



HAL
open science

Formal classification of two-dimensional neighborhoods of genus $g \geq 2$ curves with trivial normal bundle

Olivier Thom

► **To cite this version:**

Olivier Thom. Formal classification of two-dimensional neighborhoods of genus $g \geq 2$ curves with trivial normal bundle. *Annales de l'Institut Fourier*, 2020, 70 (5), pp.1825-1846. hal-01828065v2

HAL Id: hal-01828065

<https://hal.science/hal-01828065v2>

Submitted on 11 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Formal classification of two-dimensional neighborhoods of genus $g \geq 2$ curves with trivial normal bundle

Olivier Thom*

Abstract: In this paper we study the formal classification of two-dimensional neighborhoods of genus $g \geq 2$ curves with trivial normal bundle. We first construct formal foliations on such neighborhoods with holonomy vanishing along many loops, then give the formal/analytic classification of neighborhoods equipped with two foliations, and finally put this together to obtain a description of the space of neighborhoods up to formal equivalence.

1 Introduction

1.1 General setting

Let C be a complex curve of genus g . We are interested in the different 2-dimensional neighborhoods S of C . More precisely, two surfaces S, S' equipped with embeddings $C \hookrightarrow S, C \hookrightarrow S'$ define formally/analytically equivalent neighborhoods if there exists neighborhoods U, U' of C in S and S' and a formal/analytic diffeomorphism $\varphi : U \rightarrow U'$ inducing the identity on C . The equivalence of two neighborhoods is thus given by diagrams

$$\begin{array}{ccc} C & \hookrightarrow & U \subset S \\ \downarrow id & & \downarrow \varphi \\ C & \hookrightarrow & U' \subset S' \end{array}$$

We want to understand the classification of such neighborhoods up to equivalence.

*The author is supported by CNRS, ANR-16-CE40-0008 Foliage.

The first invariants in this problem are the normal bundle N_C of C in S and the self-intersection $C \cdot C = \deg(N_C)$ of the curve C . If $C \cdot C < 0$, Grauert's theorem (cf [5] or [2]) tells that if the self-intersection is sufficiently negative (more precisely, if $C \cdot C < 2(2 - 2g)$), then S is analytically equivalent to N_C (ie. a neighborhood of C in S is analytically equivalent to a neighborhood of the zero section in the total space of N_C).

In the case $C \cdot C > 0$, we can cite the works of Ilyashenko [6] on strictly positive neighborhood of elliptic curves and of Mishustin [9] for neighborhoods of genus $g \geq 2$ curves with large self-intersection ($C \cdot C > 2g - 2$). In both cases, the authors show that there is a huge family of non-equivalent neighborhoods (there are some functional invariants).

In the case $C \cdot C = 0$, the neighborhoods of elliptic curves have already been studied. Arnol'd showed in [1] that if S is a neighborhood of an elliptic curve whose normal bundle N_C is not torsion, S is formally equivalent to N_C ; if moreover N_C satisfies some diophantine condition, then S is analytically equivalent to N_C . The case when C is an elliptic curve and N_C is torsion was studied in [8]; in particular, it is shown that the formal moduli space (ie. with respect to formal classification) of such neighborhoods is a countable union of finite dimensional spaces.

The goal of this paper is to study the neighborhoods of genus $g \geq 2$ curves with trivial normal bundle under formal equivalence.

1.2 Notations

Throughout this paper, we will use the term $\text{Diff}(\mathbb{C}, 0)$ to denote the group of germs of analytic diffeomorphisms of \mathbb{C} at 0; we will write $\widehat{\text{Diff}}(\mathbb{C}, 0)$ the group of formal diffeomorphisms of \mathbb{C} at 0.

A *formal neighborhood* \widehat{S} of C is a scheme $\mathcal{X} = (X, \widehat{\mathcal{O}})$ with C as a subscheme such that there is an open covering $X = \cup U_i$ of X with $\widehat{\mathcal{O}}|_{U_i} = (\mathcal{O}_C|_{U_i})[[y_i]]$, some coordinates x_i on $C \cap U_i$ and some holomorphic functions $u_{ji}^{(k)}$ with $y_j = \sum_{k \geq 1} u_{ji}^{(k)}(x_i) y_i^k$ and $u_{ji}^{(1)}$ not vanishing on $U_i \cap U_j$.

If S is an analytic neighborhood of C , then the completion $\widehat{\mathcal{O}}$ of \mathcal{O}_S along C is the structure sheaf of a formal neighborhood \widehat{S} of C . The natural inclusion $\mathcal{O}_S \hookrightarrow \widehat{\mathcal{O}}$ gives an injection $\widehat{S} \hookrightarrow S$ and allows us to see S as a formal neighborhood. We say that two analytic neighborhoods S, S' are formally equivalent if \widehat{S} and \widehat{S}' are equivalent.

Let $S = \cup U_i$ be a covering of an analytic neighborhood S and (u_i, v_i) some analytic coordinates on U_i with $C \cap U_i = \{v_i = 0\}$. A *regular analytic foliation* on S having C as a leaf can be seen as a collection of submersive analytic functions $y_i = \sum_{k \geq 1} y_i^{(k)}(u_i) v_i^k$ on each U_i such that there exist some diffeomorphisms $\varphi_{ji} \in \text{Diff}(\mathbb{C}, 0)$ with $y_j = \varphi_{ji} \circ y_i$ on $U_i \cap U_j$. In analogy, a *regular formal foliation* on

S around C (or on a formal neighborhood \widehat{S} of C) is a collection of formal power series $y_i = \sum_{k \geq 1} y_i^{(k)}(u_i) v_i^k$ with $y_j = \varphi_{ji} \circ y_i$ for some $\varphi_{ji} \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ where the coefficients $y_i^{(k)}(u_i)$ are still analytic functions on $C \cap U_i$ and $y_i^{(1)}$ does not vanish on $C \cap U_i$ (otherwise stated, the divisor $\{y_i = 0\}$ is equal to $\{v_i = 0\} = C \cap U_i$).

1.3 Results

We will use the same strategy as in [8]: first construct two "canonical" regular formal foliations \mathcal{F}, \mathcal{G} on S having C as a leaf, then study the classification of formal/convergent bifoliated neighborhoods $(S, \mathcal{F}, \mathcal{G})$, and finally put these together to obtain the formal classification of neighborhoods.

The first step, the construction of "canonical" foliations, is explained in section 2. It has already been proved in [4] that there exist formal regular foliations in S having C as a leaf. Since we need to have some kind of unicity to be able to use these for the classification of neighborhoods, we will need to adapt the construction of [4]. The idea is to construct foliations whose holonomy is trivial along as many loops as possible. For this, we fix a family $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ of loops in C which is a symplectic basis in homology and denote A -loops the loops α_i and B -loops the β_i . We prove the following:

Theorem 1. *Let C be a curve of genus $g \geq 2$ and S a neighborhood of C with trivial normal bundle. Then there exists a unique regular formal foliation \mathcal{F} on S having C as a leaf, such that the holonomy of \mathcal{F} along A -loops is trivial.*

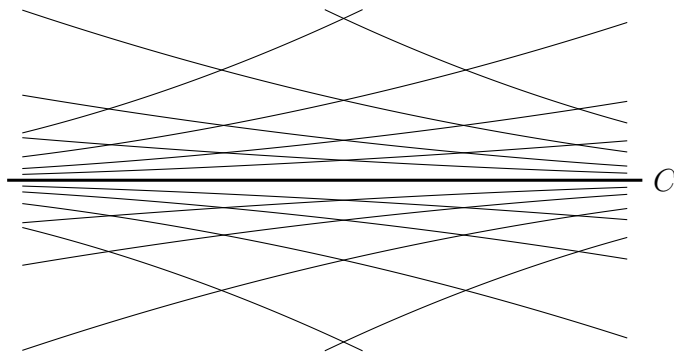


Figure 1: A bifoliated neighborhood of C

The second step, the classification of bifoliated neighborhoods, can be found in [13]. We will explain in section 3 how this classification works in the generic case and show that a bifoliated neighborhood $(S, \mathcal{F}, \mathcal{G})$ is characterised by the order of

tangency k between \mathcal{F} and \mathcal{G} along C , a 1-form ω which controls how \mathcal{F} and \mathcal{G} differ at order $k + 1$ and an additional invariant

$$\text{Inv}(S, \mathcal{F}, \mathcal{G}) \in (\text{Diff}(\mathbb{C}, 0))^{6g-3} / \sim$$

(resp. $\widehat{\text{Inv}}(S, \mathcal{F}, \mathcal{G}) \in (\widehat{\text{Diff}}(\mathbb{C}, 0))^{6g-3} / \sim$ for formal neighborhoods), where the relation \sim is given by the action of $\text{Diff}(\mathbb{C}, 0)$ on $(\text{Diff}(\mathbb{C}, 0))^{6g-3}$ by conjugacy on each factor (resp. the action of $\widehat{\text{Diff}}(\mathbb{C}, 0)$ on $(\widehat{\text{Diff}}(\mathbb{C}, 0))^{6g-3}$ by conjugacy on each factor). This invariant is given by holonomies of the foliations \mathcal{F} and \mathcal{G} computed on a tangency curve T_1 , ie. an irreducible component different from C of the set of points at which \mathcal{F} and \mathcal{G} are tangent.

Theorem 2. *Let C be a curve of genus $g \geq 2$. Let $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ be two bifoliated neighborhoods of C with same tangency order k and 1-form ω . Suppose $k \geq 1$ and that ω has simple zeroes p_1, \dots, p_{2g-2} . Denote T_1, T'_1 the tangency curves passing through p_1 and compute the invariants $\text{Inv}(S, \mathcal{F}, \mathcal{G})$ and $\text{Inv}(S', \mathcal{F}', \mathcal{G}')$ on the tangency curves T_1, T'_1 .*

Then $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ are analytically (resp. formally) diffeomorphic if and only if

$$\text{Inv}(S', \mathcal{F}', \mathcal{G}') = \text{Inv}(S, \mathcal{F}, \mathcal{G})$$

(resp. $\widehat{\text{Inv}}(S', \mathcal{F}', \mathcal{G}') = \widehat{\text{Inv}}(S, \mathcal{F}, \mathcal{G})$).

Moreover, we know which invariants come from a bifoliated neighborhood: if $((\varphi_i^1)_{i=1}^{2g}, (\varphi_i^2)_{i=1}^{2g}, (\varphi_j^3)_{j=2}^{2g-2})$ is a representant of $\text{Inv}(S, \mathcal{F}, \mathcal{G})$, then the φ_r^s must be tangent to identity at order k . Moreover, if we write $\varphi_r^s(t) = t + a_r^s t^{k+1} \pmod{t^{k+2}}$, then the periods of ω must be $(a_i^2 - a_i^1)_{i=1, \dots, 2g}$ (equation (3) in the text); a_j^3 must be equal to $\int_{p_1}^{p_j} \omega$ for $j = 2, \dots, 2g-2$ (equation (4)); and the $(\varphi_i^s)_{i=1}^{2g}$ must be representations of the fundamental group of C for $s = 1, 2$, ie. $[\varphi_1^s, \varphi_{1+g}^s] \dots [\varphi_g^s, \varphi_{2g}^s] = id$ (equation (5)).

Theorem 3. *Let $((\varphi_i^1)_{i=1}^{2g}, (\varphi_i^2)_{i=1}^{2g}, (\varphi_j^3)_{j=2}^{2g-2})$ be some analytic/formal diffeomorphisms; let k be an integer and ω a 1-form. They define a bifoliated analytic/formal neighborhood $(S, \mathcal{F}, \mathcal{G})$ with \mathcal{F} and \mathcal{G} tangent at order k and with 1-form ω if and only if every φ_r^s is tangent to identity at order (at least) k and if they satisfy the relations (3), (4) and (5).*

If the diffeomorphisms φ_r^s are only formal, then the neighborhood is a priori only a formal neighborhood of C . Note here that the relations (3) and (4) are in fact relations between jets of order $k + 1$ of the φ_r^s , so the set of bifoliated neighborhoods modulo equivalence has huge dimension. Indeed, the space of pairs of diffeomorphisms modulo common conjugacy is already infinite dimensional, even formally: if we fix one diffeomorphism $\varphi_1 \neq id$ tangent to the identity, then the

centralizer of φ_1 has dimension 1 so that the set of pairs (φ_1, φ_2) modulo common conjugacy has roughly speaking the same cardinality as the set of diffeomorphisms.

Finally, the last step (the formal classification of neighborhoods) is done in section 4. For the pair of canonical foliations constructed, the tangency order k will be the Ueda index of the neighborhood (introduced by Ueda in [14] and named by Neeman in [11]), ie. the highest order such that there is a tangential fibration on $\text{Spec}(\mathcal{O}_S/I^{k+1})$ where I is the ideal sheaf of C in S . Similarly, ω can be interpreted in term of the Ueda class of S . We will define the space $\mathcal{V}(C, k, \omega)$ of neighborhoods with trivial normal bundle, fixed Ueda index equal to k and fixed Ueda class given by ω in order to state the final theorem:

Theorem 4. *Let C be a curve of genus $g \geq 2$, $1 \leq k < \infty$ and ω a 1-form on C with simple zeroes. Then there is an injective map*

$$\Phi : \mathcal{V}(C, k, \omega) \hookrightarrow \widehat{\text{Diff}}(\mathbb{C}, 0)^g \times \widehat{\text{Diff}}(\mathbb{C}, 0)^g \times \widehat{\text{Diff}}(\mathbb{C}, 0)^{2g-3} / \sim$$

where the equivalence relation \sim is given by the action of $\widehat{\text{Diff}}(\mathbb{C}, 0)$ on $\widehat{\text{Diff}}(\mathbb{C}, 0)^N$ by conjugacy on each factor.

A tuple of diffeomorphisms $((\varphi_i^{(j)})_i)_{j=1}^3$ is in the image of Φ if and only if the $\varphi_i^{(j)}$ are tangent to the identity at order k and if they satisfy the compatibility conditions (3) and (4).

2 Construction of foliations

On the curve C we can choose loops α_i, β_i , $i = 1, \dots, g$ forming a symplectic basis of $H_1(C, \mathbb{C})$, ie. $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = 1$ if $i = j$ and 0 otherwise. We call A -loops the loops α_i and B -loops the β_i . Similarly, if ω is a 1-form on C , we will call A -period (resp. B -period) of ω any integral $\int_{\alpha_i} \omega$ (resp. $\int_{\beta_i} \omega$).

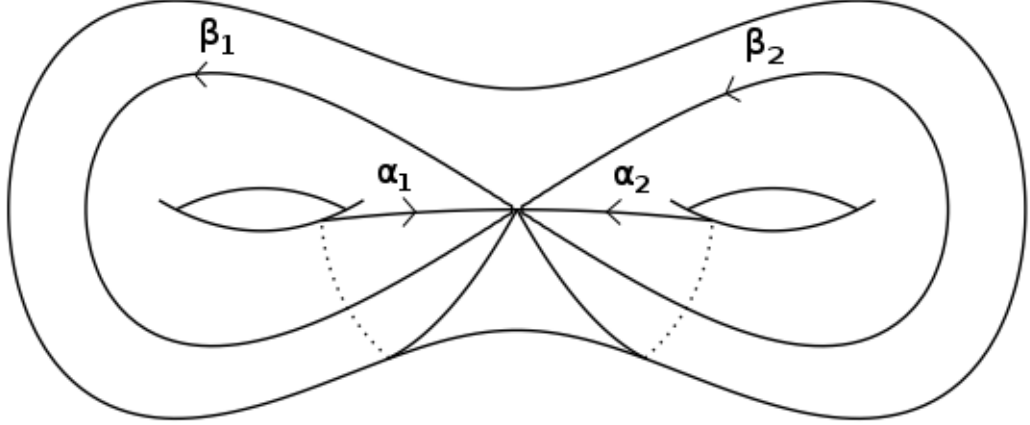


Figure 2: A - and B -loops

Definition 1. A foliation will be called A -canonical if its holonomy representation ρ satisfies $\rho(\alpha_i) = id$ for all $i = 1, \dots, g$ and if the linear part of ρ is trivial. We define the notion of B -canonicity similarly; unless otherwise stated, the term "canonical" will mean A -canonical.

Choose an open covering (U_i) of some neighborhood of C ; let $V_i = U_i \cap C$ and $\mathcal{V} = (V_i)$ the associated open covering of C . Denote by \mathbb{C} the trivial rank one local system on C and by \mathcal{O}_C the trivial line bundle on C .

First, let us give the following definitions:

Definition 2. Let (a_{ij}) be a cocycle in $Z^1(\mathcal{V}, \mathbb{C})$ and let γ be a loop on C . We define the period of (a_{ij}) along γ to be the sum

$$\int_{\gamma} (a_{ij}) = \sum_{p=1}^n a_{i_p i_{p+1}}$$

where the open sets $(V_{i_p})_{p=1}^n$ form a simple covering of γ and $V_{i_p} \cap V_{i_{p+1}} \cap \gamma \neq \emptyset$.

This application only depends on the class $[\gamma]$ of γ in the fundamental group of C ; taking periods along the α_i and β_i gives applications

$$P_A, P_B : Z^1(\mathcal{V}, \mathbb{C}) \rightarrow \mathbb{C}^g.$$

Putting these together gives an application $P : Z^1(\mathcal{V}, \mathbb{C}) \rightarrow \mathbb{C}^{2g}$ which induces an injection $P : H^1(\mathcal{V}, \mathbb{C}) \rightarrow \mathbb{C}^{2g}$.

On the other hand, the exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}_C \rightarrow \Omega^1 \rightarrow 0$$

gives the exact sequence in cohomology

$$0 \rightarrow H^0(C, \Omega^1) \xrightarrow{\delta} H^1(C, \mathbb{C}) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow 0. \quad (1)$$

The fact that the arrow $H^1(C, \mathbb{C}) \rightarrow H^1(C, \mathcal{O}_C)$ is surjective is an easy consequence of ([14], proposition 1). We have $\dim(H^0(C, \Omega^1)) = \dim(H^1(C, \mathcal{O}_C)) = g$ and $\dim(H^1(C, \mathbb{C})) = 2g$ so that $P : H^1(C, \mathbb{C}) \rightarrow \mathbb{C}^{2g}$, being injective, is bijective. It is well-known that a 1-form whose A -periods vanish is zero, so that the application $P_A \circ \delta : H^0(C, \Omega^1) \rightarrow \mathbb{C}^g$ is a bijection.

Constructing a foliation on S is equivalent to constructing functions y_i on U_i which are reduced equations of $C \cap U_i$ such that

$$y_j = \varphi_{ji}(y_i),$$

where the φ_{ji} are diffeomorphisms of $(\mathbb{C}, 0)$. As before, if γ is a loop, we can define the product

$$H_\gamma((\varphi_{ji})) = \varphi_{i_1 i_n} \circ \dots \circ \varphi_{i_3 i_2} \circ \varphi_{i_2 i_1}$$

which will be the holonomy of the foliation given by the y_i along the loop γ . To construct such functions, we are going to proceed by steps, but first, we need another definition.

Definition 3. *A set of functions (y_i) on the open sets U_i is called A -normalized at order μ if the y_i are regular functions on U_i vanishing at order 1 on C and*

$$y_j = \varphi_{ji}^{(\mu)}(y_i) + a_{ji}^{(\mu+1)} y_i^{\mu+1}, \quad (2)$$

on $U_i \cap U_j$, where $a_{ji}^{(\mu+1)}$ is a function on $U_i \cap U_j$, the $\varphi_{ji}^{(\mu)}$ are polynomials of degree μ which are also diffeomorphisms tangent to identity and the holonomies $H_{\alpha_k}((\varphi_{ji}))$ are the identity modulo $y_i^{\mu+1}$ for all $k = 1, \dots, g$.

The idea of the proof is first to construct some functions (y_i) which are A -normalized at order 1, and then to show that every A -normalized at order μ set of functions (y_i) can be transformed into an A -normalized at order $(\mu + 1)$ set of functions by changes of coordinates $y_i \mapsto y_i - b_i y_i^{\mu+1}$ for some functions b_i on U_i . At the limit, we will thus obtain a formal foliation on S with trivial holonomy along A -loops.

Lemma 1. *There exists an A -normalized at order 1 set of functions and the foliations associated to two such sets of functions coincide at order 1.*

Proof. Take any reduced equations (y_i) of C and compute y_j in the coordinate y_i :

$$y_j = a_{ji}^{(1)} y_i.$$

The cocycle $(a_{ji}^{(1)}|_C)$ defines the normal bundle $N_C = \mathcal{O}_C$ of C so is cohomologous to the trivial cocycle: there exist functions b_i on U_i such that

$$a_{ji}^{(1)}|_C = \frac{b_j|_C}{b_i|_C}.$$

Put $z_i = y_i/b_i$ to obtain

$$z_j = z_i + a_{ji}^{(2)} z_i^2.$$

for some functions $a_{ji}^{(2)}$.

For unicity, consider two sets of functions (y_i) and (z_i) A -normalized at order 1. Then (y_i) and (z_i) define two sections y^1 and z^1 on the normal bundle N_C . Necessarily, y^1 and z^1 are colinear, hence the result. \square

Lemma 2. *Let (y_i) be a set of functions A -normalized at order μ . Then there exist functions (b_i) on U_i such that the coordinates $z_i = y_i - b_i y_i^{\mu+1}$ are A -normalized at order $\mu + 1$. Moreover, two sets of functions A -normalized at order $(\mu + 1)$ which coincide at order μ define the same foliation at order $\mu + 1$.*

Proof. Since (y_i) is A -normalized at order μ , it satisfies

$$y_j = \varphi_{ji}^{(\mu)}(y_i) + a_{ji}^{(\mu+1)} y_i^{\mu+1}.$$

In the following, denote by $\text{Diff}_\mu^1(\mathbb{C}, 0) = \text{Diff}^1(\mathbb{C}, 0)/\text{Diff}^{\mu+1}(\mathbb{C}, 0)$ the group of μ -jets of diffeomorphisms tangent to the identity. The tuple $(\varphi_{ji}^{(\mu)})_{ji}$ is a cocycle in $H^1(C, \text{Diff}_\mu^1(\mathbb{C}, 0))$; it is entirely determined by its holonomy representation $H((\varphi_{ji}^{(\mu)})) : \pi_1(C) \rightarrow \text{Diff}_\mu^1(\mathbb{C}, 0)$. We would like to extend this cocycle to some cocycle in $H^1(C, \text{Diff}_{\mu+1}^1(\mathbb{C}, 0))$. Since $H_{\alpha_k}((\varphi_{ji}^{(\mu)}))$ is trivial for $k = 1, \dots, g$, extend $H_{\alpha_k}((\varphi_{ji}^{(\mu)}))$ to $\rho_{\alpha_k} = id$. Next, extend the diffeomorphisms $H_{\beta_k}((\varphi_{ji}^{(\mu)}))$ to diffeomorphisms $\rho_{\beta_k} \in \text{Diff}_{\mu+1}^1$ in any way. Then $\prod_{k=1}^g [\rho_{\alpha_k}, \rho_{\beta_k}] = id$ so the $(\rho_\gamma)_\gamma$ define a representation of $\pi_1(C)$ into $\text{Diff}_{\mu+1}^1$ which corresponds to a cocycle (ψ_{ji}) such that $H_{\alpha_k}((\psi_{ji})) = \rho_{\alpha_k}$ and $H_{\beta_k}((\psi_{ji})) = \rho_{\beta_k}$. We can then write

$$y_j = \psi_{ji}(y_i) + a'_{ji}^{(\mu+1)} y_i^{\mu+1}$$

for some $a'_{ji}^{(\mu+1)}$. Next,

$$\begin{aligned} y_k &= \psi_{kj}(y_j) + a'_{kj}^{(\mu+1)} y_j^{\mu+1} \\ &= \psi_{kj} \left(\psi_{ji}(y_i) + a'_{ji}^{(\mu+1)} y_i^{\mu+1} \right) + a'_{kj}^{(\mu+1)} \left(\psi_{ji}(y_i) + a'_{ji}^{(\mu+1)} y_i^{\mu+1} \right)^{\mu+1} \\ &= \psi_{kj}(\psi_{ji}(y_i)) + (a'_{ji}^{(\mu+1)} + a'_{jk}^{(\mu+1)}) y_i^{\mu+1} + \dots \end{aligned}$$

Since $\psi_{ki} = \psi_{kj}\psi_{ji}$, we obtain $a'_{ki}^{(\mu+1)}|_C = a'_{kj}^{(\mu+1)}|_C + a'_{ji}^{(\mu+1)}|_C$ and thus $(a'_{ji}^{(\mu+1)}|_C)$ is a cocycle in $H^1(C, \mathcal{O}_C)$. By the exact sequence (1), it is cohomologous to a constant cocycle $(c_{ji}) \in H^1(C, \mathbb{C})$: there exists functions b_i on U_i such that $a'_{ji}^{(\mu+1)}|_C - c_{ji} = b_j|_C - b_i|_C$. Still using the exact sequence (1), we see that two cocycles $(c_{ji}), (c'_{ji})$ cohomologous to $(a'_{ji}^{(\mu+1)}|_C)$ differ only by the periods of a 1-form. As noted before, $P_A \circ \delta : H^0(C, \Omega^1) \rightarrow H^1(C, \mathbb{C})$ is bijective so we can choose (c_{ji}) with trivial A -periods, and such a (c_{ji}) is unique. Put $\varphi_{ji}^{(\mu+1)}(y) = \psi_{ji}(y) + c_{ji}y^{\mu+1}$ and $z_i = y_i - b_i y_i^{\mu+1}$ to obtain

$$\begin{aligned} z_j &= \psi_{ji}(z_i) + (a'_{ji}^{(\mu+1)} - b_j + b_i) z_i^{\mu+1} + o(z_i^{\mu+1}) \\ &= \varphi_{ji}^{(\mu+1)}(z_i) + o(z_i^{\mu+1}). \end{aligned}$$

Since the choice of $(c_{ji}) \in H^1(C, \mathbb{C})$ is unique, if two sets of functions $(z_i), (z'_i)$ are both A -normalized at order $\mu+1$ and coincide at order μ , then they differ at order $\mu+1$ by a coboundary $(d_i) \in H^0(C, \mathbb{C})$: $z'_i = z_i + d_i z_i^{\mu+1} + \dots$. Hence, they define the same foliation at order $\mu+1$. \square

Putting all this together, we obtain theorem 1.

3 Classification of bifoliated neighborhoods

A *bifoliated neighborhood* of C is a tuple $(S, \mathcal{F}, \mathcal{G})$ where S is a neighborhood of C and \mathcal{F}, \mathcal{G} are distinct foliations on S having C as a common leaf. Two bifoliated neighborhoods $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ are said to be equivalent if there are two neighborhoods $U \subset S, U' \subset S'$ of C and a diffeomorphism $\phi : U \rightarrow U'$ fixing C such that

$$\phi_* \mathcal{F} = \mathcal{F}' \quad \text{and} \quad \phi_* \mathcal{G} = \mathcal{G}'.$$

In this section, we want to study the classification of bifoliated neighborhoods under this equivalence relation. We will consider here analytic equivalence, but the formal classification can be obtained by replacing the word "analytic" by "formal" everywhere.

A neighborhood will have a lot a formal foliations, and the canonical ones may diverge even though others might converge (cf. [8]). We will thus consider here a general bifoliated neighborhood $(S, \mathcal{F}, \mathcal{G})$, with the additional assumptions that \mathcal{F} and \mathcal{G} coincide at order 1 and their holomy representations are tangent to the identity. The study can be done without these assumptions (cf. [13]), but they are true for the pair of canonical foliations and simplify the results (for example, in general an affine structure is involved which under our assumptions is only a translation structure, ie. a 1-form).

3.1 First invariants

If $(S, \mathcal{F}, \mathcal{G})$ is a bifoliated neighborhood, each foliation comes with the holonomy representation of the leaf C :

$$\rho_{\mathcal{F}}, \rho_{\mathcal{G}} : \pi_1(C) \rightarrow \text{Diff}(\mathbb{C}, 0),$$

Fix a base point $p_0 \in C$, a transversal T_0 passing through C at p_0 and a coordinate t on T_0 (ie. a function $t \in (\mathbb{C}, 0) \mapsto q(t) \in T_0$). Let γ be a loop on C based at p_0 ; choose the minimal first integral F of \mathcal{F} around T_0 such that $F(q(t)) = t$. The analytic continuation F^γ of F along γ is again a first integral of \mathcal{F} , hence is of the form

$$F^\gamma = \varphi_\gamma^{-1} \circ F.$$

We define $\rho_{\mathcal{F}}(\gamma) = \varphi_\gamma$.

A second invariant is the order of tangency between \mathcal{F} and \mathcal{G} along C : take two 1-forms α and β on S defining locally the foliations \mathcal{F} and \mathcal{G} . The 2-form $\alpha \wedge \beta$ vanishes on C so the order of vanishing of $\alpha \wedge \beta$ along C gives a global invariant $k + 1$ which does not depend on the choice of α and β . The order of tangency between \mathcal{F} and \mathcal{G} is defined to be this integer k . Our assumption that the foliations coincide at order 1 exactly means that $k \geq 1$.

The next invariant is a 1-form on C associated to this pair of foliations. Choose as before a point $p_0 \in C$, a transversal T_0 at p_0 and a coordinate $t \mapsto q(t)$ on T_0 . Take local minimal first integrals F and G of \mathcal{F} and \mathcal{G} such that $F(q(t)) = G(q(t)) = t$. By definition of k , $G = F + aF^{k+1} + \dots$ in a neighborhood of p for a local function a on C . Take the analytic continuations F^γ, G^γ and a^γ of F, G and a along a loop γ . Then we can use the fact that the holonomy representations of

\mathcal{F} and \mathcal{G} are tangent to the identity to get

$$\begin{aligned} G^\gamma &= F^\gamma + (a^\gamma)(F^\gamma)^{k+1} + \dots \\ \rho_{\mathcal{G}}(\gamma)^{-1} \circ G &= \rho_{\mathcal{F}}(\gamma)^{-1} \circ F + (a^\gamma)(\rho_{\mathcal{F}}(\gamma)^{-1} \circ F)^{k+1} + \dots \\ G &= \rho_{\mathcal{G}}(\gamma) (\rho_{\mathcal{F}}(\gamma)^{-1} \circ F + a^\gamma F^{k+1} + \dots) \\ &= \rho_{\mathcal{G}}(\gamma) \circ \rho_{\mathcal{F}}(\gamma)^{-1} \circ F + a^\gamma F^{k+1} + \dots \\ &= F + aF^{k+1} + \dots \end{aligned}$$

Since $\rho_{\mathcal{G}}(\gamma) \circ \rho_{\mathcal{F}}(\gamma)^{-1}$ has constant coefficients, there exists constants c^γ such that $a^\gamma = a + c^\gamma$. Then the 1-form $\omega = da$ is a well-defined 1-form on C .

Note that we saw in the process that

$$\rho_{\mathcal{G}}(\gamma) \circ \rho_{\mathcal{F}}(\gamma)^{-1}(y) = y + \left(\int_\gamma \omega \right) y^{k+1} + \dots, \quad (3)$$

thus the form ω is entirely determined by $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{G}}$.

Note also that the holonomy representation and the form ω depend on the choice of the transversal T_0 and of a coordinate t on it. A change of coordinate $\tilde{t} = \varphi(t)$ induces conjugacies on $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{G}}$ and changes ω into some multiple of it: $\tilde{\rho}_{\mathcal{F}}(\gamma) = \varphi \circ \rho_{\mathcal{F}}(\gamma) \circ \varphi^{-1}$, $\tilde{\rho}_{\mathcal{G}}(\gamma) = \varphi \circ \rho_{\mathcal{G}}(\gamma) \circ \varphi^{-1}$ and $\tilde{\omega} = \varphi'(0)^{-k} \omega$.

3.2 Tangency set

If F and G are local minimal first integrals of \mathcal{F} and \mathcal{G} , then the tangency set between \mathcal{F} and \mathcal{G} is defined to be

$$\text{Tang}(\mathcal{F}, \mathcal{G}) = \{dF \wedge dG = 0\}.$$

This definition does not depend on the choice of F and G and gives a well-defined analytic subset of S .

Note that if we write $G = F + aF^{k+1} + \dots$, then we obtain $dF \wedge dG = F^{k+1}dF \wedge (da + \dots)$. Since $\omega = da$,

$$\text{Tang}(\mathcal{F}, \mathcal{G}) \cap C = \{\omega = 0\}.$$

In particular, the set $\text{Tang}(\mathcal{F}, \mathcal{G})$ intersects C at $2g - 2$ points counting multiplicities. In the sequel, we will suppose that we are in the generic case: ω has $2g - 2$ distinct zeroes. This also means that $\text{Tang}(\mathcal{F}, \mathcal{G})$ is the union of $2g - 2$ curves which are transverse to C .

Denote p_1, \dots, p_{2g-2} the zeroes of ω and T_i the tangency curve passing through p_i . If we fix some simple paths γ_{ij} between p_i and p_j , we can look at the holonomy transports

$$\varphi_{ij}^{\mathcal{F}}, \varphi_{ij}^{\mathcal{G}} : T_i \rightarrow T_j$$

following the leaves of \mathcal{F} and \mathcal{G} along γ_{ij} . To simplify, suppose that the γ_{1j} only intersect each other at p_1 and that $\gamma_{ij} = \gamma_{1i}^{-1} \cdot \gamma_{1j}$.

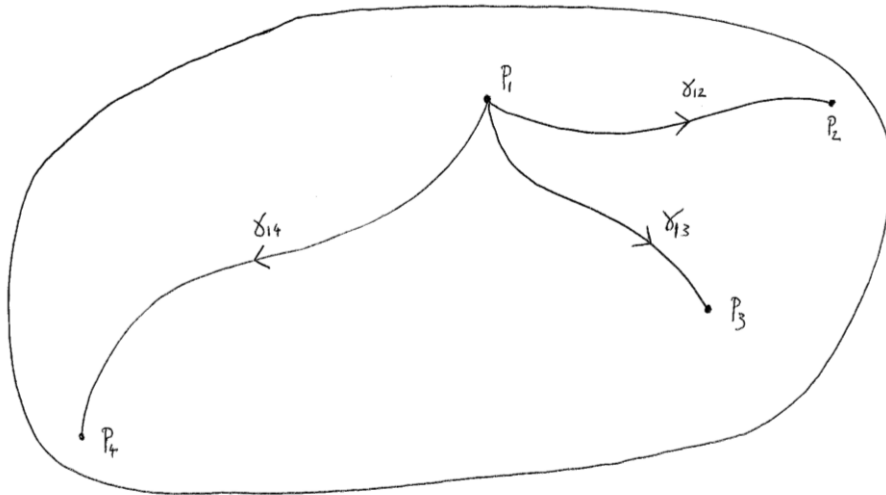


Figure 3: The paths γ_{1j}

To define this, fix some coordinates t_i, t_j on T_i and T_j ; there exists a simply connected neighborhood U of the path γ_{ij} . Let L be a leaf of \mathcal{F} on U . It intersects T_i at exactly one point (let t_i be its coordinate). In the same way, let t_j be the coordinate of $L \cap T_j$. We set $\varphi_{ij}^{\mathcal{F}}(t_i) = t_j$: this gives a germ of diffeomorphism $\varphi_{ij}^{\mathcal{F}} \in \text{Diff}(\mathbb{C}, 0)$ which depends on the choices of coordinates on T_i and T_j . Their composition

$$\varphi_{ij}^{\leftrightarrow} = (\varphi_{ij}^{\mathcal{G}})^{-1} \circ \varphi_{ij}^{\mathcal{F}}$$

is a diffeomorphism of T_i so only depends on the choice of a coordinate on T_i ; a change of coordinate $t'_i = \varphi(t_i)$ acts by conjugacy $\varphi_{ij}^{\prime \leftrightarrow} = \varphi \circ \varphi_{ij}^{\leftrightarrow} \circ \varphi^{-1}$.

We can show as in the previous subsection that the holonomy transports $\varphi_{ij}^{\leftrightarrow}$ are related to the 1-form ω by the relation:

$$\varphi_{ij}^{\leftrightarrow}(t_i) = t_i - \left(\int_{\gamma_{ij}} \omega \right) t_i^{k+1} + \dots \quad (4)$$

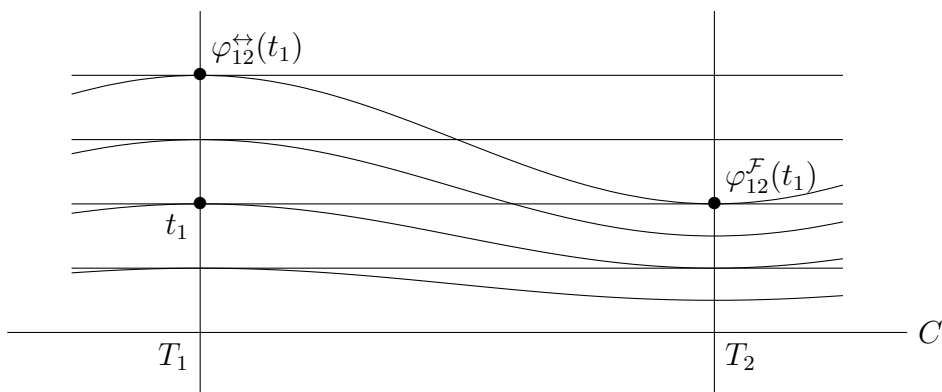


Figure 4: Holonomy transports

3.3 Classification of bifoliated neighborhoods

We say that a bifoliated neighborhood $(S, \mathcal{F}, \mathcal{G})$ is generic if there are $2g-2$ distinct tangency curves T_1, \dots, T_{2g-2} between \mathcal{F} and \mathcal{G} and if they intersect C transversely at some distinct points p_1, \dots, p_{2g-2} .

On each neighborhood, we can fix one of these points, for example p_1 , fix a coordinate t on T_1 , fix paths γ_{1j} between p_1 and p_j and compute every invariant on the transversal T_1 with coordinate t . We thus have the holonomy representations $\rho_{\mathcal{F}}, \rho_{\mathcal{G}}$ and the holonomy transports $\varphi_{1j}^{\leftrightarrow}$ between T_1 and another tangency curve T_j .

The holonomy representations $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{G}}$ are entirely determined by the images of the basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$: these are any diffeomorphisms such that

$$\begin{aligned} [\rho_{\mathcal{F}}(\alpha_1), \rho_{\mathcal{F}}(\beta_1)] \dots [\rho_{\mathcal{F}}(\alpha_g), \rho_{\mathcal{F}}(\beta_g)] &= id \\ [\rho_{\mathcal{G}}(\alpha_1), \rho_{\mathcal{G}}(\beta_1)] \dots [\rho_{\mathcal{G}}(\alpha_g), \rho_{\mathcal{G}}(\beta_g)] &= id. \end{aligned} \quad (5)$$

Every invariant diffeomorphism found $\varphi = \rho_{\mathcal{F}}(\alpha_i), \rho_{\mathcal{F}}(\beta_i), \rho_{\mathcal{G}}(\alpha_i), \rho_{\mathcal{G}}(\beta_i), \varphi_{1j}^{\leftrightarrow}$ depend on the choice of the coordinate t . A change of coordinate $t' = \psi(t)$ induces a conjugacy on φ : $\varphi' = \psi \circ \varphi \circ \psi^{-1}$. So we define the invariant of a neighborhood $(S, \mathcal{F}, \mathcal{G})$ to be

$$\begin{aligned} Inv(S, \mathcal{F}, \mathcal{G}) &= [((\rho_{\mathcal{F}}(\alpha_i))_{i=1}^g, (\rho_{\mathcal{F}}(\beta_i))_{i=1}^g, (\rho_{\mathcal{G}}(\alpha_i))_{i=1}^g, (\rho_{\mathcal{G}}(\beta_i))_{i=1}^g, (\varphi_{1j}^{\leftrightarrow})_{j=2}^{2g-2})] \\ &\in \text{Diff}(\mathbb{C}, 0)^{2g} \times \text{Diff}(\mathbb{C}, 0)^{2g} \times \text{Diff}(\mathbb{C}, 0)^{2g-3} / \sim \end{aligned}$$

where \sim is the action of $\text{Diff}(\mathbb{C}, 0)$ by conjugacy on each factor.

Theorem 2. *Let C be a curve of genus $g \geq 2$. Let $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ be two bifoliated neighborhoods of C with tangency order k and 1-form ω . Suppose*

$k \geq 1$ and that ω has simple zeroes p_1, \dots, p_{2g-2} . Denote T_1, T'_1 the tangency curves passing through p_1 and compute the invariants $\text{Inv}(S, \mathcal{F}, \mathcal{G})$ and $\text{Inv}(S', \mathcal{F}', \mathcal{G}')$ on the tangency curves T_1, T'_1 .

Then $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ are diffeomorphic if and only if

$$\text{Inv}(S', \mathcal{F}', \mathcal{G}') = \text{Inv}(S, \mathcal{F}, \mathcal{G}).$$

Before starting the proof, let us write some lemmas.

Lemma 3. *Let p be a point in C , let F, G be two reduced equations of C around p and (x, y) some local coordinates with $C = \{y = 0\}$. Suppose that F and G are tangent at order k and that the zero divisor of $dF \wedge dG$ is $(k+1)C$ (ie. there are no other tangencies). There exists a unique diffeomorphism ϕ fixing C pointwise such that*

$$(F, G) \circ \phi = (y, y + a(x)y^{k+1}).$$

The function a is unique and satisfies $da|_C = \omega$.

The proof of this lemma can be found in [8].

Lemma 4. *Let p be a point in C , let F, G be two reduced equations of C around p and (x, y) some local coordinates with $C = \{y = 0\}$. Suppose that there is a transversal T to C such that the zero divisor of $dF \wedge dG$ is $(k+1)C + T$. Then there exists a unique diffeomorphism ϕ fixing C pointwise such that*

$$(F, G) \circ \phi = (y, b(y) + a(x)y^{k+1}).$$

The function b is unique and a is the primitive of ω which is zero at p .

The function b is of course entirely determined by the equation $G|_T = b(F|_T)$.

Proof. Put $\tilde{y} = F$, b the function determined by $G|_T = b(F|_T)$, $H = G - b(\tilde{y})$ and suppose x is a reduced equation of T . Then $dF \wedge dG = (\partial_x H)dF \wedge dx$ so by the hypotheses on the tangency divisor, $\partial_x H = 2x\tilde{y}^{k+1}u$ for some invertible function u . Then $H = x^2\tilde{y}^{k+1}v$ with v invertible so for $\phi(x, \tilde{y}) = x\sqrt{v}$, we have $G = b(\tilde{y}) + \phi(x, \tilde{y})^2\tilde{y}^{k+1}$.

If $\psi = \phi|_C$, then the coordinate $\tilde{x} = \psi^{-1} \circ \phi(x, \tilde{y})$ is equal to x on C and $(F, G) = (\tilde{y}, b(\tilde{y}) + \psi(\tilde{x})^2\tilde{y}^{k+1})$. Thus the diffeomorphism $\varphi(x, y) = (\tilde{x}, \tilde{y})$ is as sought. \square

Lemma 5. *Let $(S, \mathcal{F}, \mathcal{G})$ be a bifoliated neighborhood whose 1-form ω has simple zeroes, let T_1, T_j be two tangency curves and γ_{1j} a simple path between $p_1 = T_1 \cap C$ and $p_j = T_j \cap C$. Suppose F and G are some submersive first integrals of \mathcal{F} and \mathcal{G} around p_1 such that $F|_{T_1} = G|_{T_1}$. By lemma 4, the analytic continuations of F and G along γ_{1j} can be written $F = y$ and $G = b(y) + a(x)y^{k+1}$ for some coordinates (x, y) around p_j .*

Then $b(y) = \varphi_{1j}^{\leftarrow}(y)$ if $\varphi_{1j}^{\leftarrow}$ is computed in the coordinate $t = y$ on T_1 .

Proof. Indeed, b is characterised by $G|_{T_j} = b \circ F|_{T_j}$, and $\varphi_{1j}^{\leftrightarrow}$ by the fact that the leaf of \mathcal{F} passing through T_1 at the point of coordinate $F = y_0$ intersects (tangentially) on T_j the leaf of \mathcal{G} passing through T_1 at the point of coordinate $F = \varphi_{1j}^{\leftrightarrow}(y_0)$. This means that the first integral $\varphi_{1j}^{\leftrightarrow} \circ F$ of \mathcal{F} coincides with G on T_j , ie $\varphi_{1j}^{\leftrightarrow} \circ F|_{T_j} = G|_{T_j}$, hence the result. \square

Proof of theorem 2. Take two bifoliated neighborhoods $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$ with the same tangency index k , 1-form ω and the same invariants computed in some coordinates t, t' on T_1 and T'_1 .

Begin by fixing simply connected neighborhoods Y, Y' of $\cup_{j=2}^{2g} \gamma_{1j}$ in S and S' . We begin by showing that $(Y, \mathcal{F}, \mathcal{G})$ and $(Y', \mathcal{F}', \mathcal{G}')$ are diffeomorphic, and we will then show that this diffeomorphism can be extended to S and S' .

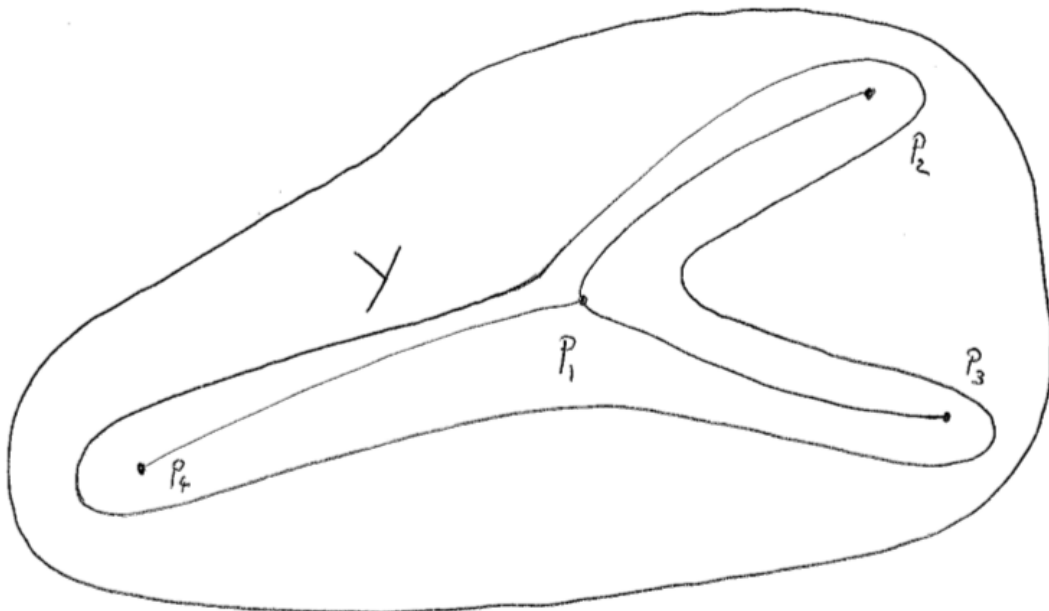


Figure 5: The neighborhood Y

Since Y and Y' are simply connected, the foliations on these sets have first integrals F, G, F', G' and we can suppose that $F(t) = G(t)$ and $F'(t') = G'(t')$ on T_1 and T'_1 . Since $\omega = \omega'$, lemma 4 tells us that there is a (unique) diffeomorphism ψ between a neighborhood of p_1 in S and a neighborhood of p_1 in S' such that $F' \circ \psi = F$ and $G' \circ \psi = G$. We can take the analytic continuation of F, G, F' and G' along one of the paths γ_{1j} . For any point p in this path, lemma 3 tells us that the pairs (F, G) and (F', G') are equivalent (by a unique diffeomorphism) if and

only if the number $a(p)$ is the same for both couples. But $a(p) = \int_{p_1}^p \omega$ where the integral is taken along the path γ_{1j} so it is the case. By unicity, the diffeomorphism ψ can be extended along the path γ_{1j} arbitrarily near the point p_j .

At the point p_j , lemmas 4 and 5 show that the pairs (F, G) and (F', G') are also conjugated by a unique diffeomorphism in a neighborhood of p_j . Hence, we can extend ψ to a diffeomorphism $\psi : Y \rightarrow Y'$ conjugating the pairs (F, G) and (F', G') .

By the lemma 3, we can also extend ψ along any simple path. Then we only need to show that ψ can be extended along a non-trivial loop. Let γ be a non-trivial loop on C based at p_1 , $\varphi_{\mathcal{F}} = \rho_{\mathcal{F}}(\gamma)$ and $\varphi_{\mathcal{G}} = \rho_{\mathcal{G}}(\gamma)$. The extensions of F and G along γ are $\varphi_{\mathcal{F}}^{-1} \circ F$ and $\varphi_{\mathcal{G}}^{-1} \circ G$; we know that $F' \circ \psi = F$ and $G' \circ \psi = G$, so $\varphi_{\mathcal{F}}^{-1} \circ F' \circ \psi = \varphi_{\mathcal{F}}^{-1} \circ F$ and $\varphi_{\mathcal{G}}^{-1} \circ G' \circ \psi = \varphi_{\mathcal{G}}^{-1} \circ G$. Hence ψ is the diffeomorphism conjugating $(\varphi_{\mathcal{F}}^{-1} \circ F, \varphi_{\mathcal{G}}^{-1} \circ G)$ with $(\varphi_{\mathcal{F}}^{-1} \circ F', \varphi_{\mathcal{G}}^{-1} \circ G')$ and by unicity this means that ψ can be extended along any loop. Thus ψ can be extended to a diffeomorphism between $(S, \mathcal{F}, \mathcal{G})$ and $(S', \mathcal{F}', \mathcal{G}')$. \square

3.4 Construction of bifoliated neighborhoods

We saw three restrictions for a set of diffeomorphisms to be an invariant of some bifoliated neighborhood: these are the compatibility relations (3), (4) and (5). These are the only restrictions; to obtain a simpler result, we will consider the 1-form ω as an invariant here.

Theorem 3. *Let $((\varphi_i^1)_{i=1}^{2g}, (\varphi_i^2)_{i=1}^{2g}, (\varphi_j^3)_{j=2}^{2g-3})$ be some diffeomorphisms; let k be an integer and ω a 1-form. They define a bifoliated neighborhood $(S, \mathcal{F}, \mathcal{G})$ with \mathcal{F} and \mathcal{G} tangent at order k and with 1-form ω if and only if every φ_r^s is tangent to identity at order (at least) k and if they satisfy the relations (3), (4) and (5).*

Proof. Denote by ρ_1 and ρ_2 the representations given by the diffeomorphisms (φ_i^1) and (φ_i^2) . Consider $\tilde{C} = \mathbb{D}_x$ the universal cover of C , X a small neighborhood of a fundamental domain, U_i a small neighborhood of p_i in X and $\check{C} = X \setminus (U_2 \cup \dots \cup U_{2g-2})$.

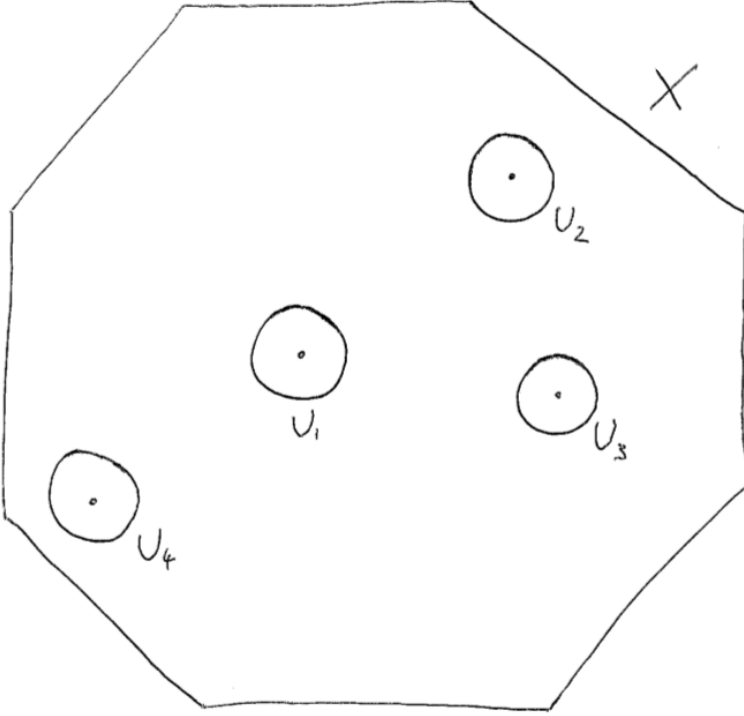


Figure 6: The neighborhood X

Consider next the trivial bundle $\check{S} = \check{C} \times \mathbb{C}_y$ along with two functions $F = y$ and $G = y + a(x)y^{k+1}$ (with $a(x) = \int_{p_1}^x \omega$). We now want to glue the borders of \check{S} together: for this, we need to show that there exists for each loop γ a diffeomorphism ψ_γ defined when it makes sense such that $\psi_\gamma|_{\check{C}} = \gamma$ and

$$(\rho_1(\gamma) \circ F, \rho_2(\gamma) \circ G) = (F \circ \psi_\gamma, G \circ \psi_\gamma).$$

Thanks to the compatibility condition (3) and lemma 3, the couples $(\rho_1(\gamma) \circ F, \rho_2(\gamma) \circ G)$ and $(F \circ \gamma, G \circ \gamma)$ are diffeomorphic so we can indeed find such a ψ_γ . We can then glue the borders of \check{S} together to obtain a surface which is a neighborhood of C with holes H_i around p_i ($i = 2, \dots, 2g-2$) and two foliations \mathcal{F} and \mathcal{G} transverse outside the holes. The holonomies of these foliations are $\rho_{\mathcal{F}} = \rho_1$ and $\rho_{\mathcal{G}} = \rho_2$ by construction.

To fill these holes, take C_i a neighborhood of p_i in X slightly larger than U_i and consider the patch $P_i = C_i \times \mathbb{C}_y$. Consider on P_i the couple

$$(\tilde{F}, \tilde{G}) = (y, \varphi_i^3(y) + (x - p_i)^2 y^{k+1}).$$

By lemma 3 and compatibility condition (4), for every point p near the boundary of the hole H_i , there exists a unique diffeomorphism ψ between a neighborhood of p in \tilde{S} and a neighborhood of p in P_i sending (F, G) to (\tilde{F}, \tilde{G}) . By unicity, these diffeomorphisms glue to a diffeomorphism between neighborhoods of the boundaries of H_i and P_i and we can then glue the patch P_i onto H_i using this diffeomorphism. By the lemma 5, we then have $\varphi_{1i}^{\leftrightarrow} = \varphi_i^3$ which concludes the proof. \square

4 Formal classification of neighborhoods

We know how to construct two canonical foliations on any neighborhood, and we know the classification of bifoliated neighborhoods, so we only need to put this together.

Denote \mathcal{F} and \mathcal{G} the A - and B -canonical foliations. Note that if $\mathcal{F} = \mathcal{G}$, then they define a fibration tangent to C , so this case can be treated by Kodaira's deformation theory. Suppose this is not the case and $\mathcal{F} \neq \mathcal{G}$, then their order of tangency k is the Ueda index of S . Moreover, let $(u_{ij}) \in H^1(C, \mathcal{O}_C)$ be the Ueda class of the neighborhood and $(a_{ij}), (b_{ij}) \in H^1(C, \mathbb{C})$ the cocycles defining the $(k+1)$ -th order holonomy of \mathcal{F} and \mathcal{G} . By definition, the images of (a_{ij}) and (b_{ij}) under the map $H^1(C, \mathbb{C}) \rightarrow H^1(C, \mathcal{O}_C)$ are both (u_{ij}) . Thus by the exact sequence (1), the cocycle $(b_{ij} - a_{ij})$ is given by a 1-form: this 1-form is exactly ω . To sum up, we have constructed an application $H^1(C, \mathbb{C}) \rightarrow H^0(C, \Omega^1)$; this is a bijection because we can find (a_{ij}) (and thus (u_{ij})) from ω as the cocycle with null A -periods and with B -periods equal to those of ω .

By extension, we will call this form the Ueda form of the neighborhood. The Ueda class (and thus the Ueda form) is well-defined only up to a multiplicative constant, but the set of its zeroes is well-defined. The situation will be quite different depending on the tangency set between \mathcal{F} and \mathcal{G} , so suppose that ω has only simple zeroes (so that the tangency set consists of $2g-2$ simple transversal tangency curves). Denote by $\mathcal{V}(C, k, \omega)$ the space of 2-dimensional formal neighborhoods of C with trivial normal bundle, Ueda index $k < \infty$ and Ueda form (a multiple of) ω modulo formal equivalence.

Theorem 4. *Let C be a curve of genus $g \geq 2$, $1 \leq k < \infty$ and ω a 1-form on C with simple zeroes. Then there is an injective map*

$$\Phi : \mathcal{V}(C, k, \omega) \hookrightarrow \widehat{\text{Diff}}(\mathbb{C}, 0)^g \times \widehat{\text{Diff}}(\mathbb{C}, 0)^g \times \widehat{\text{Diff}}(\mathbb{C}, 0)^{2g-3} / \sim$$

where the equivalence relation \sim is given by the action of $\text{Diff}(\mathbb{C}, 0)$ on $\text{Diff}(\mathbb{C}, 0)^N$ by conjugacy on each factor.

A tuple of diffeomorphisms $((\varphi_i^{(j)})_i)_{j=1}^3$ is in the image of Φ if and only if the $\varphi_i^{(j)}$ are tangent to the identity at order k and if they satisfy the compatibility conditions (3) and (4).

Proof. Fix a zero p_1 of ω , fix some loops $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ forming a symplectic basis of $H_1(C, \mathbb{C})$, fix some paths γ_{1j} between p_1 and p_j .

Let $[S] \in \mathcal{V}(C, k, \omega)$ and S be a representative of $[S]$. Let \mathcal{F} and \mathcal{G} be respectively the A -canonical and the B -canonical foliations on S . Let $\varphi_\tau^{\mathcal{F}}$ and $\varphi_\tau^{\mathcal{G}}$ be the holonomies of \mathcal{F} and \mathcal{G} along the loops $\tau = \alpha_1, \dots, \beta_g$; let $\varphi_{1j}^{\leftrightarrow} = (\varphi_{1j}^{\mathcal{G}})^{-1} \circ \varphi_{1j}^{\mathcal{F}}$ be computed along the path γ_{1j} . We put

$$\theta(S) = ((\varphi_{\beta_i}^{\mathcal{F}})_{i=1}^g, (\varphi_{\alpha_i}^{\mathcal{G}})_{i=1}^g, (\varphi_{1j}^{\leftrightarrow})_{j=2}^{2g-2}) \quad \text{and} \quad \Phi([S]) = [\theta(S)]$$

the class of $\theta(S)$ modulo common conjugacy.

Since a diffeomorphism ψ between two neighborhoods S and S' sends the A -canonical foliation \mathcal{F} of S to the A -canonical foliation \mathcal{F}' of S' (resp. the B -canonical foliations $\mathcal{G}, \mathcal{G}'$), ψ then sends the bifoliated neighborhood $(S, \mathcal{F}, \mathcal{G})$ to $(S', \mathcal{F}', \mathcal{G}')$. Thus $\theta(S)$ and $\theta(S')$ are conjugated, ie. $\Phi([S])$ is well-defined. Conversely, if $\Phi([S]) = \Phi([S'])$, then $(S, \mathcal{F}, \mathcal{G})$ is diffeomorphic to $(S', \mathcal{F}', \mathcal{G}')$ (and therefore S is diffeomorphic to S').

The realization part of the theorem is a direct consequence of theorem 2 (the relation (5) is trivial here). \square

Remark 1. About the realization of a tuple $((\varphi_i^{(j)})_i)_{j=1}^3$, remark that the conditions (3) and (4) only depend on the coefficients of $\varphi_i^{(j)}$ of order $k+1$. In this sense, we can say that the image $\Phi(\mathcal{V}(C, k, \omega))$ is of finite codimension.

5 Concluding remarks

5.1 About convergent foliations in S

In some cases, the canonical foliations do not converge even if the neighborhood is analytic. Indeed, if C is an elliptic curve, Mishustin gave in [10] an example of a neighborhood S of C with trivial normal bundle and no analytic foliations tangent to C .

We can use this example to build examples in higher genus: let p_1, p_2 be two points on C and T_1, T_2 two transversals at p_1 and p_2 . Consider the two-fold branched covering $\pi : S' \rightarrow S$ of S branching at T_1 and T_2 . Denote $C' = \pi^{-1}(C)$, α, β the A - and B -loops on C based at p_1 , and $\alpha_1, \alpha_2, \beta_1, \beta_2$ the preimages of α and β . They are the A - and B -loops on C' based at $\pi^{-1}(p_1)$. If \mathcal{F}, \mathcal{G} are the canonical foliations on S , denote \mathcal{F}' and \mathcal{G}' the preimages of \mathcal{F}, \mathcal{G} by π .

Then S' is an analytic neighborhood of the genus 2 curve C' , the canonical foliations of S' are \mathcal{F}' and \mathcal{G}' , and they do not converge.

Even with these examples, the question of the existence of an analytic neighborhood of a genus 2 curve without any convergent foliation is still open.

5.2 About analytic equivalence of neighborhoods

Let $S, S' \in \mathcal{V}(C, k, \omega)$ be two analytic neighborhoods such that the canonical foliations converge. Let $\theta = (\varphi_i), \theta' = (\varphi'_i), i = 1, \dots, 4g-3$ be the diffeomorphisms obtained in the construction, so that θ is a representative of $\Phi(S)$ and θ' is a representative of $\Phi(S')$. Consider the groups G, G' spanned by the φ_i (resp. φ'_i).

Suppose G is not abelian. Then if S and S' are formally diffeomorphic, there is a formal diffeomorphism ψ conjugating θ and θ' . This ψ realizes a conjugacy between G and G' so by Cerveau-Moussu's rigidity theorem [3], ψ is convergent. This in turn implies that θ and θ' are analytically conjugated, so that S and S' are analytically diffeomorphic. Note that since the diffeomorphisms φ_i are tangent to the identity, the group G is abelian only if the φ_i are flows of a same formal vector field [7].

This argument also works for non-canonical foliations: suppose that S and S' are analytic neighborhoods conjugated by a formal diffeomorphism ψ . Suppose that there is on S two convergent foliations \mathcal{F} and \mathcal{G} with tangency index $k \geq 1$ and 1-form ω with simple zeroes. Suppose that ψ sends \mathcal{F} and \mathcal{G} to convergent foliations \mathcal{F}' and \mathcal{G}' . Suppose finally that the group G spanned by the diffeomorphisms composing the invariant $Inv(S, \mathcal{F}, \mathcal{G})$ of theorem 2 is not abelian. Then ψ converges.

5.3 About degenerate cases

If the 1-form ω doesn't have simple zeroes, we can still obtain a classification of neighborhoods in $\mathcal{V}(C, k, \omega)$ by the same method. The problem is that in this case some non-trivial local invariants can arise. For genus $g = 2$ curves, the local situations which can be involved were classified in [12]. These local classifications can then be used to obtain a classification of bifoliated neighborhoods of genus 2 curves even in the degenerate cases (see [13]), which in turn could give a complete formal classification of neighborhoods of genus 2 curves with trivial normal bundle.

References

- [1] ARNOL'D, V. I. Bifurcations of invariant manifolds of differential equations, and normal forms of neighborhoods of elliptic curves. *Funktsional. Anal. i*

Prilozhen. 10 (1976), 1–12.

- [2] CAMACHO, C., AND MOVASATI, H. *Neighborhoods of analytic varieties*. Monografías del Instituto de Matemática y Ciencias Afines, 35. Pontificia Universidad Católica del Perú, Instituto de Matemática y Ciencias Afines, IMCA, Lima, 2003.
- [3] CERVEAU, D., AND MOUSSU, R. Groupes d’automorphismes de $(\mathbb{C}^n, 0)$ et équations différentielles $ydy + \dots = 0$. *Bulletin de la S.M.F. tome no. 4* (1988), 459–488.
- [4] CLAUDON, B., LORAY, F., PEREIRA, J. V., AND TOUZET, F. Compact leaves of codimension one holomorphic foliations on projective manifolds. to appear in *Ann. Scient. Éc. Norm. Sup.*, 2015.
- [5] GRAUERT, H. Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.* 146 (1962), 331–368.
- [6] ILYASHENKO, Y. S. Imbeddings of positive type of elliptic curves into complex surfaces. *Trudy Moskov. Mat. Obshch.* 45 (1982), 37–67.
- [7] LORAY, F. Pseudo-groupe d’une singularité de feuilletage holomorphe en dimension deux. <https://hal.archives-ouvertes.fr/hal-00016434>, 2006.
- [8] LORAY, F., THOM, O., AND TOUZET, F. Two dimensional neighborhoods of elliptic curves: formal classification and foliations. <https://hal.archives-ouvertes.fr/hal-01509317>, 2017.
- [9] MISHUSTIN, M. B. Neighborhoods of Riemann curves in complex surfaces. *Funct. Anal. Appl.* 29, no. 1 (1995), 20–31.
- [10] MISHUSTIN, M. B. On foliations in neighborhoods of elliptic curves. *Arnold Math. J.* 2, no. 2 (2016), 195–199.
- [11] NEEMAN, A. Ueda theory: theorems and problems. *Mem. Amer. Math. Soc.* 81, 415 (1989), vi+123.
- [12] THOM, O. Classification locale de bifeuilletages holomorphes sur les surfaces complexes. *Bull. Braz. Math. Soc. (N.S.)* 47, no. 4 (2016), 989–1005.
- [13] THOM, O. *Structures bifeuilletées en codimension 1*. PhD thesis, Université de Rennes 1, 2017. <https://www.theses.fr/2017REN1S064>.
- [14] UEDA, T. On the neighborhood of a compact complex curve with topologically trivial normal bundle. *J. Math. Kyoto Univ.* 22 (1982/83), 583–607.

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
Email: olivier.thom@univ-rennes1.fr