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# Non-null-controllability of the fractional heat equation and of the Kolmogorov equation

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## Abstract

We prove in this article that the Kolmogorov-type equation  $(\partial_t - \partial_v^2 + v^2 \partial_x) f(t, x, v) = \mathbb{1}_\omega u(t, x, v)$  for  $(t, x) \in \mathbb{T} \times \Omega_v$  with  $\Omega_v = \mathbb{R}$  or  $(-1, 1)$  is not null-controllable in any time if  $\omega$  is a vertical band  $\omega_x \times \Omega_v$ . The idea is to remark that, for some families of solutions, the Kolmogorov equation behaves like what we'll call the *rotated fractional heat equation*  $(\partial_t + \sqrt{i}(-\Delta)^{1/4})g(t, x) = \mathbb{1}_\omega u(t, x)$ ,  $x \in \mathbb{T}$  and to disprove the observability inequality for rotated fractional equation by looking at how coherent states evolve.

## Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction</b>   | <b>2</b>  |
| <b>2</b> | <b>Non-null-controllability of the rotated fractional heat equation</b> | <b>7</b>  |
| <b>3</b> | <b>Non-null-controllability of the Kolmogorov equation</b>              | <b>12</b> |
|          | <b>Appendix A The saddle point method</b>                               | <b>18</b> |
|          | <b>Appendix B Precise estimation of the eigenfunctions</b>              | <b>20</b> |

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# 1 Introduction

## 1.1 Problem of the null-controllability

We are interested in the following equation, which is in the terminology of Hörmander [16, Section 22.2] a generalized Kolmogorov equation, where  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ ,  $\Omega = \mathbb{T} \times \mathbb{R}$  or  $\Omega = \mathbb{T} \times (-1, 1)$  and  $\omega$  is an open subset of  $\Omega$ :

$$(\partial_t + v^2\partial_x - \partial_v^2)f(t, x, v) = \mathbb{1}_\omega u(t, x, v) \quad t \in [0, T], (x, v) \in \Omega$$

For convenience, we will just say in this paper “the Kolmogorov equation”. Note that thanks to Hörmander’s bracket condition (see previous reference), the operator  $v^2\partial_x - \partial_v^2$  is hypoelliptic.

It is a control problem with state  $f \in L^2(\Omega)$  and control  $u$  supported in  $\omega$ . More precisely, we are interested in the exact null-controllability of this equation.

**Definition 1.** We say that the Kolmogorov equation is null-controllable on  $\omega$  in time  $T > 0$  if for all  $f_0$  in  $L^2(\Omega)$ , there exists  $u$  in  $L^2([0, T] \times \omega)$  such that the solution  $f$  of:

$$\begin{aligned} (\partial_t + v^2\partial_x - \partial_v^2)f(t, x, v) &= \mathbb{1}_\omega u(t, x, v) & t \in [0, T], (x, v) \in \Omega \\ f(0, x, v) &= f_0(x, v) & (x, v) \in \Omega. \end{aligned} \quad (1)$$

with Dirichlet boundary conditions if  $\Omega = \mathbb{T} \times (-1, 1)$ , satisfies  $f(T, x, v) = 0$  for all  $(x, v)$  in  $\Omega$ .

As we will see, this Kolmogorov equation is related to the *rotated fractional heat equation*, the latter being a model of the former, and we will also investigate its null-controllability.

**Definition 2.** Let  $\alpha \in [0, 1)$  and  $z$  with  $\Re(z) > 0$ . Let  $\Omega = \mathbb{R}$  or  $\Omega = \mathbb{T}$ . We say that the rotated heat equation is null-controllable on  $\omega \subset \mathbb{R}$  in time  $T > 0$  if for all  $f_0 \in L^2(\mathbb{R})$ , there exists  $u \in L^2([0, T] \times \omega)$  such that the solution  $f$  of:

$$\begin{aligned} (\partial_t + z(-\Delta)^{\alpha/2})f(t, x) &= \mathbb{1}_\omega u(t, x) & t \in (0, T), x \in \Omega \\ f(0, x) &= f_0(x) & x \in \Omega \end{aligned} \quad (2)$$

satisfies  $f(T, x) = 0$  for all  $x \in \Omega$ . Here, we have defined  $(-\Delta)^{\alpha/2}$  with the functional calculus, that is to say,  $(-\Delta)^{\alpha/2}f = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f))$  if  $\Omega = \mathbb{R}$ , where  $\mathcal{F}$  is the Fourier transform; and  $c_n((-\Delta)^{-\alpha/2}f) = |n|^\alpha c_n(f)$  if  $\Omega = \mathbb{T}$ , where  $c_n(f)$  is the  $n$ th Fourier coefficient of  $f$ .

## 1.2 Statement of the results

We will prove that the rotated fractional heat equation is never null controllable if  $\Omega \setminus \omega$  has nonempty interior, and that the Kolmogorov equation is never null-controllable if  $\omega = \omega_x \times \mathbb{T}$  where  $\mathbb{T} \setminus \omega_x$  has nonempty interior.

**Theorem 3.** *Let  $0 \leq \alpha < 1$ ,  $z$  such that  $\Re(z) > 0$  and  $\Omega = \mathbb{R}$  or  $\Omega = \mathbb{T}$ . Let  $\omega$  be a strict open subset of  $\Omega$ . The rotated fractional heat equation (2) is not null controllable in any time on  $\omega$ .*

We can generalize this theorem to higher dimensions, with  $\Omega = \mathbb{R}^k \times \mathbb{T}^l$ , but our method seems ineffective to treat the case where  $\Omega$  is, say, an open subset of  $\mathbb{R}$ . This may be because we are using the spectral definition of the fractional Laplacian, and our method might be adapted if we used a singular kernel definition of the fractional Laplacian.

Note that if  $\alpha = 0$ , the “rotated fractional heat equation” is then just a family of decoupled ordinary differential equation, and this is completely unimpressive. At the other end, the method used in this article does not work if  $\alpha = 1$ , but we still expect non-null-controllability, even if this remains a conjecture if  $\Omega$  is not the one-dimensional torus.

**Theorem 4.** *Let  $\Omega_v = \mathbb{R}$  or  $\Omega_v = (-1, 1)$ , and  $\Omega = \mathbb{T} \times \Omega_v$ . Let  $\omega_x$  be a strict open subset of  $\mathbb{T}$ . The Kolmogorov equation (1) is never null-controllable in  $\omega = \omega_x \times \Omega_v$ .*

The theorem can be extended to higher dimension in  $x$  and  $v$  if  $\Omega_v = \mathbb{R}^d$ . If we want, say  $\Omega_v = (-1, 1)^d$ , we lack information on the eigenvalues and eigenfunctions of  $-\partial_v^2 + inv^2$  on  $(-1, 1)^d$ , but this is the only obstacle to the generalization of the Theorem to this case.

## 1.3 Bibliographical comments

### 1.3.1 Null-controllability of parabolic partial differential equations

The null-controllability of parabolic equations has been investigated for a few decades now, with Fattorini and Russel [14] proving the null controllability of the heat equation in one dimension in 1971, Lebeau and Robbiano [19] and independently Fursikov and Imanuvilov [15] proving it in any dimension, in 1995 and 1996 respectively.<sup>1</sup>

However, the interest in degenerate parabolic equations is more recent. We now understand the null-controllability of parabolic equations degenerating at the

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<sup>1</sup>Lebeau and Robbiano actually proved the null-controllability of the heat equation on a compact manifold with boundary, while Imanuvilov actually proved it for general parabolic equation  $(\partial_t + A)f = \mathbb{1}_\omega u$  on a  $C^2$  domain of  $\mathbb{R}^d$  with  $A$  a uniformly elliptic operator whose coefficients can depend on space and time, also allowing lower order terms.

boundary in dimension one [10] and two [11] (see also references therein), where the authors found that these equations were null-controllable if the degeneracy is not too strong, but might not be if the degeneracy is too strong. For equations degenerating inside the domain, we mostly are looking at individual equations at a case-by-case basis. For instance, some Kolmogorov-type equations have been investigated since 2009 [3, 1, 6], the Grushin equation is being investigated since 2014 [4, 7, 17, 5] and the heat equation on the Heisenberg group since 2017 [2]. Some parabolic equation on the real half-line, some of them related to the present work, has been shown to strongly lack controllability [13]. Apart from the parabolic equations degenerating at the boundary, the only general family of degenerate parabolic equations whose null-controllability have been investigated we are aware of are hypoelliptic quadratic differential equations [8, 9].

About the Kolmogorov equation on  $\Omega = \mathbb{T} \times (-1, 1)$ , we know that if  $\omega = \mathbb{T} \times (a, b)$  with  $0 < a < b < 1$ , it is null-controllable in large times, but not in time smaller than  $a^2/2$ , and that if  $-1 < a < 0 < b < 1$ , it is null-controllable in arbitrarily small time [1]. If in the Kolmogorov equation (1) we replace  $v^2$  by  $v$ , the null-controllability holds when the boundary conditions are of some “periodic-type”, and holds in large time only with Dirichlet boundary conditions if  $\omega = \mathbb{T} \times (a, b)$  [3, 1]. On the other hand, if we replace  $v^2$  by  $v^\gamma$  where  $\gamma$  is an integer larger than 2 and  $\omega = \mathbb{T} \times (a, b)$ , it is never null-controllable [6]. In this last article, the null-controllability of a model of the equation we are interested in, namely the equation  $(\partial_t + iv^2(-\Delta_x)^{1/2} - \partial_v^2)g = 0$ , is also investigated.

### 1.3.2 Null-controllability of fractional heat equation and the spectral inequality

For the heat equation, Lebeau and Robbiano [19, 18] used a *spectral inequality* to prove the null-controllability, which is the following: let  $M$  a compact riemannian manifold with boundary, let  $\omega$  be an open subset of  $M$ , and let  $(\phi_i)_{i \in \mathbb{N}}$  an orthonormal basis of eigenfunctions of  $-\Delta$  with associated eigenvalues  $(\lambda_i)_{i \in \mathbb{N}}$ , then there exists  $C > 0$  and  $K > 0$  such that for every sequence of complex numbers  $(a_i)_{i \in \mathbb{N}}$  and every  $\mu > 0$

$$\left| \sum_{\lambda_i < \mu} a_i \phi_i \right|_{L^2(M)} \leq C e^{K\sqrt{\mu}} \left| \sum_{\lambda_i < \mu} a_i \phi_i \right|_{L^2(\omega)} \quad (3)$$

The key point to deduce the null-controllability of the heat equation from this spectral inequality is that if one takes an initial condition of the form  $f_0 = \sum_{\lambda_i \geq \mu} a_i \phi_i$  with no component along frequencies less than  $\mu$ , the solution of the heat equation decays like  $e^{-T\mu} |f_0|_{L^2(M)}$ , and the exponent in  $\mu$  in this decay (i.e. 1) is larger than the one appearing in the spectral inequality (i.e. 1/2).

Let us discuss this kind of phenomenon in a general setting, in the spirit of Miller [21]: let  $A$  be a self-adjoint positive operator on a Hilbert space  $\mathcal{H}$ , and

$B : X \rightarrow \mathcal{H}$  a bounded control operator.<sup>2</sup> We say the low modes are observable with a spectral exponent  $\gamma$  if for all  $v \in \mathcal{H}$  and  $\mu > 0$ , we have

$$|\mathbb{1}_{A \leq \mu} v|_{\mathcal{H}} \leq C e^{c\mu^\gamma} |B \mathbb{1}_{A \leq \mu} v|_{\mathcal{H}}. \quad (4)$$

For instance, according to the previous discussion, the low modes of the heat equation are observable with a spectral exponent  $1/2$ , and it can be proved that  $1/2$  is actually the best possible spectral exponent if  $\omega$  is a strict open subset of  $M$  [18, Proposition 5.5]. Also, the low modes of the fractional heat equation  $(\partial_t + (-\Delta)^{\alpha/2})g = 0$  are observable with a spectral exponent  $1/\alpha$ . Since the Lebeau-Robbiano method allows the construction of a control if the spectral exponent is less than one, the fractional heat equation is null controllable for  $\alpha > 1$ , as already mentioned by Micu and Zuazua [20] and Miller [21].

In these two papers, the respective authors also looked at the case  $\alpha < 1$ , and even if they didn't look at *internal controls* (the kind of controls we are interested in here), Micu and Zuazua found that it was not null-controllable with *shaped controls*,<sup>3</sup> while Miller proved that the one dimensional fractional Neumann Laplacian  $(\partial_t + (-\Delta)^{\alpha/2})g = 0$  with a boundary control is not null-controllable if  $1/2 < \alpha < 1$ , and more precisely that no finite linear combination of eigenfunctions could be steered to 0 in finite time. If  $\alpha = 1$ , the non-null-controllability has been proved in dimension one [17], and the method of this reference can be adapted to treated the *rotated* half-heat equation  $(\partial_t + z\sqrt{-\Delta})g = \mathbb{1}_\omega u$ . But the null-controllability of the half-heat equation is still open if  $\Omega$  is a general analytic manifold (and even if  $\Omega = \mathbb{R}$ ).

About the equations we mentioned earlier, we can prove that the low modes of the Kolmogorov equation are observable with a spectral exponent<sup>4</sup> 2, and the low modes of the Grushin equation  $(\partial_t - \partial_x^2 - x^2 \partial_y^2)g = 0$  are observable with a spectral exponent 1 (and if  $\omega = (a, b) \times \mathbb{T}$  with  $a > 0$ , it is the best possible).

So it seems that 1 is a critical value for the spectral exponent: below, the equation is null-controllable, and above it is not. Our results tends to confirm this conjecture. Note that an equation with a spectral exponent greater than one is not *unconditionally* not null-controllable, though: for instance, if the degeneracy of the equation is contained in the control domain, we actually expect null-controllability.

---

<sup>2</sup>In our case,  $B$  is the multiplication by  $\mathbb{1}_\omega$ . We could allow  $B$  to be unbounded by invoking the notion of *admissible control operator*. We refer to Coron's book [12, Section 2.3] for the terminology of general abstract control systems.

<sup>3</sup>*Shaped controls* are right-hand-side of equation (2) of the form  $a(x)u(t)$ , where  $a$  is a fixed given function and  $u$  is the control we can choose.

<sup>4</sup>The Kolmogorov operator  $v^2 \partial_x - \partial_v^2$  is not self-adjoint, so we don't have a functional calculus, and we can't define  $\mathbb{1}_{A \leq \mu}$ , and our definition of the spectral exponent does not make sense. However, at least in the bounded case, the spectrum is discrete, so we have eigenfunctions, and we can prove a spectral inequality.

But even if it is not, some poorly understood geometric conditions on the control domain can still ensure null-controllability, at least in large enough time (it happens for the Kolmogorov equation, the Grushin equation and the heat equation on the Heisenberg group, see previous references).

## 1.4 Outline of the proof, structure of the article

As usual in controllability problems, we focus on *observability inequalities* on the *adjoint systems*, that are equivalent to the null-controllability (see [12, Theorem 2.44]). Specifically, the null-controllability of the Kolmogorov equation (1) is equivalent to the existence of  $C > 0$  such that for every solution  $g$  of<sup>5</sup>

$$(\partial_t - v^2 \partial_x - \partial_v^2)g(t, x, v) = 0 \quad t \in (0, T), (x, v) \in \Omega \quad (5)$$

with Dirichlet boundary conditions if  $\Omega = \mathbb{T} \times (-1, 1)$ ,

$$|g(T, \cdot)|_{L^2(\Omega)} \leq C |g|_{L^2((0, T) \times \omega)}. \quad (6)$$

In the same spirit, the null-controllability of the rotated fractional heat equation (2) is equivalent to the existence of  $C > 0$  such that for every solution  $g$  of

$$(\partial_t + \bar{z}(-\Delta)^{\alpha/2})g(t, x) = 0 \quad t \in (0, T), x \in \Omega \quad (7)$$

we have

$$|g(T, \cdot)|_{L^2(\Omega)} \leq C |g|_{L^2((0, T) \times \omega)} \quad (8)$$

Let us first look at the eigenfunctions of the Kolmogorov equation for  $\Omega_v = \mathbb{R}$ . The first eigenfunction of  $-\partial_v^2 + inv^2$  on  $\mathbb{R}$ , is  $e^{-\sqrt{in}v^2/2}$  (up to a normalization constant), with eigenvalue  $\sqrt{in}$ . So,  $\Phi_n(x, v) = e^{inx - \sqrt{in}v^2/2}$  is an eigenfunction of the Kolmogorov operator  $v^2 \partial_x - \partial_v^2$ , with eigenvalue  $\sqrt{in}$ . So, the solution of the Kolmogorov equation  $(\partial_t + v^2 \partial_x - \partial_v^2)f = 0$  with initial condition  $f(0, x, v) = \sum a_n \Phi_n(x, v)$  is  $f(t, x, v) = \sum a_n e^{-\sqrt{in}t} \Phi_n(x, v)$ . This suggests that, dropping the  $v$  variable for the moment, the Kolmogorov equation is close to an equation where the eigenfunctions are the  $e^{inx}$  with eigenvalue  $\sqrt{in}$ , i.e. the equation  $(\partial_t + \sqrt{i}(-\Delta_x)^{1/4})f(t, x) = 0$  with  $x \in \mathbb{T}$ .

So, we will start by looking at the rotated fractional heat equation. We will need to begin with the case of the rotated fractional heat equation on the whole real line. In Section 2.1 We will disprove the observability inequality (2) by looking at solutions of the rotated fractional heat equation (7) with coherent states as initial conditions, i.e. initial conditions of the form  $g_h(x) = e^{ix\xi_0/h - x^2/2h}$ . We will get asymptotics on the solutions thanks to the saddle point method, or more precisely, the following slight generalization we prove in Appendix A:

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<sup>5</sup>Note that this is the adjoint of the Kolmogorov equation where we reversed the time.

**Proposition 5.** *Let  $H_a^\infty$  be the set of bounded holomorphic function on  $D_a = \{|z| < a\}$ , with norm  $|u|_\infty = \sup_{|z| < a} |u(z)|$ . Let  $0 < \alpha < 1$ ,  $r \in H_a^\infty$  and  $u \in H_a^\infty$ . We have:*

$$\int_{-a}^a e^{-\xi^2/2h+r(\xi)/h^\alpha} u(\xi) d\xi = e^{\mathcal{O}(h^{-\alpha})}(u(0) + \mathcal{O}(h^{1-\alpha}|u|_\infty)) \quad (9)$$

Moreover, the first  $\mathcal{O}$  does not depend on  $u$  at all, and both of the  $\mathcal{O}$ s are locally uniform in  $r \in H_a^\infty$ .

From this non-null-controllability result of the rotated fractional heat equation on the whole real line, we prove in Section 2.2 the same result on the torus by considering periodic version of the solution on the whole real line. We treat Kolmogorov's equation on  $\Omega = \mathbb{T} \times \mathbb{R}$  by unsubtly adding the  $v$  variable to the solutions of the rotated fractional heat equation on  $\mathbb{T}$  (Section 3.2). For Kolmogorov's equation on  $\Omega = \mathbb{T} \times (-1, 1)$ , we need some information on the eigenvalues and the eigenfunctions, which are not explicit anymore. Fortunately, we already proved most of what we need in another article [17, Section 4]. We prove the non-null-controllability of Kolmogorov equation in Section 3.3.

## 2 Non-null-controllability of the rotated fractional heat equation

### 2.1 The rotated fractional heat equation on the whole real line

For  $x_0, \xi_0 \in \mathbb{R}$  and  $h > 0$ , let  $\phi_{\xi_0, h} : x \mapsto (\pi h)^{-1/4} e^{ix\xi_0/h - x^2/2h}$  be the (semiclassical) coherent state centered at  $(0, \xi_0)$ . Let us fix  $\xi_0 > 0$  and  $\chi$  a  $C^\infty(\mathbb{R}, [0, 1])$  function which is equal to 1 in  $[-\xi_0/4, \xi_0/4]$  and which is zero outside of  $[-\xi_0/2, \xi_0/2]$ . We define  $g_{0, h}$  as the semiclassical coherent state centered at  $(0, \xi_0)$  which has been bandlimited with the cutoff  $\chi(\xi_0 - \cdot)$ :

$$g_{0, h} = \mathcal{F}_h^{-1}(\chi(\xi - \xi_0)\mathcal{F}_h(\phi_{\xi_0, h})(\xi))$$

where  $\mathcal{F}_h$  is the semiclassical Fourier transform defined by  $\mathcal{F}_h(f)(\xi) = h^{-1/2}\mathcal{F}(f)(\xi/h)$ , or equivalently:

$$\mathcal{F}_h(f)(\xi) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} f(x) e^{-ix\xi/h} dx.$$

We will note  $A = z(-\Delta)^{\alpha/2}$ , where  $\Re(z) > 0$  and  $0 < \alpha < 1$ . We have  $\mathcal{F}_h(\phi_{\xi_0, h})(\xi) = (\pi h)^{-1/4} e^{-(\xi - \xi_0)^2/2h}$ , so, for all  $t > 0$ :

$$\mathcal{F}_h(e^{-tA^*} g_{0, h})(\xi) = \frac{1}{(\pi h)^{1/4}} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h - t|\xi|^\alpha/h^\alpha} \quad (10)$$



**Proposition 6** (Punctual estimates). *Let  $\epsilon > 0$ . We have uniformly in  $t > 0$  and  $|x| > \epsilon$ :*

$$e^{-tA^*} g_{0,h}(x) = \mathcal{O}(|x|^{-2} e^{-c/h-ct/h^\alpha}) \quad (11)$$

and locally uniformly in  $t > 0$  and  $|x| < \xi_0/8$ :

$$e^{-tA^*} g_{0,h}(x) = e^{ix\xi_0/h-x^2/2h+\mathcal{O}(h^{-\alpha})} \quad (12)$$

*Proof.* Thanks to the expression of  $\mathcal{F}_h(e^{-tA^*} g_{0,h})$  (Eq. (10)) we have

$$e^{-tA^*} g_{0,h}(x) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} \int_{\mathbb{R}} e^{-(\xi-\xi_0)^2/2h+ix\xi/h} \chi(\xi-\xi_0) e^{-t\bar{z}\xi^\alpha/h^\alpha} d\xi, \quad (13)$$

and noting  $\phi_x(\xi) = (\xi - ix)^2/2$ , we have  $-(\xi - \xi_0)^2/2h + ix\xi/h = -\phi_x(\xi - \xi_0)/h + i\xi_0 x/h - x^2/2h$ , so by the change of variables  $\xi \mapsto \xi - \xi_0$ , we have:

$$e^{-tA^*} g_{0,h}(x) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} e^{ix\xi_0/h-x^2/2h} \int_{\mathbb{R}} e^{-\phi_x(\xi)/h} \chi(\xi) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha} d\xi \quad (14)$$

The function  $\xi \mapsto \chi(\Re(\xi)) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha}$  is supported in  $[-\xi_0/2, \xi_0/2]$ ,  $C^\infty$  on  $\mathbb{R}$ , constant on  $\{-\xi_0/4 < \Re(\xi) < \xi_0/4\}$ . So we can deform the integration path between  $\xi = -\xi_0/4$  and  $\xi_0/4$ .

To get the first estimate, we first integrate by parts to get the decay in  $x$ : using the fact that

$$-h\partial_\xi e^{-\phi_x(\xi)/h} = (\xi - ix) e^{-\phi_x(\xi)/h}$$

we get

$$\int_{\mathbb{R}} e^{-\phi_x(\xi)/h} \chi(\xi) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha} d\xi = \int_{\mathbb{R}} e^{-\phi_x(\xi)/h} u_x(\xi) d\xi \quad (15)$$

with

$$u_x(\xi) = \left( h\partial_\xi \frac{1}{\xi - ix} \right)^2 \left( \chi(\xi) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha} \right). \quad (16)$$

Then, we deform the integration path toward  $ix$ , to increase  $\Re(\phi_x(\xi))$ . For instance, we can choose to follow an hyperbole arc of the form  $\Re(\phi_x(\xi)) = \Re(\phi_x(a)) = a^2/2 - x^2/2$  for  $a = \min(\xi_0/4, |x|/2)$  (see Fig. 1). The length of this hyperbole arc is bounded independently of  $x$ <sup>6</sup> and so we have:

$$\left| \int_{\mathbb{R}} e^{-\phi_x(\xi)/h} u_x(\xi) d\xi \right| \leq C \sup_{\xi \in \text{integration path}} |u_x(\xi)| e^{-a^2/2h+x^2/2h} \quad (17)$$

---

<sup>6</sup>We choose  $a$  less than  $|x|$  because otherwise, there is no hyperbole arc containing the integration endpoints  $\pm\xi_0/4$ : they would be on different connected components of the hyperbole  $\Re(\phi_x(\xi)) = a^2/2 - x^2/2$ . Obviously, this choice implies that the decay we prove is not the optimal one, but we will get the optimal decay in estimate (12).

and using the definition of  $u_x$  (Eq. (16)) and the fact that  $\xi$  is supported in  $[-\xi_0/2, \xi_0/2]$ , we get

$$\left| \int_{\mathbb{R}} e^{-\phi_x(\xi)/h} u_x(\xi) d\xi \right| \leq C|x|^{-2} e^{-ct/h^\alpha} e^{-a^2/2h+x^2/2h} \quad (18)$$

and with the definition of  $e^{-tA^*} g_{0,h}(x)$  (Eq. (14))

$$|e^{-tA^*} g_{0,h}(x)| \leq C|x|^{-2} h^{-3/4} e^{-a^2/2h-ct/h^\alpha} \quad (19)$$

which proves the first estimate.

To prove the second estimate (Eq. (12)), we simply use the saddle point method. First, we change the integration path in equation (14) for one that goes through the saddle point  $\xi = ix$ , and we get

$$\int_{\mathbb{R}} e^{-\phi_x(\xi)/h} \chi(\xi) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha} d\xi = \int_{-a}^a e^{-\xi^2/2h} e^{-t\bar{z}(\xi+\xi_0+ix)^\alpha/h^\alpha} d\xi + \mathcal{O}(e^{-c/h}) \quad (20)$$

where  $a > 0$  is small enough (say,  $a = \xi_0/8$ ), and the  $\mathcal{O}$ , corresponding to the part of the integral away from the saddle point, is locally uniform in  $t > 0$  and  $|x| < \xi_0/8$ . Then, we use our saddle point theorem (Proposition 5), which gives us

$$\int_{\mathbb{R}} e^{-\phi_x(\xi)/h} \chi(\xi) e^{-t\bar{z}(\xi+\xi_0)^\alpha/h^\alpha} d\xi = e^{\mathcal{O}(h^{-\alpha})} + \mathcal{O}(e^{-c/h}) = e^{\mathcal{O}(h^{-\alpha})} \quad (21)$$

where, according to the last part of Proposition 5, the  $\mathcal{O}$ s are locally uniform in  $t > 0$  and  $|x| < \xi_0/8$ . Then, equation (14) gives us the claimed estimate (12).  $\square$

We now can prove the non-null-controllability of the rotated fractional heat equation on the whole real line.

*Proof of Theorem 3 in the case  $\Omega = \mathbb{R}$ .* Since the rotated fractional heat equation is translation invariant, we can assume without loss of generality that  $\omega = \{|x| > \epsilon\}$ . Then, the functions  $g_h(t, x) = e^{-tA^*} g_{0,h}(x)$  that were defined before provide a counterexample to the observability inequality(8). Indeed we have according to the lower bound (12) of Proposition 6

$$|g_h(T, \cdot)|_{L^2(\mathbb{R})} \geq |g_h(T, \cdot)|_{L^2(|x|<\epsilon)} \geq e^{\mathcal{O}(h^{-\alpha})}$$

and according to the upper bound (11),

$$|g_h|_{L^2((0,T) \times \omega)}^2 \leq T \int_{|x|>\epsilon} \frac{C}{x^4} e^{-c/h} dx \leq C e^{-c/h}$$

and taking  $h \rightarrow 0^+$  disproves the observability inequality.  $\square$

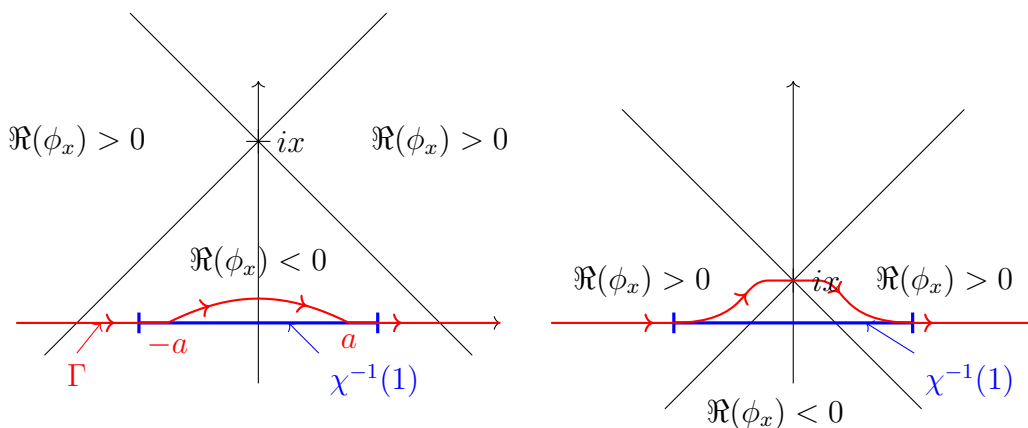


Figure 1: In blue, the interval where  $\chi = 1$ . The diagonal lines define four sectors; in the left and right ones,  $\Re(\phi_x) > 0$  and in the top and bottom ones,  $\Re(\phi_x) < 0$ . In red, the path of integration we chose in the integral defining  $e^{-tA} f_0$  (Eq. 14). Left figure: if  $x$  is not too small, we deform the integration path toward  $ix$ , by following a hyperbole arc  $\Re(\phi_x) = \text{constant}$  between  $-a$  and  $a$  ( $a$  independent of  $|x| > \epsilon$ ). Right figure: if  $|x| < \xi_0/4$ , we choose a path that goes through the saddle point  $ix$ , but that stays in  $\{\Re(\phi_x) > 0\}$ .

*Remark 7.* We can extend the proof to higher dimensions, as well as any open subset of  $\mathbb{R}^n$  (as long as  $\mathbb{R}^n \setminus \omega$  contains an open ball).

Also, we implicitly looked at the fractional heat equation with complex valued solution. This means that we proved that there exists a initial condition  $f_0$  of the rotated fractional heat equation that we cannot steer to 0, but this initial condition might not be real valued. For the (unrotated) fractional heat equation, we might be more interested in real valued solutions. But our results actually implies there exists a real valued initial condition that cannot be steered to 0, for if both the real part  $\Re(f_0)$  and the imaginary part  $\Im(f_0)$  could be steered to 0, then  $f_0$  itself could be steered to 0. Such remark stays true for the Kolmogorov-type equation.

## 2.2 The fractional heat equation on the torus

The case of the fractional heat equation on the torus is a bit different because we are not dealing with integrals, but sums. Therefore, tools like the saddle point method do not seem to be of much use. Nonetheless, with a trick, we can deduce the theorem on the torus from the theorem on the whole real line.

*Proof of Theorem 3 in the case  $\Omega = \mathbb{T}$ .* The basic idea is the trick of the proof of Poisson summation formula, namely the fact that the Fourier coefficients of a function of the form  $g_{0\text{per}}(x) = \sum_{k \in \mathbb{Z}} g_0(x + 2\pi k)$  are the value of the Fourier transform of  $g_0$  at the integers (up to a multiplication by  $\sqrt{2\pi}$ ).

So, let  $g_h \in C^\infty(\mathbb{R})$  be as in the previous section. Since the Fourier transform of  $g_h(t, \cdot)$  is  $C^\infty$  with compact support,<sup>7</sup>  $g_h(t, x)$  decays faster than any polynomials as  $|x| \rightarrow \infty$  and we can define  $g_{h\text{per}}(t, x) = \sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k)$ . According to the trick described before,  $c_n(g_{h\text{per}}(t, \cdot)) = (2\pi)^{-1/2} \mathcal{F}(g_h)(t, \cdot)(n)$ . But, by definition of  $g_h$  as the solution of the rotated fraction heat equation,  $\mathcal{F}(g_h)(t, \cdot)(\xi) = \mathcal{F}(g_h)(0, \cdot)(\xi) e^{-t\bar{z}|\xi|^\alpha}$ , so, using the trick again:

$$c_n(g_{h\text{per}}(t, \cdot)) = c_n(g_{h\text{per}}(0, \cdot)) e^{-t\bar{z}|n|^\alpha}; \quad (22)$$

So  $g_{h\text{per}}$  is solution to the rotated fractional heat equation (7) on the torus. Now we prove that the terms for  $k \neq 0$  are negligible. Indeed, we have by definition of  $g_{h\text{per}}$

$$|g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} = \left| \sum_{k \in \mathbb{Z}} g_h(T, \cdot + 2\pi k) \right|_{L^2(\mathbb{T})} \quad (23)$$

and by singling out to term for  $k = 0$  and thanks to the triangle inequality

$$|g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} \geq |g_h(T, \cdot)|_{L^2(-\pi, \pi)} - \sum_{k \neq 0} |g_h(T, \cdot)|_{L^2((2k-1)\pi, (2k+1)\pi)} \quad (24)$$

and thanks to the punctual estimates on  $g_h$  (Proposition 6)

$$|g_{h\text{per}}(T, \cdot)|_{L^2(\mathbb{T})} \geq e^{\mathcal{O}(h^{-\alpha})} - \sum_{k \neq 0} \mathcal{O}\left(\frac{1}{k^2} e^{-c/h}\right) \quad (25)$$

$$\geq e^{\mathcal{O}(h^{-\alpha})} - \mathcal{O}(e^{-c/h}). \quad (26)$$

In the same spirit, we have thanks to the triangle inequality, and identifying  $\omega = \mathbb{T} \setminus [-\epsilon, \epsilon]$  with  $(-\pi, \pi) \setminus [-\epsilon, \epsilon] \subset \mathbb{R}$

$$|g_{h\text{per}}|_{L^2([0, T] \times \omega)} \leq \sum_{k \in \mathbb{Z}} |g_h|_{L^2([0, T] \times (\omega + 2\pi k))} \quad (27)$$

and thanks again to the estimates of the previous section

$$|g_{h\text{per}}|_{L^2([0, T] \times \omega)} = \mathcal{O}(e^{-c/h}). \quad (28)$$

Taking  $h \rightarrow 0^+$  disproves the observability inequality (8) and proves the Theorem.  $\square$

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<sup>7</sup>We added the cutoff function  $\chi$  just to localize the Fourier transform away from the singularity of  $|\xi|^\alpha$  at  $\xi = 0$ .

### 3 Non-null-controllability of the Kolmogorov equation

#### 3.1 Introduction

Now, we look at the Kolmogorov equation (1) with associated observability inequality (6). As hinted in the introduction, we look for counterexamples of the observability inequality among solutions of the adjoint of the Kolmogorov equation (5) of the form  $g(t, x, v) = \sum_{n \geq 0} a_n e^{inx} g_n(v) e^{-\lambda_n t}$ , where  $g_n(v)$  is the first eigenfunction of  $-\partial_v^2 - inv^2$  and  $\lambda_n$  its associated eigenvalue, that is equal to  $\sqrt{-in}$  if  $\Omega_v = \mathbb{R}$ , and is close to  $\sqrt{-in}$  if  $\Omega_v = (-1, 1)$ .

We remark that apart from the  $g_n(v)$  term, those solutions have the same form as solutions of the rotated fractional heat equation  $(\partial_t + \sqrt{-i}(-\Delta)^{1/4})g = 0$ . So, the strategy is to prove the same estimates we proved for the rotated fractional heat equation, but with some uniformity in the parameter  $v$ . Since the computations are essentially the same, we only tell what we need to care about in comparison with the rotated fractional heat equation, but we do not give (again) the full details of the computations.

#### 3.2 The Kolmogorov equation with unbounded velocity

*Proof of Theorem 4 with  $\Omega_v = \mathbb{R}$ .* In the case  $\Omega_v = \mathbb{R}$ , the first eigenfunction of  $-\partial_v^2 + inv^2$  is  $g_n(v) = e^{-\sqrt{-in}v^2/2}$  with eigenvalue  $\lambda_n = \sqrt{-in}$ . Without loss of generality, we can assume  $\omega_x = \mathbb{T} \setminus [-\epsilon, \epsilon]$ . To mimic the proof of the non-null-controllability of the rotated heat equation on the torus, we adapt the definition of  $g_h$  in equation (14) by adding the  $v$ -variable:

$$g_{0,h}(x, v) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}v^2/2} d\xi. \quad (29)$$

Its evolution by the rotated fractional heat equation  $(\partial_t + \sqrt{-i}(-\Delta_x)^{1/4})g = 0$  is:

$$g_h(t, x, v) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(v^2/2+t)} d\xi \quad (30)$$

or equivalently

$$\mathcal{F}_h(g_h(t, \cdot, v))(\xi) = (\pi h)^{-1/4} \chi(\xi - \xi_0) e^{-(\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(v^2/2+t)} \quad (31)$$

We also define its periodic version  $g_{h\text{per}}(t, x, v) = \sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k, v)$ . Since its Fourier coefficients in  $x$  can be written as

$$c_n(g_{h\text{per}}(t, \cdot, v)) = \sqrt{h} \mathcal{F}_h(g_h)(hn) = a_{h,n} e^{-\sqrt{-in}(t+v^2/2)} = a_{h,n} e^{-\sqrt{-in}t} g_n(v), \quad (32)$$

$g_{h\text{per}}$  can be written as  $g_{h\text{per}}(t, x, v) = \sum a_{h,n} e^{\sqrt{-i}nt} g_n(v) e^{inx}$ . So  $g_{h\text{per}}$  satisfies the adjoint of the Kolmogorov equation (5).

We remark that the  $g_h$  we defined here is the same as the one we defined for the rotated fractional heat equation in equation (14), with  $\alpha = 1/2$  and  $z = \sqrt{i}$ , but with  $t + v^2/2$  instead of  $t$ . So, according to the estimates we proved on  $g_h$  in Proposition 6, we have uniformly in  $|x| > \epsilon$ ,  $t > 0$  and  $v \in \mathbb{R}$

$$g_h(t, x, v) = \mathcal{O}(|x|^{-2} e^{-c/h - c(t+v^2/2)/\sqrt{h}}) \quad (33)$$

and locally uniformly in  $t > 0$ ,  $|x| < \epsilon$  and  $v \in \mathbb{R}$

$$g_h(t, x, v) = e^{ix\xi_0/h - x^2/2h + \mathcal{O}(h^{-1/2})} \quad (34)$$

Moreover, by adapting the computations we did to get the theorem for the rotated heat equation on the torus

$$|g_{h\text{per}}|_{L^2([0,T] \times \omega)} \leq \sum_{k \in \mathbb{Z}} |g_h|_{L^2([0,T] \times (\omega + 2\pi k) \times \mathbb{R})} \quad (35)$$

and by integrating the upper bound on  $g_h$  (Eq. (33))

$$|g_{h\text{per}}|_{L^2([0,T] \times \omega)} = \sum_{k \in \mathbb{Z}} \mathcal{O}\left(\frac{1}{k^2} e^{-c/h}\right) \quad (36)$$

$$= \mathcal{O}(e^{-c/h}). \quad (37)$$

On the other hand, we have

$$|g_{h\text{per}}(T, \cdot, \cdot)|_{L^2(\Omega)} \geq |g_{h\text{per}}(T, \cdot, \cdot)|_{L^2(|x| < \epsilon, |v| < \epsilon)} \quad (38)$$

$$\geq |g_h(T, \cdot)|_{L^2(|x| < \epsilon, |v| < \epsilon)} - \sum_{k \neq 0} |g_h(T, \cdot)|_{L^2(|x+2\pi k| < \epsilon, |v| < \epsilon)} \quad (39)$$

so, integrating the lower bound (34) for the term  $k = 0$  and the upper bound (33) for the other terms, we have

$$|g_{h\text{per}}(T, \cdot, \cdot)|_{L^2(\Omega)} \geq e^{\mathcal{O}(h^{-1/2})} - \sum_{k \neq 0} \mathcal{O}\left(\frac{1}{k^2} e^{-c/h}\right) \quad (40)$$

$$\geq e^{\mathcal{O}(h^{-1/2})} - \mathcal{O}(e^{-c/h}) \quad (41)$$

Taking again  $h \rightarrow 0$  disproves the observability inequality and proves the Theorem.  $\square$

### 3.3 The Kolmogorov equation with bounded velocity

To treat the Kolmogorov equation with  $\Omega_v = (-1, 1)$ , we need some information on the first eigenfunction  $g_n$  of  $-\partial_v^2 - in v^2$  with Dirichlet boundary conditions on  $(-1, 1)$ , and with associated eigenvalue  $\lambda_n = \sqrt{-in} + \rho_n$ . Moreover, as we will do some change of integration paths, we also need some analyticity in  $n$ . We will note  $\tilde{g}_\xi$  the first<sup>8</sup> eigenfunction of  $-\partial_v^2 + (\tilde{\xi}v)^2$ , and  $\tilde{\lambda}_\xi = \tilde{\xi} + \tilde{\rho}_\xi$  the associated eigenvalue, so that, with  $\tilde{\xi} = \sqrt{-i\xi}$ , we have  $g_\xi = \tilde{g}_\xi$  and  $\rho_\xi = \tilde{\rho}_\xi$ .

In an article on the Grushin equation [17, Section 4] we proved that  $\tilde{\rho}_\xi$  and  $\tilde{g}_\xi$  exist if  $\Re(\tilde{\xi}) > 0$  and  $|\tilde{\xi}| > r(|\arg(\tilde{\xi})|)$  for some non-decreasing function  $r : (0, \pi/2) \rightarrow \mathbb{R}_+$ . We also proved the next two theorems.

**Theorem 8** (Theorem 22 and remark 23 of [17]). *Let  $0 < \theta < \pi/2$ . We have*

$$\tilde{\rho}_\xi \sim \frac{4}{\sqrt{\pi}} \tilde{\xi}^{3/2} e^{-\tilde{\xi}}$$

in the limit  $|\tilde{\xi}| \rightarrow \infty, |\arg(\tilde{\xi})| < \theta$ .

**Proposition 9** (Proposition 25 of [17]). *We normalize  $\tilde{g}_\xi$  by  $\tilde{g}_\xi(0) = 1$  instead of  $|\tilde{g}_\xi|_{L^2} = 1$ . Let  $0 < \theta < \pi/2$  and  $\epsilon > 0$ . We have for all  $v \in (-1, 1)$  and  $|\tilde{\xi}| > r(\theta)$ ,  $|\arg(\tilde{\xi})| < \theta$ :*

$$|e^{(1-\epsilon)\tilde{\xi}v^2/2} \tilde{g}_\xi(v)| \leq C_{\epsilon, \theta}.$$

Theorem 8 gives us all we need to know on the eigenvalue, while proposition 9 gives us an upper bound on the eigenfunction. We will also need the following lower bound, that we prove in appendix B.

**Proposition 10.** *Let  $0 < \theta < \pi/2$  and  $\epsilon > 0$ . We normalize  $\tilde{g}_\xi$  again by  $\tilde{g}_\xi(0) = 1$  and define  $\tilde{u}_\xi(v) = e^{\tilde{\xi}v^2/2} \tilde{g}_\xi(v)$ . Then  $\tilde{u}_\xi(v)$  converges exponentially fast to 1, as  $|\tilde{\xi}| \rightarrow \infty, |\arg(\tilde{\xi})| < \theta$ , this convergence being uniform in  $|v| < 1 - \epsilon$ .*

We now know all we need to adapt the proof of the non-null-controllability of the Kolmogorov with  $\Omega_v = \mathbb{R}$  to the case of  $\Omega_v = (-1, 1)$ .

*Proof of Theorem 4 with  $\Omega_v = (-1, 1)$ .* The counterexample to the observability inequality (6) is basically the same as in the case  $\Omega_v = \mathbb{R}$ , only with the added corrections to the eigenvalues and eigenfunctions. We define  $g_h(t, x, v)$  by:

$$g_h(t, x, v) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-ix\xi/h - (\xi - \xi_0)^2/2h - \sqrt{-i\xi/h}(t+v^2/2)} \delta_{h,t,v}(\xi) d\xi \quad (42)$$

---

<sup>8</sup>“First” in the sense that it is the analytic continuation in  $\tilde{\xi}$  of the first eigenfunction of  $-\partial_v^2 + (\tilde{\xi}v)^2$  for  $\tilde{\xi} \in \mathbb{R}_+$ , assuming it exists.

where  $\delta_{h,t,v}(\xi)$  is the ‘‘correction’’ defined by

$$\delta_{h,t,v}(\xi) = e^{\sqrt{-i\xi/h}v^2/2} g_{\xi/h}(v) e^{-t\rho_{\xi/h}} \quad (43)$$

or equivalently,

$$g_h(t, x, v) = \frac{1}{\sqrt{2}(\pi h)^{3/4}} \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-ix\xi/h - (\xi - \xi_0)^2/2h - \lambda_{\xi/h}t} g_{\xi/h}(v) d\xi. \quad (44)$$

With the notation  $\phi_x(\xi) = (\xi - ix)^2/2$  (as in the proof of Proposition 6), we rewrite it as:

$$g_h(t, x, v) = \frac{e^{ix\xi_0/h - x^2/2h}}{\sqrt{2}(\pi h)^{3/4}} I_h(t, x, v) \quad (45)$$

with

$$I_h(t, x, v) = \int_{\mathbb{R}} \chi(\xi) e^{-\phi_x(\xi)/h - \sqrt{-i(\xi + \xi_0)/h}(t + v^2/2)} \delta_{h,t,v}(\xi + \xi_0) d\xi \quad (46)$$

We also define the periodic version of  $g_h$

$$g_{h\text{per}}(t, x, v) = \sum_{k \in \mathbb{Z}} g_h(t, x + 2\pi k, v) \quad (47)$$

which is solution of the adjoint of the Kolmogorov equation (5) on  $\mathbb{T} \times (-1, 1)$ . Indeed, thanks to the definition of  $g_h$  (Eq. (44)), the Fourier transform of  $g_h$  is

$$\mathcal{F}(g_h)(\xi) = \sqrt{h} \mathcal{F}_h(g)(h\xi) = (h/\pi)^{1/4} \chi(h\xi - \xi_0) e^{-(h\xi - \xi_0)^2/2h - \lambda_{\xi}t} g_{\xi}(v) \quad (48)$$

and thanks to the trick of Poisson’s summation formula, the Fourier coefficients in  $x$  of  $g_{h\text{per}}$  are of the form

$$c_n(g_{h\text{per}}(t, \cdot, v)) = a_{h,n} e^{-\lambda_n t} g_n(v) \quad (49)$$

with  $a_{h,n} = 2^{-1/2} \pi^{-3/4} h^{1/4} \chi(hn - \xi_0) e^{-(hn - \xi_0)^2/2h}$ , and we have

$$g_{h\text{per}}(t, x, v) = \sum_{n \in \mathbb{Z}} a_{h,n} e^{-\lambda_n t} g_n(v) e^{inx} \quad (50)$$

and since  $g_n(v) e^{inx}$  is an eigenfunction of the Kolmogorov operator with eigenvalue  $\lambda_n$ , this proves the claim that  $g_{h\text{per}}$  is solution of the Kolmogorov equation.

As in the case  $\Omega_v = \mathbb{R}$ , we prove the following estimates:

**Proposition 11** (Punctual estimates). *We have uniformly in  $|x| > \epsilon$  and,  $t > 0$  and  $v \in (-1, 1)$*

$$g_h(t, x, v) = \mathcal{O}(|x|^{-2} e^{-c/h}) \quad (51)$$

and locally uniformly in  $t > 0$ ,  $|x| < \epsilon$  and  $v \in \mathbb{R}$

$$g_h(t, x, v) = e^{ix\xi_0/h - x^2/2h + \mathcal{O}(h^{-\alpha})} \quad (52)$$



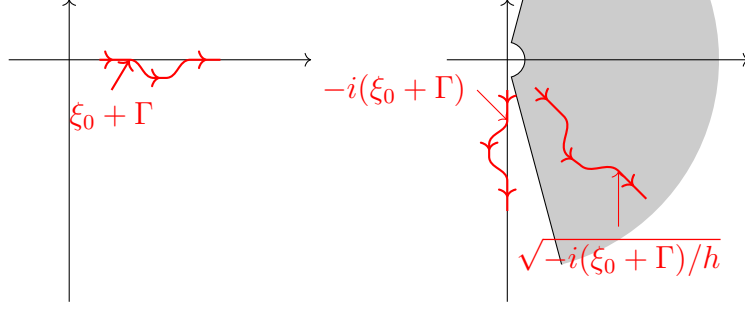


Figure 2: Left figure: an example of a path  $\Gamma$  we want to integrate the definition of  $g$  on (Eq. (45)). Right figure: what  $\Gamma$  becomes under the transformation  $\sqrt{-i(\xi_0 + \xi)}/h$  for a fixed  $h$ . In gray, (part of) the complex domain where the eigenvalue  $\tilde{\rho}_{\tilde{\xi}}$  and the eigenfunction  $\tilde{g}_{\tilde{\xi}}$  is defined. The “correction”  $\delta_{h,t,v}(\xi + \xi_0)$  is defined on  $\Gamma$  if  $\tilde{\xi} = \sqrt{-i(\xi + \xi_0)}/h$  is inside the gray domain for  $\xi$  in  $\Gamma$ .

These estimates implies, as in the case  $\Omega_v = \mathbb{R}$ , that  $g_{h_{\text{per}}}$  is a counterexample to the observability inequality (6), which in turn implies Theorem 4, and we omit this part of the proof.

The proof of these punctual estimates is basically the same as the proof of the similar (simpler) Proposition 6, but we need to make sure that the “correction” goes well with the proof, notably with the changes of integration path.

Note that in the integral defining  $g_h$  (Eq. (45) and (46)), we integrate only on  $(-\xi_0/2, \xi_0/2)$ . Now, if we want to change the integration path from  $\mathbb{R}$  to say  $\Gamma$ , we need to make sure that, as in Proposition 6, we change the path only between  $\xi = -\xi_0/4$  and  $\xi_0/4$ , where  $\chi = 1$ , but also that the “correction” is defined on this path. Note that according to the discussion at the beginning of this section, the “correction”  $\delta_{h,t,v}(\xi)$  is defined for  $|\arg(\sqrt{-i\xi/h})| < 3\pi/8$  (for instance) and  $|\sqrt{-i\xi/h}|$  large enough (see Fig. 2). This holds for example if  $\xi - \xi_0$  is in a small fixed neighborhood  $V$  of  $[-\xi_0/2, \xi_0/2]$  and  $h < h_0$  (with  $h_0$  small enough). In the rest of this proof, we will make sure the changes of integration path we will do in equation (45) are valid by ensuring they are small enough to stay inside  $V$ . Moreover, Theorem 8 and Propositions 9 and 10 translate respectively into the estimates:

$$|e^{-t\rho_{\xi/h}} - 1| \leq C e^{-c/\sqrt{h}} \quad \text{for } \xi \in \xi_0 + V, h < h_0 \text{ and } 0 < t < T \quad (53)$$

$$|e^{\sqrt{-i\xi/h}v^{2/4}} g_{\xi/h}(v)| \leq C \quad \text{for } \xi \in \xi_0 + V, h < h_0 \text{ and } |v| < 1 \quad (54)$$

$$|e^{\sqrt{-i\xi/h}v^{2/2}} g_{\xi/h}(v) - 1| \leq C e^{-c/\sqrt{h}} \quad \text{for } \xi \in \xi_0 + V, h < h_0 \text{ and } |v| < 1/2. \quad (55)$$

To get the estimate (51), we integrate by part in the definition of  $I_h$  (Eq. (46))

to get a decay in  $x$ :

$$I_h(t, x, v) = \int_{\mathbb{R}} e^{-\phi_x(\xi)/h} u_{h,t,x,v}(\xi) d\xi \quad (56)$$

with

$$\begin{aligned} u_{h,t,x,v}(\xi) &= \left( h \partial_\xi \frac{1}{\xi - ix} \right)^2 \left( \chi(\xi) e^{-\sqrt{-i(\xi+\xi_0)/h}(t+v^2/2)} \delta_{h,t,v}(\xi + \xi_0) \right) \\ &= \left( h \partial_\xi \frac{1}{\xi - ix} \right)^2 \left( \chi(\xi) e^{-\sqrt{-i(\xi+\xi_0)/h}t} g_{(\xi+\xi_0)/h}(v) e^{-t\rho(\xi+\xi_0)/h} \right) \end{aligned} \quad (57)$$

Then, we change the integration path for one  $\Gamma_h$  that follows an hyperbole arc  $\Re(\phi_x(\xi)) = a^2/2 - x^2/2 = \Re(\phi_x(a))$  with  $a$  small enough so that the hyperbole arcs for  $|x| > \epsilon$  are in the domain  $V$  (where  $\delta_{h,t,v}(\xi + \xi_0)$  is defined). Then, we get

$$I_h(t, x, v) \leq C \sup_{\xi \in \Gamma_h} |u_{h,t,x,v}(\xi)| e^{-a^2/2h + x^2/2h}. \quad (58)$$

Moreover estimates (53) and (54) imply that for  $\xi \in \Gamma_h$  and  $|x| > \epsilon$ :

$$|u_{h,t,x,v}(\xi)| \leq C_\epsilon |x|^{-2} \quad (59)$$

so, combining this estimate with equation (58) and the definition of  $g_h$  (Eq. (45)), estimate (51) holds.

To get the lower bound (52), we again use the stationary phase method. We first deform the path for one that goes through  $\xi_c = ix$ . Since we can deform the path only in the neighborhood  $V$  of  $[-\xi_0/2, \xi_0/2]$ , we can do this only if  $x$  is small enough, say  $|x| < \epsilon$ .

Then, we again use Proposition 5. Note that even though this Proposition is stated for  $u$  independent of  $h$ , the first  $\mathcal{O}$  does not depend on  $u$  and the second  $\mathcal{O}$  depends on  $u$  only via its  $H_a^\infty$ -norm  $|u|_\infty$ . So we can actually apply this Proposition with  $u_h$  depending on  $h$  assuming  $u_h \rightarrow u_0$  in  $H_a^\infty$  as  $h \rightarrow 0$ . In our case, we will apply the saddle point method to  $u_h(\xi) = \delta_{h,t,v}(\xi + \xi_0)$  (with  $\delta$  defined in equation (43)), with estimates (53) and (55) ensuring that for some  $a > 0$  small enough,  $u_h \rightarrow 1$  as  $h \rightarrow 0$  in  $H_a^\infty$ .

So the saddle point method implies that locally uniformly in  $x$  small enough, in  $v \in \mathbb{R}$  and  $t > 0$

$$I_h(t, x, v) = e^{\mathcal{O}(h^{-1/2})} (1 + \mathcal{O}(\sqrt{h})) \quad (60)$$

so, according to the definition of  $g_h$  (Eq. (45)), we have locally uniformly in  $|x| < \epsilon$ ,  $v \in \mathbb{R}$  and  $t > 0$

$$g_h(t, x, v) = e^{ix\xi_0/h - x^2/2h + \mathcal{O}(h^{-1/2})} (1 + \mathcal{O}(\sqrt{h})) \quad (61)$$

and integrating this estimate proves the last estimate of Proposition 11 and as we discussed earlier, the main Theorem 4.  $\square$

## Appendix A The saddle point method

The saddle point method (see for instance [23]) is a way to compute asymptotic expansion of integrals of the form  $\int e^{\phi(x)/h} u(x) dx$  in the limit  $h \rightarrow 0^+$ . As this is the main tool for disproving the observability inequalities of the equations we are interested in, let us take some time to briefly review it, as well as state a slightly different version than what is usually found in books.

The “standard” saddle point method deals with integrals of the form  $I(h) = \int_{-a}^a e^{\phi(x)/h} dx$ , where  $\phi$  and  $u$  are entire functions, and where (to simplify)  $\phi$  has a unique critical point, say at 0, that is nondegenerate. The basic intuition is that in the limit  $h \rightarrow 0$ , the main contribution to the integral come from where  $\Re(\phi)$  is the highest. But if the functions we integrate are analytic, we can change the integration path, and try to reduce  $\Re(\phi)$  along the integration path. The end result is a three-steps procedure to get the asymptotic expansion:

1. change the integration path for one that reduce  $\Re(\phi)$  as much as possible, while keeping the same endpoints; such a path goes through the critical point 0 of  $\phi$ ;
2. either the main contribution come from the saddle point, or from the endpoints; in the first case<sup>9</sup>, use the Morse lemma to write the integral as  $e^{\phi(x_0)/h} \int_{-a'}^{a'} e^{-x'^2/2h} \tilde{u}(x') dx' + \mathcal{O}(e^{-\delta/h})$  (the  $\mathcal{O}$  is the part of the integral that is away from the saddle point);
3. use Taylor’s formula to get  $I(h) = e^{\phi(x_0)/h} \sum_k \int_{-a'}^{a'} \tilde{u}_k x'^k e^{-x'^2/2h} dx'$ , and finally use the fact that  $\int_{\mathbb{R}} \xi^k e^{-\xi^2/2} d\xi = 0$  if  $k$  is odd and  $\frac{\sqrt{2\pi}}{k!2^k}$  if  $k$  is even to get:

$$I(h) \sim e^{\phi(0)/h} \sum_k \sqrt{2\pi} \tilde{u}_{2k} h^{k+1/2} \quad (62)$$

where the  $\tilde{u}_k$  are of the form  $A_{2k}u(0)$ , and  $A_{2k}$  are differential operators of order  $2k$  that depends on the Morse lemma, with in particular  $\tilde{u}_0 = u(0)|\phi''(0)|^{-1/2}$ .

In this “standard” saddle point method, the functions  $u$  and  $\phi$  does not depend on  $h$ , but in the rest of this article, they do, so we need Proposition 5.

*Proof of Proposition 5.* The strategy is to see  $\varphi_{h,r}(\xi) = \xi^2/2 - h^{1-\alpha}r(\xi)$  as the phase, and to do the same changes of variables and integration path that are done in the saddle point method, even if the critical point and the Morse lemma depends on  $h$ .

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<sup>9</sup>We are not interested in the second case.

Throughout this proof, we will note  $h' = h^{1-\alpha}$ , and we fix  $r_0 \in H_a^\infty$ . We want to find a neighborhood  $V$  of  $r_0$  such that the estimate of the Proposition (Eq. (9)) holds uniformly for  $r$  in  $V$ . We will also note  $I_{h,r}$  the integral we want to estimate:

$$I_{h,r}(u) = \int_{-a}^a e^{-\xi^2/2h+r(\xi)/h^\alpha} u(\xi) d\xi. \quad (63)$$

Thanks to the implicit function theorem applied to  $D_a \times \mathbb{R} \times H_a^\infty \rightarrow \mathbb{C}$ ,  $(\xi, h', r) \mapsto \varphi'_{h,r}(\xi) = \xi - h'r'(\xi)$ , the critical point of  $\varphi_{h,r}$  exists if  $h'$  is small enough and  $r$  close enough to  $r_0$ , say  $h' < h_0$  and  $r \in V$ . We will note  $\xi_{h,r}$  the critical point of  $\varphi_{h,r}$ . It depends continuously on  $(h', r)$ , and since the critical point  $\xi_0(r) = 0$  for  $h = 0$  is nondegenerate, the critical point  $\xi_{h,r}$  is still nondegenerate if  $h$  is small enough. Note that we have

$$\xi_{h,r} = h'r'(\xi_{h,r}) = h'R_{h,r} \quad (64)$$

with  $R_{h,r}$  that depends continuously on  $(h', r) \in [0, h_0) \times V$ . Therefore, the “minimum” of the phase, which we will note  $c_{h,r} = \varphi_{h,r}(\xi_{h,r})$ , satisfies

$$c_{h,r} = -\xi_{h,r}^2/2 + h'r(\xi_{h,r}) = h'R'_{h,r}. \quad (65)$$

with  $R'_{h,r}$  that depends again continuously on  $(h', r) \in [0, h_0) \times V$ .

Now that we now where the critical point is, and what the critical value of the phase is, let us look at the change of variables we need to do to get  $\varphi_{h,r}(\xi) = \eta^2/2$ . According to Taylor’s formula, if we set  $\psi_{h,r}(\xi) = 2 \int_0^1 (1-s)r''(\xi_{h,r} + s\xi) ds$ , we have:

$$\varphi(\xi_{h,r} + \xi) = c_{h,r} + (1 + h'\psi_{h,r}(\xi)) \frac{\xi^2}{2}. \quad (66)$$

So, by the change of variables/integration path  $\eta = \sqrt{1 + h'\psi_{h,r}(\xi)} \xi$ , we have:

$$I_{h,r}(u) = e^{c_{h,r}/h} \int_{-a}^a e^{-\eta^2/2h} u(\xi(\eta)) \frac{d\xi}{d\eta} d\eta + \mathcal{O}(e^{-\delta/h}|u|_\infty) \quad (67)$$

where the  $\mathcal{O}$  comes from the fact that we changed a bit the integration endpoints, and is uniform in  $r$  close enough to  $r_0$ . And since<sup>10</sup>  $\eta = \xi + \mathcal{O}(h')$ , we have  $\frac{d\xi}{d\eta} = 1 + \mathcal{O}(h')$ , and  $u(\xi(\eta)) = u(\eta) + \mathcal{O}(h'|u|_\infty)$ , so, by the standard saddle point method:

$$I_{h,r}(u) = e^{c_{h,r}/h} \sqrt{2\pi h} (u(0) + \mathcal{O}(h'|u|_\infty)) \quad (68)$$

and since  $c_{h,r}/h = R'_{h,r}/h^\alpha$  with  $R_{h,r}$  depending continuously in  $(h, r) \in [0, h_0) \times H_a^\infty$ , the Proposition is proved.  $\square$

<sup>10</sup>The  $\mathcal{O}$  is to be understood as a  $\mathcal{O}$  in  $H^\infty(D_a)$ , which implies that it is a  $\mathcal{O}$  in  $C^\infty(D_{a/2})$ .

*Remark 12.* The asymptotic expansion we stated is crude (we even wrote that  $\sqrt{2\pi h} e^{\mathcal{O}(h^{-\alpha})} = e^{\mathcal{O}(h^{-\alpha})}$ ) but we can get a precise asymptotic expansion. We actually have it in the form of equation (68), and we can get an asymptotic expansion of  $c_{h,r}$  with equations (64) and (65). We can also get an asymptotic expansion in powers of  $h$  and  $h'$  of  $u(\xi(\eta)) \frac{d\xi}{d\eta}$  to get a complete asymptotic expansion of  $I_{h,r}$  of the form

$$I_{h,r} \sim e^{c_1 h^{-\alpha} + c_2 h^{1-2\alpha} + \dots + c_k h^{k(1-\alpha)-1}} \sum_{\beta} u_{\beta} h^{\beta} \quad (69)$$

where  $k$  is the largest integer such that  $k(1-\alpha) - 1 \leq 0$  (well, we can write more terms, but we already take them into account in the sum outside of the exponential), and the sum is for  $\beta$  of the form  $\beta = 1/2 + l(1-\alpha) + m + n(k(1-\alpha) - \alpha)$  with  $k$  defined just before and  $l, m, n \geq 0$ . The  $u_{\beta}$  and  $(c_j)$  can be in principle computed explicitly, and we have in particular  $c_1 = r(0)$  and  $u_0 = \sqrt{2\pi}u(0)$ .

## Appendix B Precise estimation of the eigenfunctions

To prove Proposition 10, we will need the following theorem, which is a special case<sup>11</sup> of Theorem 18 in [17].

**Theorem 13.** *Let  $S$  be the space of holomorphic function on the domain  $\Omega = \{|z| > 1/2, \Re(z) > 0\}$  with sub-exponential growth at infinity, i.e.  $\gamma \in E$  if and only if for all  $\epsilon > 0$ ,  $p_{\epsilon}(\gamma) = \sup_{\substack{\Re(z) > 0 \\ |z| > 1/2}} |\gamma(z)e^{-\epsilon|z|}| < +\infty$ . We endow  $S$  with the seminorms family  $(p_{\epsilon})_{\epsilon > 0}$ .*

Let  $\gamma$  in  $S$  and let  $H_{\gamma}$  be the operator on polynomials with a root at zero, defined by:

$$H_{\gamma} \left( \sum_{n>0} a_n z^n \right) = \sum_{n>0} \gamma(n) a_n z^n.$$

Let  $E$  be an bounded subset of  $\mathbb{C}$ , star shaped with respect to 0. Let  $U$  be a neighborhood of  $E$ . Then there exists  $C > 0$  such that for all polynomials  $f$  with a root at 0:

$$|H_{\gamma}(f)|_{L^{\infty}(E)} \leq C |f|_{L^{\infty}(U)}.$$

Moreover, the constant  $C$  above can be chosen continuously in  $\gamma \in S$ .

*Proof of Proposition 10.* The proof is made by writing  $\tilde{u}_{\xi}(v)$  as the power series  $\tilde{u}_{\xi}(v) = \sum \tilde{u}_{\xi,2n} v^{2n}$ , and showing that the coefficients  $\tilde{u}_{\xi,n}$  of this power series are

<sup>11</sup>In the reference, the Theorem is stated with an open (bounded star-shaped) domain  $U$  instead of a arbitrary (bounded star-shaped) subset  $E$  of  $\mathbb{C}$ , but we can set  $U = E^{\delta}$ , and apply the Theorem as stated in the reference to get  $|H_{\gamma}(f)|_{L^{\infty}(E)} \leq C_{\delta} |f|_{L^{\infty}(E^{2\delta})}$ .

of the form  $\tilde{u}_{\tilde{\xi},2n} = \tilde{\rho}_{\tilde{\xi}}\gamma_{\tilde{\xi}}(n)\tilde{\xi}^n/n!$  for  $n \geq 1$ , with  $\tilde{\rho}_{\tilde{\xi}}$  defined at the beginning of Section 3.3, so that with the notation of Theorem 13:

$$\tilde{u}_{\tilde{\xi}}(v) = 1 + \tilde{\rho}_{\tilde{\xi}}H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v) \quad (70)$$

Then, Theorem 13 will allow us to conclude.

Let us write:

$$\tilde{u}_{\tilde{\xi}}(v) = \sum_{n=0}^{\infty} \tilde{u}_{\tilde{\xi},n}v^n. \quad (71)$$

Since  $\tilde{u}_{\tilde{\xi}}$  satisfies the Cauchy problem  $-\tilde{u}_{\tilde{\xi}}'' + 2\tilde{\xi}v\tilde{u}_{\tilde{\xi}}' - \tilde{\rho}_{\tilde{\xi}}\tilde{u}_{\tilde{\xi}} = 0$  with initial conditions<sup>12</sup>  $\tilde{u}_{\tilde{\xi}}(0) = 1$ ,  $\tilde{u}_{\tilde{\xi}}'(0) = 0$ , we have  $\tilde{u}_{\tilde{\xi},0} = 1$ ,  $\tilde{u}_{\tilde{\xi},2n+1} = 0$  and

$$\tilde{u}_{\tilde{\xi},n+2} = \frac{2n\tilde{\xi} - \tilde{\rho}_{\tilde{\xi}}}{(n+1)(n+2)}\tilde{u}_{\tilde{\xi},n} \quad (72)$$

so, by induction, for  $n \geq 1$

$$\tilde{u}_{\tilde{\xi},2n} = -\frac{\tilde{\rho}_{\tilde{\xi}}}{2} \frac{(4\tilde{\xi})^{n-1}(n-1)!}{(2n)!} \prod_{k=1}^{n-1} \left(1 - \frac{\tilde{\rho}_{\tilde{\xi}}}{4\tilde{\xi}k}\right). \quad (73)$$

So, by defining

$$\gamma_{\tilde{\xi}}(n) = -\frac{1}{8\tilde{\xi}n} \times \frac{4^n(n!)^2}{(2n)!} \times \prod_{k=1}^{n-1} \left(1 - \frac{\tilde{\rho}_{\tilde{\xi}}}{4\tilde{\xi}k}\right) \quad (74)$$

we have  $\tilde{u}_n = \tilde{\rho}_{\tilde{\xi}}\gamma_{\tilde{\xi}}(n)\tilde{\xi}^n$  and

$$\tilde{u}_{\tilde{\xi}}(v) = 1 + \tilde{\rho}_{\tilde{\xi}} \sum_{n \geq 1} \gamma_{\tilde{\xi}}(n) \frac{(v^2\tilde{\xi})^n}{n!} \quad (75)$$

and assuming that  $\gamma_{\tilde{\xi}}$  is in  $S$ , this is exactly the equation (70) we were claiming.

Well, let us actually prove that  $\gamma_{\tilde{\xi}}$  is in the space  $S$  defined in Theorem 13, i.e. that we can extend  $n \mapsto \gamma_{\tilde{\xi}}(n)$  to a holomorphic function on  $\Omega = \{|z| > 1/2, \Re(z) > 0\}$  with subexponential growth. This is obvious for the term  $-1/(8\tilde{\xi}n)$ . The term  $4^n(n!)^2/(2n)!$  can be extended to  $\Omega$  with the Gamma function, and Stirling's approximation gives us the subexponential growth (actually a decay in  $1/\sqrt{z}$ ). The product term is a tiny bit more tricky to extend to non-integer values. We define it with the following formula, which is inspired by [22], and where we have set  $\alpha = -\tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$ :

$$\delta_{\tilde{\xi}}(z) = \prod_{k=1}^{+\infty} \frac{1 + \frac{\alpha}{k}}{1 + \frac{\alpha}{k+z}}. \quad (76)$$

<sup>12</sup>Here we use the fact that  $\tilde{u}_{\tilde{\xi}}$  is even when  $\tilde{\xi}$  is real positive, which is well-known from Sturm-Liouville's theory.

(Both of these equalities are actually definitions.)

We now claim that if  $|\alpha| < 1/2$  and  $\Re(z) > 0$ , then  $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$ . The proof of this claim is just a few basic computations, and we postpone it after the end of the proof at hand.

Since  $\alpha = \tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$ , according to Theorem 8,  $|\alpha| < 1/2$  as soon as  $|\arg(\tilde{\xi})| < \theta$  and  $|\tilde{\xi}|$  is large enough, say  $|\tilde{\xi}| > M$  (depending on  $\theta$ ). Then, according to the claim, the term  $\delta_{\tilde{\xi}}(z)$  has subexponential growth in  $\Omega$ , and since it is holomorphic, it is in  $S$ . Moreover, this estimate also proves that  $(\delta_{\tilde{\xi}})_{|\alpha| < 1/2}$  is a bounded family of  $S$ .

So  $(\gamma_{\tilde{\xi}})$  is a bounded family of  $S$  for  $|\arg(\tilde{\xi})| < \theta$  and  $|\tilde{\xi}| > M$ . So, according to Theorem 13 for any neighborhood  $U$  of  $[-1 + \epsilon, 1 - \epsilon]$ , there exists  $C > 0$  such that for all  $v \in (-1 + \epsilon, 1 - \epsilon)$ :

$$\left| H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v) \right| \leq C(1 + |e^{\tilde{\xi}v^2}|_{L^\infty(U)}) \quad (77)$$

and if we choose  $U$  to be small enough, we have

$$\left| H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v) \right| \leq C'|e^{(1-\delta)\tilde{\xi}}|. \quad (78)$$

Finally, thanks to equation (70) and Theorem 8, we have

$$|\tilde{u}_{\tilde{\xi}}(v) - 1| \leq C_\delta |\tilde{\xi}|^{3/2} |e^{-\delta\tilde{\xi}}| \quad (79)$$

which proves the proposition.  $\square$

We now prove the claim that  $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$ .

*Proof of the claim.* We first write

$$\delta_{\tilde{\xi}}(z) = \exp \left( \sum_{k=1}^{+\infty} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right). \quad (80)$$

Let us also remind that we assume  $|\alpha| < 1/2$ ,  $|z| > 1$  and  $\Re(z) > 0$ , so that for  $k \in \mathbb{N}^*$   $|\alpha/k| < 1/2$  and  $|\alpha/(k+z+1)| < 1/2$ .

We note  $k_0 = \lfloor |z| \rfloor$ , and we separate the sum into a sum for  $k \leq k_0$  and the sum for  $k > k_0$ . About the part of a sum for  $k \leq k_0$ , we have thanks to the triangle inequality and the fact that for  $|x| < 1/2$ ,  $|\ln(1+x)| \leq c|x|$ :

$$\left| \sum_{k=1}^{k_0} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq 2c|\alpha| \sum_{k=1}^{k_0} \frac{1}{k} \quad (81)$$

and by the relation between the harmonic sum and the logarithm,

$$\left| \sum_{k=1}^{k_0} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq 2c|\alpha|(\ln(k_0) + C') \quad (82)$$

and since  $k_0 = \lfloor |z| \rfloor$ ,

$$\left| \sum_{k=1}^{k_0} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq 2c|\alpha|(\ln(|z|) + C'). \quad (83)$$

About the rest of the sum, we have by writing  $\ln(1+b) - \ln(1+a) = \int_a^b \frac{dx}{1+x}$ ,

$$\left| \sum_{k=k_0+1}^{+\infty} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq \sum_{k=k_0+1}^{\infty} \left| \int_{\alpha/k}^{\alpha/(k+z)} \frac{dx}{1+x} \right| \quad (84)$$

and since  $|\frac{1}{1+x}| \leq 2$  for  $x \in [\alpha/k, \alpha/(k+z)]$

$$\left| \sum_{k=k_0+1}^{+\infty} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq \sum_{k=k_0+1}^{+\infty} 2 \left| \frac{\alpha}{k} - \frac{\alpha}{k+z} \right| \quad (85)$$

$$\leq |\alpha z| \sum_{k=k_0+1}^{+\infty} \frac{1}{k^2} \quad (86)$$

and by comparing this sum with an integral,

$$\left| \sum_{k=k_0+1}^{+\infty} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq 2|\alpha z| \int_{k_0}^{+\infty} \frac{dx}{x^2} \quad (87)$$

$$\leq 2|\alpha| \frac{|z|}{k_0} \leq C''|\alpha| \quad (88)$$

where we again used that  $k_0 = \lfloor |z| \rfloor$ . Summing these two inequalities, we have

$$\left| \sum_{k=1}^{+\infty} \ln \left( 1 + \frac{\alpha}{k} \right) - \ln \left( 1 + \frac{\alpha}{k+z} \right) \right| \leq 2c|\alpha|(\ln(|z|) + C''') \quad (89)$$

which, with equation (80), proves the claim.  $\square$

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