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1 **LACK OF NULL-CONTROLLABILITY FOR THE FRACTIONAL HEAT**
2 **EQUATION AND RELATED EQUATIONS***

3 ARMAND KOENIG[†]

4 **Abstract.** We consider the equation $(\partial_t + \rho(\sqrt{-\Delta}))f(t, x) = \mathbb{1}_\omega u(t, x)$, $x \in \mathbb{R}$ or \mathbb{T} . We prove it is not null-
5 controllable if ρ is analytic on a conic neighborhood of \mathbb{R}_+ and $\rho(\xi) = o(|\xi|)$. The proof relies essentially on geometric
6 optics, i.e. estimates for the evolution of semiclassical coherent states.

7 The method also applies to other equations. The most interesting example might be the Kolmogorov-type equation
8 $(\partial_t - \partial_v^2 + v^2 \partial_x)f(t, x, v) = \mathbb{1}_\omega u(t, x, v)$ for $(x, v) \in \Omega_x \times \Omega_v$ with $\Omega_x = \mathbb{R}$ or \mathbb{T} and $\Omega_v = \mathbb{R}$ or $(-1, 1)$. We prove it is not
9 null-controllable in any time if ω is a vertical band $\omega_x \times \Omega_v$. The idea is to remark that, for some families of solutions,
10 the Kolmogorov equation behaves like the rotated fractional heat equation $(\partial_t + \sqrt{i}(-\Delta)^{1/4})g(t, x) = \mathbb{1}_\omega u(t, x)$,
11 $x \in \mathbb{T}$.

12 **Key words.** null controllability, observability, fractional heat equation, degenerate parabolic equations

13 **AMS subject classifications.** 93B05, 93B07, 93C20, 35K65

14 **1. Introduction.**

15 **1.1. Problem of the null-controllability.** Consider the following equation, which is called
16 the *fractional heat equation*, where $\Omega = \mathbb{R}$ or \mathbb{T} , ω is an open subset of Ω , $\alpha \geq 0$:

$$17 \quad (\partial_t + (-\Delta)^{\alpha/2})f(t, x) = \mathbb{1}_\omega u(t, x) \quad t \in [0, T], x \in \Omega$$

18 Here, we define $(-\Delta)^{\alpha/2}$ with the functional calculus, that is, $(-\Delta)^{\alpha/2}f = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f))$ if $\Omega = \mathbb{R}$,
19 where \mathcal{F} is the Fourier transform; and $c_n((-\Delta)^{-\alpha/2}f) = |n|^\alpha c_n(f)$ if $\Omega = \mathbb{T}$, where $c_n(f)$ is the n th
20 Fourier coefficient of f .

21 It is a control problem with state $f \in L^2(\Omega)$ and control u supported in ω . More precisely, we
22 are interested in the exact null-controllability of this equation.

23 **DEFINITION 1.1.** *We say that the fractional heat equation is null-controllable on ω in time $T > 0$*
24 *if for all f_0 in $L^2(\Omega)$, there exists u in $L^2([0, T] \times \omega)$ such that the solution f of:*

$$25 \quad (1.1) \quad \begin{aligned} (\partial_t + (-\Delta)^{\alpha/2})f(t, x) &= \mathbb{1}_\omega u(t, x) & t \in [0, T], x \in \Omega \\ f(0, x) &= f_0(x) & x \in \Omega. \end{aligned}$$

26 *satisfies $f(T, x, v) = 0$ for all (x, v) in Ω .*

27 The main motivation for this study, apart from studying the fractional heat equation itself, is
28 the null-controllability of a Kolmogorov-type equation. More specifically, we are interested in the
29 following equation, where $\Omega = \Omega_x \times \Omega_v$ with $\Omega_x = \mathbb{R}$ or \mathbb{T} , $\Omega_v = \mathbb{R}$ or $(-1, 1)$ and ω is an open
30 subset of Ω :

$$31 \quad (\partial_t + v^2 \partial_x - \partial_v^2)f(t, x, v) = \mathbb{1}_\omega u(t, x, v) \quad t \in [0, T], (x, v) \in \Omega.$$

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32 For convenience, we will just say in this paper “the Kolmogorov equation”. Note that thanks to
 33 Hörmander’s bracket condition [21, Section 22.2], the operator $v^2\partial_x - \partial_v^2$ is hypoelliptic. Also, this
 34 equation is well-posed. This can be proved by Hille-Yosida’s theorem (see [2, Section 4] in the case
 35 $\Omega = \mathbb{T} \times (-1, 1)$). As we will see, this Kolmogorov equation is related to the rotated fractional heat
 36 equation.

37 **DEFINITION 1.2.** *We say that the Kolmogorov equation is null-controllable on ω in time $T > 0$
 38 if for all f_0 in $L^2(\Omega)$, there exists u in $L^2([0, T] \times \omega)$ such that the solution f of:*

$$\begin{aligned}
 & (\partial_t + v^2\partial_x - \partial_v^2)f(t, x, v) = \mathbf{1}_\omega u(t, x, v) & t \in [0, T], (x, v) \in \Omega \\
 39 \quad (1.2) \quad & f(0, x, v) = f_0(x, v) & (x, v) \in \Omega \\
 & f(t, x, v) = 0 & t \in [0, T], (x, v) \in \partial\Omega \text{ (if non-empty)}
 \end{aligned}$$

40 *satisfies $f(T, x, v) = 0$ for all (x, v) in Ω .*

41 **1.2. Statement of the results.** We will prove that the rotated fractional heat equation is
 42 never null controllable if $\Omega \setminus \omega$ has nonempty interior, and that the Kolmogorov equation is never
 43 null-controllable if $\omega = \omega_x \times \Omega_v$ where $\Omega_x \setminus \omega_x$ has nonempty interior.

44 **THEOREM 1.3.** *Let $0 \leq \alpha < 1$ and $\Omega = \mathbb{R}$ or $\Omega = \mathbb{T}$. Let ω be a strict open subset of Ω . The
 45 fractional heat equation (1.1) is not null controllable in any time on ω .*

46 This Theorem still holds in higher dimension, with $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$, but our method seems
 47 ineffective to treat the case where Ω is, say, an open subset of \mathbb{R} . This may be because we are using
 48 the spectral definition of the fractional Laplacian, and our method might be adapted if instead we
 49 used a singular kernel definition of the fractional Laplacian.

50 Actually, we prove the non-null controllability of a class of equations of the form $(\partial_t +$
 51 $\rho(\sqrt{-\Delta}))f(t, x) = \mathbf{1}_\omega(t, x)$.

52 **THEOREM 1.4.** *Let $K > 0$, $\mathcal{C} = \{\xi \in \mathbb{C}, \Re(\xi) > K, |\Im(\xi)| < K^{-1}\Re(\xi)\}$ and $\rho: \mathcal{C} \cup \mathbb{R}_+ \rightarrow \mathbb{C}$ such
 53 that*

- 54 1. ρ is holomorphic on \mathcal{C} ,
- 55 2. $\rho(\xi) = o(|\xi|)$ in the limit $|\xi| \rightarrow +\infty$, $\xi \in \mathcal{C}$,
- 56 3. ρ is measurable on \mathbb{R}_+ and $\inf_{\xi \in \mathbb{R}_+} \Re(\rho(\xi)) > -\infty$.

57 *Let $\Omega = \mathbb{R}$ or \mathbb{T} , ω be a strict open subset of Ω and $T > 0$. Then the equation*

$$58 \quad (1.3) \quad (\partial_t + \rho(\sqrt{-\Delta}))f(t, x) = \mathbf{1}_\omega u(t, x), \quad t \in [0, T], \quad x \in \Omega$$

59 *is not null-controllable on ω in time T .*

60 For lack of a better name, we will call the equation (1.3) the *generalized fractional heat equation*.
 61 This Theorem can be generalized to the case $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$. The hypothesis $\inf_{\mathbb{R}_+} \Re(\rho) > -\infty$ is
 62 only used to ensure that the equation is well-posed.

63 The fractional heat equation is the case $\rho(\xi) = \xi^\alpha$. Note that if $\alpha = 0$, then the fractional
 64 heat equation is just a family of decoupled ordinary differential equations, and the conclusion of
 65 [Theorem 1.3](#) is unimpressive. At the other end, the method used in this article does not work as-is
 66 if $\alpha = 1$, but we still expect non-null-controllability, even if this remains a conjecture if Ω is not the
 67 one-dimensional torus.

68 Some equations behave like the fractional heat equation, at least in some regimes. This is the
 69 case of the Kolmogorov equation, and if the control acts on a vertical band, we will prove it is not
 70 null-controllable with the same method.

71 THEOREM 1.5. Let $\Omega_x = \mathbb{R}$ or \mathbb{T} , let $\Omega_v = \mathbb{R}$ or $(-1, 1)$, and let $\Omega = \Omega_x \times \Omega_v$. Let $T > 0$
 72 and ω_x be a strict open subset of Ω_x . The Kolmogorov equation (1.2) is not null-controllable on
 73 $\omega = \omega_x \times \Omega_v$ in time T .

74 This Theorem can be extended to higher dimension in x and v if $\Omega_v = \mathbb{R}^d$. If we want, say
 75 $\Omega_v = (-1, 1)^d$, we lack information on the eigenvalues and eigenfunctions of $-\partial_v^2 + inv^2$ on $(-1, 1)^d$,
 76 but this is the only obstacle to the generalization of the Theorem to this case. We also give a
 77 non-null-controllability result in small time for more general control region.

78 THEOREM 1.6. Let $\Omega_x = \mathbb{R}$ or $\Omega_x = \mathbb{T}$ and $\Omega_v = \mathbb{R}$ or $\Omega_v = (-1, 1)$. Let $\Omega = \Omega_x \times \Omega_v$.
 79 Let $\omega \subset \Omega$. Assume that there exist $x_0 \in \Omega_x$ and $a > 0$ such that the symmetric vertical interval
 80 $\{(x_0, v), -a < v < a\}$ is disjoint from $\bar{\omega}$. Then, the Kolmogorov equation (1.2) is not null-controllable
 81 on ω in time $T < a^2/2$.

82 Whether this condition $T < a^2/2$ is optimal or not is an open question, but we conjecture that
 83 it is optimal, at least for some geometries. If $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b$, Theorem 1.6 proves that
 84 null-controllability does not hold in time $T < a^2/2$. This special case was already known [6, Theorem
 85 1.3], it is also proved in the same reference that null-controllability holds for some $T > 0$. Our
 86 Theorem 1.6 sharpens the lower-bound on the minimal time of null-controllability if the geometry of
 87 ω is different than a cartesian product.

88 While the fractional heat equation and the Kolmogorov equation are the main focus of this
 89 article, the method can be used to treat other equations: those that behave like the fractional heat
 90 equation for $\alpha < 1$. In Appendix A, we briefly discuss the fractional Schrödinger equation, and
 91 sketch the proof for the Kolmogorov-type equation $(\partial_t - \partial_v^2 - v\partial_x)f(t, x, v) = \mathbb{1}_\omega u(t, x, v)$ (notice
 92 the v instead of the v^2), and the improved Boussinesq equation $(\partial_t^2 - \partial_x^2 - \partial_x^2 \partial_t^2)f(t, x) = \mathbb{1}_\omega u(t, x)$.

93 1.3. Bibliographical comments.

94 **1.3.1. Control of partial differential equations.** Let A be an operator on a Hilbert space
 95 H such that the equation $\partial_t f + Af = 0$ is well-posed (i.e. $-A$ generates a strongly continuous
 96 semigroup of bounded linear operators on H , see for instance [34, Ch. 2] or [15, Sec. 2.3 and
 97 Appendix A] for the definition).

98 Let U be a Hilbert space and $B: U \rightarrow H$ a bounded linear operator. With the right choice
 99 of H , A , U and B , the problems we are interested in can be stated the following way: for every
 100 $f_0 \in H$, does there exist $u \in L^2(0, T; U)$ such that the solution of $\partial_t f + Af = Bu$, $f(0) = f_0$ satisfies
 101 $f(T) = 0$?

102 For the fractional heat equation (1.1) on \mathbb{R}^n , we choose $H = L^2(\mathbb{R}^n)$, $A = (-\Delta)^{\alpha/2}$ with
 103 domain $H^\alpha(\mathbb{R}^n)$, $U = L^2(\omega)$ and $B: u \mapsto u\mathbb{1}_\omega$. For the Kolmogorov equation (1.2) on \mathbb{R}^2 , we choose
 104 $H = L^2(\mathbb{R}^2)$, $A = -\partial_v^2 + v^2\partial_x$ (the domain of A is a bit complicated to define, see [2, Sec. 4]),
 105 $U = L^2(\omega)$ and $B: u \mapsto u\mathbb{1}_\omega$.

106 Whether there exists a $u \in L^2(0, T; U)$ such that the solution of $\partial_t f + Af = Bu$, $f(0) = f_0$
 107 satisfies $f(T) = 0$ depends of course of A , B and on the spaces H and U . Let us discuss existing
 108 results when A is a parabolic operator related to the fractional heat equation or the Kolmogorov
 109 equation.

110 **1.3.2. Null-controllability of the (fractional) heat equation in dimension one: the**
 111 **moment method.** Let us first look at the heat equation in dimension one with Dirichlet boundary
 112 conditions, i.e. $H = L^2(0, \pi)$, $D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$ and $A: f \in D(A) \mapsto -\partial_x^2 f \in L^2(0, \pi)$.
 113 Let us also denote λ_n the eigenvalues of A , and assume that λ_n is increasing, so that $\lambda_n = n^2$.

114 A possible strategy to control the heat equation in dimension one is to look for a control of the

115 form $\tilde{u}(t, x) = u(t)v(x)$. In the framework of [subsection 1.3.1](#), this is the choice $U = \mathbb{R}$ and Bu the
 116 function $x \in (0, \pi) \mapsto uv(x)$. We will call this kind of controls *shaped controls*. This is the strategy
 117 pioneered by Fattorini and Russel [18]. Let us describe it briefly.

118 Let $v: (0, \pi) \rightarrow \mathbb{R}$. Let $f_0 \in L^2(0, \pi)$, let $u \in L^2(0, T)$ and let f be the solution of $(\partial_t -$
 119 $\partial_x^2)f(t, x) = u(t)v(x)$, $f(t, 0) = f(t, \pi) = 0$ with initial condition $f(0, x) = f_0(x)$. Finally, for every
 120 $g \in L^2(0, \pi)$, let $c_n(g) := \int_0^\pi g(x) \sin(nx) dx$ be the n -th Fourier coefficient of g . Then, the relation
 121 $f(T, \cdot) = 0$ is equivalent to the *moment problem*

$$122 \quad \forall n \in \mathbb{N} \setminus \{0\}, \quad \int_0^T e^{(t-T)\lambda_n} u(t) dt = -\frac{e^{-T\lambda_n} c_n(f_0)}{c_n(v)}.$$

123 Fattorini and Russel prove such a u exists by constructing a biorthogonal family to $(e^{-t\lambda_n})_{n \in \mathbb{N} \setminus \{0\}}$,
 124 i.e. a family of functions $(g_n)_{n \in \mathbb{N} \setminus \{0\}}$ such that $\int_0^T g_n(t) e^{-\lambda_m t} dt = 1$ if $n = m$ and 0 if $n \neq m$ (see
 125 also [33] for a more streamlined proof that this family exists). Then the function u defined by

$$126 \quad u(t) = -\sum_{n>0} \frac{e^{-T\lambda_n} c_n(f_0)}{c_n(v)} g_n(T-t)$$

127 formally solves the moment problem. Moreover, we can prove some estimates on the functions
 128 $(g_n)_{n>0}$, and if $c_n(v)$ does not decay too fast when $n \rightarrow +\infty$, the series that defines u actually
 129 converges.

130 This strategy can be adapted for the fractional heat equation (1.1) when $\alpha > 1$, as Micu and
 131 Zuazua [26] already remarked. Indeed, the construction and estimate on the biorthogonal family
 132 relies on the hypotheses $\sum_{n>0} |\lambda_n|^{-1} < +\infty$ and $\lambda_{n+1} - \lambda_n \geq c > 0$. These hypotheses still hold if
 133 we replace the operator $A = -\partial_x^2$ on $(0, \pi)$ by $(-\partial_x^2)^{\alpha/2}$ as long as $\alpha > 1$. Indeed, the eigenvalues
 134 are now $\lambda_n = n^\alpha$.

135 On the other hand, if $\alpha \leq 1$, this proof does not work anymore. In fact, Micu and Zuazua [26,
 136 Sec. 5] proved that if $\alpha \leq 1$, the fractional heat equation (1.1) is not null-controllable with controls
 137 of the form $u(t)v(x)$. Miller [27, Sec. 3.3] (see also [17, Appendix]) also gets similar results, with
 138 similar methods.

139 But these negative results, based on Müntz Theorem, only concern *scalar controls*, i.e. the case
 140 where the control space is $U = \mathbb{R}$ (or \mathbb{C}). If the control space is larger, say $U = L^2(\omega)$, we cannot
 141 rule out the existence of a control with Müntz Theorem. Indeed, there are many equations which
 142 are not null-controllable with scalar controls, but that are null-controllable with a larger control
 143 space. One of them is the heat equation in dimension larger than one. Let us discuss it now.

144 **1.3.3. Null-controllability of the (fractional) heat equation and the spectral inequal-**
 145 **ity.** Let Ω be a bounded open subset of \mathbb{R}^d . Let $(\lambda_n)_{n \geq 0}$ be the sequence of the eigenvalues of
 146 $-\Delta$ on Ω with Dirichlet boundary conditions.¹ According to Weyl's law, if the dimension d is
 147 greater than 1, then $\sum_{n \geq 0} \lambda_n^{-1} = +\infty$, so we cannot prove null-controllability with the moment
 148 method. To build a control for the heat equation we have to choose a control space that is infinite
 149 dimensional [15, Th. 2.79]. We choose $U = L^2(\omega)$ and $B: u \in L^2(\omega) \mapsto u\mathbb{1}_\omega \in L^2(\Omega)$. We call this
 150 kind of controls *internal controls*.

151 To prove the null-controllability of the heat equation in any dimension, Fursikov and Im-
 152 mananuvilov [20] use *parabolic Carleman inequalities*, which are weighted energy estimates, to prove

¹I.e. $D(A) = \{f \in H_0^1(\Omega), \Delta f \in L^2(\Omega)\}$ where Δf is in the sense of distributions.

153 (more or less) directly the observability inequality that is equivalent to the null-controllability (see for
154 instance [15, Th. 2.44 or Th. 2.66] for the equivalence between null-controllability and observability).

155 Independently, Lebeau and Robbiano [25, 23] developed another strategy to prove the null-
156 controllability of the heat equation. This strategy yields more insight for our purpose, so let us give
157 more details.

158 By means of *elliptic Carleman inequalities*, Lebeau and Robbiano proved a *spectral inequality*,
159 which is the following: let M be a connected compact riemannian manifold with boundary, let ω
160 be an open subset of M , and let $(\phi_i)_{i \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of $-\Delta$ with
161 associated eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$, then there exists $C > 0$ and $K > 0$ such that for every sequence of
162 complex numbers $(a_i)_{i \in \mathbb{N}}$ and every $\mu > 0$

$$163 \quad (1.4) \quad \left| \sum_{\lambda_i < \mu} a_i \phi_i \right|_{L^2(M)} \leq C e^{K\sqrt{\mu}} \left| \sum_{\lambda_i < \mu} a_i \phi_i \right|_{L^2(\omega)}.$$

164 The key point to deduce the null-controllability of the heat equation from this spectral inequality
165 is that if one takes an initial condition of the form $f_0 = \sum_{\lambda_i \geq \mu} a_i \phi_i$ with no component along
166 frequencies less than μ , the solution of the heat equation decays like $e^{-T\mu} |f_0|_{L^2(M)}$, and the exponent
167 in μ in this decay (i.e. 1) is larger than the one appearing in the spectral inequality (i.e. 1/2).²

168 For the fractional heat equation (1.1), the dissipation stays stronger than the spectral inequality
169 as long as $\alpha > 1$. Thus, for $\alpha > 1$, we can prove the null-controllability with Lebeau and Robbiano's
170 method, as already mentioned by Micu and Zuazua [26] and Miller [27] (see also [28, 17]).

171 Our Theorem 1.3 proves that the threshold $\alpha > 1$ is optimal: if $\alpha < 1$, then the fractional
172 heat equation is not null-controllable (at least for $\Omega = \mathbb{T}^n$). Note that the case $\alpha = 1$ and $\Omega = \mathbb{T}$
173 has already been proved to lack null-controllability with internal controls [22, Th. 4]. Let us also
174 mention an article where the null-controllability of an equation closely related to our fractional heat
175 equation have been investigated [11].

176 So it seems that the Lebeau-Robbiano method is in some sense optimal: if the dissipation is
177 not stronger than the spectral inequality, then we do not have null-controllability. Let us finish
178 with another class of parabolic equations for which Lebeau and Robbiano's method does not work:
179 degenerate parabolic equations.

180 **1.3.4. Null-controllability of degenerate parabolic partial differential equations.** De-
181 generate parabolic equations are equations of the form $\partial_t f(t, x) + Af(t, x) = \mathbb{1}_\omega u(t, x)$, $0 \leq t \leq T$,
182 $x \in \Omega$, where A is a second-order differential operator which is degenerate elliptic, i.e. its principal
183 symbol $P(x, \xi)$ satisfies $P(x, \xi) \geq 0$ but is zero for some $x \in \Omega$ and $\xi \neq 0$.

184 The interest in the null-controllability of degenerate parabolic equations is more recent. We now
185 understand the null-controllability of parabolic equations degenerating at the boundary in dimension
186 one [12] and two [13] (see also references therein), where the authors found that these equations
187 where null-controllable if the degeneracy is not too strong, but might not be if the degeneracy is too
188 strong. To the best of our knowledge, the only other general family of degenerate parabolic equations
189 whose null-controllability has been investigated is hypoelliptic quadratic differential equations [8, 7].

190 Other equations have been studied on a case-by-case basis. For instance, the Kolmogorov
191 equation has been investigated since 2009 [9, 2, 6]. In these papers, the authors found that if
192 $\Omega = \mathbb{T} \times (-1, 1)$ and $\omega = \mathbb{T} \times (a, b)$ with $0 < a < b < 1$, the Kolmogorov equation is null-controllable
193 in large times, but not in time smaller than $a^2/2$, and that if $-1 < a < 0 < b < 1$, it is null-
194 controllable in arbitrarily small time [2]. If in the Kolmogorov equation (1.2) we replace v^2 by

²The exponent $\frac{1}{2}$ of μ in the spectral inequality is optimal if ω is a strict open subset of M [23, Proposition 5.5].

195 v , the null-controllability holds in arbitrarily small time if $\omega = \mathbb{T} \times (a, b)$ [9, 2]. On the other
 196 hand, if we replace v^2 by v^γ where γ is an integer greater than 2 and $\omega = \mathbb{T} \times (a, b)$, it is never
 197 null-controllable [6]. In this last article, the null-controllability of a model of the equation we are
 198 interested in, namely the equation $(\partial_t + iv^2(-\Delta_x)^{1/2} - \partial_v^2)g = 0$, is also investigated.

199 Another degenerate parabolic equation is the Grushin equation $(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t, x, y) =$
 200 $\mathbb{1}_\omega u(t, x, y)$ on $\Omega = (-1, 1) \times (0, \pi)$ with Dirichlet boundary conditions. If the control domain
 201 is a vertical band $\omega = (a, b) \times (0, \pi)$ with $0 < a < b$, there exists a minimum time for the null-
 202 controllability to hold [4]. This minimum time has since been computed [5]. On the other end, if
 203 the domain control is an horizontal band $\omega = (0, \pi) \times (a, b)$ with $(a, b) \subsetneq (0, \pi)$, then the Grushin
 204 equation is not null controllable [22].

205 Let us finally just mention an article on the heat equation on the Heisenberg group since 2017 [3],
 206 and that some parabolic equations on the real half-line, some of them related to the present work,
 207 have been shown to strongly lack controllability [16].

208 **1.4. Outline of the proof, structure of the article.** As usual in controllability problems, we
 209 focus on *observability inequalities* on the *adjoint systems*, that are equivalent to the null-controllability
 210 (see [15, Theorem 2.44]).

211 Specifically, the null-controllability of the fractional heat equation (1.1) is equivalent to the
 212 existence of $C > 0$ such that for every solution g of

$$213 \quad (1.5) \quad (\partial_t + (-\Delta)^{\alpha/2})g(t, x) = 0 \quad t \in (0, T), x \in \Omega$$

214 we have

$$215 \quad (1.6) \quad |g(T, \cdot)|_{L^2(\Omega)} \leq C|g|_{L^2((0, T) \times \omega)}.$$

216 So, to disprove the null controllability, we only have to find solutions of (1.5) that are concentrated
 217 outside ω . To construct such solutions, we consider initial states that are (essentially) *semiclassical*
 218 *coherent states*, i.e. initial states of the form $g_{0,h}: x \mapsto h^{-1/4}e^{-(x-x_0)^2/2h+ix\xi_0/h}$. We will prove that
 219 solutions of Eq. (1.5) with these initial conditions stay concentrated around x_0 . More precisely, we
 220 get asymptotic expansion of these solutions thanks to the saddle point method. We do this informally
 221 at first, in section 2, then rigorously in subsection 4.1 in the case $\Omega = \mathbb{R}$ and in subsection 4.2 in the
 222 case $\Omega = \mathbb{T}$. This proof relies on some technical computations that are done in section 3. These
 223 computations are carried over in a slightly general framework, that allows to directly treat the other
 224 equations, namely the Kolmogorov-type equations and the improved Boussinesq equation. We also
 225 sketch the proof of the generalization of Theorem 1.4 in higher dimension in subsection 4.3.

226 Let us finish this introduction by explaining how the Kolmogorov equation for $\Omega_x = \Omega_v = \mathbb{R}$ and
 227 the fractional heat equation are related. The first eigenfunction of $-\partial_v^2 + i\xi v^2$ on \mathbb{R} , is³ $e^{-\sqrt{i\xi}v^2/2}$ (up to
 228 a normalization constant), with eigenvalue $\sqrt{i\xi}$. So, $\Phi_\xi: (x, v) \in \mathbb{R}^2 \mapsto e^{i\xi x - \sqrt{i\xi}v^2/2}$ is a generalized
 229 eigenfunction of the Kolmogorov operator $v^2\partial_x - \partial_v^2$, with eigenvalue $\sqrt{i\xi}$. So, the solution of the
 230 Kolmogorov equation $(\partial_t + v^2\partial_x - \partial_v^2)f = 0$ with initial condition $f(0, x, v) = \int_{\mathbb{R}} a(\xi)\Phi_\xi(x, v) d\xi$ is
 231 $f(t, x, v) = \int_{\mathbb{R}} a(\xi)\Phi_\xi(x, v)e^{-\sqrt{i\xi}t} d\xi$. This suggests that, dropping the v variable for the moment, the
 232 Kolmogorov equation behaves like an equation where the eigenfunctions are the $e^{i\xi x}$ with eigenvalue
 233 $\sqrt{i\xi}$, i.e. the equation $(\partial_t + \sqrt{i}(-\Delta_x)^{1/4})f(t, x) = 0$ with $x \in \mathbb{R}$.

³Here and in all this paper, we choose the branch of the square root with positive real part.

234 Based on this observation and the non-null-controllability result of the rotated fractional heat
 235 equation on the whole real line, we prove in [subsection 5.2](#) that the Kolmogorov equation is not null-
 236 controllable in the case $\Omega = \mathbb{R} \times \mathbb{R}$. For Kolmogorov's equation on $\Omega = \Omega_x \times (-1, 1)$, we need some
 237 information on the eigenvalues and the eigenfunctions, which are not explicit anymore. We already
 238 proved most of what we need in another article [[22](#), Section 4]. We prove the non-null-controllability
 239 of Kolmogorov equation with $\Omega_v = (-1, 1)$ in [subsection 5.4](#).

240 Finally, we sketch the proof for the Kolmogorov equation with v instead of v^2 and for the
 241 improved Boussinesq equation in [Appendix A](#).

242 **2. Informal presentation of the proof.** As we explained in [subsection 1.4](#), we will try to
 243 derive the observability inequality (1.6). We only discuss here the case $\Omega = \mathbb{R}$.

244 Since the fractional heat equation is invariant by translation, we may assume that $\omega \subset \{|x| > \delta\}$
 245 for some $\delta > 0$. Then, for $h > 0$, we consider the solution g_h of the fractional heat equation (1.5)
 246 with initial condition $g_{0,h}(x) = e^{-x^2/2h+i\xi_0/h}$ with some $\xi_0 > 0$. The solution of the fractional heat
 247 equation is then

$$248 \quad g_h(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-t|\xi|^\alpha + ix\xi} \mathcal{F}(g_{0,h})(\xi) d\xi,$$

249 where \mathcal{F} is the Fourier transform defined by $\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi} dx$. But the Fourier transform of
 250 $g_{0,h}$ has a closed-form expression. Indeed, $\mathcal{F}(e^{-x^2/2})(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$, and using the properties of
 251 the Fourier transform (scaling and translation), we find $\mathcal{F}(g_{0,h})(\xi) = \sqrt{2\pi h}e^{-(h\xi - \xi_0)^2/2h}$. Thus

$$252 \quad g_h(t, x) = \sqrt{\frac{h}{2\pi}} \int_{\mathbb{R}} e^{-(h\xi - \xi_0)^2/2h + ix\xi - t|\xi|^\alpha} d\xi.$$

253 If we make the change of variables $\xi' = h\xi$, we find

$$254 \quad (2.1) \quad g_h(t, x) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-(\xi - \xi_0)^2/2h + ix\xi/h - t|\xi|^\alpha/h^\alpha} d\xi.$$

255 Notice that if h is small, the term $e^{-(\xi - \xi_0)^2/2h}$ is concentrated around $\xi = \xi_0$, and so is the
 256 integrand. Thus, the major part of this integral comes from a neighborhood of ξ_0 . In this situation,
 257 we can compute asymptotic expansion with the saddle point method.

258 More precisely, the saddle point method (see for instance [[32](#), Ch. 2]) is a way to compute
 259 asymptotic expansion of integrals of the form $I(h) = \int e^{\phi(x)/h} u(x) dx$ in the limit $h \rightarrow 0^+$, where ϕ
 260 and u are entire functions.

261 If the main contribution in the integral $I(h)$ comes from a nondegenerate critical point of ϕ at
 262 $x = 0$, the ‘‘standard’’ saddle point method gives

$$263 \quad (2.2) \quad I(h) \sim e^{\phi(0)/h} \sum_k \sqrt{2\pi} \tilde{u}_{2k} h^{k+1/2}$$

264 where the \tilde{u}_{2k} are of the form $A_{2k}u(0)$, and A_{2k} are differential operators of order $2k$, with in
 265 particular $\tilde{u}_0 = u(0)|\phi''(0)|^{-1/2}$.

266 The term $e^{-|\xi|^\alpha/h^\alpha}$ prevents us from applying the saddle point method to Eq. (2.1) as-is, but
 267 let us pretend we can do it anyway (the rigorous computation will be carried in [section 3](#)). Since
 268 the critical point of $\xi \mapsto -(\xi - \xi_0)^2/2 + ix\xi$ is $\xi_c = \xi_0 + ix$, we get from the saddle point method

$$269 \quad g_h(t, x) \approx e^{-(\xi_c - \xi_0)^2/2h + ix\xi_c/h - t\xi_c^\alpha/h^\alpha} = e^{-x^2/2h + ix\xi_0/h - t(\xi_0 + ix)^\alpha/h^\alpha}.$$

270 By integrating this asymptotic expansion, we get the following lower bound on the left-hand side of
 271 the observability inequality (1.6):

$$272 \quad (2.3) \quad |g_h(T, \cdot)|_{L^2(\mathbb{R})}^2 \geq |g_h(T, \cdot)|_{L^2(|x| < \delta)}^2 \geq ch^{-1} \int_{|x| < \delta} e^{-x^2/h - Ch^{-\alpha}} dx \geq c' e^{-C'h^{-\alpha}}.$$

273 Again by integrating the asymptotic expansion on g_h , we get the following upper on the right-hand
 274 side of the observability inequality

$$275 \quad (2.4) \quad |g_h|_{L^2([0, T] \times \omega)}^2 \leq |g_h|_{L^2([0, T] \times \{|x| > \delta\})}^2 \leq Ch^{-1} \int_{[0, T] \times \{|x| > \delta\}} e^{-x^2/h - ct h^{-\alpha}} dt dx \leq C' e^{-\delta^2/h}.$$

276 Comparing this upper bound (2.4) and the lower bound (2.3), and taking the limit $h \rightarrow 0^+$, we see
 277 that the observability inequality (1.6) cannot be true if $\alpha < 1$. Thus the fractional heat equation is
 278 not null-controllable.

279 **3. Some technical computations.** Before we make rigorously the proof outlined in section 2,
 280 we carry here some computations as a technical preparation.

281 **3.1. Perturbation of the saddle point method.** The “standard” saddle point method can
 282 be stated in the following way [32, Th. 2.1].

283 **THEOREM 3.1.** *Let U be an open bounded neighborhood of 0 in \mathbb{R}^d . For every $N \in \mathbb{N}$, there
 284 exists $C_N > 0$ such that for every $h > 0$ and every holomorphic function u on a complex neighborhood
 285 V of \bar{U} ,*

$$286 \quad \int_U e^{-x^2/2h} u(x) dx = \sum_{j=0}^{N-1} \frac{h^{d/2+j}}{(2\pi)^{d/2} j! 2^j} (\Delta)^j u(0) + R_N(h),$$

287 where

$$288 \quad |R_N(h)| \leq C_N \lambda h^{d/2+N} \sup_{z \in V} |u(z)|.$$

289 By using the Morse lemma (see for instance [21, Lemma C.6.1]), one can often transform integral of
 290 the form $\int e^{\phi(x)/h} u(x) dx$ into integrals of the form of **Theorem 3.1**, plus some exponentially small
 291 error. Notice that in this theorem, the phase $-x^2/2$ does not depend on h . However, to rigorously
 292 treat the integral of Eq. (2.1), we need to allow the phase to depend on h .

293 **PROPOSITION 3.2** (Perturbation of the saddle point method). *Let $h_0 > 0$ and $\epsilon: (0, h_0] \rightarrow \mathbb{R}_+$
 294 such that $\epsilon(h) \rightarrow 0$ as $h \rightarrow 0$. Let $U \subset \mathbb{R}$ be an open interval around 0. Let V be a complex open
 295 bounded neighborhood of \bar{U} in \mathbb{C} .*

296 *For every $0 < h \leq h_0$, let $r_h: V \rightarrow \mathbb{C}$ such that*

- 297 1. $\forall 0 < h \leq h_0$, r_h is holomorphic on V ,
- 298 2. there exists $C > 0$ such that for every $0 < h \leq h_0$ and $\xi \in V$, $|r_h(\xi)| \leq C\epsilon(h)$.

299 *For such r_h , we define $|r|_\epsilon := \inf\{C > 0, \forall 0 < h \leq h_0, \forall \xi \in V, |r_h(\xi)| \leq C\epsilon(h)\}$.*

300 *For every $0 < h \leq h_0$, let u_h be a holomorphic bounded function on V . We consider*

$$301 \quad I_{h,r}(u) := \int_U e^{-\xi^2/2h + r_h(\xi)/h} u_h(\xi) d\xi.$$

302 *Let $M > 0$. We have uniformly in $|r|_\epsilon < M$, and uniformly in u_h holomorphic bounded on V*

$$303 \quad I_{h,r}(u) = \sqrt{2\pi h} e^{r_h(0)/h + \mathcal{O}(\epsilon(h)^2/h)} \left(u_h(0) + \mathcal{O}((h + \epsilon(h)) |u_h|_{L^\infty(V)}) \right).$$

304 *Proof of Proposition 3.2.* The strategy is to see $\varphi_{h,r}(\xi) = -\xi^2/2 + r_h(\xi)$ as the phase, and to
 305 change the integration variable and the integration path to rewrite $I_{h,r}(u)$ in the form $I_{h,r}(u) =$
 306 $\int e^{-\xi^2/2h} v_h(\xi) d\xi$, even if this change of variables depends on h , and then apply [Theorem 3.1](#).

307 In this proof, $M > 0$ is fixed. We also choose $V' \subset \mathbb{C}$ convex such that⁴ $U \Subset V' \Subset V$. Also, we
 308 use the convention that C denotes a constant, that depends only on ϵ, M, V and V' , but not on h
 309 small enough, $|r|_\epsilon \leq M$ or $\xi \in V'$, and that may be different each line.

310 *Step 1: finding the critical point.* We claim that for h small enough, for every $|r_h|_\epsilon \leq M$, there exists
 311 a unique critical point $\xi_{c,h,r}$ of $\varphi_{h,r}$ in $\overline{V'}$, and that this critical point is non-degenerate.

312 Indeed, $\xi \in \overline{V'}$ is a critical point of $\varphi_{h,r}$ if and only if $\partial_\xi r_h(\xi) = \xi$. But for every⁵ $\xi \in V'$,
 313 $|\partial_\xi r_h(\xi)| \leq C|r_h|_{L^\infty(V)} \leq C|r|_\epsilon \epsilon(h)$. Moreover, since V' is convex, according to the mean value
 314 inequality, for $\xi, \xi' \in \overline{V'}$, $0 < h \leq h_0$ and $|r|_\epsilon \leq M$,

$$315 \quad |\partial_\xi r_h(\xi) - \partial_\xi r_h(\xi')| \leq \sup_{V'} |\partial_\xi^2 r_h| |\xi - \xi'| \leq C \sup_{V'} |r_h| |\xi - \xi'| \leq C|r|_\epsilon \epsilon(h) |\xi - \xi'| \leq CM\epsilon(h) |\xi - \xi'|.$$

316 Thus, if h is small enough such that $CM\epsilon(h) < 1$, $\xi \mapsto \partial_\xi r_h(\xi)$ takes its value in $\overline{V'}$ and is a
 317 contraction. Then, according to the contraction mapping theorem, there exists a unique $\xi_{c,h,r} \in \overline{V'}$
 318 such that $\partial_\xi r_h(\xi_{c,h,r}) = \xi_{c,h,r}$. This is the unique critical point of $\varphi_{h,r}$ in $\overline{V'}$.

319 Note that we have $|\xi_{c,h,r}| \leq C|r|_\epsilon \epsilon(h)$, where C depends only on ϵ, M, V and V' . Also, the
 320 critical value $c_{h,r} := \varphi_{h,r}(\xi_{c,h,r})$ satisfies

$$321 \quad |c_{h,r} - r_h(0)| = \left| \frac{-\xi_{c,h,r}^2}{2} + r_h(\xi_{c,h,r}) - r_h(0) \right| \leq \frac{|\xi_{c,h,r}|^2}{4} + |\xi_{c,h,r}| \sup_{V'} |\partial_\xi r_h| \leq C\epsilon(h)^2 |r|_\epsilon.$$

322 Moreover, we have $|\partial_\xi^2 \varphi_{h,r}(\xi_{c,h,r}) + I| = |\partial_\xi^2 r_h(\xi_{c,h,r})| \leq C|r|_\epsilon \epsilon(h)$. So, if h is small enough, the
 323 critical point $\xi_{c,h,r}$ is nondegenerate.

324 *Step 2: change of variables.* Now that we know where the critical point is, and what the critical
 325 value of the phase is, we want to change the intergation variables to rewrite $I_{h,r}(u)$ in the form
 326 $I_{h,r}(u) = \int e^{-\xi^2/2h} \tilde{u}_h(\xi) d\xi$.

327 We define $\psi_{h,r}(\xi)$ on V' , for h small enough by

$$328 \quad \psi_{h,r}(\xi + \xi_{c,h,r}) = \xi \left(2 \int_0^1 \partial_x^2 \varphi_{h,r}(s\xi) ds \right)^{1/2}.$$

329 According to Taylor's formula, for every $\xi \in V'$, we have $\varphi_{h,r}(\xi) = c_{h,r} + \psi_{h,r}(\xi)^2/2$.

330 So, by the change of variables/integration path $\eta = \psi_{h,r}(\xi)$, we have:

$$331 \quad (3.1) \quad I_{h,r}(u) = e^{c_{h,r}/h} \int_{\psi_{h,r}(U)} e^{-\eta^2/2h} u_h(\xi(\eta)) \frac{d\xi}{d\eta} d\eta = e^{c_{h,r}/h} \int_{\psi_{h,r}(U)} e^{-\eta^2/2h} \tilde{u}_h(\xi(\eta)) d\eta,$$

332 where $\tilde{u}_h(\eta) := u_h(\xi(\eta)) d\xi/d\eta$.

333 Note that according to the definition of $\psi_{h,r}$, for $\xi \in V'$, we have $|\psi_{h,r}(\xi) - \xi| \leq C\epsilon(h)|r|_\epsilon$.

334 Thus, we have for every $\eta \in \psi_{h,r}(V')$, $|\psi_{h,r}^{-1}(\eta) - \eta| \leq C\epsilon(h)|r|_\epsilon$. Thus, $d\eta/d\xi = 1 + \mathcal{O}(\epsilon(h)|r|_\epsilon)$.

⁴We denote $A \Subset B$ for " \bar{A} compact and $\bar{A} \subset B$ ".

⁵We will frequently use the following standard consequence of the integral Cauchy formula: if f is holomorphic on $D(z_0, r)$, then $|f^{(n)}(z_0)| \leq C_{n,r} |f|_{L^\infty(D(z_0, r))}$.

335 *Step 3: conclusion.* So, by the standard saddle point method ([Theorem 3.1](#)):

$$336 \quad (3.2) \quad \begin{aligned} I_{h,r}(u) &= e^{c_{h,r}/h} \sqrt{2\pi h} (\tilde{u}_h(0) + \mathcal{O}(h|\tilde{u}_h|_{L^\infty(V)})) \\ &= e^{c_{h,r}/h} \sqrt{2\pi h} (u_h(0) + \mathcal{O}((h + \epsilon(h))|u_h|_{L^\infty(V)})) \end{aligned}$$

337 and since $c_{h,r}/h = r_h(0)/h + \mathcal{O}(\epsilon(h)^2/h)$ the Proposition is proved. \square

338 **3.2. The framework: truncated coherent states.** As we said in the introduction, our aim
339 is to prove the lack of null-controllability of several equations, that behave in some sense like the
340 fractional heat equation. However, treating these equations requires more precise estimates than the
341 fractional heat equation does.

342 To avoid making similar computations several times, we do them in a somewhat general
343 framework. This way, we will be able to treat the other equations (Kolmogorov, etc.) directly.

344 *Hypothesis 3.3.* We consider the following domain, constants and functions:

- 345 1. let $K > 0$ and $\mathcal{C} = \{\xi \in \mathbb{C}, \Re(\xi) > K, |\Im(\xi)| < K^{-1}\Re(\xi)\}$,
- 346 2. let $\xi_0 > 0$ large enough (for instance $\xi_0 = 4(K + 1)$),
- 347 3. let $\delta > 0$ small enough such that for every $\xi \in \mathbb{R}$ and $x \in \mathbb{R}$, $|\xi - \xi_0| < \delta$ and $|x| < \delta$ implies
348 $\xi + \xi_0 + ix \in \mathcal{C}$,
- 349 4. let $\chi \in C_c^\infty(-\delta, \delta)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on a neighborhood of 0, say $(-\delta_2, \delta_2)$,
- 350 5. let X be a topological space, and $0 < h_0 < 1$, and for every $\gamma \in X$ and $0 < h \leq h_0$, let
351 $\rho_{\gamma,h}: \mathcal{C} \rightarrow \mathbb{C}$ be a holomorphic function,
- 352 6. we assume that uniformly in $\gamma \in X$ and $0 < h \leq h_0$, $\rho_{\gamma,h}(\xi) = o(|\xi|)$ in the limit $|\xi| \rightarrow +\infty$,
353 $\xi \in \mathcal{C}$,
- 354 7. finally, for every $0 < h \leq h_0$, we define

$$355 \quad \epsilon(h) := \sup_{|\xi| < \delta, |x| < \delta, \gamma \in X} h \left| \rho_{\gamma,h} \left(\frac{\xi + \xi_0 + ix}{h} \right) \right|.$$

356 The goal of the next subsections is to get upper and lower bounds on the following integral,
357 where (v_h) is a family of bounded holomorphic functions on \mathcal{C} :

$$358 \quad (3.3) \quad I_{\gamma,h,v}(x) = \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi/h + \rho_{\gamma,h}(\xi/h)} v_h(\xi) d\xi.$$

- 359 *Remark 3.4.*
- 360 1. The hypothesis 6 implies that $\epsilon(h) \rightarrow 0$ as $h \rightarrow 0$.
 - 361 2. For instance, if ρ is independent of γ and h , with $\rho(\xi) = |\xi|^\alpha$ and $0 \leq \alpha < 1$, then we have
362 $C^{-1}h^{1-\alpha} < \epsilon(h) < Ch^{1-\alpha}$ for some $C > 0$.
 - 363 3. In the applications, we will typically choose $X = [0, T]$ and for $t \in X$, $\rho_{t,h}(\xi) = -t\rho(\xi)$
364 with some $\rho: \mathcal{C} \rightarrow \mathbb{C}$ such that $\rho(\xi) = o(|\xi|)$. We will also usually choose $v_h = 1$. In
365 that case, $g_h(t, x) := I_{t,h,1}(x)$ is solution of $(\partial_t + \rho(\sqrt{-\Delta}))g_h = 1$ with initial condition
366 $g_{0,h}(x) = \sqrt{2\pi h} \chi(-ih\partial_x - \xi_0) e^{-x^2/2h + ix\xi_0/h}$, which belongs in $L^2(\mathbb{R})$. However, some
applications will require a larger parameter space X and $v_h \neq 1$.

367 3.3. Asymptotics for the evolution of coherent states.

368 **PROPOSITION 3.5.** *Assuming [Hypothesis 3.3](#), we have uniformly in $\gamma \in X$, $|x|$ small enough,*

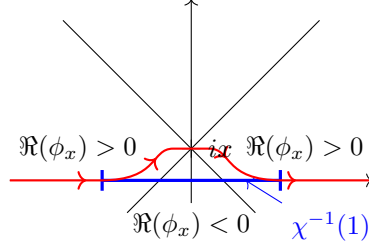


FIGURE 1. In blue, the interval where $\chi = 1$. The diagonal lines define four sectors; in the left and right ones, $\Re(\phi_x) > 0$ and in the top and bottom ones, $\Re(\phi_x) < 0$. In red, the path of integration we chose in the integral defining $I_{\gamma,h,v}(x)$ (Eq. 3.3). If $|x|$ is small enough, we choose a path that goes through the saddle point ix , but that stays in $\{\Re(\phi_x) > 0\}$.

369 $\gamma \in X$ and $v_h: \mathcal{C} \rightarrow \mathbb{C}$ holomorphic bounded

$$\begin{aligned}
 370 \quad I_{\gamma,h,v}(x) &= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi/h + \rho_{\gamma,h}(\xi/h)} v_h(\xi) d\xi \\
 371 \quad &= \sqrt{2\pi h} e^{ix\xi_0/h - x^2/2h + \rho_{\gamma,h}(\frac{\xi_0 + ix}{h})} + \mathcal{O}\left(\frac{\epsilon(h)^2}{h}\right) (v_h(\xi_0 + ix) + \mathcal{O}((h + \epsilon(h))|v_h|_{L^\infty(\mathcal{C})})), \\
 372
 \end{aligned}$$

373 in the limit $h \rightarrow 0^+$.

374 *Remark 3.6.* For most of the applications, we don't care about the term $\rho((\xi_0 + ix)/h)$, apart
 375 from the fact it is $\mathcal{O}(\epsilon(h)/h)$. Moreover, we usually have $v_h(\xi) = 1$, or at least $v_h(\xi) \rightarrow 1$ as $h \rightarrow 0$,
 376 uniformly in $\xi \in \mathcal{C}$. Under this condition, [Proposition 3.2](#) implies the slightly less precise asymptotic
 377 expansion

$$378 \quad I_{\gamma,h,v}(x) = \sqrt{2\pi h} e^{ix\xi_0/h - x^2/2h + \mathcal{O}(\frac{\epsilon(h)}{h})} (1 + o(1)),$$

379 which will be enough in most cases.

380 *Proof.* The idea is that this integral has almost the form of [Proposition 3.2](#). Let us actually
 381 rewrite it as such.

382 *Step 1: change of integration path.* Note that $\chi(\xi - \xi_0) \equiv 1$ for $\xi_0 - \delta_2 < \xi < \xi_0 + 2\delta_2$, so $\chi(\xi - \xi_0)$
 383 extends holomorphically to $|\Re(\xi) - \xi_0| < \delta_2$ by 1. Moreover, if $\xi \in \mathbb{C}$ and $|\Re(\xi) - \xi_0| < \delta_2$, $|\Im(\xi)| < \delta_2$,
 384 then, according to [Hypothesis 3.3](#) item 3, $\xi \in \mathcal{C}$. We deduce that the integrand of $I_{\gamma,h,v}(x)$ is
 385 holomorphic on $\{\xi \in \mathbb{C}, |\Re(\xi) - \xi_0| < \delta_2, |\Im(\xi)| < \delta_2\}$.

386 Thus, we can change the integration path of $I_{\gamma,h,v}(x)$, as long as we modify it only between
 387 $\xi_0 - \delta_2$ and $\xi_0 + \delta_2$, and that the modified part of the integration path stays inside $\{|\Re(\xi) - \xi_0| <$
 388 $\delta_2, |\Im(\xi)| < \delta_2\}$.

389 Let $\xi_c = \xi_0 + ix$ be the critical point of $\varphi_x(\xi) := -(\xi - \xi_0)^2/2 + ix\xi$. We choose an integration
 390 path Γ parametrized by $\Gamma(t) = t + \xi_0 + ix\chi_2(t)$, where $\chi_2 \in C_c^\infty(-\delta_2, \delta_2)$ with $0 \leq \chi_2 \leq 1$ and
 391 $\chi_2 \equiv 1$ on a neighborhood of 0 (say $(-\delta_3, \delta_3)$). Then, we have

$$\begin{aligned}
 392 \quad I_{\gamma,h,v}(x) &= \int_{\Gamma} \chi(\Re(z)) e^{\varphi_x(z)/h - \rho_{\gamma,h}(z/h)} v_h(z) dz \\
 393 \quad &= \int_{-\delta_3}^{\delta_3} e^{-t^2/2h - x^2/2h + ix\xi_0/h - \rho_{\gamma,h}(\frac{\xi_0 + ix + t}{h})} v_h(\xi_0 + ix + t) dt + R_{\gamma,h,v}(x), \\
 394
 \end{aligned}$$

395 where we used that for $\delta_3 < t < \delta_3$, $\varphi_x(\Gamma(t)) = -(t+ix)^2/2 + ix(t+\xi_0+ix) = -t^2/2 - x^2/2 + ix\xi_0$,
 396 and where $R_{\gamma,h,v}(x)$ is the part of the integral out of $(-\delta_3, \delta_3)$.

397 *Step 2: upper-bound for the remainder.* Since $\text{support}(\chi) \subset (-\delta, \delta)$, so that the integrand is 0 for
 398 $|t| > \delta$, the remainder $R_{\gamma,h,v}(x)$ is upper-bounded by

$$399 \quad |R_{\gamma,h,v}(x)| \leq 2\delta e^{-\delta_3^2/2h + \epsilon(h)/h} |\chi_2'|_{L^\infty}.$$

400 where we used the definition of ϵ ([Hypothesis 3.3](#) item 7) to bound $\rho_{\gamma,h}$. Moreover, $\epsilon(h) \rightarrow 0$, so we
 401 have (for instance)

$$402 \quad (3.4) \quad |R_{\gamma,h,v}(x)| \leq C e^{-\delta_3^2/4h} |v_h|_{L^\infty}.$$

403 *Step 3: asymptotic expansion for the integral in $(-\delta_3, \delta_3)$.* To get an asymptotic expansion of the
 404 part of the integral between $-\delta_3$ and δ_3 , we can apply [Proposition 3.2](#). Indeed, for $0 < h \leq h_0$,
 405 $\gamma \in X$, $|x| < \delta_3/2$ and ξ in a small enough complex neighborhood U of $[-\delta_3, \delta_3]$, let

$$406 \quad r_{\gamma,x,h}(\xi) = h\rho_{\gamma,h} \left(\frac{\xi_0 + ix + \xi}{h} \right).$$

407 By definition of $\epsilon(h)$, we have $|r_{\gamma,h,x}| < \epsilon(h)$, or, in other words, $|r_{\gamma,x,h}|_\epsilon \leq 1$. For the same
 408 parameters, we also define

$$409 \quad u_{x,h}(\xi) = v_h(\xi_0 + ix + \xi).$$

410 Then, according to [Proposition 3.2](#),

$$411 \quad I_{\gamma,h,v}(x) = \int_{-\delta_3}^{\delta_3} e^{-x^2/2h + ix\xi_0/h - t^2/2h + r_{\gamma,x,h}(t)/h} u_{x,h}(t) dt + R_{\gamma,h,v}(x)$$

$$412 \quad = \sqrt{2\pi h} e^{-x^2/2h + ix\xi_0/h + r_{\gamma,x,h}(0)/h} \left(u_{x,h}(0) + \mathcal{O}((h + \epsilon(h)) |u_{x,h}|_{L^\infty(U)}) \right) + R_{\gamma,h,v}(x)$$

$$413$$

414 uniformly in $\gamma \in X$, $|x| < \delta_3/2$ and v_h holomorphic bounded on \mathcal{C} .

415 *Step 4: conclusion.* With the upper-bound (3.4) on $R_{\gamma,h,v}(x)$, the claimed asymptotic expansion
 416 follows. \square

417 **3.4. Upper bounds for the evolution of coherent states.** We will also need upper bounds
 418 for $I_{\gamma,h,v}(x)$ that hold for large x .

419 **PROPOSITION 3.7.** *We assume [Hypothesis 3.3](#), except that the item 6 is only assumed to hold*
 420 *locally uniformly in $\gamma \in X$ (and uniformly in $0 < h \leq h_0$). We define for $\gamma \in X$ and $0 < h \leq h_0$*

$$421 \quad \epsilon_\gamma(x) := \sup_{|\xi| < \delta, |x| < \delta} h \Re \left(\rho_{\gamma,h} \left(\frac{\xi + \xi_0 + ix}{h} \right) \right).$$

422 *Let $\eta > 0$. For every $N > 0$, there exist $c, C > 0$ such that for every $|x| > \eta$, $\gamma \in X$ and v_h*
 423 *holomorphic bounded on \mathcal{C} ,*

$$424 \quad |I_{\gamma,h,v}(x)| \leq \frac{C}{|x|^N} e^{-c/h + \epsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}.$$

425 **Remark 3.8.** 1. In the applications, we will typically choose $X = \mathbb{R}_+$ and $\rho_{t,h}(\xi) = -t\rho(\xi)$.

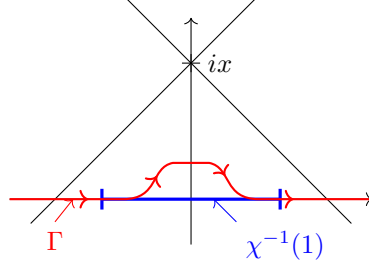


FIGURE 2. As in Figure 1, the interval where $\chi = 1$ in blue. If x is not too small, we deform a bit the integration path toward ix .

426 2. For instance, consider the case $X = \mathbb{R}_+$ and $\rho_{t,h}(\xi) = -tz\xi^\alpha$, where $\Re(z) > 0$. This choice
 427 is relevant to the equation $(\partial_t + z(-\Delta)^{\alpha/2})f(t, x) = \mathbb{1}_\omega u(t, x)$. If $K > 0$ is large enough, for
 428 every $\xi \in \mathcal{C}$, $\Re(z\rho(\xi)) \geq c|\xi|^\alpha$ for some $c > 0$.⁶ Then $\epsilon_t(h) \leq -cth^{1-\alpha}$.

429 *Proof. Step 1: integration by parts.* First, we integrate by parts to get the decay in x . As
 430 in the previous proof, we denote $\varphi_x(\xi) = -(\xi - \xi_0)^2/2 + ix\xi$. We remark that $h\partial_\xi e^{\varphi_x(\xi)/h} =$
 431 $-(\xi - \xi_0 - ix)e^{\varphi_x(\xi)/h}$. Thus, with $L_x = \frac{1}{h}\partial_\xi \frac{1}{\xi - \xi_0 - ix}$, we have

$$432 \quad I_{\gamma,h,v}(x) = \int_{-\infty}^{+\infty} e^{-\varphi_x(\xi)} L_x^N \left(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\frac{\xi}{h})} v_h(\xi) \right) d\xi.$$

433 *Step 2: change of integration path.* Next, as in the proof of Proposition 3.5, we can change the
 434 integration path between $\xi_0 - \delta_2$ and $\xi_0 + \delta_2$, as long as this modification stays inside $\{|\Re(\xi) - \xi_0| <$
 435 $\delta_2, |\Im(\xi)| < \delta_2\}$. We choose χ_2 as in the proof of Proposition 3.5, i.e. $\chi_2 \in C_c^\infty(-\delta_2, \delta_2)$, $0 \leq \chi_2 \leq 1$,
 436 $\chi_2 \equiv 1$ on $(-\delta_3, \delta_3)$. Then we choose the path $\Gamma(t) = t + i\eta_2 \operatorname{sgn}(x)\chi_2(t - \xi_0)$, where $\eta_2 > 0$ is small
 437 enough, for instance $\eta_2 = \min(\eta/2, \delta_2/2)$ (see Figure 2).

438 *Step 3: upper-bound for $e^{\varphi_x(\xi)}$.* On this path Γ , for every $|x| > \eta$ and $0 < h \leq h_0$,

$$439 \quad |e^{\varphi_x(\Gamma(t))/h}| = e^{\Re(\varphi_x(\Gamma(t)))}$$

$$440 \quad = e^{(-t^2/2 + \eta_2^2 \chi_2^2(t - \xi_0)^2/2 - |x|\eta_2 \chi_2(t - \xi_0))/h}$$

$$441 \quad \leq e^{-t^2/2h - |x|\eta_2 \chi_2(t - \xi_0)/2h},$$

443 where we used that $|x| > \eta_2$ and $0 \leq \chi_2 \leq 1$. Thus, for some $c > 0$, we have for every $|x| > \eta_2$ and
 444 $t \in \mathbb{R}$

$$445 \quad (3.5) \quad |e^{\varphi_x(\Gamma(t))/h}| \leq e^{-c/h}.$$

446 *Step 4: upper bound for the rest of the integrand.* We claim that there exists $C_N > 0$ such that for
 447 every $f \in C^\infty$ on Γ , for every $|x| > \eta$ and $\xi \in \Gamma$,

$$448 \quad (3.6) \quad |L_x^N f(\xi)| \leq \frac{C_N}{|hx|^N} \sum_{k \leq N} |\partial_\xi^k f(\xi)|.$$

⁶Indeed, if $z = r_0 e^{i\theta_0}$, then $|\theta_0| < \pi/2$. And if $\xi = r e^{i\theta}$, then $\Re(z\xi^\alpha) = r_0 r^\alpha \cos(\alpha\theta + \theta_0)$. But if $\xi \in \mathcal{C}$, then $r|\sin(\theta)| \leq rK^{-1} \cos(\theta)$, or $|\theta| \leq \arctan(K^{-1})$. So, if K is large enough, for every $\xi = r e^{i\theta} \in \mathcal{C}$, $|\alpha\theta + \theta_0| \leq \pi/2 - \tau$ for some $\tau > 0$. Then, $\Re(z\xi^\alpha) = r_0 r^\alpha \cos(\alpha\theta + \theta_0) \geq r_0 r^\alpha \cos(\pi/2 - \tau) = c|\xi|^\alpha$.

449 Indeed, according to Leibniz' rule, for any $k \geq 0$, $\xi \in \Gamma$ and $f \in C^\infty$ on Γ ,

$$450 \quad \partial_\xi^k L_x f(\xi) = h^{-1} \partial_\xi^{k+1} \frac{f(\xi)}{\xi - \xi_0 - ix} = h^{-1} \sum_{\ell \leq k+1} C_{k,\ell} \frac{\partial_\xi^\ell f(\xi)}{(\xi - \xi_0 - ix)^{k+2-\ell}}.$$

451 So, reminding that $|x| > \eta$ and $|\Im(\xi)| < \frac{\eta}{2}$

$$452 \quad |\partial_\xi^k L_x f(\xi)| \leq \frac{C_k}{|hx|} \sum_{\ell \leq k+1} |\partial_\xi^\ell f(\xi)|.$$

453 By iterating this estimate, we get the upper bound (3.6). Now, choosing $f(\xi) = \chi(\xi - \xi_0) e^{-\rho_{\gamma,h}(\xi/h)} v_h(\xi)$ ▀
454 we get

$$455 \quad |L_x^N(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi))| \leq \frac{C_N}{|hx|^N} \sum_{k \leq N} \left| \partial_\xi^k \left(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right) \right|$$

$$456 \quad \leq \frac{C'_N}{|hx|^N} \sum_{k \leq N} \left| \partial_\xi^k \left(e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right) \right|.$$

458 Moreover, for any f holomorphic on \mathcal{C} , and for any $\xi \in \Gamma$ and $r > 0$ such that $D(\xi, r) \subset \mathcal{C}$, the
459 Cauchy integral formula implies that $|\partial_\xi^k f(\xi)| \leq C_r |f|_{L^\infty(D(\xi,r))}$. So,

$$460 \quad |L_x^N(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi))| \leq \frac{C''_N}{|hx|^N} \left| e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right|_{L^\infty(D(\xi,r))}$$

$$461 \quad (3.7) \quad \leq \frac{C''_N}{|hx|^N} e^{\varepsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}.$$

463 *Step 5: conclusion.* Putting together the bounds (3.7) and (3.5), we get

$$464 \quad |I_{\gamma,h,v}(x)| \leq \frac{C_N}{|x|^N} h^{-N} e^{-c/h + \varepsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})},$$

465 which implies the claimed estimate. ▀

466 We also have the following upper-bound, which is weaker but valid for any x , even small. We
467 will need it for some applications.

468 **PROPOSITION 3.9.** *Under the same hypotheses as Proposition 3.7, there exist $c, C > 0$ such that*
469 *for every $x \in \mathbb{R}$, $\gamma \in X$ and v_h holomorphic bounded on \mathcal{C} ,*

$$470 \quad |I_{\gamma,h,v}(x)| \leq C e^{\varepsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}.$$

471 *Proof.* It is only the integral triangle inequality. ▀

472 4. Non-null-controllability of the generalized fractional heat equation.

473 4.1. The generalized fractional heat equation on the whole real line.

474 *Proof of Theorem 1.4 in the case $\Omega = \mathbb{R}$. Well-posedness.* Let us recall that $\inf_{\mathbb{R}_+} \Re(\rho) < +\infty$.
475 So, denoting M this infimum, we have for every $t \geq 0$, $\sup_{\xi \in \mathbb{R}} |e^{-t\rho(|\xi|)}| \leq e^{tM}$. Thus, for every
476 $t \geq 0$, we can define a linear bounded operator on $L^2(\mathbb{R})$ by

$$477 \quad \forall f_0 \in L^2(\mathbb{R}), \forall x \in \mathbb{R}, S(t)f_0(x) = \mathcal{F}^{-1}(e^{-t\rho(|\xi|)} \mathcal{F}f_0)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi - t\rho(|\xi|)} \mathcal{F}f_0(\xi) d\xi.$$

478 We can see that $S(t)$ is a strongly continuous semigroup of bounded operators on $L^2(\mathbb{R})$. Moreover,
 479 the infinitesimal generator of $S(t)$ is $\rho(\sqrt{-\Delta})$. Thus, the equation (1.3) is well-posed, in the sense
 480 of semigroups (see for instance [15, Def. 2.36 and Th. 2.37]).

481 *Construction of the counterexample to the observability inequality.* We remind that the null-
 482 controllability of the generalized fractional heat equation on ω and in time T is equivalent to the
 483 following *observability inequality* [15, Th. 2.44]: there exists $C > 0$ such that for every $g_0 \in L^2(\mathbb{R})$,
 484 the solution g of

$$485 \quad (4.1) \quad \partial_t g + \bar{\rho}(\sqrt{-\Delta})g = 0, \quad g(0, \cdot) = g_0$$

486 satisfies

$$487 \quad (4.2) \quad |g(T, \cdot)|_{L^2(\mathbb{R})} \leq C|g|_{L^2([0, T] \times \omega)}.$$

488 In **Hypothesis 3.3** item 1, we choose K and \mathcal{C} to be those of the statement of **Theorem 1.4**. Let
 489 ξ_0, δ and χ as in **Hypothesis 3.3** item 2–3. Then, for $h > 0$, we consider $g_{0,h} \in L^2(\mathbb{R})$ defined by

$$490 \quad g_{0,h}(x) = \sqrt{2\pi h} \chi(-ih\partial_x - \xi_0) e^{-x^2/2h + ix\xi_0/h} = h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h + ix\xi} d\xi.$$

491 The solution g_h of the generalized fractional heat equation (4.1) with this initial condition is

$$492 \quad (4.3) \quad \begin{aligned} g_h(t, x) &= h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h + ix\xi - t\bar{\rho}(\xi)} d\xi \\ &= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi_0/h - t\bar{\rho}(\xi/h)} d\xi \end{aligned}$$

493 (let us remind that according to **Hypothesis 3.3**, $\chi(h\xi - \xi_0)$ is zero for $|h\xi - \xi_0| \geq \delta$, and in particular
 494 for $\xi < 0$ if δ is chosen small enough, as in **Hypothesis 3.3**).

495 *Conclusion.* In **Hypothesis 3.3** item 5, we choose $X = [0, T]$. For $t \in X$ and $h > 0$, we choose
 496 $\rho_{t,h}: \xi \in \mathcal{C} \mapsto -t\bar{\rho}(\xi/h)$. Since ρ is holomorphic on \mathcal{C} with $\rho(\xi) = o(|\xi|)$, so is $\rho_{t,h}$ for every $t \in X$
 497 and $h > 0$. In other words **Hypothesis 3.3** item 5–6 are satisfied. Moreover, with the notations of
 498 Eq. (3.3), the function g_h given by (4.3) can be written as $g_h(t, x) = I_{t,h,1}(x)$.

499 So, according to **Proposition 3.5** (or more precisely **Remark 3.6**), there exists $C, c > 0$ such that
 500 for $t \in [0, T]$, x small enough (say $|x| < \eta'$) and $h > 0$ small enough

$$501 \quad (4.4) \quad |g_h(t, x)| \geq \frac{1}{2} e^{-x^2/2h - C\epsilon(h)/h}.$$

502 Moreover, according to **Proposition 3.7**, there exists $C, c > 0$ such that for $t \geq 0$, $|x| > \eta$ and $h > 0$
 503 small enough,

$$504 \quad (4.5) \quad g_h(t, x) \leq \frac{C}{|x|^2} e^{-c/h}.$$

505 Thus, we have according to the lower bound (4.4)

$$506 \quad (4.6) \quad |g_h(T, \cdot)|_{L^2(\mathbb{R})} \geq |g_h(T, \cdot)|_{L^2(|x| < \eta')} \geq ch^{1/4} e^{-C\epsilon(h)/h}$$

508 and according to the upper bound (4.5),

$$509 \quad (4.7) \quad |g_h|_{L^2((0,T) \times \omega)}^2 \leq T \int_{|x| > \eta} \frac{C}{x^4} e^{-c/h} dx \leq C e^{-c/h}.$$

510

511 and since $\epsilon(h) \rightarrow 0$ as $h \rightarrow 0$, taking $h \rightarrow 0$ disproves the observability inequality. \square

512 *Remark 4.1.* We implicitly looked at the generalized fractional heat equation with complex
513 valued solutions. This means that we proved that there exists an initial condition f_0 of the generalized
514 fractional heat equation that we cannot steer to 0, but this initial condition might not be real
515 valued. In the case where $\rho(\mathbb{R}_+) \subset \mathbb{R}_+$, we might be more interested in real valued solutions. But
516 our results actually implies there exists a real valued initial condition that cannot be steered to 0,
517 for if both the real part $\Re(f_0)$ and the imaginary part $\Im(f_0)$ could be steered to 0, then f_0 itself
518 could be steered to 0. A similar arguments stays valid for the Kolmogorov-type equation.

519 **4.2. The generalized fractional heat equation on the torus.** The case of the generalized
520 fractional heat equation on the torus is a bit different because we are not dealing with integrals, but
521 sums. Therefore, tools like the saddle point method do not seem to be of much use. Nonetheless,
522 with a trick, we can deduce the theorem on the torus from the theorem on the whole real line.

523 *Proof of Theorem 1.4 in the case $\Omega = \mathbb{T}$.* The basic idea is the trick of the proof of Poisson
524 summation formula, namely the fact that the Fourier coefficients of a function of the form $g_{0\text{per}}(x) =$
525 $\sum_{k \in \mathbb{Z}} g_0(x + 2\pi k)$ are the values of the Fourier transform of g_0 evaluated at the integers (up to a
526 multiplication by $\sqrt{2\pi}$).

527 So, let $g_h \in C^\infty(\mathbb{R})$ be as in the previous section. Since the Fourier transform of $g_h(t, \cdot)$ is
528 C^∞ with compact support,⁷ $g_h(t, x)$ decays faster than any polynomials as $|x| \rightarrow \infty$ and we can
529 define $g_{h\text{per}}(t, x) = \sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k)$. According to the trick described before, $c_n(g_{h\text{per}}(t, \cdot)) =$
530 $(2\pi)^{-1/2} \mathcal{F}(g_h)(t, \cdot)(n)$. But, by definition of g_h as the solution of the rotated fraction heat equation,
531 $\mathcal{F}(g_h)(t, \cdot)(\xi) = \mathcal{F}(g_h)(0, \cdot)(\xi) e^{-t\bar{\rho}(|\xi|)}$, so, using the trick again:

$$532 \quad (4.8) \quad c_n(g_{h\text{per}}(t, \cdot)) = c_n(g_{h\text{per}}(0, \cdot)) e^{-t\bar{\rho}(|n|)}.$$

533 So $g_{h\text{per}}$ is a solution to the generalized fractional heat equation (4.1) on the torus. Now we prove
534 that the terms for $k \neq 0$ are negligible. Indeed, we have by definition of $g_{h\text{per}}$

$$535 \quad (4.9) \quad |g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} = \left| \sum_{k \in \mathbb{Z}} g_h(T, \cdot + 2\pi k) \right|_{L^2(\mathbb{T})}$$

536

537 and by singling out to term for $k = 0$ and thanks to the triangle inequality

$$538 \quad (4.10) \quad |g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} \geq |g_h(T, \cdot)|_{L^2(-\pi, \pi)} - \sum_{k \neq 0} |g_h(T, \cdot)|_{L^2((2k-1)\pi, (2k+1)\pi)}$$

539

540 and thanks to the pointwise estimates on g_h (Eq. (4.5) and (4.4))

$$541 \quad (4.11) \quad |g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} \geq ch^{1/4} e^{-C\epsilon(h)/h} - \sum_{k \neq 0} \frac{C}{k^2} e^{-c/h} \geq ch^{1/4} e^{-C\epsilon(h)/h} - C e^{-c/h}.$$

542

⁷We added the cutoff function χ just to localize the Fourier transform away from the singularity of $|\xi|^\alpha$ at $\xi = 0$.

543 In the same spirit, we have thanks to the triangle inequality, and identifying $\omega = \mathbb{T} \setminus [-\epsilon, \epsilon]$ with
 544 $(-\pi, \pi) \setminus [-\epsilon, \epsilon] \subset \mathbb{R}$

$$545 \quad (4.12) \quad |g_{h,\text{per}}|_{L^2([0,T] \times \omega)} \leq \sum_{k \in \mathbb{Z}} |g_h|_{L^2([0,T] \times (\omega + 2\pi k))}$$

546

547 and according to the estimate (4.5),

$$548 \quad (4.13) \quad |g_{h,\text{per}}|_{L^2([0,T] \times \omega)} = \mathcal{O}(e^{-c/h}).$$

549

550 Taking $h \rightarrow 0^+$ disproves the observability inequality (4.2) and proves the Theorem. \square

551 **4.3. Higher dimension.** Theorem 1.4 can be generalized to take into account the case
 552 $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$. Indeed, the Propositions of section 3 are still valid in higher dimension. The
 553 computations are carried essentially the same way, only with the added technicalities of the higher
 554 dimension, for instance:

- 555 • in Proposition 3.2, U and V are assumed to be open, bounded and convex subset of \mathbb{R}^d and
 556 \mathbb{C}^d respectively,
- 557 • in the proof of Proposition 3.2, the change of variables of step 2 is given by a Morse Lemma
 558 with parameter, in the spirit of [21, Lemma C.6.1],
- 559 • in Hypothesis 3.3 χ is chosen to be $C_c^\infty(B(0, \delta))$ (open ball in \mathbb{R}^d),
- 560 • $\rho(|\xi|)$ has to be replaced by $\rho\left[\left(\sum_i \xi_i^2\right)^{1/2}\right]$ (i.e. what happens to be holomorphic in $\xi \in \mathbb{C}^d$
 561 and that is equal to $\rho(|\xi|)$ if $\xi \in \mathbb{R}^d$),
- 562 • in all the complex integrals that follows, we integrate against $d\xi_1 \wedge \dots \wedge d\xi_d$,
- 563 • also, the power of $2\pi h$ in front of the asymptotic expansion is $(2\pi h)^{d/2}$ (but it does not
 564 matter).

565 Then, the construction of the counterexample to the observability inequality if $\Omega = \mathbb{R}^d$ is the same.
 566 For the case $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$, we first consider a counter example in $\mathbb{R}^{d+d'}$, and we periodize the last
 567 d' components with the method of subsection 4.2.

568 5. Non-null-controllability of the Kolmogorov equation.

569 **5.1. Introduction.** Now, we look at the Kolmogorov equation (1.2). As for the fractional heat
 570 equation, the null-controllability of the Kolmogorov equation (1.2) is equivalent to the existence of
 571 $C > 0$ such that for every solution g of⁸

$$572 \quad (5.1) \quad (\partial_t - v^2 \partial_x - \partial_v^2)g(t, x, v) = 0 \quad t \in (0, T), (x, v) \in \Omega$$

573 with Dirichlet boundary conditions if $\Omega_v = (-1, 1)$,

$$574 \quad (5.2) \quad |g(T, \cdot)|_{L^2(\Omega)} \leq C|g|_{L^2((0,T) \times \omega)}.$$

575 As hinted in the introduction, we look for counterexamples of the observability inequal-
 576 ity among solutions of the adjoint of the Kolmogorov equation (5.1) of the form $g(t, x, v) =$
 577 $\int_{\mathbb{R}} a(\xi) e^{ix\xi} g_\xi(v) e^{-\lambda_\xi t} d\xi$, where $g_\xi(v)$ is the first eigenfunction of $-\partial_v^2 - i\xi v^2$ and λ_ξ its associated
 578 eigenvalue. Let us remind that⁹ $\lambda_\xi = \sqrt{-i\xi}$ if $\Omega_v = \mathbb{R}$, and is close to $\sqrt{-i\xi}$ if $\Omega_v = (-1, 1)$.

⁸Note that this is the adjoint of the Kolmogorov equation where we reversed the time.

⁹We choose the branch of the square root with positive real part.

579 We remark that apart from the $g_\xi(v)$ term, those solutions have the same form as solutions
 580 of the rotated fractional heat equation $(\partial_t + \sqrt{-i}(-\Delta)^{1/4})g = 0$. So, the strategy is to prove the
 581 same estimates we proved for the rotated fractional heat equation, but with some uniformity in the
 582 parameter v . Since the computations are essentially the same, we only tell what we need to care
 583 about in comparison with the rotated fractional heat equation, but we do not give the full details of
 584 the computations again.

585 5.2. The Kolmogorov equation with unbounded velocity.

586 *Proof of Theorem 1.5 with $\Omega_x = \Omega_v = \mathbb{R}$.* In the case $\Omega_v = \mathbb{R}$, the first eigenfunction of $-\partial_v^2 -$
 587 $i\xi v^2$ is $g_\xi(v) = e^{-\sqrt{-i\xi}v^2/2}$ with eigenvalue $\lambda_\xi = \sqrt{-i\xi}$. Without loss of generality, we may assume
 588 $\omega_x = \mathbb{R} \setminus [-\eta, \eta]$ (let us remind that $\omega = \omega_x \times \mathbb{R}$).

589 Thus, we consider the function $g_h: \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{C}$ defined by

$$590 \quad (5.3) \quad \begin{aligned} g_h(t, x, v) &= h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{ix\xi - (h\xi - \xi_0)^2/2h - \sqrt{-i\xi}(v^2/2+t)} d\xi \\ &= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(v^2/2+t)} d\xi. \end{aligned}$$

591 Since every $(t, x, v) \mapsto e^{ix\xi - \sqrt{-i\xi}(t+v^2/2)}$ is a generalized solution to the Kolmogorov equation (5.1),
 592 the function g_h is also solution to the Kolmogorov equation. We also remark that $g_h(0, \cdot, \cdot) \in L^2(\mathbb{R}^2)$.
 593 Notice that these solutions are of the form $\tilde{g}_h(t + v^2/2, x)$, where \tilde{g}_h is solution to the “rotated
 594 fractional heat equation” $(\partial_t + \sqrt{-i}(-\Delta_x)^{1/4})\tilde{g}_h(t, x) = 0$. Thus we have asymptotic expansion on
 595 \tilde{g}_h similar to (4.4) and (4.5).

596 For $t \geq 0$, $h > 0$ and $\Re(\xi) > 0$, we define $\rho_{t,h}(\xi) = -t\sqrt{-i\xi}$. Thus, with any $K > 0$, and with
 597 $X = [0, T + 1/2]$, Hypothesis 3.3 holds. Thus, with the notations of Hypothesis 3.3 and Eq. (3.3),
 598 we have for h small enough

$$599 \quad g_h(t, x, v) = I_{t+v^2/2, h, 1}(x).$$

600 Moreover, still with the notations of Hypothesis 3.3, we have for some $C > 0$ and for every $0 < h < 1$,
 601 $C^{-1}h^{1/2} \leq \epsilon(h) \leq Ch^{1/2}$.

602 Thus, according to Proposition 3.5 (or more precisely Remark 3.6), there exist $C, c > 0$ such
 603 that for every $h > 0$ small enough, $0 \leq t \leq T$, $|x|$ small enough (say $|x| < \eta'$) and $|v| < 1$

$$604 \quad (5.4) \quad |g_h(t, x, v)| \geq ce^{-x^2/2h - Ch^{-1/2}}.$$

Moreover, assuming K large enough and choosing $X = \mathbb{R}_+$, according to Proposition 3.7 (see also
 606 Remark 3.8), there exist $C, c > 0$ such that for every $h > 0$ small enough, $|x| > \eta$, $t \geq 0$ and $v \in \mathbb{R}$

$$607 \quad (5.5) \quad |g_h(t, x, v)| \leq C|x|^{-2}e^{-c/h - c(t+v^2/2)h^{-1/2}},$$

609 So, integrating these estimates, we have

$$610 \quad (5.6) \quad |g_h|_{L^2([0, T] \times \omega)} \leq Ce^{-c/h}$$

611 and

$$612 \quad (5.7) \quad |g_h(T, \cdot, \cdot)|_{L^2(\Omega)} \geq |g_h(T, \cdot, \cdot)|_{L^2(|x| < \eta', |v| < 1)} \geq ce^{-Ch^{-1/2}}.$$

613 Taking again $h \rightarrow 0$ disproves the observability inequality and proves the Theorem. \square

614 For the Kolmogorov equation with $\Omega_x = \mathbb{T}$ and $\Omega_v = \mathbb{R}$, we define $g_{h\text{per}}(t, x, v) = \sum_{k \in \mathbb{Z}} g_h(t, x +$
 615 $2\pi k, v)$, and as in subsection 4.2, all but the term for $k = 0$ are $\mathcal{O}(e^{-c/h})$. We let the careful reader
 616 work out the details.

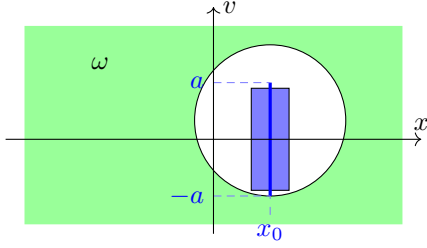


FIGURE 3. In green, the control domain ω . If there is a vertical line, symmetrical with respect to $\{v = 0\}$, that does not intersect $\bar{\omega}$ (in dark blue), for every $a' < a$, there exists a rectangle of the form $\{|x - x_0| < b, -a' < v < a'\}$ that does not intersect $\bar{\omega}$ (in lighter blue).

617 5.3. The Kolmogorov equation with non-rectangular control domain.

618 *Proof of Theorem 1.6 in the case $\Omega_x = \Omega_v = \mathbb{R}$.* If $0 < a' < a$, there exists $b > 0$ such that the
 619 rectangle $R = \{|x - x_0| < b, |v| < a'\}$ does not intersect $\bar{\omega}$ (see Figure 3). Since the equation is
 620 invariant by translation in the x direction, we may assume without loss of generality that $x_0 = 0$.

621 We will use the same functions g_h as in the previous proof. But while we used only the terms
 622 of order h^{-1} in the exponent of estimate of Proposition 3.5, we will now use the next term. More
 623 precisely, according to Proposition 3.5 with $X = [0, T + a^2/2]$ and $\rho_{t,h}(\xi) = -t\sqrt{-i\xi}$, we have
 624 uniformly in $|x|$ small enough, $0 \leq t \leq T$ and $|v| < a$:

$$625 \quad (5.8) \quad g_h(t, x, v) = I_{t+v^2/2, h, 1}(x) = \sqrt{2\pi h} e^{\phi(t, x, v)/h} (1 + \mathcal{O}_h(\sqrt{h}))$$

626 with

$$627 \quad (5.9) \quad \phi(t, x, v) = ix\xi_0 - \frac{x^2}{2} - \sqrt{-i\xi_0 - x} \left(t + \frac{v^2}{2} \right) h^{1/2} + \mathcal{O}_h(h).$$

628 The idea is that, when computing $\int_{\Omega} |g_h(T, x, v)|^2 dx dv$, the dominant part of this integral is
 629 around $x = v = 0$, and when computing $\int_{[0, T] \times \omega} |g_h(t, x, v)|^2 dt dx dv$, the dominant part is around
 630 $t = 0, x = x_0 = 0$ and $v = a$ or $-a$. So, noting $c_0 = \Re(\sqrt{-i\xi_0}) > 0$ and ignoring the error terms for
 631 the moment, we have

$$632 \quad (5.10) \quad \int_{\Omega} |g_h(T, x, v)|^2 dx dv \approx 2\pi h \int_{\substack{|x| < \epsilon \\ |v| < \epsilon}} e^{-x^2/h - c_0(2T+v^2)/\sqrt{h}} dx dv \approx Ch^{7/4} e^{-2Tc_0/\sqrt{h}}$$

633 and

634

$$635 \quad (5.11) \quad \int_{[0, T] \times \omega} |g_h(t, x, v)|^2 dt dx dv \approx 2\pi h \int_{\substack{|x| < \epsilon \\ a < |v| < a + \epsilon \\ t < \epsilon}} e^{-x^2/h - c_0(2t+v^2)/\sqrt{h}} dt dx dv \approx Ch^{5/2} e^{-c_0 a^2/\sqrt{h}}.$$

636

637 So, if $2T < a^2$, the observability inequality cannot hold. Now, let us rigorously prove this. Let $\epsilon > 0$
 638 and $T = a^2/2 - \epsilon$. We have $\Re(\sqrt{-i\xi_0 - x}) = c_0 + \mathcal{O}_x(x)$. So, for x small enough, say, $|x| < \delta_{\xi_0}$, we
 639 have

$$640 \quad (5.12) \quad c_0 - \epsilon \leq \Re\left(\sqrt{-i\xi_0 - x}\right) \leq c_0 + \epsilon.$$

641 So, we have locally uniformly in $|x| < \delta_{\xi_0}$, $t \geq 0$ and $v \in \mathbb{R}$:

$$642 \quad (5.13) \quad \Re(\phi(t, x, v)) \geq -\frac{x^2}{2} - (c_0 + \epsilon) \left(t + \frac{v^2}{2} \right) h^{1/2} - \mathcal{O}_h(h)$$

643 and

$$644 \quad (5.14) \quad \Re(\phi(t, x, v)) \leq -\frac{x^2}{2} - (c_0 - \epsilon) \left(t + \frac{v^2}{2} \right) h^{1/2} + \mathcal{O}_h(h).$$

645 Now, let us get a lower bound for the left-hand side of the observability inequality (5.2). We
646 have:

$$647 \quad (5.15) \quad |g_h(T, \cdot, \cdot)|_{L^2(\Omega)}^2 \geq |g_h(T, \cdot, \cdot)|_{L^2(|x| < b, |v| < a')}^2$$

649 and thanks to the asymptotic of Eq. (5.8):

$$650 \quad (5.16) \quad |g_h(T, \cdot, \cdot)|_{L^2(\Omega)}^2 \geq 2\pi h \int_{\substack{|x| < b \\ |v| < a'}} e^{2\Re(\phi(T, x, v))/h} dx dv (1 + \mathcal{O}(\sqrt{h}))$$

652 and with the lower bound above (Eq. (5.13)):

$$653 \quad (5.17) \quad |g_h(T, \cdot, \cdot)|_{L^2(\Omega)}^2 \geq 2\pi h e^{\mathcal{O}_h(1)} \int_{\substack{|x| < b \\ |v| < a'}} e^{-x^2/h - (c_0 + \epsilon)(2T + v^2)/\sqrt{h}} dx dv (1 + \mathcal{O}(\sqrt{h})).$$

654

655 The integral in x is

$$656 \quad (5.18) \quad \int_{|x| < b} e^{-x^2/h} dx = \sqrt{\pi h} + \mathcal{O}(e^{-c/h})$$

657 while the integral in v is

$$658 \quad (5.19) \quad \int_{|v| < a'} e^{-(c_0 + \epsilon)v^2/\sqrt{h}} dv = \sqrt{\frac{\pi}{c_0 + \epsilon}} h^{1/4} + \mathcal{O}(e^{-c/\sqrt{h}}).$$

659 So, we have

$$660 \quad (5.20) \quad |g_h(T, \cdot, \cdot)|_{L^2(\mathbb{R}^2)}^2 \geq c \frac{2\pi^2}{\sqrt{c_0 + \epsilon}} h^{7/4} e^{-(c_0 + \epsilon)2T/\sqrt{h}} (1 + \mathcal{O}(\sqrt{h}))$$

661

662 and for h small enough:

$$663 \quad (5.21) \quad |g_h(T, \cdot, \cdot)|_{L^2(\Omega)}^2 \geq ch^{7/4} e^{-(c_0 + \epsilon)2T/\sqrt{h}}.$$

665 Now, let us bound the right-hand side of the observability inequality (5.2). Let us remind that
666 ω is a subset of $\Omega \setminus \{|x| < b, |v| < a'\}$. Let $a'' > a'$, that will be chosen large enough afterwards. We
667 define

$$668 \quad (5.22) \quad \omega_0 = \{|x| < \delta_{\xi_0}, a' < |v| < a''\}$$

$$669 \quad (5.23) \quad \omega_1 = \{|x| \geq \delta_{\xi_0}\}$$

$$670 \quad (5.24) \quad \omega_2 = \{|x| < \delta_{\xi_0}, |v| > a''\}.$$

671

672 With these definitions, if $\delta_{\xi_0} < b$, we have $\omega \subset \omega_0 \cup \omega_1 \cup \omega_2$. So,

$$673 \quad (5.25) \quad |g_h|_{L^2([0,T] \times \omega)}^2 \leq |g_h|_{L^2([0,T] \times \omega_0)}^2 + |g_h|_{L^2([0,T] \times \omega_1)}^2 + |g_h|_{L^2([0,T] \times \omega_2)}^2$$

674 First, according to [Proposition 3.9](#), we have for every $t \geq 0, v \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$675 \quad (5.26) \quad |g_h(t, x, v)| \leq C e^{-c'(t+v^2/2)h^{-1/2}}$$

676 So, integrating this estimate, we have

$$677 \quad (5.27) \quad |g_h|_{L^2([0,T] \times \omega_2)}^2 = \mathcal{O} \left(\int_{|v| \geq a''} e^{-c'v^2/\sqrt{h}} dv \right) = \mathcal{O} \left(e^{-c'a''^2/\sqrt{h}} \right).$$

678 We choose a'' large enough so that $c'a''^2 > (c_0 - \epsilon)a'^2$. That way, we have

$$679 \quad (5.28) \quad |g_h|_{L^2([0,T] \times \omega_2)}^2 = \mathcal{O} \left(e^{-(c_0 - \epsilon)a'^2/\sqrt{h}} \right).$$

680 We have already seen in [subsection 5.2](#) that

$$681 \quad (5.29) \quad |g_h|_{L^2([0,T] \times \omega_1)}^2 = \mathcal{O}(e^{-c/h}).$$

682 Finally, thanks to Eq. (5.8) with upper bound (5.14), we have uniformly in $0 \leq t \leq T, |x| < \delta_{\xi_0}$ and
683 $a' < |v| < a''$

$$684 \quad (5.30) \quad |g_h(t, x, v)|^2 \leq 2\pi h e^{-x^2/h - (c_0 - \epsilon)(2t + v^2)/\sqrt{h} + \mathcal{O}_h(1)}.$$

685 So, we have

$$686 \quad (5.31) \quad |g_h|_{L^2([0,T] \times \omega_0)}^2 = \mathcal{O} \left(\int_{a' < |v| < a''} e^{-(c_0 - \epsilon)v^2/\sqrt{h}} dv \right) = \mathcal{O} \left(e^{-(c_0 - \epsilon)a'^2/\sqrt{h}} \right).$$

687 So, putting the three upper bounds (5.28) (5.29) and (5.31) together, we have

$$688 \quad (5.32) \quad |g_h|_{L^2([0,T] \times \omega)}^2 = \mathcal{O} \left(e^{-(c_0 - \epsilon)a'^2/\sqrt{h}} \right).$$

689 Let us assume that $(c_0 - \epsilon)a'^2 > 2T(c_0 + \epsilon)$. Then, considering the previous upper bound (5.32)
690 and the lower bound (5.21), and taking $h \rightarrow 0$ disproves the observability inequality. So, the
691 Kolmogorov equation is not null-controllable in time $T < \frac{c_0 - \epsilon}{c_0 + \epsilon} a'^2/2$. This is true for every $a' < a$
692 and $\epsilon > 0$, so the Kolmogorov equation is not null-controllable in time $T < a^2/2$. \square

693 The case $\Omega_x = \mathbb{T}, \Omega_v = \mathbb{R}$ is similar. We look at $g_{h\text{per}}(t, x, v) = \sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k, v)$. In this
694 sum, as in [subsection 4.2](#), only the term for $k = 0$ matters, as the other are $\mathcal{O}(e^{-c/h})$.

695 **5.4. The Kolmogorov equation with bounded velocity.** To treat the Kolmogorov equation
696 with $\Omega_v = (-1, 1)$, we need some information on the first eigenfunction g_ξ of $-\partial_v^2 - i\xi v^2$ with
697 Dirichlet boundary conditions on $(-1, 1)$, and with associated eigenvalue $\lambda_\xi = \sqrt{-i\xi} + \rho_\xi$. Moreover,
698 as we will use [Theorems 3.5–3.9](#), we also need some analyticity in ξ . We will denote \tilde{g}_ξ the first¹⁰

¹⁰“First” in the sense that it is the analytic continuation in $\tilde{\xi}$ of the first eigenfunction of $-\partial_v^2 + (\tilde{\xi}v)^2$ for $\tilde{\xi} \in \mathbb{R}_+$, assuming it exists.

699 eigenfunction of $-\partial_v^2 + (\tilde{\xi}v)^2$, and $\tilde{\lambda}_\xi = \tilde{\xi} + \tilde{\rho}_\xi$ the associated eigenvalue, so that, with $\tilde{\xi} = \sqrt{-i\xi}$, we
 700 have $g_\xi = \tilde{g}_\xi$ and $\rho_\xi = \tilde{\rho}_\xi$, when this is defined.

701 In an article on the Grushin equation [22, Section 4] we proved that $\tilde{\rho}_\xi$ and \tilde{g}_ξ exist if $\Re(\tilde{\xi}) > 0$
 702 and $|\tilde{\xi}| > r(|\arg(\tilde{\xi})|)$ for some non-decreasing function $r : (0, \pi/2) \rightarrow \mathbb{R}_+$. We also proved the next
 703 two theorems.

704 **THEOREM 5.1** (Theorem 22 and Remark 23 of [22]). *Let $0 < \theta < \pi/2$. With $\tilde{\rho}_\xi$ defined above,*
 705 *we have*

$$706 \quad \tilde{\rho}_\xi \sim \frac{4}{\sqrt{\pi}} \tilde{\xi}^{3/2} e^{-\tilde{\xi}}$$

707 *in the limit $|\tilde{\xi}| \rightarrow \infty$, $|\arg(\tilde{\xi})| < \theta$.*

708 **PROPOSITION 5.2** (Proposition 25 of [22]). *Let \tilde{g}_ξ be defined above and normalized by $\tilde{g}_\xi(0) = 1$*
 709 *(instead of $|\tilde{g}_\xi|_{L^2} = 1$). Let $0 < \theta < \pi/2$ and $\epsilon > 0$. We have for all $v \in (-1, 1)$ and $|\tilde{\xi}| > r(\theta)$,*
 710 *$|\arg(\tilde{\xi})| < \theta$:*

$$711 \quad |e^{(1-\epsilon)\tilde{\xi}v^2/2} \tilde{g}_\xi(v)| \leq C_{\epsilon, \theta}.$$

712 **Theorem 5.1** gives us all we need to know on the eigenvalue, while **Proposition 5.2** gives us an
 713 upper bound on the eigenfunction. We will also need the following lower bound, that we prove in
 714 **Appendix B**.

715 **PROPOSITION 5.3.** *Let $0 < \theta < \pi/2$ and $\epsilon > 0$. We normalize \tilde{g}_ξ again by $\tilde{g}_\xi(0) = 1$ and define*
 716 *$\tilde{u}_\xi(v) = e^{\tilde{\xi}v^2/2} \tilde{g}_\xi(v)$. Then $\tilde{u}_\xi(v)$ converges exponentially fast to 1, as $|\tilde{\xi}| \rightarrow \infty$, $|\arg(\tilde{\xi})| < \theta$, this*
 717 *convergence being uniform in $|v| < 1 - \epsilon$.*

718 With this, we know all we need to adapt the proof of the non-null-controllability of the
 719 Kolmogorov equation with $\Omega_v = \mathbb{R}$ to the case of $\Omega_v = (-1, 1)$.

720 *Proof of **Theorem 1.5** with $\Omega_v = (-1, 1)$.* We start with the case $\Omega_x = \mathbb{R}$.

721 *Step 1: construction of the counterexample to the observability inequality.* The counterexample we
 722 build to the observability inequality (5.2) is basically the same as in the case $\Omega_v = \mathbb{R}$, only with
 723 the added corrections to the eigenvalues and eigenfunctions. We define $g_h(t, x, v)$ for $t \geq 0$, $x \in \mathbb{R}$,
 724 $v \in (-1, 1)$ and $h > 0$ small enough by:

$$725 \quad (5.33) \quad g_h(t, x, v) = \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-ix\xi/h - (\xi - \xi_0)^2/2h - \lambda_{\xi/h} t} g_{\xi/h}(v) d\xi,$$

726 where $\xi_0 > 0$ and $\chi \in C_c^\infty(\mathbb{R})$ are chosen as follows.

727 First note that according to the discussion at the top of this subsection, λ_ξ and g_ξ are defined
 728 and holomorphic with respect to $\xi \in \mathbb{C}$ such that $|\arg(\xi)| < 3\pi/8$ (for instance) and $|\xi|$ large enough.
 729 Let then $K > 0$ be large enough so that for any $\xi \in \mathcal{C} := \{\Re(\xi) > K, |\Im(\xi)| < K^{-1}\Re(\xi)\}$, λ_ξ and g_ξ
 730 are defined and holomorphic with respect to $\xi \in \mathcal{C}$. Finally, let $\xi_0 > 0$ and χ as in **Hypothesis 3.3**
 731 (see **Figure 4**). With these choices, g_h is well-defined for $0 < h \leq 1$, $t \geq 0$, $x \in \mathbb{R}$ and $v \in (-1, 1)$.

732 We remark that each function $(t, x, v) \mapsto e^{-ix\xi - \lambda_\xi t} g_\xi(v)$ is solution to the Kolmogorov equa-
 733 tion (5.1). So g_h is solution of the Kolmogorov equation.

734 *Step 2: estimates on g_h .* Note that **Theorem 5.1** and **Propositions 5.2** and **5.3** (with the choice

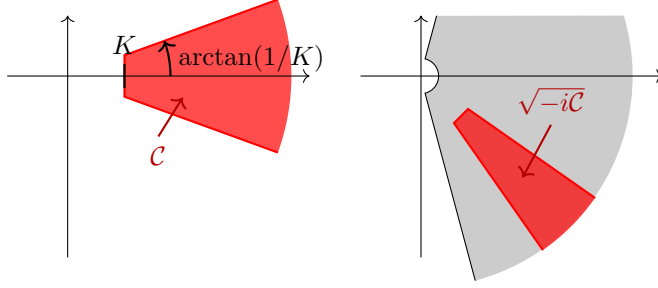


FIGURE 4. Left figure: in red, shape of \mathcal{C} for some K . Right figure: if $\tilde{\xi}$ is in the gray domain, the eigenvalue $\tilde{\rho}_{\tilde{\xi}}$ and the eigenfunction $\tilde{g}_{\tilde{\xi}}$ of $-\partial_v^2 + (\tilde{\xi}v)^2$ are defined. Thus, the eigenvalue λ_{ξ} and eigenfunction g_{ξ} of $-\partial_v^2 + i\xi v^2$ are defined for $\xi \in \mathcal{C}$ if $\sqrt{-i\mathcal{C}}$ lies inside the gray domain.

735 $\epsilon = 1/2$) translate respectively into the estimates:

$$736 \quad (5.34) \quad |e^{-t\rho_{\xi}} - 1| \leq Ce^{-c\sqrt{|\xi|}} \quad \text{for } \xi \in \mathcal{C} \text{ and } 0 \leq t \leq T$$

$$737 \quad (5.35) \quad |e^{\sqrt{-i\xi}v^2/4}g_{\xi}(v)| \leq C \quad \text{for } \xi \in \mathcal{C} \text{ and } |v| < 1$$

$$738 \quad (5.36) \quad |e^{\sqrt{-i\xi}v^2/2}g_{\xi}(v) - 1| \leq Ce^{-c\sqrt{|\xi|}} \quad \text{for } \xi \in \mathcal{C} \text{ and } |v| < 1/2.$$

740 for some $C, c > 0$.

741 *Step 2a: lower bound on g_h .* We want to write $g_h(t, x, v)$ in the form of Eq. (3.3). Let $X = [0, T+1/2]$,
 742 and for $t \in X$, $0 < h \leq 1$ and $\xi \in \mathcal{C}$, let $\rho_{t,h}(\xi) = -t\sqrt{-i\xi}$. Finally, for $0 < h \leq 1$, $t \geq 0$, $v \in (-1, 1)$
 743 and $\xi \in \mathcal{C}$, let

$$744 \quad (5.37) \quad \delta_{h,t,v}(\xi) := e^{\sqrt{-i\xi/h}v^2/2}g_{\xi/h}(v)e^{-t\rho_{\xi}/h}.$$

745 Then, according to the definition of g_h (Eq. (5.33)),

$$746 \quad (5.38) \quad g_h(t, x, v) = \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(t+v^2/2)}\delta_{h,t,v}(\xi) d\xi = I_{t+v^2/2,h,\delta}(x).$$

747 Moreover, according to estimates Eq. (5.34) and (5.36), we have, for some $C, c > 0$:

$$748 \quad (5.39) \quad |\delta_{h,t,v}(\xi) - 1| \leq Ce^{-ch^{-1/2}} \quad \text{for } \xi \in \mathcal{C}, |v| < 1/2 \text{ and } 0 < h \leq 1.$$

749 So, according to Proposition 3.5, there exists $C, c > 0$ such that for every $t \in [0, T]$, $|v| < 1/2$, x
 750 small enough and h small enough,

$$751 \quad (5.40) \quad |g_h(t, x, v)| \geq ce^{-x^2/2h - Ch^{-1/2}}.$$

752 *Step 2b: upper bound on g_h .* The estimate (5.36) does not extend up to the boundary. Thus, we have
 753 to use the less precise upper bound (5.35). To this end, we define $\tilde{\delta}_{h,t,v}(\xi) = e^{-\sqrt{-i\xi/h}v^2/2}\delta_{h,t,v}(\xi)$.
 754 Then, according to the definition of g_h (Eq. (5.33)) and δ (Eq. (5.37)),

$$755 \quad (5.41) \quad g_h(t, x, v) = \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(t+v^2/4)}\tilde{\delta}_{h,t,v}(\xi) d\xi = I_{t+v^2/4,h,\tilde{\delta}}(x).$$

756 Moreover, according to estimates (5.34) and (5.35), there exist $C, c > 0$ such that

757 (5.42)
$$|\tilde{\delta}_{h,t,v}(\xi)| \leq C \quad \text{for } \xi \in \mathcal{C}, \quad |v| < 1 \text{ and } 0 < h \leq 1.$$

758 So, according to Proposition 3.7, there exist $C, c > 0$ such that for every $t \in [0, T]$, $|v| < 1$,
759 $|x| > \eta$ (where we assume without loss of generality $\omega = \{|x| > \eta\} \times (-1, 1)$) and h small enough,

760 (5.43)
$$|g_h(t, x, v)| \leq \frac{C}{|x|^2} e^{-c/h}.$$

761 *Step 3: conclusion.* From this point on, the proof is the same as in subsection 5.2: integrating the
762 estimates (5.40) and (5.43) proves that g_h is a counterexample to the observability inequality (5.2)
763 in the case $\Omega = \mathbb{R} \times (-1, 1)$ and $\omega = \omega_x \times (-1, 1)$.

764 In the case $\Omega_x = \mathbb{T}$, we again look at the periodic version of g_h , that is $g_{h\text{per}}(t, x, v) =$
765 $\sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k, v)$. As in subsection 4.1 (and 5.2), $g_{h\text{per}}$ is a solution to the Kolmogorov
766 equation, and it is a counterexample to the observability inequality. \square

767 The proof of Theorem 1.6 in the case $\Omega_v = (-1, 1)$ is similar to the case $\Omega_v = \mathbb{R}$ with the
768 adaptation of the previous proof. Let us just sketch it.

769 We choose $a' < a$ and $b > 0$ as in subsection 5.3. We consider the functions g_h of the previous
770 proof (Eq. (5.33)).

771 We compute the next order in the estimate (5.40). With Proposition 3.5, we can prove that
772 locally uniformly in $v \in (-1, 1)$, $|x|$ small enough and $t > 0$,

773 (5.44)
$$g_h(t, x, v) = \sqrt{2\pi h} e^{ix\xi_0/h - x^2/2h - \sqrt{\xi_0 + ix(t+v^2/2)}/\sqrt{h} + \mathcal{O}_h(1)} \left(1 + \mathcal{O}(\sqrt{h})\right).$$

774 Also, thanks to Proposition 3.7, we prove that uniformly in $|x| > b$, $t > 0$ and $v \in (-1, 1)$,

775 (5.45)
$$g_h(t, x, v) = \mathcal{O}(|x|^{-2} e^{-c/h}).$$

776 We choose $a' < a'' < 1$ and we define ω_0 , ω_1 and ω_2 as in equations (5.22), (5.23) and (5.24) (the
777 δ_{ξ_0} is the same in the cases $\Omega_v = (-1, 1)$ and $\Omega_v = \mathbb{R}$).

778 With the estimate (5.44), we can prove an estimate similar to the lower bound (5.21). We can
779 also prove an upper bound similar to (5.31). With the estimate (5.45), we can prove an estimate
780 similar to (5.29). And with the help of Proposition 5.2 to manage the terms for $|v| > a''$, we can
781 prove an upper bound similar to (5.28). The rest of the proof is a copy-paste.

782 **Appendix A. Other equations.** In this appendix, we explain how we can use the method
783 of section 4 to prove the lack of null-controllability for some other equations.

784 **A.1. Fractional Schrödinger equations.** Let $0 \leq \alpha < 1$. If we consider $\rho(\xi) = i\xi^\alpha$ (defined
785 e.g. for $\Re(\xi) \geq 0$), the hypotheses of Theorem 1.4 hold. Thus, we have:

786 **COROLLARY A.1.** *Let $0 \leq \alpha < 1$. Let $T > 0$ and ω be a strict open subset of \mathbb{R} . The fractional*
787 *Schrödinger equation $(\partial_t + i(-\Delta)^{\alpha/2})f(t, x) = \mathbf{1}_\omega u(t, x)$, $t \geq 0$, $x \in \mathbb{R}$ is not null-controllable on ω*
788 *in time T .*

789 Since $i(-\Delta)^{\alpha/2}$ generates a strongly-continuous group of bounded operators on $L^2(\mathbb{T})$, it seems likely
790 that this corollary can be extended to any riemannian manifold (and not only $\mathbb{R}^d \times \mathbb{T}^d$).¹¹ But this

¹¹The fact that $i(-\Delta)^{\alpha/2}$ generates a strongly continuous group also implies that null-controllability is equivalent to exact-controllability [15, Th. 2.41].

791 is outside the scope of this article. Also, we conjecture that the threshold $\alpha < 1$ is optimal. Indeed,
 792 it seems that if ω satisfies the Geometric Control Condition of Bardos, Lebeau and Rauch [1], then
 793 the methods of [24] could be used to prove exact controllability in any time if $\alpha > 1$ (see [19] for the
 794 case $\alpha \geq 2$) and with a minimal time if $\alpha = 1$ (better known as “the half-wave equation”), but this
 795 is again outside the scope of this article. See also [10] for a variant of the fractional Schrödinger
 796 equation.

797 **A.2. Another Kolmogorov-type equation.** Our method can also be used to prove that the
 798 Kolmogorov-type equation $(\partial_t - \partial_v^2 + v\partial_x)f(t, x, v) = \mathbb{1}_\omega u(t, x, v)$ is not null-controllable on vertical
 799 bands (notice that is is Eq. (1.2) where we replaced v^2 by v).

800 **THEOREM A.2.** *Let $\Omega = (0, +\infty) \times \Omega_x$, and $\Omega_x = \mathbb{R}$ or \mathbb{T} . Let ω_x be a strict open subset of Ω_x*
 801 *and $\omega = \omega_x \times (0, +\infty)$, and let $T > 0$. Then the equation*

$$\begin{aligned} & (\partial_t + v\partial_x - \partial_v^2)f(t, x, v) = \mathbb{1}_\omega u(t, x, v) & t \in [0, T], (x, v) \in \Omega \\ 802 \text{ (A.1)} \quad & f(t, x, 0) = 0 & t \in [0, T], x \in \Omega_x \\ & f(0, x, v) = f_0(x, v) & (x, v) \in \Omega \end{aligned}$$

803 *is not null-controllable on ω in time T .*

804 *Sketch of the proof.* We consider Ai the standard Airy function (see for instance [30, Ch. 9]).
 805 Let $-\mu_0$ the first zero of Ai. We denote $\lambda_0 = e^{i\pi/3}\mu_0$. For $\xi > 0$, let $u_\xi: \mathbb{R} \rightarrow \mathbb{C}$ defined
 806 by $u_\xi(v) = \text{Ai}(\xi^{1/3}e^{-i\pi/6}v - \mu_0)$. Using the ODE satisfied by Ai ([30, §9.2(i)]), we see that
 807 $(-\partial_v^2 - i\xi v)u_\xi = \xi^{2/3}\lambda_0 u_\xi$. Moreover, $u_\xi(0) = 0$, and according to the asymptotic expansion satisfied
 808 by Ai ([30, §9.7ii]), u_ξ decays exponentially at ∞ , as well as its derivatives. So u_ξ is an eigenfunction
 809 of $-\partial_v^2 - i\xi v$ on $(0, +\infty)$ with Dirichlet boundary condition at $v = 0$.

810 Let $\xi_0 > 0$ and $\chi \in C_c^\infty(-\xi_0, \xi_0)$. For $h > 0$ we consider the function $g_h: \mathbb{R}_+ \times \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{C}$
 811 defined by

$$\begin{aligned} 812 \text{ (A.2)} \quad g_h(t, x, v) &= h \int_{\mathbb{R}_+} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h + ix\xi - t\lambda_0 \xi^{2/3}} u_\xi(v) d\xi \\ &= \int_{\mathbb{R}_+} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi/h - t\lambda_0 \xi^{2/3} h^{-2/3}} u_{\xi/h}(v) d\xi. \end{aligned}$$

813 Since u_ξ is an eigenfunction of $-\partial_v^2 - i\xi v$, g_h is solution to $(\partial_t - v\partial_x - \partial_v^2)g_h = 0$. Moreover, using
 814 the asymptotic expansion of Ai [30, §9.7(ii)], we have uniformly in $v > 1$, in the limit $|\xi| \rightarrow +\infty$,
 815 $|\arg(\xi)| < \pi/2$ (for instance)

$$816 \quad u_\xi(v) = \exp\left(-\frac{2}{3}(e^{-i\pi/6}\xi^{1/3}v - \mu_0)^{3/2}\right) \tilde{u}_\xi(v) \quad \text{with} \quad \tilde{u}_\xi(v) = C\xi^{-1/12}v^{-1/4}(1 + \mathcal{O}(\xi^{-1/2})).$$

817 Thus, we can rewrite Eq. (A.2) as

$$818 \text{ (A.3)} \quad g_h(t, x, v) = \int_{\mathbb{R}_+} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h + ix\xi/h + \rho_{t,v,h}(\xi/h)} \tilde{u}_{\xi/h}(v) d\xi$$

819 with

$$820 \quad \rho_{t,v,h}(\xi) = -t\lambda_0 \xi^{2/3} - \frac{2}{3}(e^{-i\pi/6}\xi^{1/3}v - \mu_0)^{3/2}.$$

821 Thus, $g_h(t, x, v)$ can be written in the form (3.3). Moreover, if we choose $K > 0$, then we can
 822 choose $\xi_0 > 0$ and $\chi \in C_c^\infty(\mathbb{R})$ such that Hypothesis 3.3 holds with $X = \{(t, v), 0 \leq t \leq T, 1 \leq v \leq 2\}$.
 823 Then, Proposition 3.5 can be used to prove the lower-bound

$$824 \quad |g_h(t, x, v)| \geq ce^{-x^2/2h - Ch^{-2/3}}, \quad t \in [0, T], |x| \text{ small enough, } v \in [1, 2].$$

825 To get an upper-bound, we choose K large enough in Hypothesis 3.3 so that $|\tilde{u}_\xi(v)| \leq C$ for
 826 some $C > 0$ and every $\xi \in \mathcal{C}$ and $v \in (0, +\infty)$. Then, with the choice $X = [0, T] \times (0, +\infty)$ in
 827 Hypothesis 3.3, the Proposition 3.7 can be used to prove the upper-bound

$$828 \quad |g_h(t, x, v)| \leq \frac{C}{|x|^2} e^{-c/h - cv^{3/2}}.$$

829 As for the Kolmogorov equation (1.2), these two estimates prove that the observability inequality
 830 associated with the control problem (A.1) does not hold if $\Omega_x = \mathbb{R}$. For the case $\Omega_x = \mathbb{T}$, we
 831 periodize the solutions as in subsection 4.2. \square

832 We refer to subsection 1.3.4 for references related to the equation (1.2). It seems Theorem A.2
 833 could be extended to the case $\Omega = \Omega_x \times (a, b)$, as it is only a perturbation of the case $\Omega = \Omega_x \times (0, +\infty)$.

834 **A.3. Improved Boussinesq equation.** Finally, we mention another equation whose null-
 835 controllability can be treated with our method.

836 PROPOSITION A.3. *Let $\Omega = \mathbb{R}$ or \mathbb{T} . Let ω be a strict open subset of Ω and let $T > 0$. The*
 837 *equation*

$$838 \quad (\text{A.4}) \quad (\partial_t^2 - \partial_x^2 - \partial_x^2 \partial_t^2) f(t, x) = \mathbb{1}_\omega u(t, x), \quad t \in [0, T], x \in \Omega$$

839 *is not null-controllable on ω in time T .*

840 This equation has been studied by Cerpa and Crépeau [14], where it is called «improved
 841 Boussinesq equation». They prove that, when posed on $\Omega = (0, 1)$, it is not null-controllable with
 842 boundary control at $x = 1$. They also prove that if $\Omega = \mathbb{T}$, it is null-controllable with moving
 843 internal control on $\omega + ct$ ¹² if the speed c is large enough. But while their results suggest the
 844 improved Boussinesq equation is not null-controllable with (non-moving) internal control, they do
 845 not prove it. Here, we provide a proof of this fact.

846 *Sketch of the proof.* Let $\xi_0 > 0$ and $\chi \in C_c^\infty$ to be chosen later. For $\xi \in \mathbb{R}$ we define $\lambda_\xi =$
 847 $\xi^2(1 + \xi^2)^{-1}$. For $h > 0$, we consider

$$848 \quad g_h(t, x) = h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h + ix\xi - it\sqrt{\lambda_\xi}} d\xi.$$

849 Elementary computations prove that g_h is solution of $(\partial_t^2 - \partial_x^2 - \partial_x^2 \partial_t^2)g_h(t, x) = 0$ (it is related to
 850 the fact that this equation can be rewritten as $(\partial_t^2 - (I - \partial_x^2)^{-1} \partial_x^2)g_h(t, x) = 0$, and to the spectral
 851 analysis of this operator, see [14]).

852 With the notation of Eq. (3.3), $g_h(t, x) = I_{t,h,1}(x)$ with ρ independant of (t, h) defined by
 853 $\rho(\xi) = it\sqrt{\lambda_\xi}$. We can choose $K > 0$, $\xi_0 > 0$, and $\chi \in C_c^\infty$ such that Hypothesis 3.3 holds with
 854 $X = [0, T]$.

855 Then, with Proposition 3.5 and Proposition 3.7, we prove that $(g_h)_{h>0}$ is a counterexample to
 856 the observability inequality associated to the control problem (A.4) in the case $\Omega = \mathbb{R}$. In the case
 857 $\Omega = \mathbb{T}$, we periodize g_h as in subsection 4.2. \square

¹²In other words, the right-hand side is $\mathbb{1}_\omega(x - ct)u(t, x)$ instead of $\mathbb{1}_\omega(x)u(t, x)$.

858 **Appendix B. Precise estimation of the eigenfunctions.** To prove [Proposition 5.3](#), we
859 will need the following theorem, which is a special case¹³ of Theorem 18 in [22].

860 **THEOREM B.1.** *Let \mathcal{S} be the space of holomorphic function on the domain $\Omega = \{\Re(z) >$
861 $1\}$ with sub-exponential growth at infinity, i.e. $\gamma \in \mathcal{S}$ if and only if for all $\epsilon > 0$, $p_\epsilon(\gamma) =$
862 $\sup_{\Re(z) > 1} |\gamma(z)e^{-\epsilon|z|}| < +\infty$. We endow \mathcal{S} with the seminorms family $(p_\epsilon)_{\epsilon > 0}$.*

863 *Let γ in \mathcal{S} and let H_γ be the operator on polynomials with a double root at zero, defined by:*

$$864 \quad H_\gamma \left(\sum_{n>1} a_n z^n \right) = \sum_{n>1} \gamma(n) a_n z^n.$$

865 *Let E be an bounded subset of \mathbb{C} , star shaped with respect to 0. Let U be a neighborhood of \bar{E} . Then*
866 *there exists $C > 0$ such that for all polynomials f with a double root at 0:*

$$867 \quad (\text{B.1}) \quad |H_\gamma(f)|_{L^\infty(E)} \leq C|f|_{L^\infty(U)}.$$

868 *Moreover, the constant C above can be chosen continuously in $\gamma \in \mathcal{S}$.*

869 Note that according to the estimate of the previous [Theorem B.1](#), and assuming U is star-shaped
870 with respect to 0, the operators H_γ extend by density to every holomorphic function¹⁴ on U (with a
871 double zero at 0). So, we will apply this estimate (B.1) on entire functions (with a double zero at 0).

872 *Proof of [Proposition 5.3](#).* The proof is made by writing $\tilde{u}_\xi(v)$ as the power series $\tilde{u}_\xi(v) =$
873 $\sum \tilde{u}_{\xi,2n} v^{2n}$, and showing that the coefficients $\tilde{u}_{\xi,n}$ of this power series are of the form $\tilde{u}_{\xi,2n} =$
874 $\tilde{\rho}_\xi \gamma_\xi(n) \tilde{\xi}^n / n!$ for $n \geq 1$, with $\tilde{\rho}_\xi$ defined at the beginning of [subsection 5.4](#), so that with the notation
875 of [Theorem B.1](#):

$$876 \quad (\text{B.2}) \quad \tilde{u}_\xi(v) = 1 + \tilde{\rho}_\xi H_{\gamma_\xi}(e^{\tilde{\xi}v^2} - 1)(v)$$

877 Then, [Theorem B.1](#) will allow us to conclude.

878 Let us write $\tilde{u}_\xi(v) = \sum_{n=0}^{+\infty} \tilde{u}_{\xi,n} v^n$. Since \tilde{u}_ξ satisfies the Cauchy problem $-\tilde{u}_\xi'' + 2\tilde{\xi}v\tilde{u}_\xi' - \tilde{\rho}_\xi\tilde{u}_\xi = 0$
879 with initial conditions¹⁵ $\tilde{u}_\xi(0) = 1$, $\tilde{u}_\xi'(0) = 0$, we have $\tilde{u}_{\xi,0} = 1$, $\tilde{u}_{\xi,2n+1} = 0$ and

$$880 \quad (\text{B.3}) \quad \tilde{u}_{\xi,n+2} = \frac{2n\tilde{\xi} - \tilde{\rho}_\xi}{(n+1)(n+2)} \tilde{u}_{\xi,n}$$

881 so, by induction, for $n \geq 1$

$$882 \quad (\text{B.4}) \quad \tilde{u}_{\xi,2n} = -\frac{\tilde{\rho}_\xi}{2} \frac{(4\tilde{\xi})^{n-1}(n-1)!}{(2n)!} \prod_{k=1}^{n-1} \left(1 - \frac{\tilde{\rho}_\xi}{4\tilde{\xi}k} \right).$$

883 So, by defining

$$884 \quad (\text{B.5}) \quad \gamma_\xi(n) = -\frac{1}{8\tilde{\xi}n} \times \frac{4^n(n!)^2}{(2n)!} \times \prod_{k=1}^{n-1} \left(1 - \frac{\tilde{\rho}_\xi}{4\tilde{\xi}k} \right)$$

¹³In the reference, the Theorem is stated with an open (bounded star-shaped) domain U instead of a arbitrary (bounded star-shaped) subset E of \mathbb{C} , but we can set $U = E^\delta$, and apply the Theorem as stated in the reference to get $|H_\gamma(f)|_{L^\infty(E)} \leq C_\delta |f|_{L^\infty(E^{2\delta})}$.

¹⁴According to Runge's theorem [31, Theorem 13.9], the polynomials are dense in the space of holomorphic functions on U with the topology of the convergence on every compact.

¹⁵Here we use the fact that \tilde{u}_ξ is even when $\tilde{\xi}$ is real positive, which is well-known from Sturm-Liouville's theory.

885 we have $\tilde{u}_{\tilde{\xi}, 2n} = \tilde{\rho}_{\tilde{\xi}} \gamma_{\tilde{\xi}}(n) \tilde{\xi}^n / n!$. So, $\tilde{u}_{\tilde{\xi}}(v) = 1 + \tilde{\rho}_{\tilde{\xi}} \sum_{n \geq 1} \gamma_{\tilde{\xi}}(n) \frac{1}{n!} (v^2 \tilde{\xi})^n$. Assuming that $\gamma_{\tilde{\xi}}$ is in \mathcal{S} ,
 886 this is exactly the equation (B.2) we were claiming.

887 Well, let us actually prove that $\gamma_{\tilde{\xi}}$ is in the space \mathcal{S} defined in [Theorem B.1](#), i.e. that we can
 888 extend $n \mapsto \gamma_{\tilde{\xi}}(n)$ to a holomorphic function on $\Omega = \{\Re(z) > 1\}$ with subexponential growth. This is
 889 obvious for the term $-1/(8\tilde{\xi}n)$. The term $4^n(n!)^2/(2n)!$ can be extended to Ω with Euler's Gamma
 890 function, and Stirling's approximation gives us the subexponential growth (actually an equivalent in
 891 $\sqrt{\pi z}$). The product term is a tiny bit more tricky to extend to non-integer values. We define it with
 892 the following formula, which is inspired by [29], and where we have set $\alpha = -\tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$:

$$893 \quad (\text{B.6}) \quad \delta_{\tilde{\xi}}(z) = \prod_{k=1}^{+\infty} \frac{1 + \frac{\alpha}{k}}{1 + \frac{\alpha}{k+z-1}}.$$

894 This product converges if $|\alpha| < 1/2$ and $\Re(z) > 1$. And if n is integer, $\delta_{\tilde{\xi}}(n)$ is a telescopic
 895 product, and we have $\delta_{\tilde{\xi}}(n) = \prod_{k=1}^{n-1} (1 + \frac{\alpha}{k})$. Moreover, $\delta_{\tilde{\xi}}$ is holomorphic on Ω . We also claim that
 896 there exists $c, C > 0$ such that if $|\alpha| < 1/2$ and $\Re(z) > 1$, $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$. The proof of this claim is
 897 just a few basic computations, and we postpone it after the end of the proof at hand.

898 Since $\alpha = \tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$, according to [Theorem 5.1](#), $|\alpha| < 1/2$ as soon as $|\arg(\tilde{\xi})| < \theta$ and $|\tilde{\xi}|$ is
 899 large enough, say $|\tilde{\xi}| > M$ (depending on θ). Then, according to the claim, the term $\delta_{\tilde{\xi}}(z)$ has
 900 subexponential growth in Ω , and since it is holomorphic, it is in \mathcal{S} . Moreover, this estimate also
 901 proves that $(\delta_{\tilde{\xi}})_{|\alpha| < 1/2}$ is a bounded family of \mathcal{S} .

902 So $(\gamma_{\tilde{\xi}})$ is a bounded family of \mathcal{S} for $|\arg(\tilde{\xi})| < \theta$ and $|\tilde{\xi}| > M$. So, according to [Theorem B.1](#)
 903 and the following remark, for any neighborhood U of $[-1 + \epsilon, 1 - \epsilon]$ that is star-shaped with respect
 904 to 0, there exists $C > 0$ such that for all $|\tilde{\xi}| > M$ with $|\arg(\tilde{\xi})| < \theta$ and for every $v \in (-1 + \epsilon, 1 - \epsilon)$:

$$905 \quad (\text{B.7}) \quad \left| H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v) \right| \leq C(1 + |e^{\tilde{\xi}v^2}|_{L^\infty(U)})$$

906 and if we choose U to be small enough, we have $|H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v)| \leq C'|e^{(1-\delta)\tilde{\xi}}|$. Finally, thanks to
 907 equation (B.2) and [Theorem 5.1](#), we have

$$908 \quad (\text{B.8}) \quad |\tilde{u}_{\tilde{\xi}}(v) - 1| \leq C_\delta |\tilde{\xi}|^{3/2} |e^{-\delta\tilde{\xi}}|. \quad \square$$

909 *Proof of the claim that $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$.* We first write

$$910 \quad (\text{B.9}) \quad \delta_{\tilde{\xi}}(z) = \exp \left(\sum_{k=1}^{+\infty} \ln \left(1 + \frac{\alpha}{k} \right) - \ln \left(1 + \frac{\alpha}{k+z-1} \right) \right).$$

911 Let us also remind that we assume $|\alpha| < 1/2$ and $\Re(z) > 1$, so that for $k \in \mathbb{N}^*$ $|\alpha/k| < 1/2$ and
 912 $|\alpha/(k+z-1)| < 1/2$. We denote $k_0 = \lfloor |z| \rfloor$, and we separate the sum into two parts:

$$913 \quad S_{\leq k_0} = \sum_{k=1}^{k_0} \ln \left(1 + \frac{\alpha}{k} \right) - \ln \left(1 + \frac{\alpha}{k+z-1} \right) \quad S_{> k_0} = \sum_{k=k_0+1}^{+\infty} \ln \left(1 + \frac{\alpha}{k} \right) - \ln \left(1 + \frac{\alpha}{k+z-1} \right)$$

914 About the part of a sum for $k \leq k_0$, we have thanks to the triangle inequality and the fact that
 915 for $|x| < 1/2$, $|\ln(1+x)| \leq c|x|$:

$$916 \quad (\text{B.10}) \quad |S_{\leq k_0}| \leq 2c|\alpha| \sum_{k=1}^{k_0} \frac{1}{k} \leq 2c|\alpha|(\ln(k_0) + C') \leq 2c|\alpha|(\ln(|z|) + C'),$$

917 where we used the relation between the harmonic sum and the logarithm and the fact that $k_0 = \lfloor |z| \rfloor$.
 918 About the rest of the sum, we have by writing $\ln(1+b) - \ln(1+a) = \int_a^b \frac{dx}{1+x}$,

$$919 \quad (\text{B.11}) \quad |S_{>k_0}| \leq \sum_{k=k_0+1}^{+\infty} \left| \int_{\alpha/k}^{\alpha/(k+z-1)} \frac{dx}{1+x} \right| \leq \sum_{k=k_0+1}^{+\infty} 2 \left| \frac{\alpha}{k} - \frac{\alpha}{k+z-1} \right| \leq 2|\alpha(z-1)| \sum_{k=k_0+1}^{+\infty} \frac{1}{k^2},$$

920 where we used the fact that for $x \in [\frac{\alpha}{k}, \frac{\alpha}{k+z-1}]$, $|\frac{1}{1+x}| \leq 2$. By comparing this sum with an integral,

$$921 \quad (\text{B.12}) \quad |S_{>k_0}| \leq 2|\alpha(z-1)| \int_{k_0}^{+\infty} \frac{dx}{x^2} \leq 2|\alpha| \frac{|z-1|}{k_0} \leq C''|\alpha|,$$

922 where we again used that $k_0 = \lfloor |z| \rfloor$. Summing the two inequalities (B.10) and (B.12), and plugging
 923 this into equation (B.9) proves the claim. \square

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