p)$ contains $m = 11p + 7 = \lceil \frac{1}{2}(3 \times (8p + 5) - \lfloor \frac{8p+5}{4} \rfloor) \rceil$ edges.

From Property 5.6 we know that $\mathcal{H}^{+2}(p)$ is expandable.

Let $xy \notin E$. If xy is also a non-edge of $\mathcal{H}^{+2}(p)$ and the corresponding 2-factor contains a_1b_1 then we substitute the sequence a_1b_1 for a_1lrkb_1 ; else if a_1b_1 is not in the 2-factor, so we add the cycle (k, l, r, k) to the 2-factor. In both cases we obtain a 2-factor for $\mathcal{H}^{+5}(p)$.

Now let $xy = a_1b_1$: $(a_1, b_2, a_2, b_3, a_3, \dots, b_p, a_p, b_1), (c_1, j, i, e_1, g_1, h_1, f_1, d_1, c_1), (c_i, e_i, g_i, h_i, f_i, d_i, c_i), 2 \leq i \leq p$, and (k, l, r, k) is a 2-factor.

Let $x = k$ (the case where $x = l$ is symmetric). If $y \neq a_1$: a_1y can be extended in $\mathcal{H}^{+2}(p)$; the corresponding 2-factor contains the path $a_1 \cdots y$; we substitute it for $krla_1 \cdots y$ and we have 2-factor for $\mathcal{H}^{+5}(p)$. If $y = a_1$: $(k, r, l, a_1), (c_1, d_1, d_p, c_p, d_{p-1}, c_{p-1}, \dots, d_2, c_2, e_2, g_2, h_2, f_2, b_2, a_2, b_3, f_3, h_3, g_3, e_3, a_3, b_4, f_4, h_4, g_4, e_4, a_4, \dots, b_p, f_p, h_p, g_p, e_p, a_p, b_1, f_1, h_1, g_1, e_1, i, j, c_1)$ is a 2-factor.

Let $x = r$. Remark that all 2-factors containing the non-edge ky contain also kr ; to extend ry we replace kr by rk in these 2-factors. \square

5.4 Meg(n) for $10 \leq n \leq 13$

The constructions given in the previous sections do not cover the cases $10 \leq n \leq 13$. We give constructions for all the situations.

We start from the graph G_{10} shown in Figure 9. Note that G_{10} and $G_{10} - a$ are hamiltonian.

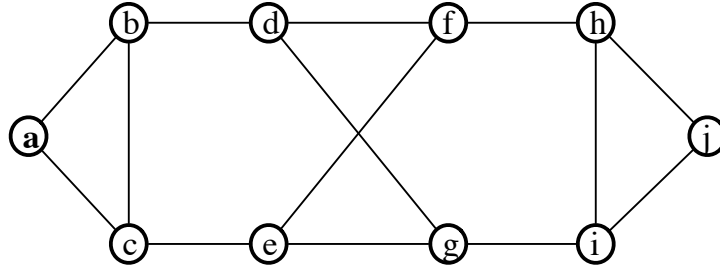


Figure 9: The graph G_{10} is a $meg(10)$.

Property 5.10 G_{10} is a $meg(10)$.

Proof: G_{10} contains $m = 14 = \lceil \frac{1}{2}(3 \times 10 - \lfloor \frac{10}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$.

- $xy = ad$: $(a, b, c, e, f, h, j, i, g, d)$ yields a 2-factor; by symmetry ae, jf, jg can be extended;
- $xy = af$: $(a, b, c, e, g, d, f), (h, i, j, h)$ yields a 2-factor; by symmetry ag, jd, je can be extended;

- $xy = ah$: $(a, b, c, e, f, d, g, i, j, h)$ yields a 2-factor; by symmetry ai, jb, jc can be extended;
- $xy = aj$: $(a, b, c, e, f, d, g, i, h, j)$ yields a 2-factor;
- $xy = be$: $(b, a, c, e), (d, g, i, j, h, f, d)$ yields a 2-factor; by symmetry cd, hg, if can be extended;
- $xy = bf$: $(b, a, c, e, g, d, f), (h, i, j, h)$ yields a 2-factor; by symmetry cg, hd, ie can be extended;
- $xy = bg$: $(b, a, c, e, f, d, g), (h, i, j, h)$ yields a 2-factor; by symmetry cf, he, id can be extended;
- $xy = bh$: $(b, a, c, e, f, d, g, i, j, h)$ yields a 2-factor; by symmetry ci can be extended;
- $xy = bi$: $(b, a, c, e, g, d, f, h, j, i)$ yields a 2-factor; by symmetry ch can be extended;
- $xy = de$: $(d, f, h, j, i, g, e), (a, b, c, a)$ yields a 2-factor; by symmetry fg can be extended.

□

The graph G_{11} shown in Figure 10 is obtained from G_{10} by the addition of the vertex k , the removal of dg, ef and the addition of dk, ek, fk, gk to E .

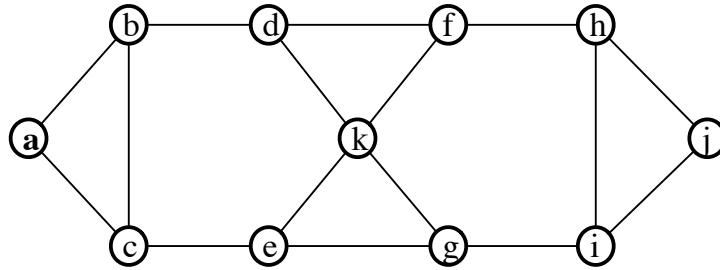


Figure 10: The graph G_{11} is a $meg(11)$.

Property 5.11 G_{11} is a $meg(11)$.

Proof: G_{11} contains $m = 16 = \lceil \frac{1}{2}(3 \times 11 - \lfloor \frac{11}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$. If $xy \notin \{dg, ef, kz \mid z \in \{a, \dots, j\}\}$ from Property 5.10 xy can be extended to a 2-factor F in G_{10} . Since at least one of dg, ef, df, eg is an edge of F , F can easily be modified to a 2-factor F' of G_{11} by replacing the edge by the 3-path passing through k .

- $xy = dg$: $(d, f, k, e, g), (a, b, c, a), (h, i, j, h)$ is a 2-factor; by symmetry ef can be extended;

- $xy = ak$: $(a, b, c, e, g, i, j, h, f, d, k)$ is a 2-factor; by symmetry jk can be extended;
- $xy = bk$: $(b, a, c, e, g, i, j, h, f, d, k)$ is a 2-factor; by symmetry ck, hk, ik can be extended.

□

The graph G_{12} shown in Figure 11 is obtained from G_{10} by the addition of the vertices k, l , the removal of dg, ef and the addition of dk, ek, fl, gl, kl to E .

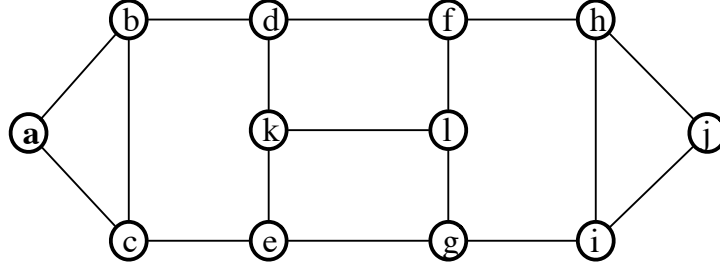


Figure 11: The graph G_{12} is a $meg(12)$.

Property 5.12 G_{12} is a $meg(12)$.

Proof: G_{12} contains $m = 17 = \lceil \frac{1}{2}(3 \times 12 - \lfloor \frac{12}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$. If $xy \notin \{dg, ef, kz, lz \mid z \in \{a, \dots, j\}\}$ from Property 5.10 xy can be extended to a 2-factor F in G_{10} . Since at least one of dg, ef, df, eg is an edge of F , F can easily be modified to a 2-factor F' of G_{12} by replacing the edge by the 4-path passing through k and l .

- $xy = dg$: $(d, b, a, c, e, k, l, f, h, j, i, g)$ is a 2-factor; by symmetry ef can be extended;
- $xy = kf$: $(k, l, g, e, c, a, b, d, f), (h, i, j, h)$ is a 2-factor; by symmetry kg, dl, el can be extended;
- $xy = ka$: $(k, d, f, l, g, e, c, b, a), (h, i, j, h)$ is a 2-factor; by symmetry lj can be extended;
- $xy = kb$: $(k, d, f, l, g, e, c, a, b), (h, i, j, h)$ is a 2-factor; by symmetry kc, lh, li can be extended;
- $xy = kj$: $(k, l, g, e, c, a, b, d, f, h, i, j)$ is a 2-factor; by symmetry la can be extended;
- $xy = ki$: $(k, l, g, e, c, a, b, d, f, h, j, i)$ is a 2-factor; by symmetry kh, lb, lc can be extended.

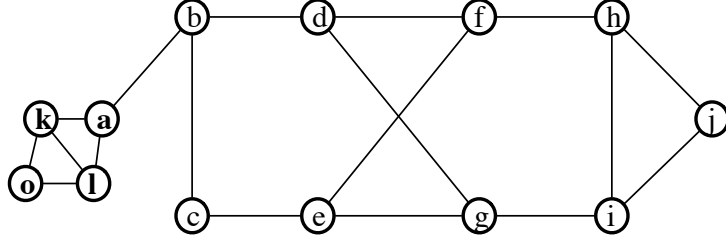


Figure 12: The graph G_{13} is a $meg(13)$.

□

The graph G_{13} shown in Figure 12 is obtained from G_{10} by the addition of the vertices k, l, o , the removal of ac . $\{a, k, l, o\}$ induce the diamond with $ao \notin E$.

Property 5.13 G_{13} is a $meg(13)$.

Proof: G_{13} contains $m = 18 = \lceil \frac{1}{2}(3 \times 13 - \lfloor \frac{13}{4} \rfloor) \rceil$ edges.

Let $xy \notin E$. If $x, y \notin \{a, k, l, o\}$ then we take F the corresponding 2-factor of G_{10} . F has bac as subsequence. Substituting bac for bc and adding (a, k, o, l, a) we have a 2-factor for G_{13} . If $xy = ao$ then (a, k, l, o) and an hamiltonian cycle of $G_{10} - a$ is a 2-factor for G_{13} .

Let $x = a$. If $y \in \{d, f, h, j\}$ we know from Property 5.10 that there exists a 2-factor F for G_{10} which does not contain ac . So adding (k, l, o, k) we have a 2-factor for G_{13} .

- $y = c$: $(a, b, d, f, h, j, i, g, e, c), (k, l, o, k)$ is a 2-factor;
- $y = e$: $(a, b, c, e), (d, g, i, j, h, f, d), (k, l, o, k)$ is a 2-factor;
- $y = g$: $(a, b, c, e, f, d, g), (h, i, j, h), (k, l, o, k)$ is a 2-factor;
- $y = i$: $(a, b, c, e, g, d, f, h, j, i), (k, l, o, k)$ is a 2-factor.

Now when $x \in \{k, l, o\}$ we use the 2-factors we obtained above for $x = a$: the first sequence begins with $kola$ or $loka$ or $okla$ instead of a . □

6 Conclusion

We have determined the values of $Exp_2(n)$ for all values of n . It could be interesting to characterize the $meg(n)$, for instance by a (finite ?) collection of forbidden induced subgraphs. Furthermore one could consider a generalization of the basic problem: the edges of a complete graph K_n are coloured in blue and in red. We want to color a minimal set of red edges in white so that any red edge uv can be extended to a 2-factor using only white edges. In our case, we had only white and red edges. Furthermore instead of just one red edge we could require that any appropriate

subset of d red edges could be extended to a 2-factor by adding only white edges. More generally we could concentrate on k -factors, $k \geq 3$, or even other structured set of edges.

References

- [1] B. Bai, H. Lu, Q. Yu, *Generalization of matching extensions in graphs (III)*, Discrete Applied Mathematics, 159-8 (2011) 727-732.
- [2] C. Berge, *Graphes*, Gauthier-Villars, (Paris, 1983).
- [3] J.A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, (2008).
- [4] M.-C. Costa, D. de Werra, C. Picouleau, *Minimal graphs for matching extensions*, Discrete Applied Mathematics, Available online.
- [5] S. Grünewald, E. Steffen, *Independent sets and 2-factors in edge-chromatic-critical graphs*, Journal of Graph Theory, 45-2 (2004) 113-118.
- [6] L. Lovász, M.D. Plummer, *Matching Theory*, Annals of discrete mathematics 29, North Holland, (1986).
- [7] M.D. Plummer, *Extending matchings in graphs: A survey*, Discrete Mathematics, 127(1994) 277-292.
- [8] J. Vandenbussche, D. B. West, *Extensions to 2 -Factors in Bipartite Graphs*, The electronic Journal of Combinatorics, 20-3 (2013) Paper #P11.
- [9] X. Wang, Z. Zhang, Y. Lin, *Bipartite matching extendable graphs*, Discrete Mathematics, 308-23 (2008) 5334-5341.