



**HAL**  
open science

## Slightly degenerate categories and $Z$ -modular data

Abel Lacabanne

► **To cite this version:**

Abel Lacabanne. Slightly degenerate categories and  $Z$ -modular data. International Mathematics Research Notices, 2019, 2021 (Issue 12), pp. 9340-9374. 10.1093/imrn/rnz105 . hal-01827846v3

**HAL Id: hal-01827846**

**<https://hal.science/hal-01827846v3>**

Submitted on 22 Oct 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

# SLIGHTLY DEGENERATE CATEGORIES AND $\mathbb{Z}$ -MODULAR DATA

by

Abel Lacabanne

---

An  $\mathbb{N}$ -modular datum over  $\mathbb{C}$  is a finite set  $I$ , a distinguished element  $i_0 \in I$ , a square matrix  $S \in \mathcal{M}_I(\mathbb{C})$ , a diagonal matrix  $T \in \mathcal{M}_I(\mathbb{C})$  such that

1. for any  $i \in I$ ,  $S_{i_0,i} \neq 0$ ,
2.  $S$  is unitary, symmetric and  $S^4 = \text{Id}$ ,  $(ST)^3 = \lambda \text{Id}$  and  $[S^2, T] = \text{Id}$ ,
3. for any  $i, j, k \in I$ ,

$$N_{i,j}^k = \sum_{l \in I} \frac{S_{i,l} S_{j,l} \overline{S_{k,l}}}{S_{i_0,l}}$$

belongs to  $\mathbb{N}$ .

Modular categories naturally give rise to modular data [EGNO15, Section 8.16] and given a modular datum, one can ask the question of finding a modular category with this modular datum. In [Lus94], Lusztig gives a slightly more restrictive definition of modular datum, and associate a modular datum to each dihedral group.

In order to generalize Lusztig's work for imprimitive complex reflection groups, Malle [Mal95] defines a modular datum, but such that the integers  $N_{i,j}^k$  are in  $\mathbb{Z}$ , which we will call  $\mathbb{Z}$ -modular datum. The question of finding a categorification of these data is much more complicated, as a modular category always defines an  $\mathbb{N}$ -modular datum: the integers  $N_{i,j}^k$  are the multiplicities of the object  $k$  in the tensor product  $i \otimes j$ .

In [BR17] Bonnafé and Rouquier gave a categorification of the Malle  $\mathbb{Z}$ -modular datum associated with cyclic groups, by constructing a tensor triangulated category with extra structure.

In this article, we explain how slightly degenerate categories [EGNO15, Definition 9.15.3] give rise to  $\mathbb{Z}$ -modular datum. This is also related to braided pivotal superfusion categories. Note that there exist two pivotal structures on supervector spaces, one of which is unitary (and therefore all simple objects have positive dimension), and one of which is not (the two simple objects are of dimension 1 and  $-1$ ). We will show that with the non-unitary structure, a slightly degenerated category gives rise to a  $\mathbb{Z}$ -modular datum.

As an application, we will reinterpret the example of Bonnafé and Rouquier in this setting of slightly degenerate categories. This approach will be generalized in [Lac18].

This paper is organized as follows. Section 1 extends well-known results on modular categories to pivotal braided fusion categories which are not necessarily spherical. Section 2 introduce slightly degenerate categories, and we explain how such a category gives rise to a  $\mathbb{Z}$ -modular datum. We introduce in Section 3 the notion of a supercategory, and

explain how to produce a supercategory from a slightly degenerate category with extra structure. Finally, Section 4 is devoted to the example of Bonnafé-Rouquier in the setting of slightly degenerate categories.

**Acknowledgements.** — I warmly thank my advisor C. Bonnafé for many fruitful discussions and his constant support. The paper is partially based upon work supported by the NSF under grant DMS-1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester.

## 1. Categorical preliminaries

Let  $\mathbb{k}$  be an algebraically closed field of characteristic 0 and  $\mathcal{C}$  be a tensor category over  $\mathbb{k}$ , as defined in [EGNO15, Definition 4.1.1]:  $\mathcal{C}$  is a locally finite  $\mathbb{k}$ -linear abelian rigid monoidal category (with unit object denoted by  $\mathbf{1}$ ) such that the bifunctor  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is  $\mathbb{k}$ -bilinear on morphisms and  $\text{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{k}$ . We denote by  $a_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$  the associativity constraint, but will often omit it. The left (resp. right) dual of an object  $X \in \mathcal{C}$  is denoted by  $X^*$  (resp.  ${}^*X$ ) with evaluation and coevaluation morphism

$$\text{ev}_X: X^* \otimes X \rightarrow \mathbf{1} \quad \text{and} \quad \text{coev}_X: \mathbf{1} \rightarrow X \otimes X^*$$

(resp.

$$\text{ev}'_X: X \otimes {}^*X \rightarrow \mathbf{1} \quad \text{and} \quad \text{coev}'_X: \mathbf{1} \rightarrow {}^*X \otimes X)$$

such that the following compositions are identities

$$X \xrightarrow{\text{coev}_X \otimes \text{id}_X} X \otimes X^* \otimes X \xrightarrow{\text{id}_X \otimes \text{ev}_X} X,$$

$$X^* \xrightarrow{\text{id}_{X^*} \otimes \text{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\text{ev}_X \otimes \text{id}_{X^*}} X^*.$$

One can define, for any  $f: X \rightarrow Y$  with  $X$  and  $Y$  having left duals, the left dual of  $f$  as the map  $f^*: Y^* \rightarrow X^*$  given by the composition

$$Y^* \xrightarrow{\text{id}_{Y^*} \otimes \text{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{\text{id}_{Y^*} \otimes f \otimes \text{id}_{X^*}} Y^* \otimes Y \otimes X^* \xrightarrow{\text{ev}_Y \otimes \text{id}_{X^*}} X^*,$$

and similarly there exists the right dual of a map.

We assume that  $\mathcal{C}$  is equipped with a pivotal structure [EGNO15, Definition 4.7.8]: there is a family of natural isomorphisms  $a_X: X \rightarrow X^{**}$  such that  $a_{X \otimes Y} = a_X \otimes a_Y$ . For  $f \in \text{Hom}_{\mathcal{C}}(X, X)$ , we can define two traces which are elements of  $\text{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{k}$ . The left quantum trace  $\text{Tr}_X^+(f)$  is given by the composition

$$\mathbf{1} \xrightarrow{\text{coev}_X} X \otimes X^* \xrightarrow{(a_X \circ f) \otimes \text{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\text{ev}_{X^*}} \mathbf{1},$$

and the right quantum trace  $\text{Tr}_X^-(f)$  is given by the composition

$$\mathbf{1} \xrightarrow{\text{coev}_{X^*}} X^* \otimes X^{**} \xrightarrow{\text{id}_{X^*} \otimes (f \circ a_X^{-1})} X^* \otimes X \xrightarrow{\text{ev}_X} \mathbf{1}.$$

It is well known that for any  $f \in \text{End}_{\mathcal{C}}(X)$ ,  $\text{Tr}_{X^*}^+(f^*) = \text{Tr}_X^-(f)$ . We also define the partial traces of  $f \in \text{End}_{\mathcal{C}}(X \otimes Y)$  by

$$\text{id}_X \otimes \text{Tr}_Y^+(f): X \xrightarrow{\text{coev}_Y} X \otimes Y \otimes Y^* \xrightarrow{((\text{id}_X \otimes a_Y) \circ f) \otimes \text{id}_{Y^*}} X \otimes Y^{**} \otimes Y \xrightarrow{\text{ev}_Y} X$$

and

$$\mathrm{Tr}_X^- \otimes \mathrm{id}_Y(f): Y \xrightarrow{\mathrm{coev}_{X^*}} X^* \otimes X^{**} \otimes Y \xrightarrow{\mathrm{id}_{X^*} \otimes (f \circ (a_X^{-1} \otimes \mathrm{id}_Y))} X^* \otimes X \otimes Y \xrightarrow{\mathrm{ev}_X} Y.$$

Finally, denote by  $\mathrm{Tr}_{X \otimes Y}^+(f)$  the endomorphism of  $\mathbf{1}$  given by  $\mathrm{Tr}_X^-(\mathrm{id}_X \otimes \mathrm{Tr}_Y^+(f))$  which is then equal to  $\mathrm{Tr}_Y^+(\mathrm{Tr}_X^- \otimes \mathrm{id}_Y(f))$ .

**Lemma 1.1.** — *Let  $\mathcal{C}$  be a pivotal rigid monoidal category,  $X$  and  $Y$  two objects of  $\mathcal{C}$  and  $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, X \otimes Y)$ . Then  $\mathrm{Tr}_{Y^* \otimes X^*}^-(f^*) = \mathrm{Tr}_{X \otimes Y}^+(f)$ .*

*Proof.* — In order to simplify notations, we omit tensor product signs. We denote by  $a$  the pivotal structure. By definition:

$$\begin{aligned} \mathrm{id}_{Y^*} \otimes \mathrm{Tr}_{X^*}^+(f) &= (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_{X^{**}}) \circ (\mathrm{id}_{Y^*} \otimes a_{X^*} \otimes \mathrm{id}_{X^{**}}) \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^* X^* X^{**}}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{Y^* X^* X^{**}}) \\ &\circ (\mathrm{id}_{Y^* X^*} \otimes f \otimes \mathrm{id}_{Y^* X^* X^{**}}) \circ (\mathrm{id}_{Y^* X^* X} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^* X^{**}}) \circ (\mathrm{id}_{Y^* X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_{X^{**}}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_{X^*}). \end{aligned}$$

Using the functoriality of the tensor product, we have:

$$\begin{aligned} \mathrm{id}_{Y^*} \otimes \mathrm{Tr}_{X^*}^+(f) &= (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{Y^* Y^*}) \circ (\mathrm{id}_{Y^* X^*} \otimes f \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^* X^* X} \otimes \mathrm{coev}_Y) \\ &\circ (\mathrm{id}_{Y^* X^*} \otimes ((\mathrm{id}_X \otimes \mathrm{ev}_{X^{**}}) \circ (\mathrm{id}_X \otimes a_{X^*} \otimes \mathrm{id}_{X^{**}}) \circ (\mathrm{coev}_X \otimes \mathrm{id}_{X^{**}}))) \circ \mathrm{id}_{Y^*} \otimes \mathrm{coev}_{X^*}. \end{aligned}$$

But  $a_{X^*} = (a_X^{-1})^*$  (c.f. [EGNO15, Exercise 4.7.9]) and therefore  $(\mathrm{id}_X \otimes a_{X^*}) \circ \mathrm{coev}_X = (a_X^{-1} \otimes \mathrm{id}_{X^{**}}) \circ \mathrm{coev}_{X^{**}}$ . Using the definition of the duality, one has  $(\mathrm{id}_X \otimes \mathrm{ev}_{X^{**}}) \circ (\mathrm{id}_X \otimes a_{X^*} \otimes \mathrm{id}_{X^{**}}) \circ (\mathrm{coev}_X \otimes \mathrm{id}_{X^{**}}) = a_X^{-1}$  and consequently

$$\mathrm{id}_{Y^*} \otimes \mathrm{Tr}_{X^*}^+(f) = (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_Y^* \otimes (\mathrm{Tr}_X^- \otimes \mathrm{id}_Y)(f) \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y) = [(\mathrm{Tr}_X^- \otimes \mathrm{id}_Y)(f)]^*.$$

Finally,

$$\mathrm{Tr}_{Y^* \otimes X^*}^-(f^*) = \mathrm{Tr}_{Y^*}^-([(\mathrm{Tr}_X^- \otimes \mathrm{id}_Y)(f)]^*) = \mathrm{Tr}_Y^+((\mathrm{Tr}_X^- \otimes \mathrm{id}_Y)(f)) = \mathrm{Tr}_{X \otimes Y}^+(f),$$

as expected.  $\square$

The *left* and *right quantum dimensions* are

$$\dim^+(X) = \mathrm{Tr}_X^+(\mathrm{id}_X) \quad \text{and} \quad \dim^-(X) = \mathrm{Tr}_X^-(\mathrm{id}_X)$$

which therefore satisfy  $\dim^+(X^*) = \dim^-(X)$ . Define the squared norm of an object  $X$  by

$$|X|^2 = \dim^+(X) \dim^-(X) = \dim^+(X) \dim^+(X^*).$$

It is a totally positive number if  $X$  is simple [EGNO15, Proposition 7.21.14]: for any embedding  $\iota$  of the subfield  $\mathbb{k}_{\mathrm{alg}}$  of algebraic elements of  $\mathbb{k}$  in  $\mathcal{C}$ , one has  $\iota(|X|^2) > 0$ . The dimension of the category  $\mathcal{C}$  is

$$\dim(\mathcal{C}) = \sum_{X \in \mathrm{Irr}(\mathcal{C})} |X|^2,$$

where  $\mathrm{Irr}(\mathcal{C})$  denotes the set of isomorphism classes of simple objects of  $\mathcal{C}$ .

We further assume that  $\mathcal{C}$  is braided: there exists a family of bifunctorial isomorphisms  $c_{X,Y}: X \otimes Y \rightarrow Y \otimes X$  such that the hexagon axioms are satisfied [EGNO15, Definition 8.1.1].

For a rigid braided tensor category, there exists a natural isomorphism  $u_X: X \rightarrow X^{**}$ , called the Drinfeld morphism, defined as the composition

$$X \xrightarrow{\mathrm{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*}} X^* \otimes X \otimes X^{**} \xrightarrow{\mathrm{ev}_X} X^{**}.$$

It satisfies for all  $X, Y \in \mathcal{C}$ ,

$$u_X \otimes u_Y = u_{X \otimes Y} \circ c_{Y,X} \circ c_{X,Y}.$$

To give a pivotal structure  $a$  on  $\mathcal{C}$  is therefore equivalent to give a *twist* on  $\mathcal{C}$ , which is a natural isomorphism  $\theta_X: X \rightarrow X$  satisfying for all  $X, Y \in \mathcal{C}$

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}.$$

The pivotal structure and the twist are related by  $a = u\theta$ . We will often endow the braided pivotal category  $\mathcal{C}$  with the twist given by the pivotal structure.

**1.1. Semi-simplification.** — We recall the procedure of semi-simplification for pivotal categories (which are not necessarily spherical) which is given in [EO18]. Let  $\mathcal{C}$  be a braided pivotal tensor category over  $\mathbb{k}$ . Denote by  $a_X: X \rightarrow X^{**}$  the pivotal structure.

A morphism  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  is said to be *left* (resp. *right*) *negligible* if for all  $g \in \text{Hom}_{\mathcal{C}}(Y, X)$  one has  $\text{Tr}_X^+(g \circ f) = 0$  (resp.  $\text{Tr}_X^-(g \circ f) = 0$ ). An application of [Bru00, Proposition 1.5.1] shows that the notions of left and right negligible morphisms coincide because  $\mathcal{C}$  is braided. Therefore the left quantum dimension of an object is zero if and only if its right quantum dimension is zero: the assumption (2) of [EO18, Theorem 2.6] is satisfied. We then denote by  $\text{Hom}_{\text{negl}}(X, Y)$  the subspace of negligible morphisms. Define a category  $\mathcal{C}^{\text{ss}}$  with the same objects as  $\mathcal{C}$  and  $\text{Hom}_{\mathcal{C}^{\text{ss}}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y) / \text{Hom}_{\text{negl}}(X, Y)$ .

**Proposition 1.2** ([EO18, Theorem 2.6]). — *Let  $\mathcal{C}$  be a braided pivotal tensor category. The category  $\mathcal{C}^{\text{ss}}$  is a semisimple braided pivotal tensor category whose simple objects are the indecomposable objects of  $\mathcal{C}$  with left quantum dimension 0.*

**1.2. S-matrices and symmetric center.** — The following definition is due to Müger [Müg03, Definition 2.9].

**Definition 1.3.** — *The symmetric center  $\mathcal{Z}_{\text{sym}}(\mathcal{C})$  of a braided monoidal category  $\mathcal{C}$  is the full subcategory of  $\mathcal{C}$  with objects  $X$  such that*

$$\forall Y \in \mathcal{C}, c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$

*We say that  $\mathcal{C}$  is non-degenerate if  $\mathbf{1}$  is the unique simple object in  $\mathcal{Z}_{\text{sym}}(\mathcal{C})$ .*

**Hypothesis and notations.** We suppose until the end of this Section that the category  $\mathcal{C}$  is a braided pivotal fusion category. Denote by  $\text{Irr}(\mathcal{C})$  the set of isomorphism classes of simple objects in  $\mathcal{C}$  and by  $\text{Gr}(\mathcal{C})$  its Grothendieck ring which admits  $([X])_{X \in \text{Irr}(\mathcal{C})}$  as a basis. For  $X, Y, Z \in \text{Irr}(\mathcal{C})$ , we denote by  $N_{X,Y}^Z$  the multiplicity of  $Z$  in the tensor product  $X \otimes Y$ . Then  $\text{Gr}(\mathcal{C})$  is a free  $\mathbb{Z}$ -algebra with basis  $\text{Irr}(\mathcal{C})$  and the structure constants are given by the positive integers  $N_{X,Y}^Z$ :

$$[X] \cdot [Y] = \sum_{Z \in \text{Irr}(\mathcal{C})} N_{X,Y}^Z [Z].$$

For any simple object  $X$ , its left and right quantum dimensions are non-zero [EGNO15, Proposition 4.8.4]. If  $X$  and  $Y$  are objects of  $\mathcal{C}$ , we set

$$s_{X,Y}^+ = (\text{id}_X \otimes \text{Tr}_Y^+)(c_{Y,X} \circ c_{X,Y}) \in \text{End}_{\mathcal{C}}(X)$$

and

$$s_{X,Y}^- = (\mathrm{Tr}_X^- \otimes \mathrm{id}_Y)(c_{Y,X} \circ c_{X,Y}) \in \mathrm{End}_{\mathcal{C}}(Y).$$

These induce two morphisms of abelian groups

$$s_X^+ : \begin{cases} \mathrm{Gr}(\mathcal{C}) & \longrightarrow & \mathrm{End}_{\mathcal{C}}(X) \\ Y & \longmapsto & s_{X,Y}^+ \end{cases} \quad \text{and} \quad s_Y^- : \begin{cases} \mathrm{Gr}(\mathcal{C}) & \longrightarrow & \mathrm{End}_{\mathcal{C}}(Y) \\ X & \longmapsto & s_{X,Y}^- \end{cases}.$$

**Proposition 1.4** ([EGNO15, Proposition 8.3.11]). — *Let  $\mathcal{C}$  be a braided pivotal fusion category. If  $X \in \mathcal{C}$  is simple then  $s_X^+ : \mathrm{Gr}(\mathcal{C}) \rightarrow \mathbb{k}$  and  $s_X^- : \mathrm{Gr}(\mathcal{C}) \rightarrow \mathbb{k}$  are morphisms of rings.*

We now consider the matrices  $S^{++}$ ,  $S^{-+}$  and  $S^{--}$  in  $\mathrm{Mat}_{\mathrm{Irr}(\mathcal{C})}(\mathbb{k})$  defined by

$$S_{X,Y}^{++} = \mathrm{Tr}_{X \otimes Y}^+(c_{Y,X} \circ c_{X,Y}) = \mathrm{Tr}_X^+(s_{X,Y}^+),$$

$$S_{X,Y}^{--} = \mathrm{Tr}_{X \otimes Y}^-(c_{Y,X} \circ c_{X,Y}) = \mathrm{Tr}_Y^-(s_{X,Y}^-),$$

$$S_{X,Y}^{-+} = \mathrm{Tr}_Y^+(s_{X,Y}^-) = \mathrm{Tr}_X^-(s_{X,Y}^+).$$

These three matrices are related as follow

$$\frac{\dim^-(X)}{\dim^+(X)} S_{X,Y}^{++} = S_{X,Y}^{-+} = \frac{\dim^+(Y)}{\dim^-(Y)} S_{X,Y}^{--}.$$

**Remark.** — The matrices  $S^{++}$  and  $S^{--}$  are symmetric but  $S^{-+}$  is not in general; if the pivotal structure is spherical, these three matrices are equal.

As for any  $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$  we have  $\mathrm{Tr}_X^+(f) = \mathrm{Tr}_{X^*}^-(f^*)$ , the following relations are satisfied:

$$\begin{aligned} S_{X^*,Y^*}^{++} &= S_{X,Y}^{--} \\ S_{X^*,Y^*}^{-+} &= S_{Y,X}^{-+}. \end{aligned}$$

Hence the matrix  $S = (S_{X,Y^*}^{-+})_{X,Y \in \mathrm{Irr}(\mathcal{C})}$  is symmetric.

It is clear that if  $X \in \mathrm{Irr}(\mathcal{Z}_{\mathrm{sym}}(\mathcal{C}))$  then for all  $Y \in \mathrm{Irr}(\mathcal{C})$  we have  $S_{X,Y}^{?,?'} = \dim^?(X) \dim^{?'}(Y)$ , for  $(?, ?') \in \{(+, +), (-, +), (-, -)\}$ .

**Proposition 1.5** ([EGNO15, Proposition 8.20.5]). — *Let  $\mathcal{C}$  be a braided pivotal fusion category and  $X$  be a simple object in  $\mathcal{C}$ . The following are equivalent:*

1.  $X \in \mathcal{Z}_{\mathrm{sym}}(\mathcal{C})$ ,
2. for all  $Y \in \mathrm{Irr}(\mathcal{C})$  we have  $S_{X,Y}^{-+} = \dim^-(X) \dim^+(Y)$ ,
3. for all  $Y \in \mathrm{Irr}(\mathcal{C})$  we have  $S_{X,Y}^{++} = \dim^+(X) \dim^+(Y)$ ,
4. for all  $Y \in \mathrm{Irr}(\mathcal{C})$  we have  $S_{X,Y}^{--} = \dim^-(X) \dim^-(Y)$ .

The category  $\mathcal{C}$  can be endowed with another braiding, namely the reverse braiding. We denote it by  $c^{\mathrm{rev}}$  and it is defined by  $c_{X,Y}^{\mathrm{rev}} = c_{Y,X}^{-1}$ . We denote by  $\mathcal{C}^{\mathrm{rev}}$  the category  $\mathcal{C}$  equipped with the reverse braiding, and by  $S^{\mathrm{rev},++}$ ,  $S^{\mathrm{rev},-+}$  and  $S^{\mathrm{rev},--}$  the corresponding  $S$ -matrices. Note that we use the same pivotal structure on  $\mathcal{C}$  and  $\mathcal{C}^{\mathrm{rev}}$  for the computation of the traces.

**Proposition 1.6.** — *Let  $\mathcal{C}$  a braided pivotal fusion category. Then for any  $X$  and  $Y$  simple objects we have*

$$S_{X,Y}^{\mathrm{rev},-+} = S_{Y,X^*}^{++}.$$

*Proof.* — We start with a lemma, which is a direct consequence of [EGNO15, Proposition 8.9.1].

**Lemma 1.7.** — *Let  $\mathcal{C}$  be a braided rigid tensor category. Then for every  $X$  and  $Y$  objects in  $\mathcal{C}$  we have:*

1.  $(\text{ev}_X \otimes \text{id}_Y) \circ (\text{id}_{X^*} \otimes c_{Y,X}) = (\text{id}_Y \otimes \text{ev}_X) \circ (c_{Y,X^*}^{-1} \otimes \text{id}_X)$ ,
2.  $(c_{X,Y} \otimes \text{id}_{Y^*}) \circ (\text{id}_X \otimes \text{coev}_Y) = (\text{id}_Y \otimes c_{X,Y^*}^{-1}) \circ (\text{coev}_Y \otimes \text{id}_X)$ .

Now, by definition

$$\begin{aligned} S_{X,Y}^{\text{rev},-+} &= (\text{ev}_X \otimes \text{ev}_{Y^*}) \circ (\text{id}_{X^* \otimes X} \otimes a_Y \otimes \text{id}_{Y^*}) \circ (\text{id}_{X^*} \otimes c_{X,Y}^{-1} \otimes \text{id}_{Y^*}) \\ &\quad \circ (\text{id}_{X^*} \otimes c_{Y,X}^{-1} \otimes \text{id}_{Y^*}) \circ (\text{id}_{X^*} \otimes a_X^{-1} \otimes \text{id}_{Y \otimes Y^*}) \circ (\text{coev}_{X^*} \otimes \text{coev}_Y). \end{aligned}$$

By naturality of the braiding and using Lemma 1.7 we have

$$\begin{aligned} (\text{ev}_X \otimes \text{id}_{Y^*}) \circ (\text{id}_{X^* \otimes X} \otimes a_Y) \circ (\text{id}_{X^*} \otimes c_{X,Y}^{-1}) &= (\text{ev}_X \otimes \text{id}_{Y^*}) \circ (\text{id}_{X^*} \otimes c_{X,Y^*}^{-1}) \circ (\text{id}_{X^*} \otimes a_Y \otimes \text{id}_X) \\ &= (\text{id}_{Y^*} \otimes \text{ev}_X) \circ (c_{X^*,Y^*} \otimes \text{id}_X) \circ (\text{id}_{X^*} \otimes a_Y \otimes \text{id}_X). \end{aligned}$$

Similarly,

$$(\text{id}_{X^*} \otimes c_{Y,X}^{-1}) \circ (\text{id}_{X^*} \otimes a_X^{-1} \otimes \text{id}_Y) \circ (\text{coev}_{X^*} \otimes \text{id}_Y) = (\text{id}_{X^* \otimes Y} \otimes a_X^{-1}) \circ (c_{Y,X^*} \otimes \text{id}_{X^*}) \circ (\text{id}_Y \otimes \text{coev}_{X^*}).$$

Therefore

$$\begin{aligned} S_{X,Y}^{\text{rev},-+} &= \text{ev}_{X \otimes Y^*} \circ (c_{X^*,Y^*} \otimes \text{id}_{X \otimes Y^*}) \circ (\text{id}_{X^*} \otimes a_Y \otimes a_X^{-1} \otimes \text{id}_{Y^*}) \circ (c_{Y,X^*} \otimes \text{id}_{X^* \otimes Y^*}) \circ \text{coev}_{Y \otimes X^*} \\ &= \text{ev}_{X \otimes Y^*} \circ (a_Y \otimes \text{id}_X^* \otimes a_X^{-1} \otimes \text{id}_{Y^*}) \circ (c_{X^*,Y} \otimes \text{id}_{X^* \otimes Y^*}) \circ (c_{Y,X^*} \otimes \text{id}_{X^* \otimes Y^*}) \circ \text{coev}_{Y \otimes X^*}. \end{aligned}$$

Finally, using that  $(a_X^{-1})^* = a_{X^*}$  and that for any  $f: W \rightarrow Z$  we have  $\text{ev}_W \circ (\text{id}_W \otimes f^*) = \text{ev}_Z \circ (f \otimes \text{id}_{Z^*})$ , we obtain  $S_{X,Y}^{\text{rev},-+} = S_{Y,X^*}^{++}$ .  $\square$

**1.3. Twists and Gauss sums.** — We now suppose that the category  $\mathcal{C}$  is equipped with a twist  $\tilde{\theta}$ , the twist associated to the pivotal structure is denoted by  $\theta$ . On a simple object  $X$ , the twist is the multiplication by a scalar, and we will identify  $\tilde{\theta}_X$  with this scalar.

**Proposition 1.8.** — *Let  $\mathcal{C}$  be a braided pivotal fusion category. We consider  $\theta$  the twist associated to the pivotal structure. For any simple object  $X$ , we have  $\theta_{X^*} \dim^+(X) = \dim^-(X) \theta_X$ .*

*Proof.* — Similarly to [EGNO15, Proposition 8.10.14], we have  $\dim^+(X) = \theta_X \text{ev}_X \circ c_{X,X^*} \circ \text{coev}_X$ . We show that  $\text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1}) = \text{ev}_X \circ c_{X,X^*} \circ \text{coev}_X$ . By definition,

$$\begin{aligned} \text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1}) &= (\text{ev}_X \otimes \text{ev}_{X^*}) \circ (\text{id}_{X^* \otimes X} \otimes a_X \otimes \text{id}_{X^*}) \circ (\text{id}_{X^*} \otimes c_{X,X}^{-1} \otimes \text{id}_{X^*}) \\ &\quad \circ (\text{id}_{X^*} \otimes a_X^{-1} \otimes \text{id}_{X \otimes X^*}) \circ (\text{coev}_{X^*} \otimes \text{coev}_X). \end{aligned}$$

Using the naturality of the braiding, we get rid of the pivotal structure:

$$\text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1}) = (\text{ev}_X \otimes \text{ev}_{X^*}) \circ (\text{id}_{X^*} \otimes c_{X,X^*}^{-1} \otimes \text{id}_{X^*}) \circ (\text{coev}_{X^*} \otimes \text{coev}_X).$$

From the Lemma 1.7 we deduce that  $(\text{id}_X \otimes \text{ev}_{X^*}) \circ (c_{X,X^*}^{-1} \otimes \text{id}_{X^*}) = (\text{ev}_{X^*} \otimes \text{id}_X) \circ (\text{id}_{X^*} \otimes c_{X,X^*})$ . Therefore, using the properties of the duality, we obtain  $\text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1}) = \text{ev}_X \circ c_{X,X^*} \circ \text{coev}_X$ , which leads to the conclusion that  $\dim^+(X) = \theta_X \text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1})$ .

But the Lemma 1.1 shows that  $\text{Tr}_{X^* \otimes X^*}^+ ((c_{X,X}^{-1})^*) = \text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1})$  and as  $(c_{X,X})^* = c_{X^*,X^*}$  (c.f. [EGNO15, Exercise 8.9.2]) we finally obtain

$$\theta_X^{-1} \dim^+(X) = \text{Tr}_{X \otimes X}^+ (c_{X,X}^{-1}) = \text{Tr}_{X^* \otimes X^*}^+ (c_{X^*,X^*}^{-1}) = \theta_{X^*}^{-1} \dim^+(X^*),$$

as expected.  $\square$

Therefore, for a simple object  $X$ , its positive and negative quantum dimensions are equal if and only if  $\theta_{X^*} = \theta_X$ .

**Corollary 1.9.** — *Let  $\mathcal{C}$  be a braided pivotal fusion category. The pivotal structure is spherical if and only if the associated twist  $\theta$  is a ribbon, that is satisfy  $\theta_{X^*} = (\theta_X)^*$  for any object  $X$ .*

*Proof.* — The category being semisimple, for an object  $X = \bigoplus_{Z \in \text{Irr}(\mathcal{C})} Z^{\oplus n_Z}$ , we have  $\dim^\pm(X) = \sum_{Z \in \text{Irr}(\mathcal{C})} n_Z \dim^\pm(Z)$ ,  $\theta_{X^*} = \bigoplus_{Z \in \text{Irr}(\mathcal{C})} (\theta_{Z^*} \text{id}_{Z^*})^{\oplus n_Z}$  and  $(\theta_X)^* = \bigoplus_{Z \in \text{Irr}(\mathcal{C})} (\theta_Z \text{id}_{Z^*})^{\oplus n_Z}$ . The result follows immediately from the Proposition 1.8.  $\square$

**Definition 1.10.** — *Let  $\mathcal{C}$  be a braided pivotal fusion category equipped with a twist  $\tilde{\theta}$ . The Gauss sums of the category  $\mathcal{C}$  are*

$$\tau^\pm(\mathcal{C}, \tilde{\theta}) = \sum_{X \in \text{Irr}(\mathcal{C})} \tilde{\theta}_X^\pm |X|^2.$$

*If the twist  $\tilde{\theta}$  is the one coming from the pivotal structure, we simply denote these sums by  $\tau^\pm(\mathcal{C})$ .*

The relation  $\tilde{\theta}_{X \otimes Y} = (\tilde{\theta}_X \otimes \tilde{\theta}_Y) \circ c_{Y,X} \circ c_{X,Y}$  gives, by taking the positive quantum trace,

$$(1.1) \quad \tilde{\theta}_X \tilde{\theta}_Y S_{X,Y}^{++} = \sum_{Z \in \text{Irr}(\mathcal{C})} N_{X,Y}^Z \dim^+(Z) \tilde{\theta}_Z,$$

for  $X$  and  $Y$  simple objects of  $\mathcal{C}$ .

**Lemma 1.11.** — *Let  $Y$  be a simple object of a braided pivotal fusion category  $\mathcal{C}$  equipped with a twist  $\tilde{\theta}$ . Then*

$$(1.2) \quad \sum_{X \in \text{Irr}(\mathcal{C})} \tilde{\theta}_X \dim^-(X) S_{X,Y}^{++} = \tilde{\theta}_Y^{-1} \dim^+(Y) \tau^+(\mathcal{C}, \tilde{\theta}).$$

*Proof.* — The proof is essentially the same as the one of [EGNO15, Lemma 8.15.2]. Using (1.1), we have

$$\begin{aligned} \sum_{X \in \text{Irr}(\mathcal{C})} \tilde{\theta}_X \dim^-(X) S_{X,Y}^{++} &= \tilde{\theta}_Y^{-1} \sum_{X,Z \in \text{Irr}(\mathcal{C})} N_{X,Y}^Z \dim^+(Z) \dim^-(X) \tilde{\theta}_Z \\ &= \tilde{\theta}_Y^{-1} \sum_{Z \in \text{Irr}(\mathcal{C})} \dim^+(Z) \tilde{\theta}_Z \sum_{X \in \text{Irr}(\mathcal{C})} N_{Z^*,Y}^{X^*} \dim^+(X^*) \\ &= \tilde{\theta}_Y^{-1} \dim^+(Y) \sum_{Z \in \text{Irr}(\mathcal{C})} \tilde{\theta}_Z |Z|^2. \end{aligned}$$

$\square$

We have a similar formula for  $\theta^{-1}$  using simultaneously the Propositions 1.6 and 1.8.

**Lemma 1.12.** — *Let  $Y$  be a simple object of a braided pivotal fusion category  $\mathcal{C}$  equipped with the twist  $\theta$  associated to the pivotal structure. Then*

$$(1.3) \quad \sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{-1} \dim^+(X) S_{X,Y}^{++} = \theta_Y \dim^+(Y) \tau^-(\mathcal{C}).$$

*Proof.* — Using the fact that  $\theta^{\text{rev}} = \theta^{-1}$  is a twist for the category  $\mathcal{C}^{\text{rev}}$ , we deduce from Lemma 1.11 that

$$\sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{\text{rev}} \dim^-(X) S_{X,Y}^{\text{rev},++} = (\theta_Y^{\text{rev}})^{-1} \dim^+(Y) \tau^+(\mathcal{C}^{\text{rev}}, \theta^{\text{rev}}).$$



From the proposition 1.6, one has  $\dim^-(X)S_{X,Y}^{\text{rev},++} = \dim^+(X)S_{X^*,Y}^{++}$  and as  $\mathcal{C}$  and  $\mathcal{C}^{\text{rev}}$  have the same simple objects, we have  $\tau^+(\mathcal{C}^{\text{rev}}, \theta^{\text{rev}}) = \tau^-(\mathcal{C})$ . Therefore

$$\begin{aligned} \theta_Y \dim^+(Y) \tau^-(\mathcal{C}) &= \sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{-1} \dim^-(X) S_{X,Y}^{\text{rev},++} \\ &= \sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{-1} \dim^+(X) S_{X^*,Y}^{++} \\ &= \sum_{X \in \text{Irr}(\mathcal{C})} \theta_{X^*}^{-1} \dim^-(X) S_{X^*,Y}^{++}, \end{aligned}$$

the last equality being the Proposition 1.8. As  $X \mapsto X^*$  is a bijection of  $\text{Irr}(\mathcal{C})$ , we conclude using  $\dim^-(X) = \dim^+(X^*)$ .  $\square$

**1.4. Non-degenerate pivotal categories.** — It is well known that a modular category gives rise to a projective representation of  $SL_2(\mathbb{Z})$ . We aim to generalize this result to categories with a pivotal structure which is not spherical.

**Hypothesis.** *In this section, we suppose that the category  $\mathcal{C}$  is a non-degenerate braided pivotal fusion category.*

All the characters of the ring  $\text{Gr}(\mathcal{C})$  are then of the form  $s_X^+$  for  $X$  a simple object of  $\mathcal{C}$ . The map  $Y \mapsto s_X^+(Y^*)$  is a character of  $\text{Gr}(\mathcal{C})$  hence equal to  $s_{\bar{X}}^+$  for some  $\bar{X} \in \text{Irr}(\mathcal{C})$ . This defines an involution  $\bar{\phantom{x}}$  on  $\text{Irr}(\mathcal{C})$ . Note that if the pivotal structure is spherical, this involution is nothing more than the duality.

**Proposition 1.13.** — *The object  $\bar{\mathbf{1}}$  is invertible and  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}}$ .*

*Proof.* — Let  $X$  be a simple object. We compute  $s_X^+(\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*)$ :

$$\begin{aligned} s_X^+(\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*) &= s_X^+(\bar{\mathbf{1}}) s_X^+(\bar{\mathbf{1}}^*) = \frac{S_{X,\bar{\mathbf{1}}}^{++} S_{X,\bar{\mathbf{1}}^*}^{++}}{\dim^+(X)^2} \\ &= \frac{\dim^+(\bar{\mathbf{1}})}{\dim^+(X)^2} s_{\bar{\mathbf{1}}}^+(X) S_{X^*,\bar{\mathbf{1}}}^- \\ &= \frac{\dim^+(\bar{\mathbf{1}})}{\dim^+(X)^2} s_{\bar{\mathbf{1}}}^+(X^*) \frac{\dim^-(X^*) \dim^-(\bar{\mathbf{1}})}{\dim^+(X^*) \dim^+(\bar{\mathbf{1}})} S_{X^*,\bar{\mathbf{1}}}^{++} \\ &= \frac{\dim^+(\bar{\mathbf{1}}) \dim^-(\bar{\mathbf{1}})}{\dim^+(X)} s_{\bar{\mathbf{1}}}^+(X^*) \\ &= \dim^+(\bar{\mathbf{1}}) \dim^-(\bar{\mathbf{1}}). \end{aligned}$$

The element  $[\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*] - |\bar{\mathbf{1}}|^2 [\mathbf{1}]$  is then killed by any character of  $\mathbb{k} \otimes_{\mathbb{Z}} \text{Gr}(\mathcal{C})$ . In  $\mathbb{k} \otimes_{\mathbb{Z}} \text{Gr}(\mathcal{C})$  we have  $[\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*] = \sum_{X \in \text{Irr}(\mathcal{C})} N_{\bar{\mathbf{1}}, \bar{\mathbf{1}}^*}^X [X]$  which implies that  $N_{\bar{\mathbf{1}}, \bar{\mathbf{1}}^*}^X = 0$  for all  $X$  simple different from  $\mathbf{1}$  and that  $N_{\bar{\mathbf{1}}, \bar{\mathbf{1}}^*}^{\mathbf{1}} = |\bar{\mathbf{1}}|^2$  since  $\text{Irr}(\mathcal{C})$  is a basis of the  $\mathbb{k}$ -vector space  $\mathbb{k} \otimes_{\mathbb{Z}} \text{Gr}(\mathcal{C})$ . But  $N_{X, X^*}^{\mathbf{1}} = 1$  for any simple object  $X$  and we can conclude that  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^* \simeq \mathbf{1}$ , hence  $\bar{\mathbf{1}}$  is invertible.

Now, as  $\bar{\mathbf{1}}$  is invertible,  $X \otimes \bar{\mathbf{1}}$  is simple for any simple object  $X$ . Showing that  $s_{X^* \otimes \bar{\mathbf{1}}}^+(Y) = s_X^+(Y^*)$  for any simple object  $Y$  ends the proof:

$$\begin{aligned} s_{X^* \otimes \bar{\mathbf{1}}}^+(Y) &= \frac{\dim^+(Y)}{\dim^+(X^* \otimes \bar{\mathbf{1}})} s_Y^+(X^* \otimes \bar{\mathbf{1}}) = \frac{\dim^+(Y)}{\dim^+(X^* \otimes \bar{\mathbf{1}})} s_Y^+(X^*) s_Y^+(\bar{\mathbf{1}}) \\ &= \frac{S_{Y, X^*}^{++}}{\dim^+(X^*) \dim^+(Y)} s_{\bar{\mathbf{1}}}^+(Y) \\ &= \frac{S_{Y^*, X}^-}{\dim^+(X^*) \dim^+(Y)} \dim^+(Y^*) \\ &= \frac{S_{Y^*, X}^{++}}{\dim^+(X)} \\ &= s_X^+(Y^*). \end{aligned}$$

□

**Corollary 1.14.** — *Under the same hypothesis, for any simple objects  $X$  and  $Y$  we have  $S_{X, Y}^{++} = S_{X, \bar{Y}}^{++}$ .*

*Proof.* — By definition of  $s_{\bar{X}}^+$ , we have  $S_{\bar{X}, Y}^{++} = \dim^+(\bar{X}) s_{\bar{X}}^+(Y)$ . But  $\dim^+(\bar{X}) = \dim^+(\bar{\mathbf{1}}) \dim^+(X^*)$  and  $\dim^+ = s_{\bar{\mathbf{1}}}^+$  so that

$$\dim^+(\bar{\mathbf{1}}) \dim^+(X^*) = \dim^+(\bar{\mathbf{1}}) s_{\bar{\mathbf{1}}}^+(X) = S_{\bar{\mathbf{1}}, X}^{++} = \dim^+(X) s_X^+(\bar{\mathbf{1}}).$$

Hence we have  $S_{\bar{X}, Y}^{++} = \dim^+(X) s_X^+(Y^* \otimes \bar{\mathbf{1}})$  which leads to the conclusion. □

If the pivotal structure is spherical, the square of the  $S$ -matrix is well known: up to a scalar multiple, it is the permutation matrix given by the duality on simple objects (see [EGNO15, 8.14] for further details). Let  $E$  be the square matrix such that  $E_{X, Y} = \delta_{X, \bar{Y}}$ .

**Proposition 1.15.** — *Let  $\mathcal{C}$  be a non-degenerate braided pivotal fusion category. Then  $(S^{++})^2 = \dim(\mathcal{C}) \dim^+(\bar{\mathbf{1}}) E$ .*

*Proof.* — Since  $\mathcal{C}$  is non-degenerate, for  $X, Y \in \text{Irr}(\mathcal{C})$ , the equality  $s_X^+ = s_Y^+$  as characters of  $\text{Gr}(\mathcal{C})$  holds if and only if  $X = Y$ .

Suppose  $Y \neq \bar{Z}$ . We have, thanks to the orthogonality of characters [EGNO15, Lemma 8.14.1],

$$\sum_{X \in \text{Irr}(\mathcal{C})} S_{Y, X}^{++} S_{X, Z}^{++} = \dim^+(Y) \dim^+(Z) \sum_{X \in \text{Irr}(\mathcal{C})} s_Y^+(X) s_Z^+(X^*) = 0.$$

It remains to compute  $(S^{++})_{Y, \bar{Y}}^2$ :

$$\begin{aligned} \sum_{X \in \text{Irr}(\mathcal{C})} S_{Y, X}^{++} S_{X, \bar{Y}}^{++} &= \sum_{X, W \in \text{Irr}(\mathcal{C})} N_{Y, \bar{Y}}^W \dim^+(X) S_{X, W}^{++} \\ &= \sum_{W \in \text{Irr}(\mathcal{C})} \dim^+(W) N_{Y, \bar{Y}}^W \sum_{X \in \text{Irr}(\mathcal{C})} \dim^+(X) s_W^+(X). \end{aligned}$$

As  $\dim^+(X) = s_{\bar{\mathbf{1}}}^+(X) = s_{\bar{\mathbf{1}}}^+(X^*)$ , the second sum is zero unless  $W = \bar{\mathbf{1}}$  and is equal to

$$\sum_{X \in \text{Irr}(\mathcal{C})} \dim^+(X) s_{\bar{\mathbf{1}}}^+(X) = \sum_{X \in \text{Irr}(\mathcal{C})} \dim^+(X) \dim^-(X) = \dim(\mathcal{C}).$$

Moreover, as  $\bar{Y} \simeq Y^* \otimes \bar{\mathbf{1}}$ , we have  $N_{Y, \bar{Y}}^{\bar{\mathbf{1}}} = 1$  and  $(S^{++})_{Y, \bar{Y}}^2 = \dim^+(\bar{\mathbf{1}}) \dim(\mathcal{C})$ . □

**Corollary 1.16 (Verlinde formula).** — Let  $\mathcal{C}$  be a non-degenerate braided pivotal fusion category and  $X, Y, Z \in \text{Irr}(\mathcal{C})$ . The structure constants of  $\text{Gr}(\mathcal{C})$  are given by

$$N_{X,Y}^Z = \frac{1}{\dim(\mathcal{C})\dim^+(\bar{\mathbf{1}})} \sum_{W \in \text{Irr}(\mathcal{C})} \frac{S_{W,X}^{++} S_{W,Y}^{++} S_{W,Z}^{++}}{\dim^+(W)}.$$

Recall that giving a pivotal structure on a braided monoidal category is equivalent to endowing the category with a twist.

**Lemma 1.17.** — Let  $\mathcal{C}$  be a non-degenerate braided pivotal fusion category. Then for  $X$  simple,  $\theta_{\bar{X}} = \theta_{\bar{\mathbf{1}}}\theta_X$ .

*Proof.* — Taking the positive quantum trace of the morphism  $\theta_{X^* \otimes \bar{\mathbf{1}}} = \theta_{X^*} \otimes \theta_{\bar{\mathbf{1}}} \circ c_{\bar{\mathbf{1}}, X^*} \circ c_{X^*, \bar{\mathbf{1}}}$  we obtain

$$\begin{aligned} \theta_{\bar{X}} \dim^+(\bar{X}) &= \theta_{X^*} \theta_{\bar{\mathbf{1}}} S_{X^*, \bar{\mathbf{1}}}^{++} = \theta_{X^*} \theta_{\bar{\mathbf{1}}} \dim^+(\bar{\mathbf{1}}) s_{\bar{\mathbf{1}}}^+(X^*) \\ &= \theta_{X^*} \theta_{\bar{\mathbf{1}}} \dim^+(\bar{\mathbf{1}}) \dim^+(X). \end{aligned}$$

The equality  $\theta_{\bar{X}} = \theta_{\bar{\mathbf{1}}}\theta_X$  follows then immediately from the fact that  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}}$  and from the Proposition 1.8.  $\square$

**Remark.** — Taking for  $X$  the simple object  $\bar{\mathbf{1}}$ , we find that  $\theta_{\bar{\mathbf{1}}}^2 = 1$ .

As in the case of a spherical category [EGNO15, Proposition 8.15.4], the Gauss sums satisfy  $\tau^+(\mathcal{C})\tau^-(\mathcal{C}) = \dim(\mathcal{C})$  and hence are non-zero.

**Proposition 1.18.** — Let  $\mathcal{C}$  be a non-degenerate braided pivotal fusion category. Then  $\theta_{\bar{\mathbf{1}}} = 1$ , where  $\theta$  is the twist associated to the pivotal structure. Therefore for any simple object  $X$ , one has  $\theta_{\bar{X}} = \theta_X$ .

*Proof.* — Using the fact that  $\mathcal{C}$  is non-degenerate, we have, as in [EGNO15, Corollary 8.15.5]

$$\sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{-1} \dim^+(X) S_{X,Y}^{++} = \theta_{\bar{Y}} \dim^-(\bar{Y}) \tau^-(\mathcal{C}) \dim^+(\bar{\mathbf{1}}).$$

As  $\bar{Y} \simeq Y^* \otimes \bar{\mathbf{1}}$ ,  $\theta_{\bar{Y}} = \theta_{\bar{\mathbf{1}}}\theta_Y$  and  $|\mathbf{1}|^2 = 1$  we have

$$\sum_{X \in \text{Irr}(\mathcal{C})} \theta_X^{-1} \dim^+(X) S_{X,Y}^{++} = \theta_{\bar{\mathbf{1}}}\theta_Y \dim^+(Y) \tau^-(\mathcal{C}).$$

This equality for  $Y = \mathbf{1}$ , together with Lemma 1.12, show that  $\theta_{\bar{\mathbf{1}}} = 1$ .  $\square$

The modular group is  $SL_2(\mathbb{Z})$  and has presentation

$$\langle \mathfrak{s}, \mathfrak{t} \mid \mathfrak{s}^4 = 1, (\mathfrak{st})^3 = \mathfrak{s}^2 \rangle$$

by choosing

$$\mathfrak{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We now choose an embedding  $\mathbb{k}_{\text{alg}} \rightarrow \mathcal{C}$ . The categorical dimension of  $\mathcal{C}$  being a totally positive number [EGNO15, Theorem 7.21.12], we denote its positive square root for the chosen embedding by  $\sqrt{\dim(\mathcal{C})}$ . We moreover choose a square root  $\sqrt{\dim^+(\bar{\mathbf{1}})}$  of  $\dim^+(\bar{\mathbf{1}})$ . The  $T$ -matrix of  $\mathcal{C}$  is the diagonal matrix given by the action of the inverse of the twist  $\theta$  on simple objects. We have the non-spherical analogue of [EGNO15, Theorem 8.16.1] :

**Theorem 1.19.** — *Let  $\mathcal{C}$  be a non-degenerate braided pivotal fusion category. We have  $(S^{++}T)^3 = \tau^-(\mathcal{C})(S^{++})^2$  and  $(S^{++})^4 = (\dim(\mathcal{C})\dim^+(\bar{\mathbf{1}}))^2 \text{id}$ . Therefore*

$$\mathfrak{s} \mapsto \frac{1}{\sqrt{\dim^+(\bar{\mathbf{1}})}\sqrt{\dim(\mathcal{C})}}S^{++} \quad \text{and} \quad \mathfrak{t} \mapsto T$$

define a projective representation of  $SL_2(\mathbb{Z})$ .

We can show that that  $S^{++}$  and  $T$  satisfy  $(S^{++}T^{-1})^3 = \tau^+(\mathcal{C})\dim(\mathcal{C})\dim^+(\bar{\mathbf{1}})^2 \text{id}$ . Indeed, we have

$$S^{++}TS^{++} = \tau^-(\mathcal{C})T^{-1}S^{++}T^{-1}.$$

Multiplying by  $S^{++}$  on both sides, using Proposition 1.15 and the fact that  $\theta_{\bar{X}} = \theta_X$ , we have

$$\dim(\mathcal{C})^2 \dim^+(\bar{\mathbf{1}})^2 T = \tau^-(\mathcal{C})S^{++}T^{-1}S^{++}T^{-1}S^{++},$$

which gives  $(S^{++}T^{-1})^3 = \tau^+(\mathcal{C})\dim(\mathcal{C})\dim^+(\bar{\mathbf{1}})^2 \text{id}$  since  $\tau^+(\mathcal{C})\tau^-(\mathcal{C}) = \dim(\mathcal{C})$ .

Define  $\xi(\mathcal{C}) = \frac{\tau^+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}}\sqrt{\dim^+(\bar{\mathbf{1}})}$  so that the images of  $\mathfrak{s}$  and  $\mathfrak{t}$  satisfy

$$\mathfrak{s}^4 = \text{id}, \quad (\mathfrak{st})^3 = \xi(\mathcal{C})^{-1}\mathfrak{s}^2 \quad \text{and} \quad (\mathfrak{st}^{-1})^3 = \xi(\mathcal{C})\text{id}.$$

## 2. Slightly degenerate fusion category

The main object of study of this section is slightly degenerate fusion categories. These are braided fusion categories with symmetric center equivalent to superspaces. We give an analogue of the Verlinde formula, the structure constants involved are the ones of a quotient of the Grothendieck ring of  $\mathcal{C}$ ; these structure constants can be negative.

**Hypothesis.** *In this section, we assume that  $\mathcal{C}$  is a slightly degenerate braided pivotal fusion category.*

Denote by  $\mathfrak{e}$  the invertible object generating the symmetric center of  $\mathcal{C}$ . As  $\mathcal{C}' \simeq \text{sVect}$ , the twist of  $\mathfrak{e}$  is either 1 or  $-1$ . In the first case,  $\mathfrak{e}$  is of quantum dimension  $-1$  whereas in the second case, it is of quantum dimension 1.

Tensoring by  $\mathfrak{e}$  gives an involution on the set of isomorphism classes of simple objects. According to [EGNO15, Proposition 9.15.4], this involution has no fixed points. We choose  $J \subseteq \text{Irr}(\mathcal{C})$  a set of representatives of orbits of this involution such that  $\mathbf{1} \in J$ . The  $S$ -matrix of  $\mathcal{C}$  is then of rank half its size by [EGNO15, Theorem 8.20.7] and  $s_X^+ = s_Y^+$  if and only if  $X \simeq Y$  or  $X \simeq Y \otimes \mathfrak{e}$  by [EGNO15, Lemma 8.20.8].

The following is related to [BGH<sup>+</sup>17, Question 2.8]:

**Conjecture 2.1.** — *If  $X, Y, Z$  are simple objects then  $N_{X,Y}^Z N_{X,Y}^{Z \otimes \mathfrak{e}} = 0$ .*

**Lemma 2.2.** — *The  $S$ -matrix gives the characters of the quotient ring  $\text{Gr}(\mathcal{C})/([\mathfrak{e}] - \dim(\mathfrak{e})[\mathbf{1}])$ .*

*Proof.* — Denote by  $A$  the ring  $\text{Gr}(\mathcal{C})/([\mathfrak{e}] - \dim(\mathfrak{e})[\mathbf{1}])$ . It has a  $\mathbb{Z}$ -basis given by the elements of  $J$  and for  $X, Y, Z \in J$ , the structure constants in  $A$  are given by

$$sN_{X,Y}^Z = N_{X,Y}^Z + \dim(\mathfrak{e})N_{X,Y}^{Z \otimes \mathfrak{e}}.$$

As  $\boldsymbol{\epsilon}$  is in the symmetric center,  $S_{X,Y \otimes \boldsymbol{\epsilon}}^{++} = \dim(\boldsymbol{\epsilon})S_{X,Y}^{++}$ . Therefore, for  $W, X, Y \in J$

$$\begin{aligned} s_W^+(X \otimes Y) &= \sum_{Z \in \text{Irr}(\mathcal{C})} N_{X,Y}^Z s_W^+(Z) \\ &= \sum_{Z \in J} (N_{X,Y}^Z s_W^+(Z) + N_{X,Y}^Z s_W^+(Z \otimes \boldsymbol{\epsilon})) \\ &= \sum_{Z \in J} s N_{X,Y}^Z s_W^+(Z), \end{aligned}$$

and  $s_W^+$  is indeed a character of  $A$ .  $\square$

As in section 1.4, we define an involution  $\bar{\cdot}$  on  $J$ : for any  $X \in J$ , there exists a unique  $\bar{X} \in J$  such that for all  $Y \in J$

$$s_X^+(Y^*) = s_{\bar{X}}^+(Y).$$

Again, if the pivotal structure is spherical, then  $\bar{X} \simeq X^*$  or  $\bar{X} \simeq X^* \otimes \boldsymbol{\epsilon}$  whether  $X^*$  is in  $J$  or not.

For a slightly degenerate category, we define its superdimension by

$$\text{sdim}(\mathcal{C}) = \sum_{X \in J} |X|^2 = \frac{1}{2} \dim(\mathcal{C}).$$

Note that this does not depend on the choice of  $J$ .

**Proposition 2.3.** — *Let  $\mathcal{C}$  be a slightly degenerate braided pivotal fusion category satisfying Conjecture 2.1. Then the simple object  $\bar{\mathbf{1}}$  is invertible and for  $X \in J$  we have  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}}$  or  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}} \otimes \boldsymbol{\epsilon}$ .*

*Proof.* — The same computations as in the proof of Proposition 1.13 show that  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^* = |\bar{\mathbf{1}}|^2 \mathbf{1}$  in  $\text{Gr}(\mathcal{C})/([\boldsymbol{\epsilon}] - \dim(\boldsymbol{\epsilon})[\mathbf{1}])$ . But  $s N_{X,X^*}^{\bar{\mathbf{1}}} = N_{X,X^*}^{\bar{\mathbf{1}}} + \dim(\boldsymbol{\epsilon}) N_{X,X^*}^{\boldsymbol{\epsilon}}$ ,  $N_{X,X^*}^{\bar{\mathbf{1}}} = 1$  and  $N_{X,X^*}^{\boldsymbol{\epsilon}} = N_{X,X^* \otimes \boldsymbol{\epsilon}}^{\bar{\mathbf{1}}} = 0$  as  $X^* \not\in X^* \otimes \boldsymbol{\epsilon}$ . Hence in  $\text{Gr}(\mathcal{C})/([\boldsymbol{\epsilon}] - \dim(\boldsymbol{\epsilon})[\mathbf{1}])$  we have  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^* = \mathbf{1}$ .

Therefore in  $\text{Gr}(\mathcal{C})$  we have

$$\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^* = \mathbf{1} + \sum_{X \in J \setminus \{\mathbf{1}\}} n_X (X - \dim(\boldsymbol{\epsilon})X \otimes \boldsymbol{\epsilon}),$$

with  $n_X \in \mathbb{Z}$ .

If  $\dim(\boldsymbol{\epsilon}) = 1$ , the object  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*$  being in  $\mathcal{C}$ , the image of  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^*$  in  $\text{Gr}(\mathcal{C})$  is in the monoid generated by  $\text{Irr}(\mathcal{C})$ . Hence we necessarily have  $\bar{\mathbf{1}} \otimes \bar{\mathbf{1}}^* = \mathbf{1}$  in  $\text{Gr}(\mathcal{C})$  and therefore  $\bar{\mathbf{1}}$  is invertible.

If  $\dim(\boldsymbol{\epsilon}) = -1$ , then the hypothesis made on  $\mathcal{C}$  shows that  $n_X = 0$  for all  $X \in J \setminus \{\mathbf{1}\}$ .

As in the proof of 1.13, a simple calculation shows that  $s_{X^* \otimes \bar{\mathbf{1}}}^+(Y) = s_X^+(Y^*)$  for any simple object  $Y$ . Therefore  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}}$  if  $X^* \otimes \bar{\mathbf{1}} \in J$  and  $\bar{X} \simeq X^* \otimes \bar{\mathbf{1}} \otimes \boldsymbol{\epsilon}$  otherwise.  $\square$

**Hypothesis.** *In the following, we suppose that the object  $\bar{\mathbf{1}}$  is invertible.*

Let  $E$  be the square matrix such that  $E_{X,Y} = (\dim(\boldsymbol{\epsilon}))^{\delta_{X^* \otimes \bar{\mathbf{1}} \neq J}} \delta_{X,\bar{Y}}$  for  $X$  and  $Y$  in  $J$ .

**Proposition 2.4.** — *Let  $\mathcal{C}$  be a slightly degenerate pivotal braided fusion category. The matrix  $\mathbf{S} = (S_{X,Y}^{++})_{X,Y \in J}$  satisfies  $\mathbf{S}^2 = \text{sdim}(\mathcal{C}) \dim^+(\bar{\mathbf{1}})E$ .*

*Proof.* — First, note that if  $\chi_1$  and  $\chi_2$  are two different characters of  $\text{Gr}(\mathcal{C})$  such that  $\chi_1(\mathfrak{e}) = \chi_2(\mathfrak{e})$  then

$$\sum_{W \in J} \chi_1(W) \chi_2(W^*) = 0.$$

Indeed, by the usual orthogonality of characters,

$$0 = \sum_{W \in J} \chi_1(W) \chi_2(W^*) + \sum_{W \in J} \chi_1(W \otimes \mathfrak{e}) \chi_2(W^* \otimes \mathfrak{e}) = 2 \sum_{W \in J} \chi_1(W) \chi_2(W^*).$$

Using the fact that for  $X, Y \in J$ ,  $s_X^+ = s_Y^+$  if and only if  $X = Y$  we show as in the non-degenerate case that  $(\mathbf{S}^2)_{X,Y} = 0$  and that  $(\mathbf{S}^2)_{X,\bar{X}} = \text{sdim}(\mathcal{C}) \dim^+(\bar{\mathbf{1}}) s N_{X,\bar{X}}^{\bar{\mathbf{1}}}$ . It is then easy to see that  $s N_{X,\bar{X}}^{\bar{\mathbf{1}}} = 1$  if  $X^* \otimes \bar{\mathbf{1}} \in J$  and  $s N_{X,\bar{X}}^{\bar{\mathbf{1}}} = \dim(\mathfrak{e})$  if  $X^* \otimes \bar{\mathbf{1}} \notin J$ .  $\square$

**Corollary 2.5 (Verlinde formula).** — *Let  $\mathcal{C}$  be a slightly degenerate braided pivotal fusion category and  $X, Y, Z \in J$ . The structure constants of  $\text{Gr}(\mathcal{C})/([1] - \dim(\mathfrak{e})[\mathfrak{e}])$  are given by*

$$s N_{X,Y}^Z = \frac{(\dim(\mathfrak{e}))^{\delta_{Z^* \otimes \mathbf{1} \notin J}}}{\text{sdim}(\mathcal{C}) \dim^+(\bar{\mathbf{1}})} \sum_{W \in J} \frac{\mathbf{S}_{W,X} \mathbf{S}_{W,Y} \mathbf{S}_{W,\bar{Z}}}{\dim^+(W)}.$$

We now study the  $T$ -matrix of a slightly degenerate braided pivotal fusion category. The equality  $\theta_{X^*} \dim^+(X) = \theta_X \dim^-(X)$  has been proven without assumption on the degeneracy of the category  $\mathcal{C}$ . As in the non-degenerate setting, it is easy to prove that  $\theta_{\bar{X}} = \theta_X \theta_{\bar{\mathbf{1}}}$  if  $X^* \otimes \bar{\mathbf{1}} \in J$  and  $\theta_{\bar{X}} = \theta_X \theta_{\bar{\mathbf{1}}} \theta_{\mathfrak{e}}$  otherwise. Moreover,  $\theta_{\bar{\mathbf{1}}}^2 = 1$ . For  $X$  and  $Y \in J$ , we have the relation

$$\begin{aligned} \theta_X \theta_Y \mathbf{S}_{X,Y} &= \sum_{Z \in \text{Irr}(\mathcal{C})} N_{X,Y}^Z \dim^+(Z) \theta_Z \\ &= \sum_{Z \in J} (N_{X,Y}^Z + \dim(\mathfrak{e}) \theta_{\mathfrak{e}} N_{X,Y}^{Z \otimes \mathfrak{e}}) \dim^+(Z) \theta_Z. \end{aligned}$$

But  $\dim(\mathfrak{e}) \theta_{\mathfrak{e}} = -1$  by definition of  $\mathfrak{e}$ .

**Hypothesis.** *From now on, we suppose that  $\dim(\mathfrak{e}) = -1$  and  $\theta_{\mathfrak{e}} = 1$ :  $\text{sVect}$  is equipped with its non-unitary pivotal structure.*

With these assumptions, the structure constants of  $\text{Gr}(\mathcal{C})/([1] + [\mathfrak{e}])$  appear naturally:

$$(2.1) \quad \theta_X \theta_Y \mathbf{S}_{X,Y} = \sum_{Z \in J} s N_{X,Y}^Z \dim^+(Z) \theta_Z$$

for any  $X, Y \in J$ .

We define the Gauss supersums of the slightly degenerate category  $\mathcal{C}$  as

$$s \tau^{\pm}(\mathcal{C}) = \sum_{X \in J} |X|^2 \theta_X^{\pm 1} = \frac{1}{2} \tau^{\pm}(\mathcal{C}).$$

Note that these are independent of the choice of  $J$  since  $\theta_{\mathfrak{e}} = 1$ .

**Proposition 2.6.** — *The twists and the  $S$ -matrix satisfy for all  $Y \in J$*

$$\sum_{X \in J} \theta_X \dim^-(X) \mathbf{S}_{X,Y} = \theta_Y^{-1} \dim^+(Y) s \tau^+(\mathcal{C})$$

and

$$\sum_{X \in J} \theta_X^{-1} \dim^+(X) \mathbf{S}_{X,Y} = \theta_{\bar{\mathbf{1}}} \theta_Y \dim^+(Y) s \tau^-(\mathcal{C}).$$

*Proof.* — Using (1.2), we have

$$\sum_{X \in \text{Irr}(\mathcal{C})} \theta_X \dim^-(X) S_{X,Y}^{++} = \theta_Y^{-1} \dim^+(Y) s \tau^+(\mathcal{C}).$$

But  $\theta_{\varepsilon \otimes X} \dim^-(\varepsilon \otimes X) S_{\varepsilon \otimes X, Y}^{++} = \theta_X \dim^-(X) S_{X,Y}^{++}$  and then

$$\sum_{X \in \text{Irr}(\mathcal{C})} \theta_X \dim^-(X) S_{X,Y}^{++} = 2 \sum_{X \in J} \theta_X \dim^-(X) \mathbf{S}_{X,Y}.$$

We conclude by definition of  $s \tau^+(\mathcal{C})$ . We do the same for the second assertion, starting from (1.3).  $\square$

**Corollary 2.7.** — *Let  $\mathcal{C}$  be a slightly degenerate braided pivotal fusion category. Then  $\theta_{\mathbf{1}} = 1$ .*

*Proof.* — Same as Proposition 1.18.  $\square$

We finally conclude this section by giving an analogue of Theorem 1.19 in the setting of slightly degenerate braided pivotal fusion category. We denote by  $\mathbf{T}$  the diagonal matrix with entries  $\theta_X^{-1}$  for  $X \in J$ . We again fix an embedding  $\mathbb{k}_{\text{alg}} \rightarrow \mathbb{C}$  and denote the positive square root of  $\text{sdim}(\mathcal{C})$  for this embedding by  $\sqrt{\text{sdim}(\mathcal{C})}$ . We moreover choose a square root  $\sqrt{\dim^+(\mathbf{1})}$  of  $\dim^+(\mathbf{1})$ .

**Theorem 2.8.** — *Let  $\mathcal{C}$  be a slightly degenerate braided pivotal fusion category. We have  $(\mathbf{ST})^3 = s \tau^-(\mathcal{C}) \mathbf{S}^2$  and  $\mathbf{S}^4 = (\text{sdim}(\mathcal{C}) \dim^+(\mathbf{1}))^2 \text{id}$ . Therefore*

$$s \mapsto \frac{1}{\sqrt{\dim^+(\mathbf{1})} \sqrt{\text{sdim}(\mathcal{C})}} \mathbf{S} \quad \text{and} \quad t \mapsto \mathbf{T}$$

define a projective representation of  $SL_2(\mathbb{Z})$ .

**Remark.** — If  $\dim(\varepsilon) = 1$  and  $\theta_\varepsilon = -1$ , the  $S$  and  $T$ -matrices do not necessarily give a representation of  $SL_2(\mathbb{Z})$ .

Consider the Verlinde modular category  $\mathcal{C}(\mathfrak{sl}_2, q)$  where  $q$  is a 16-th root of unity [EGNO15, Section 8.18.2]. It has 7 simple objects  $V_0 = \mathbf{1}, \dots, V_6$ . Let  $\mathcal{C}$  be the full subcategory of  $\mathcal{C}(\mathfrak{sl}_2, q)$  generated by  $V_0, V_2, V_4, V_6$ . The  $S$ -matrix and the  $T$ -matrix of  $\mathcal{C}$  are

$$S = \begin{pmatrix} 1 & [3] & [3] & 1 \\ [3] & -1 & -1 & [3] \\ [3] & -1 & -1 & [3] \\ 1 & [3] & [3] & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where  $[3] = q^{-2} + 1 + q^2$  and  $i = q^4$  is a primitive fourth root of unity. It is immediate that the symmetric center of  $\mathcal{C}$  is generated by  $V_6$  as a tensor category, and  $V_6$  is of dimension 1 and of twist  $-1$ . The symmetric center of  $\mathcal{C}$  is then equivalent to  $\text{sVect}$ , the matrices  $\mathbf{S}$  and  $\mathbf{T}$  are

$$\mathbf{S} = \begin{pmatrix} 1 & [3] \\ [3] & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

and they do not define a projective representation of  $SL_2(\mathbb{Z})$ .

### 3. Braided fusion supercategory associated to a slightly degenerate braided fusion category

In this section, we recall the notion of a superfusion category, which is a fusion category enriched over super vector spaces. We then construct, following [BE17] and [BCK17], a superfusion category  $\hat{\mathcal{C}}$  associated to a slightly degenerate fusion category  $\mathcal{C}$ . The braiding and the pivotal structure of  $\mathcal{C}$  naturally endow the supercategory  $\hat{\mathcal{C}}$  with a braiding and a pivotal structure. The  $S$ -matrix of  $\mathcal{C}$  gives then the structure constants of a quotient of the super Grothendieck ring of  $\hat{\mathcal{C}}$ .

**3.1. Superfusion categories.** — By a superspace, we mean a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. We denote by  $|v|$  the parity of an homogeneous element  $v$  of a superspace, by  $\text{sVect}$  the category of superspaces of finite dimension with morphisms even linear maps (*i.e.* mapping even degree to even degree and odd one to odd one). This category is monoidal with  $(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1$  and  $(V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0$ , braided with braiding defined by

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v,$$

for  $v$  and  $w$  homogeneous elements; this braiding is symmetric.

*Remark.* — The category of superspaces with even and odd morphisms is not monoidal. This is due to the following relation, known as *superinterchange law*

$$(g \otimes g') \circ (f \otimes f') = (-1)^{|g'||f|} (g \circ f) \otimes (g' \circ f'),$$

for  $f$  and  $g$  homogeneous morphisms. The definition of a monoidal supercategory will naturally satisfy this superinterchange law.

A *supercategory* is a  $\text{sVect}$ -enriched category; a *superfunctor* between two supercategories is a  $\text{sVect}$ -enriched functor (see [Kel05, Section 1.2] for more details); a *supernatural transformation*  $\alpha: F \rightarrow G$  between two superfunctors  $F$  and  $G$  is a collection of morphisms  $\alpha_X: F(X) \rightarrow G(X)$  satisfying a certain supernaturality condition. A supernatural transformation is said to be *even* if all its components are even.

Now, given two supercategories  $\mathcal{A}$  and  $\mathcal{B}$ , we define  $\mathcal{A} \boxtimes \mathcal{B}$  as the supercategory whose objects are pairs  $(X, Y)$  with  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$  and whose morphisms are given by  $\text{Hom}_{\mathcal{A} \boxtimes \mathcal{B}}((X, X'), (Y, Y')) = \text{Hom}_{\mathcal{A}}(X, X') \otimes \text{Hom}_{\mathcal{B}}(Y, Y')$ , the tensor product being the one of superspaces. The composition in  $\mathcal{A} \boxtimes \mathcal{B}$  is defined using the braiding in  $\text{sVect}$ :

$$(g \otimes g') \circ (f \otimes f') = (-1)^{|g'||f|} (g \circ f) \otimes (g' \circ f').$$

**Definition 3.1** ([BE17, Definition 1.4]). — A monoidal supercategory is a sextuple  $(\mathcal{C}, \otimes, \mathbf{1}, l, r)$  where  $\mathcal{C}$  is a supercategory,  $\otimes: \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$  is a superfunctor,  $\mathbf{1}$  is the unit object, and  $a: (- \otimes -) \otimes - \rightarrow - \otimes (- \otimes -)$ ,  $l: \mathbf{1} \otimes - \rightarrow -$  and  $r: - \otimes \mathbf{1} \rightarrow -$  are even supernatural isomorphisms satisfying axioms analogous to the ones of a monoidal category.

A monoidal superfunctor between two monoidal supercategories  $\mathcal{C}$  and  $\mathcal{D}$  is a pair  $(F, J)$  where  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a superfunctor and  $J: F(-) \otimes F(-) \rightarrow F(- \otimes -)$  is an even supernatural isomorphism satisfying axioms analogous to the ones for a monoidal functor and such that  $F(\mathbf{1}_{\mathcal{C}})$  is evenly isomorphic to  $\mathbf{1}_{\mathcal{D}}$ .

We now define the notion of *braided monoidal supercategory* [BCK17, Section 2.2]. First, let  $\tau: \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$  denote the superfunctor sending  $(X, Y)$  on  $Y \otimes X$  and  $f \otimes g$  on  $(-1)^{|f||g|} g \otimes f$ . A *braiding* on  $\mathcal{C}$  is then an even supernatural isomorphism  $c: - \otimes - \rightarrow \tau$  satisfying the usual hexagon axioms of a braided monoidal category.



In a monoidal supercategory  $\mathcal{C}$ , a *left dual* of an object  $X \in \mathcal{C}$  is a triple  $(X^*, \text{ev}_X, \text{coev}_X)$  where  $X^* \in \mathcal{C}$ ,  $\text{ev}_X: X^* \otimes X \rightarrow \mathbf{1}$  and  $\text{coev}_X: \mathbf{1} \rightarrow X \otimes X^*$  are even morphisms satisfying the same axioms of duality in a rigid monoidal category. Similarly, there is a notion of right dual. A monoidal supercategory in which any object has a left and right dual is called *rigid*. In a rigid monoidal supercategory, we define the dual  $f^* \in \text{Hom}_{\mathcal{C}}(Y^*, X^*)$  of  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  as in any rigid monoidal category. Note that we have  $(g \circ f)^* = (-1)^{|f||g|} f^* \circ g^*$  for two homogeneous morphisms  $f$  and  $g$ , the sign being due to the superinterchange law.

Finally, a *superfusion category* is an abelian semisimple rigid monoidal supercategory with finite number of simple objects, finite dimensional spaces of morphisms and simple unit object  $\mathbf{1}$ .

Before giving a definition of a pivotal structure on a rigid monoidal supercategory, let us have a closer look to the case of the supercategory of superspaces. On  $\text{sVect}$ , we have two pivotal structures.

The first one is given by the Drinfeld morphism. It is indeed a pivotal structure since the braiding is symmetric. With this pivotal structure, the quantum traces are known as supertrace. One can show that for any morphism  $f$  between superspaces  $V$  and  $W$ , the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ a_V \downarrow & & \downarrow a_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

is commutative.

The second one is given by a slight modification of the previous one by a tensor auto-functor of  $\text{id}$  given by  $v \mapsto (-1)^{|v|}$  on homogeneous elements. With this pivotal structure, one recovers the usual trace as the quantum trace. One can show that for any homogeneous morphism  $f$  between superspaces  $V$  and  $W$ , the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ a_V \downarrow & & \downarrow a_W \\ V^{**} & \xrightarrow{(-1)^{|f|} f^{**}} & W^{**} \end{array}$$

is commutative.

For a rigid monoidal supercategory, we define two even superfunctors  $\text{Bid}$  and  $\text{sBid}$ . Both are given by  $X \mapsto X$  on objects. On morphisms,  $\text{Bid}$  sends  $f$  to  $f^{**}$  whereas  $\text{sBid}$  sends  $f$  to  $(-1)^{|f|} f^{**}$ . The functor  $\text{sBid}$  has been considered in [ALW17, Section 8.3].

A *pivotal structure* on a rigid monoidal supercategory is an even monoidal supernatural isomorphism  $a: - \rightarrow \text{Bid}$ . As in the usual case of pivotal monoidal categories, we have the notion of left and right quantum traces. Note that for  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  homogeneous morphisms we have

$$\text{Tr}_X^+(g \circ f) = (-1)^{|f||g|} \text{Tr}_Y^+(f \circ g)$$

and similarly for the right quantum trace.

A *superpivotal structure* on a rigid monoidal supercategory is an even monoidal supernatural isomorphism  $a: - \rightarrow \text{sBid}$ . As in the usual case of pivotal monoidal categories, we have the notion of left and right quantum traces. Note that for  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$

homogeneous morphisms we have

$$\mathrm{Tr}_X^+(g \circ f) = (-1)^{|f|(|g|+1)} \mathrm{Tr}_Y^+(f \circ g)$$

and similarly for the right quantum trace.

**3.2. Super Grothendieck ring.** — Let  $\mathbb{Z}_\varepsilon = \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1)$ . The *super Grothendieck group*  $\mathrm{sGr}(\mathcal{C})$  of an abelian supercategory  $\mathcal{C}$  is the  $\mathbb{Z}_\varepsilon$ -module generated by isomorphism classes  $[X]$  of objects in  $\mathcal{C}$  modulo the following relations: if  $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$  is a short exact sequence with homogeneous maps, then  $[Y] = \varepsilon^{|f|}[X] + \varepsilon^{|g|}[Z]$ .

There is a map  $\mathrm{sGr}(\mathcal{C})/(\varepsilon - 1) \rightarrow \mathrm{Gr}(\mathcal{C})$  which is clearly surjective but not necessarily injective.

If  $\mathcal{C}$  is monoidal, then tensor product in  $\mathcal{C}$  endow  $\mathrm{sGr}(\mathcal{C})$  with a structure of an associative algebra. If moreover  $\mathcal{C}$  is braided, the multiplication is commutative.

In a superfusion category, the super Grothendieck ring is not necessarily a free  $\mathbb{Z}_\varepsilon$ -algebra as it may exist simple objects with odd automorphisms. It is nevertheless free as a  $\mathbb{Z}$ -algebra.

**3.3. Braided monoidal supercategory associated to a slightly degenerate braided category.** — In this section, we fix a slightly degenerate pivotal braided fusion category  $\mathcal{C}$ . We denote by  $\mathbf{e}$  the unique non-unit simple object of  $\mathcal{C}'$ . We fix an isomorphism  $\xi: \mathbf{e} \otimes \mathbf{e} \rightarrow \mathbf{1}$ .

Define a supercategory  $\hat{\mathcal{C}}$  with the same objects as  $\mathcal{C}$  and superspaces of morphisms being:

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)_0 = \mathrm{Hom}_{\mathcal{C}}(X, Y) \quad \text{and} \quad \mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)_1 = \mathrm{Hom}_{\mathcal{C}}(X, \mathbf{e} \otimes Y).$$

The composition of  $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)$  and  $g \in \mathrm{Hom}_{\hat{\mathcal{C}}}(Y, Z)$  is obviously defined except when  $f$  and  $g$  are odd morphisms. In this case,  $g \circ f$  is the map given by the following composition

$$X \xrightarrow{f} \mathbf{e} \otimes Y \xrightarrow{\mathrm{id}_{\mathbf{e}} \otimes g} \mathbf{e} \otimes \mathbf{e} \otimes Z \xrightarrow{\xi \otimes \mathrm{id}_Z} Z.$$

By a case-by-case checking, we see that the composition above is indeed associative.

The category  $\hat{\mathcal{C}}$  is equipped with a tensor product. On objects, the tensor product is the same as in  $\mathcal{C}$ . On morphisms, we define the tensor product  $f \hat{\otimes} f'$  on homogeneous morphisms  $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)$  and  $f' \in \mathrm{Hom}_{\hat{\mathcal{C}}}(X', Y')$  as follows:

- if  $f$  and  $f'$  are both even,  $f \hat{\otimes} f' = f \otimes f'$ ,
- if  $f$  is even and  $f'$  odd,  $f \hat{\otimes} f' = (c_{Y, \mathbf{e}} \otimes \mathrm{id}_{Y'}) \circ f \otimes f'$ ,
- if  $f$  is odd and  $f'$  even,  $f \hat{\otimes} f' = f \otimes f'$ ,
- if  $f$  and  $f'$  are both odd,  $f \hat{\otimes} f' = (-\xi \otimes \mathrm{id}_{Y \otimes Y'}) \circ (\mathrm{id}_{\mathbf{e}} \otimes c_{Y, \mathbf{e}} \otimes \mathrm{id}_{Y'}) \circ f \otimes f'$ .

As for the associativity, we check case-by-case, that  $\hat{\otimes}$  satisfies the superinterchange law.

The braiding in  $\mathcal{C}$  gives an even morphism  $c_{X, Y} \in \mathrm{Hom}_{\hat{\mathcal{C}}}(X \otimes Y, Y \otimes X)$ .

**Proposition 3.2.** — *Let  $\mathcal{C}$  be a slightly degenerate braided category. The braiding in  $\mathcal{C}$  endows  $\hat{\mathcal{C}}$  with a structure of a braided supercategory.*

*Proof.* — We have to check that for any two homogeneous morphisms  $f \in \text{Hom}_{\hat{\mathcal{C}}}(X, X')$  and  $g \in \text{Hom}_{\hat{\mathcal{C}}}(Y, Y')$  the diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \\ \downarrow f \hat{\otimes} g & & \downarrow g \hat{\otimes} f \\ X' \otimes Y' & \xrightarrow{c_{X',Y'}} & Y' \otimes X' \end{array}$$

is  $(-1)^{|f||g|}$ -commutative in  $\hat{\mathcal{C}}$ . If  $X' = X$  and  $f = \text{id}_X$ , the commutativity follows from the naturality of the braiding in  $\mathcal{C}$  and from one of the hexagon axioms. If  $Y = Y'$ ,  $g = \text{id}_Y$  and  $f$  is of even degree it is trivially commutative; if  $f$  is of odd degree, we have to show that

$$\begin{array}{ccc} \mathbf{\epsilon} \otimes X' \otimes Y & \xrightarrow{c_{\mathbf{\epsilon} \otimes X', Y}} & Y \otimes \mathbf{\epsilon} \otimes X' \\ \searrow \text{id}_{\mathbf{\epsilon}} \otimes c_{X', Y} & & \downarrow c_{Y, \mathbf{\epsilon}} \otimes \text{id}_{X'} \\ & & \mathbf{\epsilon} \otimes Y \otimes X' \end{array}$$

is commutative in  $\mathcal{C}$ , which follows from one of the hexagon axioms and the fact that  $\mathbf{\epsilon}$  is in the symmetric center of  $\mathcal{C}$ . The general case now follows from the superinterchange law.  $\square$

The category  $\hat{\mathcal{C}}$  is also rigid, the evaluations and coevaluations being the same as in  $\mathcal{C}$ . The dual of a morphism of odd degree  $f \in \text{Hom}_{\hat{\mathcal{C}}}(X, Y)$  is then given by the following map  $f^{\otimes} \in \text{Hom}_{\hat{\mathcal{C}}}(Y^*, \mathbf{\epsilon} \otimes X^*)$

$$\begin{array}{ccccccc} Y^* & \xrightarrow{\text{coev}_X} & Y^* \otimes X \otimes X^* & \xrightarrow{f} & Y^* \otimes \mathbf{\epsilon} \otimes Y \otimes X^* & \xrightarrow{c_{Y^*, \mathbf{\epsilon}}} & \mathbf{\epsilon} \otimes Y^* \otimes Y \otimes X^* \\ & & & & & & \downarrow \text{ev}_Y \\ & & & & & & \mathbf{\epsilon} \otimes X^* \end{array}$$

**Proposition 3.3.** — *If  $a_{\mathbf{\epsilon}} = u_{\mathbf{\epsilon}}$  then the pivotal structure on  $\mathcal{C}$  gives rise to a pivotal structure on  $\hat{\mathcal{C}}$ . If  $a_{\mathbf{\epsilon}} = -u_{\mathbf{\epsilon}}$  then the pivotal structure on  $\mathcal{C}$  gives rise to a superpivotal structure on  $\hat{\mathcal{C}}$ .*

*Proof.* — Let  $\hat{f} \in \text{Hom}_{\hat{\mathcal{C}}}(X, Y)$  of odd degree equal to  $f \in \text{Hom}_{\mathcal{C}}(X, \mathbf{\epsilon} \otimes Y)$ . Denote  $\theta$  the twist in  $\mathcal{C}$  associated to the pivotal structure. Using its naturality, we have  $(\text{theta}_{\mathbf{\epsilon}} \otimes \theta_Y) \circ f = f \circ \theta_X$  as morphisms in  $\mathcal{C}$ . Then, in  $\hat{\mathcal{C}}$ , we find that  $\theta_{\mathbf{\epsilon}} \theta_Y \hat{\otimes} \hat{f} = \hat{f} \hat{\otimes} \theta_X$ , identifying  $\theta_X$  and  $\theta_Y$  to even morphisms in  $\hat{\mathcal{C}}$  and  $\theta_{\mathbf{\epsilon}}$  to the scalar  $\pm 1$ . By definition of the twist, we have  $a_Z = u_Z \circ \theta_Z$  in  $\mathcal{C}$  for any object  $Z$  and therefore we have in  $\hat{\mathcal{C}}$

$$\hat{a}_Y \hat{\otimes} \hat{f} = \theta_{\mathbf{\epsilon}} u_Y \hat{\otimes} \hat{f} \hat{\otimes} \theta_X.$$

But in  $\hat{\mathcal{C}}$ ,  $u_Y \hat{\otimes} \hat{f} = \hat{f}^{**} \hat{\otimes} u_X$ , exactly as in a pivotal rigid monoidal category. Finally, we have

$$\hat{a}_Y \hat{\otimes} \hat{f} = \theta_{\mathbf{\epsilon}} \hat{f}^{**} \hat{\otimes} \hat{a}_X,$$

which gives us a pivotal or a superpivotal structure on  $\hat{\mathcal{C}}$ .  $\square$

We end this section with the super Grothendieck ring. The quotient ring  $\text{Gr}(\mathcal{C})/([\mathbf{1}] - \dim(\mathbf{\epsilon})[\mathbf{\epsilon}])$  can be seen as a quotient of the super Grothendieck ring of  $\hat{\mathcal{C}}$

$$\text{Gr}(\hat{\mathcal{C}})/([\mathbf{1}] - \dim(\mathbf{\epsilon})[\mathbf{\epsilon}]) \simeq \text{sGr}(\hat{\mathcal{C}})/(1 - \dim(\mathbf{\epsilon})\mathbf{\epsilon}).$$

The quotient ring  $\text{sGr}(\hat{\mathcal{C}})/(1 - \varepsilon)$  is isomorphic to the usual Grothendieck ring of  $\hat{\mathcal{C}}$  and to the quotient ring  $\text{Gr}(\mathcal{C})/([\mathbf{1}] - [\boldsymbol{\varepsilon}])$ . Therefore, if  $\dim(\boldsymbol{\varepsilon}) = -1$ , we are in the following situation:

$$\begin{array}{ccc} & \text{sGr}(\hat{\mathcal{C}}) & \\ \varepsilon=1 \swarrow & & \searrow \varepsilon=-1 \\ \text{Gr}(\hat{\mathcal{C}}) \simeq \text{Gr}(\mathcal{C})/([\mathbf{1}] - [\boldsymbol{\varepsilon}]) & & \text{Gr}(\mathcal{C})/([\mathbf{1}] + [\boldsymbol{\varepsilon}]) \end{array}$$

The ring  $\text{sGr}(\hat{\mathcal{C}})$  is free as a  $\mathbb{Z}$ -module and of rank  $|\text{Irr}(\hat{\mathcal{C}})|$ . The two quotient rings  $\text{Gr}(\mathcal{C})/([\mathbf{1}] - [\boldsymbol{\varepsilon}])$  and  $\text{Gr}(\mathcal{C})/([\mathbf{1}] + [\boldsymbol{\varepsilon}])$  are also free as  $\mathbb{Z}$ -modules and of rank  $\frac{|\text{Irr}(\hat{\mathcal{C}})|}{2}$ . If the Conjecture 2.1 is true, we remark that the structure constants of the ring  $\text{Gr}(\mathcal{C})/([\mathbf{1}] - [\boldsymbol{\varepsilon}])$  relatively to a choice of a basis  $J$  as in Section 2 are equal to the absolute values of the structure constants of the ring  $\text{Gr}(\mathcal{C})/([\mathbf{1}] + [\boldsymbol{\varepsilon}])$  relatively to the same basis  $J$ .

#### 4. An application to Bonnafé-Rouquier asymptotic cell category

In this section, we consider the example of Bonnafé and Rouquier [BR17] and give another interpretation of their categorification of a  $\mathbb{Z}$ -fusion datum. We fix an integer  $d \geq 2$  and  $\zeta$  a primitive  $d$ -th root of unity in  $\mathbb{C}$ . The algebra  $D(B)$  is the algebra with generators  $K, z, E, F$  satisfying the following relations:

$$\begin{aligned} K^d &= z^d = 1, \\ E^d &= F^d = 0, \\ [z, E] &= [z, F] = [z, K] = 0, \\ KE &= \zeta EK, \\ KF &= \zeta^{-1} FK, \\ [E, F] &= K - zK^{-1}. \end{aligned}$$

This algebra is the Drinfeld double of the Taft algebra, a finite dimensional version of the quantum enveloping algebra of a Borel of  $\mathfrak{sl}_2$ . The category  $D(B)\text{-mod}$  of finite dimensional  $D(B)$ -modules has  $d^2$  simple modules. For each integer  $1 \leq l \leq d$  there exists  $d$  simple modules of dimension  $l$  denoted by  $M_{l,p}$ ,  $p \in \mathbb{Z}/d\mathbb{Z}$  (see [BR17, 2.A] for further details).

As the algebra  $D(B)$  is a braided Hopf algebra, the category  $D(B)\text{-mod}$  is a braided tensor category. We endow it with a pivotal structure whose pivot is given by  $z^{-1}K$ . We denote by  $\mathcal{C}$  the semi-simplification of the category  $D(B)\text{-mod}$ . The simple objects in this category are the indecomposable finite dimensional  $D(B)$ -modules with non-zero positive and negative quantum dimension. In particular, the modules  $M_{l,p}$  for  $1 \leq l < d$  and  $p \in \mathbb{Z}/d\mathbb{Z}$  are simple in  $\mathcal{C}$  and the modules  $M_{d,p}$  are isomorphic to 0.

**Proposition 4.1.** — *The full subcategory  $\mathcal{D}$  of  $\mathcal{C}$  additively generated by the simple modules  $M_{l,p}$  for  $1 \leq l < d$  and  $p \in \mathbb{Z}/d\mathbb{Z}$  is stable by tensor product.*

*Proof.* — As the category  $\mathcal{C}$  is semi-simple and  $M_{l,p} \simeq M_{l,0} \otimes M_{1,p}$ , it suffices to show that, for  $1 \leq l \leq l' < d$  and  $p \in \mathbb{Z}/p\mathbb{Z}$ , the simple constituents in  $\mathcal{C}$  of  $M_{l,0} \otimes M_{l',p}$  are of the form  $M_{n,q}$ . We proceed by induction on  $l$ , the case  $l = 1$  being trivial. For  $l = 2$ , this follows

from [BR17, Theorem 3.3]:

$$M_{2,0} \otimes M_{l',p} \simeq \begin{cases} M_{2,p} & \text{if } l' = 1, \\ M_{l'+1,p} \oplus M_{l'-1,p+1} & \text{if } 1 < l' < d-1, \\ M_{d-2,p+1} & \text{if } l' = d-1. \end{cases}$$

Now, for  $l \geq 3$  and  $l' \geq l$ , the module  $M_{l,0} \otimes M_{l',p}$  is a direct summand of  $(M_{l,0} \oplus M_{l-2,1}) \otimes M_{l',p} \simeq M_{2,0} \otimes M_{l-1,0} \otimes M_{l',p}$ . By induction, every simple constituent of the module  $M_{l-1,0} \otimes M_{l',p}$  is of the form  $M_{n,q}$  and therefore so is every simple constituent of  $(M_{l,0} \oplus M_{l-2,1}) \otimes M_{l',p}$ .  $\square$

The category  $\mathcal{D}$  is then a fusion category with  $d(d-1)$  simple objects. The positive S-matrix  $S^{++}$  has been computed by Bonnafé-Rouquier [BR17, Corollary 5.5]:

$$S_{(l,p),(l',p')}^{++} = \frac{\zeta}{1-\zeta} \zeta^{-ll'-lp'-p'l'-2pp'} (1-\zeta^{ll'}).$$

The twist is given by

$$\theta_{l,p} = \zeta^{p(l+p)}.$$

We denote by  $\mathbf{\epsilon}$  the object  $M_{d-1,1}$  which is of positive and negative quantum dimension  $-1$  and of twist 1. An easy calculation shows that

$$S_{(l,p),(d-1,1)}^{++} = -\dim^+(M_{l,p}),$$

and therefore  $\mathbf{\epsilon}$  lies in the symmetric center of the category  $\mathcal{D}$ . As this S-matrix has rank  $\frac{d(d-1)}{2}$ ,  $\mathbf{\epsilon}$  is the only non-trivial simple object in the symmetric center. Therefore

$$\mathcal{X}_{\text{sym}}(\mathcal{D}) \simeq \text{sVect},$$

so that the category  $\mathcal{D}$  is slightly degenerate. We then compute the action of tensorisation by  $\mathbf{\epsilon}$  on the set of simple objects.

**Proposition 4.2.** — *In  $\mathcal{D}$  we have  $\mathbf{\epsilon} \otimes M_{l,p} \simeq M_{d-l,l+p}$ .*

*Proof.* — We proceed by induction on  $l$ , the case  $l = 1$  being trivial. For  $l = 2$ , the computation is done in the proof of Proposition 4.1. Suppose that  $l \geq 3$ . We embed  $\mathbf{\epsilon} \otimes M_{l,p}$  in  $\mathbf{\epsilon} \otimes (M_{l,p} \oplus M_{l-2,p+1}) \simeq \mathbf{\epsilon} \otimes M_{2,0} \otimes M_{l-1,p}$  as in the proof of 4.1. Therefore, by induction,  $\mathbf{\epsilon} \otimes M_{l,p} \otimes \mathbf{\epsilon}$  embeds in  $M_{d-l,l+p} \oplus M_{d-l+2,l-1+p}$ . As  $\mathbf{\epsilon} \otimes M_{l-2,p+1} \simeq M_{d-l+2,p+1}$  we finally deduce that  $M_{l,p} \otimes \mathbf{\epsilon} \simeq M_{d-l,l+p}$ .  $\square$

We may choose  $\{M_{l,p} \mid 0 \leq p < l+p < d\}$  as a set of representatives of the action on simples given by tensorisation by  $\mathbf{\epsilon}$ .

We now compute explicitly the object  $\bar{X}$ , which can be reduced to the computation of the invertible object  $\bar{\mathbf{1}}$ .

The dual of  $M_{l,p}$  is given by  $M_{l,1-l-p}$  and therefore

$$s_{\bar{\mathbf{1}}}^+(M_{l,p}^*) = \zeta^p \frac{\zeta^l - 1}{\zeta - 1} = s_{M_{d-1,0}}^+(M_{l,p})$$

shows that  $\bar{\mathbf{1}} = M_{d-1,0}$  and  $\dim^+(\bar{\mathbf{1}}) = -\zeta$ .

Following section 2, we define the normalized S-matrix of the slightly degenerate pivotal braided fusion category  $\mathcal{D}$  by

$$\mathbb{S}_{X,Y} = \frac{S_{X,Y}^{++}}{\sqrt{\text{sdim}(\mathcal{D})} \sqrt{\dim(\bar{\mathbf{1}})}}.$$

As  $\text{sdim}(\mathcal{D}) = \frac{-\zeta d^2}{(1-\zeta)^2}$  (see [BR17, Section 5.C]), we have

$$\mathbb{S}_{(l,p),(l',p')} = \frac{\zeta^{-ll'-lp'-p'l'-2pp'}(\zeta^{ll'} - 1)}{d}.$$

We therefore recover the  $S$  and  $T$ -matrices of Bonnafé-Rouquier by the means of a slightly degenerate fusion category.

**Theorem 4.3.** — *The braided pivotal superfusion category  $\hat{\mathcal{D}}$  categorifies the Malle  $\mathbb{Z}$ -fusion datum associated to the non-trivial family of the cyclic group.*

This example gives us a slightly degenerate braided pivotal fusion category which is not of the form  $\mathcal{D}_0 \boxtimes \text{sVect}$  for a non-degenerate braided pivotal fusion category  $\mathcal{D}_0$ .

**Proposition 4.4.** — *If  $d$  is even, the category  $\mathcal{D}$  is not of the form  $\mathcal{D}_0 \boxtimes \text{sVect}$ .*

*Proof.* — Let us suppose that  $\mathcal{D} = \mathcal{D}_0 \boxtimes \text{sVect}$ . We consider the objects  $M_{\frac{d}{2},0}$  and  $\epsilon \otimes M_{\frac{d}{2},0} \simeq M_{\frac{d}{2},\frac{d}{2}}$ . One of these two objects is in  $\mathcal{D}_0$ .

An easy induction shows that  $(M_{2,0})^{\otimes 2k} \otimes M_{\frac{d}{2},0}$  has  $M_{\frac{d}{2},k}$  as a simple constituent. Therefore, if  $M_{2,0}$  belongs to  $\mathcal{D}_0$ , both  $M_{\frac{d}{2},0}$  and  $\epsilon \otimes M_{\frac{d}{2},0}$  are simple objects of  $\mathcal{D}_0$ , which is a contradiction.

Then the object  $M_{2,0}$  does not belong to  $\hat{\mathcal{D}}$  and therefore  $\epsilon \otimes M_{2,0}$  does. But  $(\epsilon \otimes M_{2,0})^{\otimes 2k} \otimes M_{\frac{d}{2},0} \simeq (M_{2,0})^{\otimes 2k} \otimes M_{\frac{d}{2},0}$  for any integer  $k$ , which leads to the same contradiction.  $\square$

## References

- [ALW17] D. Aasen, E. Lake, and K. Walker. Fermion condensation and super pivotal categories. *ArXiv e-prints*, September 2017, 1709.01941.
- [BCK17] J. Brundan, J. Comes, and J. R. Kujawa. A basis theorem for the degenerate affine oriented Brauer-Clifford supercategory. *ArXiv e-prints*, June 2017, 1706.09999.
- [BE17] J. Brundan and A. P. Ellis. Monoidal supercategories. *Comm. Math. Phys.*, 351(3):1045–1089, 2017.
- [BGH<sup>+</sup>17] P. Bruillard, C. Galindo, T. Hagge, S.-H. Ng, J. Y. Plavnik, E. C. Rowell, and Z. Wang. Fermionic modular categories and the 16-fold way. *J. Math. Phys.*, 58(4):041704, 31, 2017.
- [BR17] C. Bonnafé and R. Rouquier. An asymptotic cell category for cyclic groups. *ArXiv e-prints*, August 2017, 1708.09730.
- [Bru00] A. Bruguières. Tresses et structure entière sur la catégorie des représentations de  $\text{SL}_N$  quantique. *Comm. Algebra*, 28(4):1989–2028, 2000.
- [EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
- [EO18] P. Etingof and V. Ostrik. On semisimplification of tensor categories. *ArXiv e-prints*, January 2018, 1801.04409.
- [Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005.
- [Lac18] A. Lacabanne. Drinfeld double of quantum groups, tilting modules and  $\mathbb{Z}$ -modular data associated to complex reflection groups. *ArXiv e-prints*, July 2018, 1807.00770.
- [Lus94] G. Lusztig. Exotic Fourier transform. *Duke Math. J.*, 73(1):227–241, 243–248, 1994. With an appendix by Gunter Malle.
- [Mal95] G. Malle. Unipotente Grade imprimitiver komplexer Spiegelungsgruppen. *J. Algebra*, 177(3):768–826, 1995.

- [Müg03] M. Müger. On the structure of modular categories. *Proc. London Math. Soc.* (3), 87(2):291–308, 2003.

---

*August 14, 2018*

ABEL LACABANNE • *E-mail* : `abel.lacabanne@umontpellier.fr`, Institut Montpelliérain Alexander Grothendieck (CNRS: UMR 5149), Université de Montpellier, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE