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# SOME SINGULAR CURVES AND SURFACES ARISING FROM INVARIANTS OF COMPLEX REFLECTION GROUPS

by

CÉDRIC BONNAFÉ

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If  $V$  is an  $n$ -dimensional  $\mathbb{C}$ -vector space and  $W$  is a finite subgroup of  $\mathbf{GL}_{\mathbb{C}}(V)$ , then the algebra  $\mathbb{C}[V]^W$  of  $W$ -invariant polynomial functions on  $V$  is finitely generated. If  $f \in \mathbb{C}[V]^W$  is homogeneous, then it defines a projective hypersurface  $\mathcal{Z}(f)$  in  $\mathbf{P}(V)$  which is acted on by  $W$  and it is natural to determine the geometric structure of  $\mathcal{Z}(f)$ .

Let us illustrate this on an example due to Sarti [Sar3]. Assume for the moment that  $W$  is a Weyl group of type  $H_4$  acting on a 4-dimensional  $\mathbb{C}$ -vector space  $V$ . Then  $\mathbb{C}[V]^W = \mathbb{C}[q, f, g, h]$  where  $q, f, g, h$  are homogeneous of respective degrees 2, 12, 20, 30 ( $q$  is the invariant quadratic form). If  $f$  is chosen randomly, any other polynomial of the form  $f - tq^6$ , with  $t \in \mathbb{C}$ , is homogeneous of degree 12 and can replace  $f$  in the list of generators. But, according to [Sar3], there exists a unique  $t_0 \in \mathbb{C}$  such that the projective surface in  $\mathbf{P}(V) \simeq \mathbf{P}^3(\mathbb{C})$  defined by  $f - t_0q^6$  has 600 nodal points (this is *Sarti dodecic*) For generic values of  $t$ , the corresponding surface is smooth, while for three other values of  $t$ , the surface has “only” 60, 300 and 360 nodes respectively. This example, after *Barth sextic* [Bar], shows that one can obtain surfaces with many singularities from this method.

In this paper, we stick to the case where  $W$  is generated by (pseudo-)reflections (i.e. elements fixing an hyperplane pointwise) and we will construct several singular curves and surfaces in this way. Recall for instance that we have already obtained a surface of degree 8 (resp. 24) with 44 (resp. 1440) quotient singularities of type  $D_4$ , starting from invariants of degree 24 of the complex reflection group denoted by  $G_{32}$  in Shephard-Todd classification. Since we are interested in *exceptional* curves and surfaces, it is natural to expect to obtain them using invariants of *exceptional* complex reflection groups: we will mostly stick to these groups.

Contrary to previous constructions, the singular points of our curves or surfaces are not all real (even though most of them are defined over  $\mathbb{Q}$ ). By contrast, note also that, using a theorem of Marin-Michel on automorphisms of reflection groups [MaMi], we can show that Sarti dodecic can be defined over  $\mathbb{Q}$ : this was still an open question.

## 1. Notation, preliminaries

We fix an  $n$ -dimensional  $\mathbb{C}$ -vector space  $V$  and a finite subgroup  $W$  of  $\mathbf{GL}_{\mathbb{C}}(V)$ . We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}}(V^s) = n - 1\}.$$

**Hypothesis.** *We assume throughout this paper that*

$$W = \langle \mathrm{Ref}(W) \rangle.$$

*In other words,  $W$  is a **complex reflection group**. We also assume that  $W$  acts **irreducibly** on  $V$ . The number  $n$  is called the **rank** of  $W$ .*

**1.A. Invariants.** — We denote by  $\mathbb{C}[V]$  the ring of polynomial functions on  $V$  (identified with the symmetric algebra  $S(V^*)$  of the dual  $V^*$  of  $V$ ) and by  $\mathbb{C}[V]^W$  the ring of  $W$ -invariant elements of  $\mathbb{C}[V]$ . By Shephard-Todd/Chevalley-Serre Theorem [Bro, Theorem 4.1], there exist  $n$  algebraically independent homogeneous elements  $f_1, f_2, \dots, f_n$  of  $\mathbb{C}[V]^W$  such that

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, f_2, \dots, f_n].$$

Let  $d_i = \deg(f_i)$ . We will assume that  $d_1 \leq d_2 \leq \dots \leq d_n$ . A family  $(f_1, f_2, \dots, f_n)$  satisfying the above property is called a *family of fundamental invariants* of  $W$ . Whereas such a family is not uniquely defined, the list  $(d_1, d_2, \dots, d_n)$  is well-defined and is called the list of *degrees* of  $W$ . If  $f \in \mathbb{C}[V]$  is homogeneous, we will denote by  $\mathcal{Z}(f)$  the projective (possibly reduced) hypersurface in  $\mathbf{P}(V) \simeq \mathbf{P}^{n-1}(\mathbb{C})$  defined by  $f$ . Its singular locus will be denoted by  $\mathcal{Z}_{\mathrm{sing}}(f)$ . An homogeneous element  $f \in \mathbb{C}[V]$  is called a *fundamental invariant* if it belongs to a family of fundamental invariants. As we will be interested in the singular locus of projective hypersurfaces, recall the following very easy fact:

**Lemma 1.1.** — *Let  $g \in \mathbb{C}[V]$  be homogeneous. If  $\dim \mathcal{Z}_{\mathrm{sing}}(g) \leq n - 4$ , then  $\mathcal{Z}(g)$  is irreducible.*

*Proof.* — Indeed, if  $\mathcal{Z}(g)$  has (at least) two distinct irreducible components  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$ , then  $\dim(\mathcal{Z}_1 \cap \mathcal{Z}_2) = n - 3$  (because it is the intersection of two hypersurfaces in the projective space  $\mathbf{P}^{n-1}(\mathbb{C})$ ), and all the points in  $\mathcal{Z}_1 \cap \mathcal{Z}_2$  are singular.  $\square$

Recall that a subgroup  $G$  of  $\mathbf{GL}_{\mathbb{C}}(V)$  is called *primitive* if there does not exist a decomposition  $V = V_1 \oplus \dots \oplus V_r$  with  $V_i \neq 0$  and  $r \geq 2$  such that  $G$  permutes the  $V_i$ 's. We will be mainly interested in *primitive* complex reflection groups, and we will refer to Shephard-Todd numerology [ShTo] for such groups (there are 34 isomorphism classes, named  $G_i$  for  $4 \leq i \leq 37$ ). Almost all the computations<sup>(1)</sup> have been done using the software MAGMA [Magma].

<sup>(1)</sup>Some Milnor and Tjurina numbers were computed with SINGULAR [DGPS].

**1.B. Marin-Michel Theorem.** — Let  $\overline{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$  and we set  $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . Using the classification of finite reflection groups, Marin-Michel [MaMi] proved that there exists a  $\mathbb{Q}$ -structure  $V_{\mathbb{Q}}$  of  $V$  such that:

- (1)  $V_{\overline{\mathbb{Q}}} = \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$  is stable under the action of  $W$  (so that  $W$  might be viewed as a subgroup of  $\mathbf{GL}_{\overline{\mathbb{Q}}}(V_{\overline{\mathbb{Q}}})$ ).
- (2) The action of  $\Gamma$  on  $\mathbf{GL}_{\overline{\mathbb{Q}}}(V_{\overline{\mathbb{Q}}})$  induced by the  $\mathbb{Q}$ -form  $V_{\mathbb{Q}}$  stabilizes  $W$ .

This implies that  $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$  is a  $\overline{\mathbb{Q}}$ -form of  $\mathbb{C}[V]$  stable under the action of  $W$  and that the action of  $\Gamma$  on  $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$  induced by the  $\mathbb{Q}$ -form  $\mathbb{Q}[V_{\mathbb{Q}}]$  stabilizes the invariant ring  $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$ .

**Proposition 1.2.** — *Sarti dodecic can be defined over  $\mathbb{Q}$ .*

*Proof.* — Assume here that  $W$  is a Coxeter group of type  $H_4$  acting on a vector space  $V$  of dimension 4. We fix a  $\mathbb{Q}$ -form  $V_{\mathbb{Q}}$  as above. Then the homogeneous invariant  $f$  of  $W$  of degree 12 defining the Sarti dodecic belongs to  $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$ . But, if  $\gamma \in \Gamma$ , then  ${}^{\gamma}f$  is also an invariant of  $W$  of degree 12 defining an irreducible projective surface with 600 nodes. By the unicity of such an invariant [Sar3], this forces  ${}^{\gamma}f = f$ , and so  $f \in \mathbb{Q}[V_{\mathbb{Q}}]$ .  $\square$

## 2. Strategy for finding some “canonical” invariants in rank $n \geq 3$

**Hypothesis and notation.** *From now on, and until the end of this paper, we assume that  $n \geq 3$  and that  $W$  is primitive. We denote by  $r$  the minimal natural number such that the space of homogeneous invariants of degree  $d_r$  has dimension  $\geq 2$ .*

Note that this implies that  $W$  is one of the groups  $G_i$ , with  $23 \leq i \leq 37$ , in Shephard-Todd classification. We recall in Table 2.1 the degrees  $(d_1, d_2, \dots, d_n)$  of the fundamental invariants of these groups. We also give the following information: the order of  $W$ , the order of  $W/Z(W)$  (which is the group which acts faithfully on  $\mathbf{P}(V)$ ) and, whenever  $W$  is a Coxeter group, we recall its type ( $W(X_i)$  denotes the Coxeter group of type  $X_i$ ). Recall from general theory that  $|W| = d_1 d_2 \cdots d_n$  and  $|Z(W)| = \text{Gcd}(d_1, d_2, \dots, d_n)$ .

Using MAGMA, we first determine by computer calculations some fundamental invariants  $f_1, \dots, f_r$ . By the definition of  $r$ , the fundamental invariants  $f_1, \dots, f_{r-1}$  are uniquely determined up to scalar. By inspection of Table 2.1, we see that there is a unique  $f$  of the form  $f_1^{m_1} \cdots f_{r-1}^{m_{r-1}}$  which has degree  $d_r$ . So the space of homogeneous invariants of degree  $d_r$  has dimension 2, and is spanned by  $f_r$  and  $f$ . Moreover, all fundamental invariants of degree  $d_r$  are of the form  $f_r - t f$ , for some  $t \in \mathbb{C}$ .

This means that we need to determine the values of  $t$  such that  $\mathcal{Z}(f_r - t f)$  is singular. For this, we use the basis  $(x_1, \dots, x_n)$  of  $V^*$  chosen by MAGMA and we set

$$F_t = f_r - t f \quad \text{and} \quad F_t^{\text{aff}}(x_1, \dots, x_{n-1}) = F_t(x_1, \dots, x_{n-1}, 1).$$

This basis allows to identify  $\mathbf{P}(V)$  with  $\mathbf{P}^{n-1}(\mathbb{C})$  and we denote by  $\mathbf{A}^{n-1}(\mathbb{C})$  the affine open subset of  $\mathbf{P}^{n-1}(\mathbb{C})$  defined by  $x_n \neq 0$ . Then  $\mathcal{Z}^{\text{aff}}(F_t^{\text{aff}})$  denotes the affine open subset of

$n$	$W$	$ W $	$ W/Z(W) $	$(d_1, d_2, \dots, d_n)$
3	$G_{23} = W(H_3)$	120	60	2, 6, 10
	$G_{24}$	336	168	4, 6, 14
	$G_{25}$	646	108	6, 9, 12
	$G_{26}$	1 296	216	6, 12, 18
	$G_{27}$	2 160	360	6, 12, 30
4	$G_{28} = W(F_4)$	1 152	576	2, 6, 8, 12
	$G_{29}$	7 680	1 920	4, 8, 12, 20
	$G_{30} = W(H_4)$	14 400	7 200	2, 12, 20, 30
	$G_{31}$	46 080	11 520	8, 12, 20, 24
	$G_{32}$	155 520	25 920	12, 18, 24, 30
5	$G_{33}$	51 840	25 920	4, 6, 10, 12, 18
6	$G_{34}$	39 191 040	6 531 840	6, 12, 18, 24, 30, 42
	$G_{35} = W(E_6)$	51 840	25 920	2, 5, 6, 10, 12, 14, 18
7	$G_{36} = W(E_7)$	2 903 040	1 451 520	2, 6, 8, 10, 12, 14, 18
8	$G_{37} = W(E_8)$	696 729 600	348 364 800	2, 8, 12, 14, 18, 20, 24, 30

TABLE 2.1. Degrees of primitive complex reflection groups in rank  $\geq 3$ 

$\mathcal{Z}(F_t)$  defined by  $x_n \neq 0$ . Note the following fact, which will be useful for saving many time during computations:

$$(2.2) \quad \mathcal{Z}(F_t) \text{ is singular if and only if } \mathcal{Z}^{\text{aff}}(F_t^{\text{aff}}) \text{ is singular.}$$

*Proof.* — Indeed, if  $\mathcal{Z}(F_t)$  contains a singular point  $\xi = [\xi_1, \dots, \xi_n] \in \mathbf{P}^{n-1}(\mathbb{C})$  such that  $\xi_n = 0$ , then the  $W$ -orbit of  $\xi$  consists of singular points of  $\mathcal{Z}(F_t)$  and then it also meets the affine open subset  $\mathcal{Z}^{\text{aff}}(F_t^{\text{aff}})$  because  $W$  acts irreducibly on  $V$  (so it cannot stabilize the hyperplane orthogonal to  $x_n$ ).  $\square$

Let

$$\mathcal{X} = \{(\xi, t) \in \mathbf{A}^{n-1}(\mathbb{C}) \times \mathbf{A}^1(\mathbb{C}) \mid F_t^{\text{aff}}(\xi) = 0\}.$$

We denote by  $\phi : \mathcal{X} \rightarrow \mathbf{A}^1(\mathbb{C})$  the second projection. Then the fiber  $\phi^{-1}(t)$  is the variety  $\mathcal{Z}^{\text{aff}}(F_t^{\text{aff}})$ . We can then define

$$\mathcal{X}_{\text{sfib}} = \{(\xi, t) \in \mathcal{X} \mid \frac{\partial F_t^{\text{aff}}}{\partial x_1}(\xi) = \dots = \frac{\partial F_t^{\text{aff}}}{\partial x_{n-1}}(\xi) = 0\}.$$

Then  $\mathcal{X}_{\text{sfib}}$  is not necessarily the singular locus of  $\mathcal{X}$ , but the points in  $\phi(\mathcal{X}_{\text{sfib}})$  are the values of  $t$  for which the fiber  $\phi^{-1}(t) = \mathcal{Z}^{\text{aff}}(F_t^{\text{aff}})$  (or, equivalently,  $\mathcal{Z}(F_t)$ ) is singular. This provides an algorithm for finding these values of  $t$ : it turns out that generally  $\phi$  is not dominant, so that there are only finitely such values of  $t$ . We will then study more precisely these finite number of cases (number of singular points, nature of singularities, Milnor number, ...).

Let us see on a simple example how it works:

**Example 2.3 (Coxeter group of type  $H_3$ ).** — Assume here, and only here, that  $W = G_{23} = W(H_3)$ . Then  $(d_1, d_2, d_3) = (2, 6, 10)$  so that  $r = 2$  and  $d_r = 6$ . Then  $F_t = f_2 - t f_1^3$ . We first define  $W$  and the fundamental invariants  $f_1$  and  $f_2$ :

```

> W:=ShephardTodd(23);
> K<a>:=CoefficientRing(W);
> R:=InvariantRing(W);
> P<x1,x2,x3>:=PolynomialRing(R);
> f1:=InvariantsOfDegree(W,2)[1];
> f2:=InvariantsOfDegree(W,6)[1];
> Gcd(f1,f2);
1

```

Note that the last command shows that the invariant  $f_2$  of degree 6 we have chosen is indeed a fundamental invariant. We now define  $F_t^{\text{aff}}$  and  $\mathcal{X}_{\text{sfb}}$  and then determine the set  $T_{\text{sing}}$  of values of  $t$  such that  $\mathcal{Z}(F_t)$  is singular:

```

> P2:=Proj(P);
> A2xA1<xx1,xx2,t>:=AffineSpace(K,3);
> A1<T>:=AffineSpace(K,1);
> phi:=map<A2xA1->A1 | [t]>;
> flaff:=Evaluate(f1,[xx1,xx2,1]);
> f2aff:=Evaluate(f2,[xx1,xx2,1]);
> Ftaff:=f2aff - t * flaff^3;
> X:=Scheme(A2xA1,Ftaff);
> Xsfb:=Scheme(X,[Derivative(Ftaff,i) : i in [1,2]]);
> Tsing:=MinimalBasis(phi(Xsfb));
> # Tsing;
1
> Factorization(Tsing[1]);
[
  <T - 1, 1>,
  <T - 9/10, 1>,
  <T - 63/64, 1>
]
> Tsing:=[1, 9/10, 63/64];

```

We next determine for which values  $t \in T_{\text{sing}} = \{t_1, t_2, t_3\}$  the curve  $\mathcal{Z}(F_t)$  is irreducible:

```

> F:=[f2-ti*f1^3 : ti in Tsing]; // the polynomials F_{t_i}
> Z:=[Curve(P2,f2-ti*f1^3) : ti in Tsing];
> [IsAbsolutelyIrreducible(i) : i in Z];
[ true, true, false ]

```

We then study the singular locus of the irreducible curves  $\mathcal{Z}(F_t)$  for  $t = t_1$  or  $t_2$ . Let us see how we do in for  $t = t_1$ :

```

> Zlsing:=SingularSubscheme(Z[1]);
> Zlsing:=ReducedSubscheme(Zlsing);
> Zlsingirr:=IrreducibleComponents(Zlsing);
> &+ [Degree(i) : i in Zlsingirr]
10
> p1:=P2 ! [1,0,0];
> p1 in Zsing[1];
true (1 : 0 : 0)
> # ProjectiveOrbit(W,p1);
10
> IsNode(Z[1],p1);
true

```

The command `&+ [Degree(i) : i in Zlsingirr]` shows that  $\mathcal{Z}(F_{t_1})$  contains 10 singular points. The command `# ProjectiveOrbit(W, p1)` shows that they are all in the same  $W$ -orbit (the function `ProjectiveOrbit` has been defined by the author for computing orbits in projective spaces (see [Bon] for the code). So all these singularities are equivalent and the command `IsNode(Z[1], p1)` shows that they are all nodes.

One can check similarly that  $\mathcal{Z}(F_{t_2})$  has 6 nodes, all belonging to the same  $W$ -orbit. ■

In the next sections, we will give tables of singular curves and surfaces obtained in this way: inspection of these tables lead to the following result:

**Proposition 2.4.** — *Apart from the two singular curves  $\mathcal{C}$  and  $\mathcal{C}'$  of degree 8 with 80 nodes defined by invariants of  $G_{29}$ , all the singular curves and surfaces described in Tables 3.2, 3.6 and 4.2 can be defined over  $\mathbb{Q}$ . The singular curves  $\mathcal{C}$  and  $\mathcal{C}'$  are Galois conjugate.*

*Proof.* — This follows from the same argument as in Proposition 1.2, based on Marin-Michel Theorem, by using the fact that all these singular curves and surfaces are characterized by their number of singular points or their type. □

### 3. Singular curves from groups of rank 3

**Hypothesis.** *We still assume that  $W$  is primitive but, in this section, we assume moreover that  $n = 3$ .*

This means that  $W$  is one of the groups  $G_i$ , for  $23 \leq i \leq 27$ . We denote by  $(f_1, f_2, f_3)$  a set of fundamental invariants provided by MAGMA and we denote by  $T_{\text{sing}}^{\text{irr}}$  the set of elements  $t \in \mathbb{C}$  such that  $\mathcal{Z}(F_t)$  is irreducible and singular. It turns out that the following fact can be checked case-by-case:

**Proposition 3.1.** — *Assume that  $n = 3$  and  $W$  is primitive. Then  $W$  acts transitively on  $\mathcal{Z}_{\text{sing}}(F_t)$  for all  $t \in T_{\text{sing}}^{\text{irr}}$ .*

Table 3.2 gives the list of curves obtained through the methods detailed in Section 2. We do not provide details of the computations, as they follow the same lines as the ones in Example 2.3. This table contains the degree  $d_r$ , the number of values of  $t$  such that  $\mathcal{Z}(F_t)$  is irreducible and singular, the number of singular points and informations about the singularity (since all singular points belong to the same  $W$ -orbit by Proposition 3.1, they are all equivalent singularities): the number  $m$  (resp.  $\mu$ , resp.  $\tau$ ) denotes the multiplicity (resp. the Milnor number, resp. the Tjurina number) of the singularity. A singularity is called *ordinary* if its tangent cone is reduced.

$W$	$d_r$	$ T_{\text{sing}}^{\text{irr}} $	$t_i$	$ \mathcal{Z}_{\text{sing}}(F_{t_i}) $	Singularity
$G_{23} = W(H_3)$	6	2	$t_1$	10	$A_1$
			$t_2$	6	$A_1$
$G_{24}$	14	3	$t_1$	28	$A_2$
			$t_2$	21	$A_2$
			$t_3$	42	$A_2$
$G_{25}$	12	2	$t_1$	12	Ordinary, $m = 3, \mu = 4, \tau = 4$
			$t_2$	36	$A_2$
$G_{27}$	12	2	$t_1$	45	$A_1$
			$t_2$	36	$A_1$

TABLE 3.2. Singularities of the curves  $\mathcal{Z}(F_t)$  for  $t \in T_{\text{sing}}^{\text{irr}}$ 

**Remark 3.3.** — A plane curve is called *cuspidal* if all its singular points are of type  $A_2$ . By [Sak, (0.4)], a cuspidal plane curve of degree 14 has at most 55 singular points of type  $A_2$ . But it is not known if this is the sharpest bound: at the best of our knowledge, no cuspidal plane curve of degree 14 with 42 or more singular points of type  $A_2$  was known before the above example of  $\mathcal{Z}(F_{t_3})$  for  $W = G_{24}$ .

Also, a cuspidal plane curve of degree 12 can have at most 40 points [Sak, (0.4)], but it is not known if this bound can be achieved. However, there exists at least one cuspidal curve of degree 12 with 39 cusps [C-Ali, Example 6.3]. ■

**Remark 3.4.** — Note that  $G_{26}$  does not appear in Table 3.2. The reason is the following: if  $W = G_{26}$ , then  $d_r = 12$  but  $G_{26}$  contains  $W' = G_{25}$  as a normal subgroup of index 2 and it turns out that invariants of degree 12 of  $G_{25}$  and  $G_{26}$  coincide. This makes the computation for  $G_{26}$  unnecessary in this case. Note, however, the next Example 3.5, where we construct singular curves of degree 18 using invariants of  $G_{26}$ . ■

**Example 3.5 (The group  $G_{26}$ ).** — We assume in this example that  $W = G_{26}$ . Recall that  $(d_1, d_2, d_3) = (6, 12, 18)$ . Any fundamental invariant of degree 18 of  $W$  is of the form  $F_{t,u} = f_3 - t f_1 f_2 - u f_2^3$  for some  $(t, u) \in \mathbf{A}^2(\mathbb{C})$ . Using MAGMA, one can check the following facts. First, the set  $\mathcal{C}$  of  $(t, u) \in \mathbf{A}^2(\mathbb{C})$  such that  $\mathcal{Z}(F_{t,u})$  is singular is a union of three affine lines  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$  and a smooth curve  $\mathcal{E}$  isomorphic to  $\mathbf{A}^1(\mathbb{C})$ . The singular locus  $\mathcal{C}_{\text{sing}}$  of  $\mathcal{C}$  consists into 6 points and it turns out that there are only 4 points  $(t_i, u_i)_{1 \leq i \leq 4}$  in  $\mathcal{C}_{\text{sing}}$  such that  $\mathcal{Z}(F_{t_i, u_i})$  is irreducible. Table 3.6 gives the information about singularities of the varieties  $\mathcal{Z}(F_{t_i, u_i})$  (with the numbering used in our MAGMA programs).

Note that a cuspidal curve of degree 18 has at most 94 singularities of type  $A_2$  [Sak, (0.3)]. Note also that there exists a cuspidal curve of degree 18 with 81 cusps [Iv]. ■



$(t, u)$	$ \mathcal{L}_{\text{sing}}(F_{t,u}) $	$W$ -orbits	Singularity
$(t_1, u_1)$	21	9	Ordinary, $m = 4, \mu = 9, \tau = 9$
		12	Ordinary, $m = 3, \mu = 4, \tau = 4$
$(t_2, u_2)$	45	9	Ordinary, $m = 4, \mu = 9, \tau = 9$
		36	$A_2$
$(t_3, u_3)$	36	36	Non-ordinary, $m = 3, \mu = 6, \tau = 6$
$(t_4, u_4)$	84	12	$m = 3, \mu = 4, \tau = 4$
		72	$A_2$

TABLE 3.6. Some singular curves of degree 18 defined by invariants of  $G_{26}$ 

#### 4. Singular surfaces from groups of rank 4

**Hypothesis.** *We still assume that  $W$  is primitive but, in this section, we assume moreover that  $n = 4$ .*

This means that  $W$  is one of the groups  $G_i$ , for  $28 \leq i \leq 32$ . We denote by  $(f_1, f_2, f_3, f_4)$  a set of fundamental invariants provided by MAGMA and we denote by  $T_{\text{sing}}^{\text{irr}}$  the set of elements  $t \in \mathbb{C}$  such that  $\mathcal{Z}(F_t)$  is irreducible and singular. It turns out that we have again the same result as in rank 3 about  $W$ -orbit of singular points:

**Proposition 4.1.** — *Assume that  $n = 4$  and  $W$  is primitive. Then  $W$  acts transitively on  $\mathcal{L}_{\text{sing}}(F_t)$  for all  $t \in T_{\text{sing}}^{\text{irr}}$ .*

Table 4.2 gives the list of curves obtained through the methods detailed in Section 2. We do not provide details of the computations for all groups: the surface of degree 24 admitting 1440 quotient singularities of type  $D_4$  defined by an invariant of  $G_{32}$  is detailed in [Bon]. This table contains the degree  $d_r$ , the number of values of  $t$  such that  $\mathcal{Z}(F_t)$  is irreducible and singular, the number of singular points and informations about the singularity (since all singular points belong to the same  $W$ -orbit by Proposition 3.1, they are all equivalent singularities).

**Remark 4.3 (Coxeter groups of rank 4).** — The case of Coxeter groups of type  $F_4$  and  $H_4$  (i.e. primitive reflection groups  $G_{28}$  and  $G_{30}$ ) was dealt with by Sarti [Sar1]. ■

**Remark 4.4 (The group  $G_{29}$ ).** — Recall that *Endraß octic* [End] has degree 8 and 168 nodes and its automorphism group has order 16. As shown in Table 4.2, invariants of  $G_{29}$  provide an irreducible surface in  $\mathbf{P}^3(\mathbb{C})$  with 160 nodes with a group of automorphisms of order at least 1920, thus approaching Endraß' record but with more symmetries. It is still an open question to determine whether one can find a surface of degree 8 in  $\mathbf{P}^3(\mathbb{C})$  with more than 168 nodes (being aware that the maximal number of nodes cannot exceed 174, see [Miy]). ■

$W$	$d_r$	$ T_{\text{sing}}^{\text{irr}} $	$t_i$	$ \mathcal{Z}_{\text{sing}}(F_{t_i}) $	Singularity
$G_{28} = W(F_4)$	6	4	$t_1$	12	$A_1$
			$t_2$	12	$A_1$
			$t_3$	48	$A_1$
			$t_4$	48	$A_1$
$G_{29}$	8	5	$t_1$	40	$A_1$
			$t_2$	20	Ordinary, $m = 3, \mu = 11, \tau = 10$
			$t_3$	160	$A_1$
			$t_4$	80	$A_1$
			$t_5$	80	$A_1$
$G_{30} = W(H_4)$	12	4	$t_1$	300	$A_1$
			$t_2$	60	$A_1$
			$t_3$	360	$A_1$
			$t_4$	600	$A_1$
$G_{31}$	20	5	$t_1$	480	$A_1$
			$t_2$	960	$A_1$
			$t_3$	1 920	$A_1$
			$t_4$	640	$A_1$
			$t_5$	1 440	$A_1$
$G_{32}$	24	40	$t_1$	40	Ordinary, $m = 6, \mu = 125, \tau = 125$
			$t_2$	360	Non-ordinary, $m = 3, \mu = 18, \tau = 18$
			$t_3$	1 440	$D_4$
			$t_4$	540	Non-simple, non-ordinary, $m = 2, \mu = 9, \tau = 9$

TABLE 4.2. Singularities of the surfaces  $\mathcal{Z}(F_t)$  for  $t \in T_{\text{sing}}^{\text{irr}}$ 

**Remark 4.5 (The group  $G_{31}$ ).** — Recall that *Chmutov surface* [Chm] of degree 20 has 2926 nodes and that a surface in  $\mathbf{P}^3(\mathbb{C})$  of degree 20 cannot have more than 3 208 nodes [Miy]. The third surface associated with  $G_{31}$  in Table 4.2 has “only” 1 920 nodes and most of them are not real (contrarily to Chmutov surface). However, it has a big group of automorphisms (of order at least 11 520). ■

## 5. Examples in higher dimension

**Example 5.1 (The group  $G_{33}$ ).** — Computations with MAGMA show that there are no fundamental invariant  $f$  of degree 10 of  $G_{33}$  such that  $\mathcal{Z}(f)$  is singular. ■

**Example 5.2 (Coxeter group of type  $E_6$ ).** — Assume in this Example, and only in this Example, that  $W = G_{35}$  is a Coxeter group of type  $E_6$ . Then  $r = 3$  and  $(d_1, d_2, d_3) = (2, 5, 6)$ , so that any fundamental invariant of degree 6 of  $W$  is of the form  $F_t = f_3 - t f_1^3$  for some  $t \in \mathbb{C}$ . Computations with MAGMA show that:

- (a)  $T_{\text{sing}} = T_{\text{sing}}^{\text{irr}}$  has cardinality 8.

- (b) For each  $t \in T_{\text{sing}}$ ,  $\mathcal{Z}(F_t)_{\text{sing}}$  has dimension 0,  $W$  acts transitively on  $\mathcal{Z}(F_t)_{\text{sing}}$ , and all these singular points are nodes.
- (c) The hypersurfaces  $\mathcal{Z}(F_t)$ ,  $t \in T_{\text{sing}}^{\text{irr}}$ , have respectively 27, 36, 135, 216, 360, 432, 1080 and 1080 singular points. ■

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