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Minimum colored maximum matching is NP-hard on trees

Jonas S enizergues

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Vertex-colored graphs

Definition 1. A vertex-colored graph is a couple $G^c = (G, c)$ where $G = (V, E)$ is a graph and c a coloration on V (ie a function giving a color to each vertex).

Remark. The coloration doesn't need to be a *proper* coloration, two adjacent vertices can be of the same color.

Definition 2. (Subgraph of vertex-colored graph) H^d is said to be a (vertex-colored) subgraph of G^c a vertex-colored graph when H is a subgraph of G and d is c restricted to $V(H)$.

Remark. Given the definition, H^d can be alternatively written H^c , or H when it's clearly stated that it is a subgraph of G^c .

Definition 3. (Tropical subgraph) A subgraph H of a vertex-colored graph G^c is said to be a *tropical subgraph* if $|c(V(G))| = |c(V(H))|$, that is if every color of G appears in H .

Vertex colored Maximum Matching

The problem of finding a maximum matching is a classical one, but let's define that properly so we can extend it to the colored case.

Definition 4. (Matching) If $G = (V, E)$ is a graph, $M \subset E$ is said to be a *matching* if no edges in M are adjacent.

Definition 5. *Maximum matching* is the problem where given $G = (V, E)$, we search for $M \subset E$ such that M is a matching with the maximum number of edges.

The decision problem associated to this optimisation problem is known to be polynomial, but what happens when we add some constraint on the colors to the problem ? For example one could think about the "tropical" version of the problem :

Definition 6. *Tropical maximum matching* is then the problem where given $G^c = (V, E)^c$ a vertex-colored graph, we search for $M \subset E$ such that M is a maximum matching and the subgraph of G^c induced by M is tropical.

In [1], the authors handle this case, giving a polynomial-time algorithm. Using their Theorem 2.2, an immediate corollary is that we still have a polynomial time algorithm when we replace "tropical" with "maximum colored" :

Definition 7. *Maximum edge-colored maximum matching* is the problem where given $G^c = (V, E)^c$ a vertex-colored graph, we search for $M \subset E$ such that M is a maximum matching and the subgraph of G^c induced by M has the maximum number of colors for that property.

An other natural variation is to consider the minimization of the number of colors instead of maximizing it :

Definition 8. *Minimum edge-colored maximum matching* is then the problem where given $G^c = (V, E)^c$ a vertex-colored graph, we search for $M \subset E$ such that M is a maximum matching and the subgraph of G^c induced by M has the minimum number of colors for that property.

That problem, however, is not as easy to solve, as we will prove. The corresponding decision problem, presented below, is indeed NP-hard.

Definition 9. *Less than k -edge-colored maximum matching* is the problem of deciding, for a given vertex-colored graph $G^c = (V, E)^c$ and an integer k , if there exists M a maximum matching of G^c such that the subgraph of G^c induced by M has at most k colors.

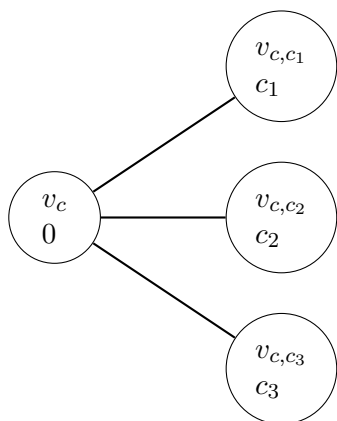
Theorem. *Less than k -edge-colored maximum matching is NP-hard on trees*

Construction

To prove this, we are going to reduce *3-SAT* to *Less than k -edge-colored maximum matching*.

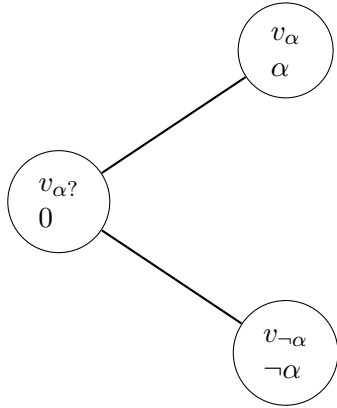
Let's take an instance of 3-SAT, given by a set \mathcal{C} of clause, and let's denote by \mathcal{V} the set of variable of the clauses in \mathcal{C} and \mathcal{L} the corresponding set of literals $\mathcal{L} = \{a, \neg a | a \in \mathcal{V}\}$. We will then build a vertex-colored graph using $\mathcal{L} \uplus \{0\}$ as colors.

For $c = c_1 \vee c_2 \vee c_3$ a clause of \mathcal{C} , where $(c_1, c_2, c_3) \in \mathcal{L}^3$, we construct the following gadget (in each vertex, the upper notation stand for the name of the vertex, the lower for the color) :

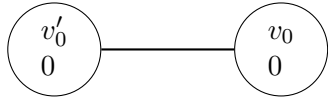


The idea is to force a matching to choose between one of the three colors (ie one of the literals).

Also, for each variable $\alpha \in \mathcal{V}$, to be sure to chose at least one color between α and $\neg\alpha$ we construct :



We add one last gadget to ensure that the "neutral" color 0 will not interfere with the construction :



Finally, we need to make a tree out of those gadgets : our vertex-colored graph will be the union of those gadgets where we add the following set of edges : $\{v_0v_c | c \in \mathcal{C}\} \cup \{v_0v_{\alpha?} | \alpha \in \mathcal{V}\}$.

This vertex-colored tree is of size $(7 + 1)|\mathcal{C}| + (5 + 1)|\mathcal{V}| + 3 \leq 14|\mathcal{C}| + 3$, polynomial in the size of our 3-SAT problem.

From now on, we will be referring to this tree as $G^c = (V, E)$.

Proof

The first thing to see in G is that we know already the size of the maximum matching (that is $|\mathcal{C}| + |\mathcal{V}| + 1$).

Lemma 1. $\{v'_0v_0\} \cup \{v_c v_{c_1} | c \in \mathcal{C}\} \cup \{v_{\alpha?} v_{\alpha} | \alpha \in \mathcal{V}\}$ is a maximum matching of G

Proof. One can easily see that there is no augmenting path, since all paths that go from an unmatched vertex to another are either of length 2 or 4. □

Moreover, no maximum matching can use an edge that doesn't touch a leaf, since this would create an augmenting path.

Lemma 2. If M is a matching of G and $M \cap (\{v_0v_c | c \in \mathcal{C}\} \cup \{v_0v_{\alpha?} | \alpha \in \mathcal{V}\}) \neq \emptyset$, then M is not a maximum matching.

Proof. Let M be a matching of G .

Let's suppose that $v_0v_c \in M$ for some $c \in \mathcal{C}$. Then since M is a matching, v'_0v_0 and $v_c v_{c_1}$ are not in M , so $v'_0v_0v_c v_{c_1}$ is an augmenting chain and M is not maximal therefore not maximum.

By the same argument, if $v_0v_{\alpha?} \in M$ for some $\alpha \in \mathcal{V}$, M is not maximum. □

Then, the only thing we have to worry about is which edge is taken in each gadget.

If \mathcal{C} is satisfiable, then there is a maximum matching of G^c with $|\mathcal{V}| + 1$ colors or less

In this section we suppose that \mathcal{C} is satisfiable. Let's take then $\phi : \mathcal{V} \rightarrow \{T, \perp\}$ an affectation that satisfies \mathcal{C} .

Let's define, for $c = c_1 \vee c_2 \vee c_3$ a clause on the variables of \mathcal{V} :

$$f_\phi(c) = \begin{cases} c_1 & \text{if } \phi \vdash c_1 \\ c_2 & \text{if } \phi \vdash c_2 \text{ and } \phi \not\vdash c_1 \\ c_3 & \text{if } \phi \vdash c_3 \text{ and } \phi \not\vdash c_1, \phi \not\vdash c_2 \\ \perp & \text{otherwise} \end{cases}$$

Lemma 3. $\forall c \in \mathcal{C}, f_\phi(c) \neq \perp$

Proof. Since ϕ satisfies \mathcal{C} , $\phi \vdash A \vee B \vee C$, therefore $\phi \vdash A$ or $\phi \vdash B$ or $\phi \vdash C$ □

Then we can define $M = \{v_0'v_0\} \cup \left\{ v_c v_{c, f_\phi(c)} \mid c \in \mathcal{C} \right\} \cup \{v_\alpha v_\alpha \mid \alpha \in \mathcal{V}, \phi(\alpha) = T\} \cup \{v_\alpha v_{-\alpha} \mid \alpha \in \mathcal{V}, \phi(\alpha) = \perp\}$

M is a maximum matching of G (it is a matching, of size maximum size $|\mathcal{C}| + |\mathcal{V}| + 1$).

Lemma 4. *The vertex-colored subgraph G'^c induced by M has a color $\ell \in im(c) \setminus \{0\}$ if and only if $\phi \vdash \ell$.*

Proof. Let $\ell \in im(c) \setminus \{0\}$, since it's not 0 it's a literal.

If $\phi \vdash \ell$ then $v_\alpha v_\ell \in M$, so ℓ appears in G'^c .

If $\phi \not\vdash \ell$, then $v_\alpha v_\ell \notin M$ (since $v_\alpha v_{\bar{\ell}} \in M$ and M is a matching). The only way ℓ could be a color of G'^c would then be that $v_c v_{c, \ell} \in M$ for some $c \in \mathcal{C}$. But by construction that would mean that $\phi \vdash \ell$ wich contradicts $\phi \not\vdash \ell$. Therefore, ℓ does not appear in G'^c . □

Corollary 1. *The vertex-colored subgraph induced by M has $|\mathcal{V}| + 1$ colors.*

If there is a maximum matching of G^c with $|\mathcal{V}| + 1$ colors or less, \mathcal{C} is satisfiable

First, we can observe that, since we must take for each variable an edge in the corresponding gadget, we have at least one color per variable, plus 0, in the subgraph of G induced by a maximum matching M , which makes at least $|\mathcal{V}| + 1$ colors, so it's equivalent to take an exactly $|\mathcal{V}| + 1$ colors maximum matching as hypothesis.

Let's then suppose that there is maximum matching M of G^c with $|\mathcal{V}| + 1$ colors (G'^c is then the vertex-colored subgraph induced by M).

We define $\phi_M : \mathcal{V} \rightarrow \{T, \perp\}$, for M a maximum matching of G^c as follows :

$$\phi_M(\alpha) = \begin{cases} T & \text{if } v_\alpha v_\alpha \in M \\ \perp & \text{if } v_\alpha v_{-\alpha} \in M \end{cases}$$

Since a maximum matching in G^c must take exactly one edge in the gadget corresponding to a variable, the function is well-defined.

Lemma 5. ϕ_M satisfies \mathcal{C}

Proof. Let's take $c \in \mathcal{C}$. Since M is a maximum matching, $\exists \ell \in \mathcal{L}$ such that $v_c v_{c,\ell}$, and the color ℓ appears in G'^c .

But since G'^c has only $|\mathcal{V}| + 1$ colors, this means that, taking α the variable associated with the literal ℓ , $v_\alpha v_\ell \in M$ (otherwise there would be at least $|\mathcal{V}| + 2$ colors in G'). Therefore $\phi_M \vdash \ell$, and $\phi_M \vdash c$.

Thus, $\forall c \in \mathcal{C}$, $\phi_M \vdash c$, ie ϕ_M satisfies \mathcal{C} □

Conclusion

Lemma 6. *\mathcal{C} is satisfiable if and only if there is a maximum matching of G^c with $|\mathcal{V}| + 1$ colors or less.*

Proof. From Corollary 1. and Lemma 5. □

We can now go back to our theorem :

Theorem. *Less than k -edge-colored maximum matching is NP-hard on trees*

Proof. By lemma 6, we constructed a polynomial reduction of 3 -SAT to *Less than k -edge-colored maximum matching* on a tree. Since 3 -SAT is NP-hard (since it is NP-C), *Less than k -edge-colored maximum matching* is indeed NP-hard on trees. □

References

- [1] Johanne Cohen, Yannis Manoussakis, Hong Phong Pham, and Zsolt Tuza. Tropical matchings in vertex-colored graphs. *Electronic Notes in Discrete Mathematics*, 62:219–224, 2017.