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► **To cite this version:**

Matthieu Romagny, Giulia Battiston. REPRESENTATIONS OF AFFINE GROUP SCHEMES OVER GENERAL RINGS. 2008. hal-01827260v1

HAL Id: hal-01827260

<https://hal.science/hal-01827260v1>

Preprint submitted on 2 Jul 2018 (v1), last revised 19 Jul 2018 (v3)

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REPRESENTATIONS OF AFFINE GROUP SCHEMES OVER GENERAL RINGS

GIULIA BATTISTON AND MATTHIEU ROMAGNY

In memory of Michel Raynaud

ABSTRACT. Among all affine, flat, finitely presented group schemes, we focus on those that are *pure*; this includes all groups which are extensions of a finite locally free group by a group with connected fibres. We prove that over an arbitrary base ring, pure group schemes have a classifying space satisfying the resolution property, an embedding into some GL_n , a tensor generator for their category of finite type representations, and can be reconstructed from their category of projective finite type representations. In the case of an Artinian base ring, the same is true for all affine, flat, finitely presented group schemes; this answers a question of Conrad. We also prove that quotients of pure groups by closed pure subgroups over an arbitrary base scheme are Zariski-locally quasi-projective. This answers a question of Raynaud, in the case of affine groups. We give various applications.

1. INTRODUCTION

In this paper we are interested in some properties of affine group schemes that are related to their categories of representations. To make this more precise, we introduce the category of affine schemes (Aff), the fibred category of affine, flat, finitely presented group schemes $\mathcal{C} \rightarrow (\mathrm{Aff})$, and the following full subcategories:

- $\mathcal{C}_{\mathrm{lin}}$: group schemes $G \rightarrow S$ which are linear, in other words admit a closed embedding $G \hookrightarrow \mathrm{GL}_{n,R}$ for some $n \geq 0$.
- $\mathcal{C}_{\mathrm{mono}}$: group schemes whose category $\mathrm{Rep}(G)$ of finite type representations possesses a tensor generator (we shall call them *monogenic*).
- $\mathcal{C}_{\mathrm{recons}}$: group schemes that satisfy Tannaka reconstruction, that is can be reconstructed from the category $\mathrm{PRep}(G)$ of representations whose underlying module is projective of finite type.
- $\mathcal{C}_{\mathrm{resol}}$: group schemes G whose classifying stack BG satisfies the resolution property (see beginning of Section 2 for the definitions).

Following custom, for a subcategory $\mathcal{D} \subset \mathcal{C}$ we write $\mathcal{D}(R)$ instead of $\mathcal{D}(S)$ when $S = \mathrm{Spec}(R)$. One thing we want to know is whether $\mathcal{D}(R)$ is “big” for some rings R . For example $\mathcal{D}(k) = \mathcal{C}(k)$ for all four categories in the simplest case where $R = k$ is a field; we review the current state of our knowledge on this question below.

In order to gain better understanding of these categories, it is natural to exploit the group-theoretic and algebro-geometric features of the situation. For example, the following properties may help in producing objects of \mathcal{D} by flat descent, approximation, or extensions:

- (α) \mathcal{D} is a stack for the fpqc topology.
- (β) \mathcal{D} is limit-preserving, or in other words locally of finite presentation.
- (γ) \mathcal{D} is stable under group extensions.

Date: July 2, 2018

2010 Mathematics Subject Classification: Primary 14L15; Secondary 14M17, 20G05, 20G35

Keywords: affine group schemes, resolution property, linear embedding, homogeneous space

However, the relations between Representation Theory and Geometry are often unpredictable, and it is difficult in general to decide whether the categories \mathcal{C}_* with $*$ \in $\{\text{lin, mono, recons, resol}\}$ satisfy these properties. In fact, it is not even clear a priori if the \mathcal{C}_* are *fibred* subcategories of \mathcal{C} .

∴

Let us come to our contributions. We propose the subcategory $\mathcal{C}_p \subset \mathcal{C}$ of *pure* group schemes as the “good” object of study in relation with the questions raised. The notion of purity was introduced by Raynaud and Gruson in [RG71] in order to understand the geometric meaning of projectivity of modules; it is defined in terms of a kind of “valuative criterion at generic points” (see 3.1 for the precise definition). While flat morphisms have fibres enjoying some continuity properties, flat *and pure* morphisms have fibres even closer to each other. The power and relevance of this notion is proved by the following landmark result of Raynaud and Gruson.

Theorem. *Let $R \rightarrow A$ be flat, finitely presented morphism of rings. Then the morphism on spectra is pure if and only if A is a projective R -module.*

Proof. See [RG71, Première partie, Th. 3.3.5]. □

This gives a way to think about groups in \mathcal{C}_p that may be more appealing to algebraists, and the index ‘p’ in \mathcal{C}_p may be interpreted as meaning ‘pure’ or ‘projective’ according to the reader’s preference.

Our first main theorem builds on the results of Raynaud and Gruson; it gives the first reasons why \mathcal{C}_p is arguably a good object, namely, it enjoys some geometric properties and is big enough.

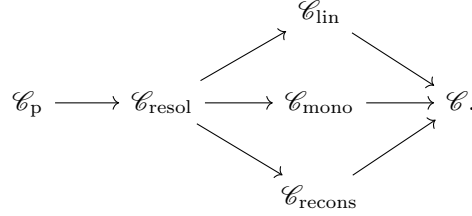
Theorem A. (See 3.4, 3.5, 3.7, 3.8)

- (1) \mathcal{C}_p satisfies properties (α) , (β) , (γ) ;
- (2) \mathcal{C}_p contains all group schemes with connected fibres, all finite flat group schemes, and all extensions of such groups;
- (3) $\mathcal{C}_p(R) = \mathcal{C}(R)$ if R is a zero-dimensional ring.

Next we enquire about the relations between \mathcal{C}_p and the other four categories presented at the outset. Over fields or Noetherian base rings, classical arguments show that $\mathcal{C}_{\text{resol}}$ is a subcategory of \mathcal{C}_{lin} , $\mathcal{C}_{\text{mono}}$ and $\mathcal{C}_{\text{recons}}$. This can be extended to arbitrary rings by calling upon recent results of Rydh and Schäppi (see 2.1, 2.3, 2.4). Thus the key question becomes: which groups have a classifying space satisfying the resolution property, or otherwise said how big is $\mathcal{C}_{\text{resol}}$? Known results are available mostly when R is Noetherian, regular of dimension at most 2; in this case $\mathcal{C}_p(R) = \mathcal{C}_{\text{resol}}(R)$ (Thomason [Tho87, 2.5]) and $\mathcal{C}_{\text{lin}}(R) = \mathcal{C}(R)$ ([SGA3_I, Exp. VI_B, Prop. 13.2]). Thomason also showed that $\mathcal{C}_{\text{resol}}$ contains all semisimple groups over a Noetherian ring, as well as groups of multiplicative type and reductive groups over a Noetherian normal ring, see [Tho87, 2.16, 2.17, 2.18]. Not much is known apart from these cases; for instance, the linearity question as raised in [SGA3_{II}, Exp. XI, Rem. 4.3] remains open for base rings as simple as the ring of dual numbers, even for smooth group schemes, see Conrad [Con14, Rem. 2.3.3] and the MathOverflow post [Con12].

Our second main theorem gives another reason why \mathcal{C}_p is nice: it is included in $\mathcal{C}_{\text{resol}}$, hence a good lower approximation to our four favourite categories. For completeness, we include all known inclusions in the diagram.

Theorem B. (See 4.3) *We have the following diagram where all arrows feature inclusions:*



The proof that any $G \in \mathcal{C}_{\text{p}}(R)$ lies in $\mathcal{C}_{\text{resol}}(R)$ derives from the key case where $\mathcal{O}(G)$ is a free R -module, together with the result of Bass on freeness of nonfinitely generated projective modules over Noetherian rings (see [Bas63], recalled in 4.2). Let us emphasize that together with A(2) this answers the linearity question (and others), for arbitrary base rings, and all groups which are extensions of finite locally free group schemes by fibrewise connected group schemes. Also, together with A(3) this answers the most general version of Conrad's question: all affine, flat, finite type group schemes over a zero-dimensional (for instance Artinian) ring have an embedding into GL_n for some $n \geq 0$.

Our third main theorem is about homogeneous spaces. Recall that Raynaud asked the following questions for a smooth (possibly non-affine) finitely presented S -group scheme G and a flat, finitely presented, closed subgroup scheme H .

- (i) Is G/H representable by a scheme, at least if all residue characteristics of S are 0 or if H has connected fibres?
- (ii) Is G/H quasi-projective if S is normal, Noetherian, integral, and G has connected fibres?

See [Ray70, XV, 6] and [Ray70, XV, 2.ii)]. We stick to the case where G is affine, as both questions are already difficult in this situation. Positive answers were given when S is Noetherian of dimension at most 1 (see Anantharaman [An73, Chap. IV] who does not assume affineness of G, H) or when S is affine excellent Noetherian regular of dimension 2, and G, H are smooth fibrewise connected (see Pappas and Zhu [PZ13, Cor. 11.5]). In general, Raynaud gave a counterexample to representability by a scheme ([Ray70, X, 13]) where $S = \mathbb{A}_k^2$ is the affine plane over a field k of characteristic 2, the group $G = (\mathbb{G}_{a,S})^2$ is the square of the additive group, and H is a closed étale subgroup scheme. Our result improves on Pappas and Zhu's in two directions: we allow an arbitrary base scheme S , and we allow arbitrary groups $G, H \in \mathcal{C}_{\text{p}}(S)$.

Theorem C. (See 5.2) *Let S be a scheme and $G, H \in \mathcal{C}_{\text{p}}(S)$ such that $H \subset G$ is a closed subgroup. Then the fppf quotient G/H is representable and Zariski-locally quasi-projective over S . More precisely, for every affine open $U \subset S$ the restriction $(G/H)_U \rightarrow U$ is quasi-projective.*

The method of proof is to establish the analogue of Chevalley's theorem realizing H as the stabilizer of a line in a suitable representation of G , see 5.1. We deduce a criterion of descent for homogeneous spaces that have a section after a finite locally free surjective morphism (see 5.3). Finally, we mention various applications to multiplicative groups. In particular, we prove that a finitely presented group of multiplicative type is always locally isotrivial (see 6.3), and correct the unproven claimed counterexample that appears in [SGA3II, Exp. X, 1.6 and Exp. XI, Rem. 4.6].

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In most of the paper, we work over affine base schemes. In some places however, when it does not cost more, we allow more general bases. Sometimes slight modifications are necessary; for example, for an arbitrary scheme S the category $\mathcal{C}_{\text{lin}}(S)$ should be defined as the category of affine, flat, finitely presented group schemes $G \rightarrow S$ which admit a closed embedding $G \hookrightarrow \text{GL}(\mathcal{V})$ for some finite, locally free \mathcal{O}_S -module \mathcal{V} . We added such precision when needed.

The organization of the article follows the order in which the results were presented, so that we do not need to say much. In Section 2 we recall the arguments showing that $\mathcal{C}_{\text{resol}}$ is a subcategory of \mathcal{C}_{lin} , $\mathcal{C}_{\text{mono}}$ and $\mathcal{C}_{\text{recons}}$. In Section 3 we prove Theorem A; in Section 4 we prove Theorem B; in Section 5 we prove Theorem C. Section 6 is then devoted to the applications of these results to the study of groups of multiplicative type. Finally, Section 7 collects some questions left open by our work.

Acknowledgements. The second author would like to thank the executive and administrative staff of IRMAR and of the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment.

2. CONSEQUENCES OF THE RESOLUTION PROPERTY

Let X be an algebraic stack, then X is said to *satisfy the resolution property* if for every quasi-coherent module of finite type M on X there exists a locally free \mathcal{O}_X -module of finite rank P and an epimorphism $P \rightarrow M$. To satisfy the resolution property is quite strong; if X is quasi-compact quasi-separated and the stabilizers at closed points are affine, it is actually equivalent (see [Tot04] and [Gro17]) for the algebraic stack X to be isomorphic to $[U/GL_n]$ for some quasi-affine scheme U .

We will be interested in the classifying stack $X = BG$ for a group scheme $G \in \mathcal{C}$. In this case the resolution property translates into the more down-to-earth property that every G -module of finite type over S is the quotient under a G -equivariant map of a locally free G -module of finite rank over S .

Note that if BG has the resolution property then so does S , but while there exist affine group schemes over some separated k -variety for k a field not satisfying the resolution property (see [SGA3_I, Exp. X, Sec. 1.6]) to our knowledge there is no example of a separated scheme of finite type over a field not satisfying the resolution property (the diagonal of a scheme satisfying the resolution property needs to be an affine morphism and hence there are non-separated counterexamples, see [Tot04, Prop. 1.3]).

2.1. Resolution property implies Linearity. If the base S is Noetherian then it is well known that if BG satisfies the resolution property then G is linear, that is it is isomorphic to a closed subgroup scheme of $\text{GL}(P)$, for some P locally free of finite rank over S . The Noetherian hypothesis on S though can be relaxed:

Proposition 2.1. *Let G be an affine, flat, finitely presented group scheme over a quasi-compact quasi-separated scheme S . If every G -module of finite type is the G -equivariant quotient of a locally free G -module of finite rank (that is, if BG satisfies the resolution property) then G admits a closed group embedding in $\text{GL}(P)$ over S , where P is a locally free module of finite rank over S .*

Proof. If S is Noetherian the result is due to Thomason when G is essentially free over S ([Tho87, Thm. 3.1]) and is generalized in [SGA3_I, Exp. VI_B, Prop. 13.5]) without the extra assumption on G . In general, the only missing part is some approximation theorem for G -quasi-coherent modules. But as S is quasi-compact and quasi-separated so is BG : the surjective map $S \rightarrow BG$ ensures quasi-compactness, while the diagonal morphism of BG is quasi-compact as its base change through the

fppf map $S \times_S S \rightarrow [S/G] \times_S [S/G]$ is simply $G \rightarrow S$ which is quasi-compact. The Main Theorem of [Ryd16] says that every quasi-coherent G -module is the union of its sub- G -modules of finite type. As S is quasi-compact, there exists a finite open covering by affine opens on which G is affine and its algebra is finitely generated. As S is quasi-separated, every open embedding of an affine open subscheme is quasi-compact. Hence by [EGA1_{new}, Cor. 6.9.3] every finite type quasi-coherent sub-module of the restriction of $h_*\mathcal{O}_G$ on these open affines extends to a finite type coherent submodule of $h_*\mathcal{O}_G$ on the whole S . In particular, by the approximation theorem for BG , there exists a finitely generated sub- G -module M of $h_*\mathcal{O}_G$, containing a generating set of $h_*\mathcal{O}_G$ as a \mathcal{O}_S -algebra. If P is a locally free G -module of finite rank and $P \rightarrow M$ a G -equivariant epimorphism, it induces a S group morphism $G \rightarrow \mathrm{GL}(P)$. The rest of the proof goes as in [SGA3_I, Exp. VI_B, Prop. 13.5]: in order to check that it is a closed embedding it suffices to do it locally on S hence we can assume that S is affine and P is free, but then the classical argument (see for example the proof of [Mil17, Th. 4.9]) carries over giving the surjectivity of the map on the level of Hopf algebras. \square

2.2. Resolution property implies Monogenicity. Let R be a ring and $G \in \mathcal{C}(R)$ an affine, flat, finitely presented R -group scheme. We want to extend some classical results about the category of representations of G (we refer to [Mil17, Section 4e] for the same results when R is a field).

Lemma 2.2. *Let G be an affine, flat group scheme over an affine scheme $\mathrm{Spec} R$. Then every free finite dimensional representation V of G embeds in a finite sum of copies of the regular representation.*

Proof. This is [Mil17, Prop. 4.12] which is stated over a base field but carries over verbatim provided $\mathcal{O}(G)$ is flat. \square

Proposition 2.3. *Let G be a closed subgroup of $\mathrm{GL}_{n,R}$, flat over R and let V be the induced free faithful representation of G . Then every free finite dimensional representation of G over R is a subquotient of a finite sequence of direct sums, tensor and duals of V . In particular, if BG satisfies the resolution property, then the same holds for every representation of finite type over R .*

Proof. Because of Proposition 2.1, the second part of the proposition follows from the first, for which we recall the proof of [Mil17, Th. 4.14]. By Lemma 2.2 it is enough to prove the proposition for each free representation of finite rank $V \subset \mathcal{O}(G)^m$ seen as direct sum of m copies of the regular representation. Let B be the ring of global sections of $\mathrm{GL}_{n,R}$, that is

$$B = R[X_{ij}, \det^{-1} \mid 1 \leq i, j \leq n],$$

as G is a closed subgroup scheme of the latter we have a surjection

$$\pi : B^m \rightarrow \mathcal{O}(G)^m.$$

Moreover, as V has finite rank, there exist $s, r \in \mathbb{N}$ such that V is contained in the image of $F_{r,s}^m$, where $F_{r,s} = \det^{-r}\{P(X_{ij}) \mid \deg(P) \leq s\}$. In particular, we can assume without loss of generality that $V = F_{r,s}^m$.

The rest of the proof goes as in [Mil17, Th. 4.14]: if $\{v_j\}$ is a basis of R^n , then the morphism $V = R^n \rightarrow B$, $v_j \mapsto X_{ij}$ is a morphism of comodules, in particular $F_{0,1} \simeq V^n$ as R -comodules. The module of homogeneous polynomials of degree s is a quotient of $F_{0,1}^s$ and $F_{0,s}$ is hence obtained as direct product of the homogeneous terms. Moreover $F_{0,n}$ contains the determinant, hence the 1-dimensional representation $M \mapsto \det M$, whose dual is $M \mapsto (\det M)^{-1}$ and $F_{r-1,s}$ with the latter gives $F_{r,s}$. \square

2.3. Resolution property implies Reconstruction. Let $\mathcal{O}(G)$ be the Hopf algebra associated to an affine group finitely presented over $S = \text{Spec } R$, then it is a classical result that $\mathcal{O}(G)$ can be uniquely reconstructed from its category of $\mathcal{O}(G)$ -comodule finitely presented over R (see for example [Mil17, Note 9.4]). If one wants to reconstruct G only from the projective $\mathcal{O}(G)$ -comodules of finite type, though, this may not always be true. Fortunately, the resolution property for BG implies that this is the case for G , hence the following.

Theorem 2.4 ([Sch13, Thm. 1.3.2]). *Let R be any ring and G be an affine flat R -group scheme. If BG satisfies the resolution property, then G can be uniquely reconstructed, up to isomorphism, from its category of representations that are free and of finite rank over R . Namely, $G \rightarrow \underline{\text{Aut}}^{\otimes}(\text{for})$ is an isomorphism of group functors on R -algebras, where $\text{for} : \text{PRep}_R(G) \rightarrow \text{Mod}_R$ is the forgetful functor, and $\text{PRep}_R(G)$ is the category of $\mathcal{O}(G)$ -comodules whose underlying R -module is projective and of finite type over R .*

Proof. This is a rephrasing of [Sch13, Thm. 1.3.2], where the condition of the theorem is satisfied by [Sch13, Corollary 7.5.2]: here, *Cauchy comodules* are simply the comodules whose underlying R -module is projective and of finite type (see [Sch13, Def. 1.3.1]) and to *generate* the category of all comodules corresponds to the resolution property (see the beginning of the proof on [Sch13, Corollary 7.5.2]). \square

3. THE SUBCATEGORY OF PURE GROUPS

In this section, we introduce the subcategory of pure group schemes, and give its basic properties. The notion of purity elucidates the geometric content of the notion of projectivity of modules. We first recall the definition from and the main facts we shall use; the reference is [RG71, Première partie, § 3.3].

Definition 3.1. A morphism of schemes $X \rightarrow S$ locally of finite type is called *pure* if for each point $s \in S$ with henselization $(\tilde{S}, \tilde{s}) \rightarrow (S, s)$, and each point $\tilde{x} \in \tilde{X} := X \times_S \tilde{S}$ which is an associated point in its fibre, the closure of \tilde{x} in \tilde{X} meets the special fibre $X \otimes k(\tilde{s})$.

For a scheme S , we let $\mathcal{C}_p(S) \subset \mathcal{C}(S)$ denote the full subcategory whose objects are the pure group schemes. An example of pure morphism is given in [RG71, Première partie, Ex. 3.3.4(iii)]. Given its importance for us, we reproduce it here and provide a proof.

Lemma 3.2. *Let $f : X \rightarrow S$ be a morphism which is flat, of finite type, universally open, with geometrically irreducible fibres without embedded components. Then X is S -pure.*

Proof. By definition, replacing S by its henselization at an arbitrary point s , we may assume that S is local Henselian and we have to prove that the closure of a point $x' \in X$ which is associated in its fibre $X_{s'}$, $s' = f(x')$, meets the special fibre X_s . Let Z be the closure of s' in S . Since S is local, Z meets s and hence we may replace S by Z and assume that S is irreducible with generic point s' . Since $X \rightarrow S$ is open with irreducible fibres, it follows that X is irreducible, see [SP18, Tag 004Z]. Now the fibre $X_{s'}$ is irreducible without embedded component, hence the assassin $\text{Ass}(X_{s'})$ is a single point, that is $\text{Ass}(X_{s'}) = \{x'\}$. This means that x' is the generic point of the generic fibre, hence the generic point of X . It follows that its closure is equal to X , and meets the special fibre (note that irreducible implies nonempty by definition). \square

One key fact is the relation between purity and projectivity of modules given by Raynaud and Gruson's theorem quoted in the introduction. Let $\mathcal{C}_{\text{proj}} \subset \mathcal{C}$

be the full subcategory composed of the group schemes whose Hopf algebra is a projective module over the base; the previous theorem implies that $\mathcal{C}_{\text{proj}} = \mathcal{C}_{\text{p}}$. We will hence denote both categories by \mathcal{C}_{p} and we shall think about it in a way or in the other, depending on whether we wish to emphasize the algebraic aspect (projectivity) or the geometric one (purity). Also let $\mathcal{C}_{\text{conn}}$ resp. \mathcal{C}_{fin} be the full subcategories of group schemes which have connected fibres, resp. which are finite. We shall establish basic properties of these categories. Beforehand, it is helpful to start with \mathcal{C} itself.

Lemma 3.3. *The fibred category \mathcal{C} satisfies:*

- (α) \mathcal{C} is a stack for the fpqc topology.
- (β) \mathcal{C} is limit-preserving.
- (γ) \mathcal{C} is stable under group extensions in the category of fpqc sheaves.

Proof. (α) Follows from fpqc descent of relative affine schemes, and the fact that flatness and finite presentation are fpqc local on the base.

(β) We need to prove that for all filtering direct systems of rings $(R_i)_{i \in I}$, the natural functor $\varinjlim \mathcal{C}(R_i) \rightarrow \mathcal{C}(\varinjlim R_i)$ is an equivalence. This follows from the usual arguments on limits found in [EGA4.3] and the Stacks Project Section on Limits of Schemes [SP18, Tag 01YT]. We give some detail now and will be more sketchy when these arguments come again later. References are to [SP18]. Write $S_i = \text{Spec}(R_i)$. There exists some index i and an S_i -scheme of finite presentation G_i (Tag 01ZM) which can be chosen affine (Tag 01ZN) and flat (Tag 04AI), and morphisms $m_i : G_i \times_{S_i} G_i \rightarrow G_i$ and $e_i : S_i \rightarrow G_i$ (Tag 01ZM). The index i can be chosen so that the group axioms are satisfied (Tag 01ZM(3)). The fact that these choices can be made simultaneously is granted by Tag 05FL.

(γ) This follows from (α) because if $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is an exact sequence of sheaves with $G', G'' \in \mathcal{C}$, then the map $G \rightarrow G''$ is an fpqc torsor under the group G' , hence G is representable by an object of \mathcal{C} . \square

We can build on this to study $\mathcal{C}_{\text{p}}, \mathcal{C}_{\text{conn}}, \mathcal{C}_{\text{fin}}$.

Lemma 3.4. *If R is a zero-dimensional ring, we have $\mathcal{C}_{\text{p}}(R) = \mathcal{C}(R)$.*

Proof. Let $G \in \mathcal{C}(R)$. Since purity is checked at henselizations at the points of $\text{Spec}(R)$, we may assume that R is Henselian. Being zero-dimensional, its spectrum then is a point, and the condition of Definition 3.1 is trivially satisfied. \square

Lemma 3.5. *The fibred category \mathcal{C}_{p} satisfies (α), (β), (γ).*

Proof. (α) Descent of projectivity of modules through faithfully flat morphisms is proved by Raynaud and Gruson; see [RG71, Seconde partie, (3.1.4).1] and the complements in Perry [Per10] and the Stacks Project [SP18, Tag 05A9].

(β) In view of the equivalence between projectivity and “flatness plus purity” [RG71, Première partie, Th. 3.3.5], descent of projectivity through limits is the combination of descent of flatness together with [RG71, Première partie, 3.3.10].

(γ) We start with a remark: if $A \rightarrow B$ is a ring homomorphism which makes B a projective A -module, and P is a projective B -module, then P is projective as an A -module. Indeed, we can find an A -module C , and index set I , and an isomorphism of A -modules $B \oplus C \simeq A^{(I)}$. Similarly we can find a B -module Q , index set J , and an isomorphism of B -modules $P \oplus Q \simeq B^{(J)}$. Then

$$P \oplus Q \oplus C^{(J)} \simeq B^{(J)} \oplus C^{(J)} \simeq (A^{(I)})^{(J)} \simeq A^{(I \times J)}$$

is a free A -module, proving that P is projective over A .

Now we prove (γ). Note that according to 3.3(γ), extensions in the category of sheaves or in \mathcal{C} are the same. We need to prove that if $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is an

exact sequence in $\mathcal{C}(R)$ for some ring R , then $G', G'' \in \mathcal{C}_p(R)$ implies $G \in \mathcal{C}_p(R)$. Given that $\mathcal{O}(G'')$ is projective over R , by the preliminary remark it is enough to prove that the ring map $\mathcal{O}(G'') \rightarrow \mathcal{O}(G)$ makes $\mathcal{O}(G)$ a projective $\mathcal{O}(G'')$ -module. By faithfully flat descent of projectivity, it is enough to check this after the faithfully flat extension $\mathcal{O}(G'') \rightarrow \mathcal{O}(G)$. Since $G \rightarrow G''$ is a G' -torsor, we have an isomorphism $\mathcal{O}(G) \otimes_{\mathcal{O}(G'')} \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes_R \mathcal{O}(G')$. Since $\mathcal{O}(G')$ is projective over R , this ring is projective as an $\mathcal{O}(G)$ -module, as desired. \square

Lemma 3.6. *The fibred category $\mathcal{C}_{\text{conn}}$ satisfies (α) , (β) , (γ) .*

Proof. (α) and (γ) are clear. For (β) recall that for group schemes over a field, connectedness and geometric connectedness are the same because of the existence of the neutral rational point. The condition to have geometrically connected fibres descends through filtering directed limits by [SP18, Tag 05F1]. \square

Lemma 3.7. *We have $\mathcal{C}_{\text{conn}} \subset \mathcal{C}_p$.*

Proof. A connected group scheme over a field is geometrically irreducible [SGA3_I, Exp. VI_A, Thm. 2.6.5] and any group scheme over a field is locally complete intersection [SGA3_I, Exp. VII_B, Cor. 5.5.1] hence without embedded points. It follows from Lemma 3.2 that any group $G \in \mathcal{C}_{\text{conn}}(S)$ is pure. \square

Lemma 3.8. *The fibred category \mathcal{C}_{fin} satisfies (α) , (β) , (γ) , and $\mathcal{C}_{\text{fin}} \subset \mathcal{C}_p$.*

Proof. These are standard facts. \square

4. THE RESOLUTION PROPERTY

The main result of this section is that when the base S is affine and $G \in \mathcal{C}(S)$ is a *pure* S -group scheme, then its classifying stack BG satisfies the resolution property; see Corollary 4.3. The key to this is that this holds as soon as the Hopf algebra of the group scheme is free over the base, as we now prove.

Theorem 4.1. *Let R be any ring and G an affine group scheme over $S = \text{Spec } R$ such that $\mathcal{O}(G)$ is free over R as a module. Let V be an $\mathcal{O}(G)$ -comodule, finitely generated over R . Then there exists an $\mathcal{O}(G)$ -comodule W free and of finite rank over R and a surjective $\mathcal{O}(G)$ -invariant map $\varphi : W \rightarrow V$. Said otherwise, BG satisfies the resolution property by free objects.*

Proof. Let $\rho : V \rightarrow V \otimes \mathcal{O}(G)$ denote the coaction and x_1, \dots, x_n a system of generators for V . We will construct W_1, \dots, W_n free $\mathcal{O}(G)$ -comodules of finite rank and $\mathcal{O}(G)$ -equivariant maps $\varphi_i : W_i \rightarrow V$ such that x_i is in the image of φ_i . Taking $W = \bigoplus_{i=1}^n W_i$ and $\varphi = \bigoplus \varphi_i$ will yield the desired object.

What we will actually prove is that for every $x \in V$ there exists W_x an $\mathcal{O}(G)$ -comodule, free of finite rank over R and a $\mathcal{O}(G)$ -equivariant map $\phi_x : W_x \rightarrow V$ such that x is in the image of ϕ_x .

Fix hence $x \in V$ and a R -basis $\{g_i\}_{i \in I}$ of $\mathcal{O}(G)$, then we can write

$$\rho(x) = \sum_i v_i(x) \otimes g_i,$$

where the $v_i(x)$ are in V . It is a classical calculation (see for example the proof of [Mil17, Prop. 4.7]) that if $\Delta(g_i) = \sum_{jk} \alpha_{jk}^i g_j \otimes g_k$ then

$$(1) \quad \rho(v_k(x)) = \sum_{ij} \alpha_{jk}^i v_i(x) \otimes g_j$$

and in particular $\langle v_i(x) \rangle$ is a finitely generated $\mathcal{O}(G)$ -submodule of V , containing $x = \sum_i v_i \varepsilon(g_i)$.

Let $I_x \subset I$ be the set consisting of all i such that $v_i(x)$ is nonzero. It is a finite set and we define W_x to be the free R -module with basis $\{w_i\}_{i \in I_x}$. We want to put a structure of comodule on W_x so that the R -module morphism $w_i \mapsto v_i(x)$ is actually a morphism of $\mathcal{O}(G)$ -comodules. We define

$$\rho_x(w_k) = \sum_{ij} \alpha_{jk}^i w_i \otimes g_j,$$

and we need to check the two axioms of comodules: that $(id \otimes \Delta) \circ \rho_x = (\rho_x \otimes id) \circ \rho_x$ and that $(id \otimes \varepsilon) \circ \rho_x = id$.

Let's start with the first one: we need to exploit the similar axiom that holds on $\mathcal{O}(G)$ when considered as a comodule over itself. If we write $\Delta(g_i) = \sum_{jk} \alpha_{jk}^i h_j \otimes h_k$ and we apply $\Delta \otimes id$ then we get

$$(\Delta \otimes id)(\Delta(g_i)) = \sum_{jkmn} \alpha_{jk}^i \alpha_{mn}^j h_m \otimes h_n \otimes h_k.$$

On the other hand, if we write $\Delta(g_i) = \sum_{uv} \alpha_{uv}^i h_u \otimes h_v$ then we have

$$(id \otimes \Delta)(\Delta(g)) = \sum_{uvzw} \alpha_{uv}^i \alpha_{wz}^v h_u \otimes h_w \otimes h_z.$$

Then the two need to be equal by the comultiplication axiom on $\mathcal{O}(G)$, hence by looking at the coefficient of $h_m \otimes h_n \otimes h_k$ we get that

$$(2) \quad \sum_j \alpha_{jk}^i \alpha_{mn}^j = \sum_j \alpha_{mj}^i \alpha_{nk}^j.$$

Now to prove the axioms for ρ_x , let us write down the two sides of $(id \otimes \Delta) \circ \rho_x = (\rho_x \otimes id) \circ \rho_x$: if $\rho_x(w_k) = \sum_{ij} \alpha_{jk}^i w_i \otimes g_j$ then

$$(id \otimes \Delta) \circ \rho_f(w_k) = \sum_{ij} \alpha_{jk}^i w_i \otimes \Delta(g_j) = \sum_{ijmn} \alpha_{jk}^i \alpha_{mn}^j w_i \otimes g_m \otimes g_n$$

while if we rename the indices and $\rho_f(w_k) = \sum_{uv} \alpha_{uk}^v w_v \otimes g_u$ then

$$(\rho_f \otimes id) \circ \rho_f(w_k) = \sum_{uvwz} \alpha_{uk}^v \alpha_{zv}^w w_w \otimes g_z \otimes g_u,$$

hence by looking at the coefficients of $w_i \otimes g_m \otimes g_n$ we get the axiom is fulfilled if and only if

$$\sum_j \alpha_{jk}^i \alpha_{mn}^j = \sum_j \alpha_{nk}^j \alpha_{mj}^i,$$

but this is exactly (2).

Now the counit axiom. We need to show that $(id \otimes \varepsilon) \circ \rho_f(w_k) = w_k$, but

$$(id \otimes \varepsilon) \circ \rho_f(w_k) = \sum_{ij} \alpha_{jk}^i w_i \varepsilon(h_j)$$

hence it is equal to w_k if and only if $\sum_j \alpha_{jk}^i \varepsilon(h_j) = \delta_{ik}$, because the w_i form a free basis.

But $\Delta(h_i) = \sum_{jk} \alpha_{jk}^i h_j \otimes h_k$ hence by the counit axiom on $\mathcal{O}(G)$ we have $h_i = \sum_{jk} \alpha_{jk}^i \varepsilon(h_j) h_k$ and hence $\sum_j \alpha_{jk}^i \varepsilon(h_j) = \delta_{ik}$. It is clear from the definitions that $w_i \mapsto v_i(x)$ is a morphism of $\mathcal{O}(G)$ -comodules, in particular $W_x \rightarrow \langle v_i(x) \rangle$ is a surjective morphism of $\mathcal{O}(G)$ -comodules, with W_x free and of finite rank. \square

To draw consequences of this result for a group $G \in \mathcal{C}_p$, we will use a theorem of Bass which we recall for the convenience of the reader.

Theorem 4.2. *Let R be a Noetherian ring with connected spectrum. Then any nonfinitely generated projective R -module is free.*

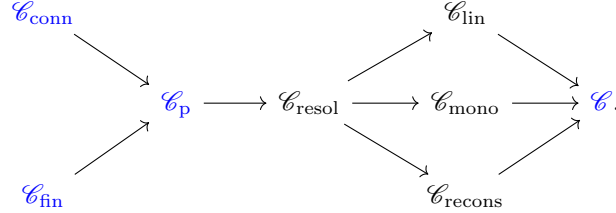
Proof. See [Bas63, Cor. 4.5]. \square

Corollary 4.3. *Let $G \in \mathcal{C}_p(R)$ be an affine, flat, finitely presented R -group scheme whose Hopf algebra is projective. Then BG satisfies the resolution property.*

Proof. First assume that $\mathcal{O}(G)$ is a finite R -module, that is G is finite locally free. Let M be a G - R -module of finite type and x_1, \dots, x_n a system of R -module generators. Then M is a G -equivariant quotient of $\mathcal{O}(G)^n$ where $\mathcal{O}(G)$ is the regular representation. Since $\mathcal{O}(G)$ is R -projective, there is an R -module C such that $F := \mathcal{O}(G) \oplus C$ is finite free. Endow C with the trivial G -module structure. Then the projection $F^n \rightarrow \mathcal{O}(G)^n \rightarrow M$ resolves M by an R -finite free G -module.

Now assume that $\mathcal{O}(G)$ is not finite. Write R as the directed union of its subrings R_i finitely generated over \mathbb{Z} . Using that \mathcal{C}_p is limit preserving (Lemma 3.5), we see that there exists an index i and a group scheme $G_i \in \mathcal{C}_p(R_i)$ whose base change to R is G . In this way we reduce to the case where R is Noetherian. Working on the individual connected components, we reduce to the case where R is Noetherian with connected spectrum. Then it follows from Bass's Theorem 4.2 that $\mathcal{O}(G)$ is free over R . Now Theorem 4.1 applies and gives the conclusion. \square

Let us take up the picture from the introduction, illustrating the inclusion relations between our categories of group schemes. We write in black the categories with representation-theoretic interest, with a priori few geometric properties; and we write in blue the categories we introduced, because they *do* have such properties, to study the preceding ones.



When S is not affine, it is less frequently the case that the sheaf of functions of G is free. Still if that happens and S has an ample line bundle (in the sense of [EGA2, Def. 4.5.3]) the same result holds:

Corollary 4.4. *Let S be a quasi-compact scheme having an ample line bundle and let $h : G \rightarrow S$ be a group scheme, affine over S and such that $h_*\mathcal{O}_G$ is globally free over S . Let V be a $\mathcal{O}(G)$ -module, coherent over S . Then there exists a locally free coherent G -comodule W and a surjective G -invariant map $\varphi : W \rightarrow V$. Said otherwise, BG satisfies the resolution property.*

Proof. Recall that the direct image of the structure sheaf gives an equivalence between affine morphisms over S and sheaves of \mathcal{O}_S -algebras on S . In particular, once fixed an isomorphism $h_*\mathcal{O}_G \simeq \bigoplus_i \mathcal{O}_S g_i$, then the multiplication $m : G \times_S G \rightarrow G$ is uniquely determined by the corresponding map of \mathcal{O}_S -algebra sheaves $\Delta : h_*\mathcal{O}_G \rightarrow h_*\mathcal{O}_G \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G$. On the global sections Δ induces

$$H^0(\Delta) : H^0(S, h_*\mathcal{O}_G) \rightarrow H^0(S, h_*\mathcal{O}_G \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G).$$

Now, as $h_*\mathcal{O}_G \simeq \bigoplus_i \mathcal{O}_S g_i$, we have that

$$\begin{aligned} H^0(S, h_*\mathcal{O}_G \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G) &= H^0(S, \bigoplus_{i,j} \mathcal{O}_S \cdot g_i \otimes g_j) = \bigoplus_{i,j} H^0(S, \mathcal{O}_S) \cdot g_i \otimes g_j \\ &= (\bigoplus_i H^0(S, \mathcal{O}_S) \cdot g_i) \otimes_{H^0(S, \mathcal{O}_S)} (\bigoplus_j H^0(S, \mathcal{O}_S) \cdot g_j) \\ &= H^0(S, h_*\mathcal{O}_G) \otimes_{H^0(S, \mathcal{O}_S)} H^0(S, h_*\mathcal{O}_G) \end{aligned}$$

where the equalities follow by the fact that global sections commute with direct sums. Writing $H^0(S, h_*\mathcal{O}_G) = \bigoplus_i H^0(S, \mathcal{O}_S) g_i$ we have that

$$H^0(\Delta)(g_i) = \sum_{jk} \alpha_{jk}^i g_j \otimes g_k,$$

with $\alpha_{jk}^i \in H^0(S, \mathcal{O}_S)$. As the g_i generates h_*G , the α_{jk}^i uniquely determine the group structure of G . Similarly, the unit map $e : S \rightarrow G$ is uniquely determined by the counit map $\varepsilon : h_*\mathcal{O}_G \rightarrow \mathcal{O}_S$ on the global sections, that is by $H^0(\varepsilon)(g_i) \in H^0(S, \mathcal{O}_S)$. Now let $M \in \text{Coh}(\mathcal{O}_S)$ be a G -module. Similarly as before, giving an action $G \times_S M \rightarrow M$, where M is the functor $M(f : T \rightarrow S) = (f^*M)$, is the same as giving a coaction $\rho : M \rightarrow M \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G$. Let $x \in H^0(M, \mathcal{O}_S)$, then as

$$H^0(\rho) : H^0(S, M) \rightarrow H^0(S, M \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G) = \bigoplus_i H^0(S, M) g_i,$$

we can write $H^0(\rho)(x) = \sum s_i(x) g_i$, with $s_i(x) \in H^0(S, M)$. Let I_x be the finite set of i such that $s_i(x)$ is not zero. Then we define $W_x = \sum_{i \in I_x} w_i$ with comodule structure $\rho_x(w_k) = \sum_{ij} \alpha_{jk}^i w_i \otimes g_j \in W_x \otimes_{\mathcal{O}_S} h_*\mathcal{O}_G$ and $W_x \rightarrow M$ given by $w_i \rightarrow s_i(x)$. To check that ρ_x is a coaction it suffices to do so on an open covering of S , hence it follows from Theorem 4.1.

If M is generated by its global sections we are done. If not, let L be an ample line bundle on S , then for some $n \in \mathbb{N}$ we have that $M \otimes L^n$ is generated by its global sections. Endowing L with the structure of trivial G -module, $M \otimes L^n$ has a natural structure of G -module and as it is generated by its global sections, there exists a free G -module W of finite rank and a G -equivariant epimorphism $W \rightarrow M \otimes L^n$. Tensoring the latter with L^{-n} , again endowed with the structure of trivial G -module, produces a G -equivariant epimorphism $W \otimes L^{-n} \rightarrow M$. \square

5. QUOTIENTS AND HOMOGENEOUS SPACES

In this section, we give some results of representability and quasi-projectivity for homogeneous spaces. We briefly recall some definitions; a general reference is Raynaud's book [Ray70]. Let S be a scheme, G an fppf sheaf in groups over S , and X an fppf sheaf with an action of G . We say that X is a *homogeneous sheaf (under G)* if the morphisms $X \rightarrow S$ and $G \times_S X \rightarrow X \times_S X$, $(g, x) \mapsto (gx, x)$ are covering for the fppf topology (that is, have sections fppf locally). We say that X is *trivial* if it has a section $x_0 \in X(S)$; in this case the morphism $G \rightarrow X$, $g \mapsto gx_0$ induces a G -isomorphism of sheaves $G/H \xrightarrow{\sim} X$ where H is the stabilizer of x_0 . We say that X is a *homogeneous space* if it is representable by an algebraic space or a scheme.

Let us start with trivial homogeneous spaces. Recall that over a field, Chevalley's theorem ([Mil17, Thm. 4.27]) says that every closed subgroup $H \subset G$ is the stabilizer of a line V in some representation W of G . When trying to extend this to a general base ring, one encounters similar problems as for proving linearity, namely the lack of enough projective (or free) representations. We prove that the construction that in Theorem 4.1 produces a projective cover for any $\mathcal{O}(G)$ -comodule can be refined to give a version of Chevalley's theorem in the category \mathcal{C}_p .

Theorem 5.1. *Let R be a ring and $H, G \in \mathcal{C}_p(R)$ such that H is a closed subgroup scheme of G . Then there exists a faithful free G -representation of finite rank $W = V \oplus V'$ for some free R -submodules $V, V' \subset W$ where V has rank 1, such that H is the stabilizer of V in G .*

Proof. As in Corollary 4.3, we can without loss of generality assume that R is Noetherian and connected. Also, by taking exterior powers as in the classical case, it is enough to find a faithful finite free G -representation $W = V \oplus V'$ for some free R -submodules $V, V' \subset W$ of arbitrary rank; see for example [Mil17, Lemma 4.28].

Let I be the ideal associated with H , that is the kernel of $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$. Recall that as I is a Hopf ideal, it satisfies the following two axioms (see [Mil17, Def. 3.10]):

$$(3) \quad \Delta(I) \subset I \otimes \mathcal{O}(G) + \mathcal{O}(G) \otimes I$$

$$(4) \quad \varepsilon(I) = 0.$$

Moreover, as $H \in \mathcal{C}_p$ the exact sequence of R -modules $I \rightarrow \mathcal{O}(G) \rightarrow \mathcal{O}(H)$ splits, hence I is projective as well.

Let us assume first that I and $\mathcal{O}(H)$ are projective R -modules of infinite rank. Then by Bass's Theorem 4.2 they are free and, as R -modules, $\mathcal{O}(G) = I \oplus C$ for some free complement C . Fix a basis $\{h_i\}$ of I and complete it with a basis $\{g_j\}$ of C to a basis of $\mathcal{O}(G)$.

Consider a finite set of generators a_1, \dots, a_n of the ideal I . As in the proof of Theorem 4.1, for every $a = a_m$ we can consider

$$(5) \quad \Delta(a) = \sum_i s_i(a) \otimes h_i + \sum_j t_j(a) \otimes g_j$$

and construct a free module W_a with basis σ_i^a and τ_j^a for all i , respectively j , such that $s_i(a)$, respectively $t_j(a)$, are nonzero. Moreover we can put on W_a a structure of comodule such that the map $\phi_a : W_a \rightarrow \mathcal{O}(G)$ given by $\sigma_i^a \mapsto s_i(a)$ and $\tau_j^a \mapsto t_j(a)$ is G -equivariant. Now remark the following two things: firstly, by (3), in the expression (5) the $t_j(a)$ are actually in I . Secondly, while it is clear that a is in the span of the $t_j(a)$ and $s_i(a)$, thanks to (4) we have

$$a = (id \otimes \varepsilon)(\Delta(a)) = \sum_i s_i(a) \otimes \varepsilon(h_i) + \sum_j t_j(a) \otimes \varepsilon(g_j) = \sum_j t_j(a) \otimes \varepsilon(g_j).$$

Hence a is in actually in the span of the $t_j(a)$.

Consider $W_\tau = \bigoplus_m W_{a_m}^\tau$, then it is a free direct summand of the G -representation $W = \bigoplus_m W_{a_m}$ and we claim that H is the stabilizer of V in W_τ (which is representable by [SGA3II, Exp. VIII, 6.5.e]). In order to prove this claim, write:

$$\Delta(h_i) = \sum_{jk} \alpha_{jk}^i h_j \otimes g_k + \sum_{jk} \beta_{jk}^i g_j \otimes h_k + \sum_{jk} \gamma_{jk}^i h_i \otimes h_k$$

where there is no $g_j \otimes g_k$ component by (3). Then for $a = a_m$, we have, similarly as in (1):

$$\Delta(t_k(a)) = \sum_i s_i(a) \otimes \left(\sum_j h_j \alpha_{jk}^i \right) + \sum_j t_j(a) \otimes \mu_{jk}^a$$

with some $\mu_{jk}^a \in \mathcal{O}(G)$ whose explicit expression can be given but is not needed. Hence, by definition of the G -action on W_a , we have:

$$\Delta(\tau_k^a) = \sum_i \sigma_i^a \otimes \left(\sum_j h_j \alpha_{jk}^i \right) + \sum_j \tau_j^a \otimes \mu_{jk}^a.$$

The stabilizer of W_τ in G is represented by the quotient of $\mathcal{O}(G)$ by the Hopf ideal generated by the $\sum_j h_j \alpha_{jk}^i$ that is by the $\sigma_i^{a_m}$ -coefficients in the expression

of $\Delta(\tau_k^{a_m})$, for $m = 1, \dots, n$ and i, j, k accordingly (generalize for example [Mil17, Prop. 4.3], using that everything is free and W_τ is a free direct summand of W).

Let hence I' be the ideal generated by the elements $\sum_j h_j \alpha_{jk}^i$; we want to show that $I = I'$. As the elements $\sum_j h_j \alpha_{jk}^i$ are linear combinations of elements in I , it is clear that $I' \subset I$. On the other hand we saw that a is a linear combination of $t_k(a)$ and:

$$\begin{aligned} t_k(a) &= (\varepsilon \otimes id)(\Delta(t_k(a))) \\ &= \sum_i \varepsilon(s_i(a)) \otimes \left(\sum_j h_j \alpha_{jk}^i \right) + \sum_j \varepsilon(t_j(a)) \otimes \beta_{jk}^a \\ &= \sum_i \varepsilon(s_i(a)) \otimes \left(\sum_j h_j \alpha_{jk}^i \right) \end{aligned}$$

because $t_j(a) \in I$ implies $\varepsilon(t_j(a)) = 0$ by (4). So we proved that for every m if $a = a_m$ then $t_k(a)$ is in the span of the $\sum_j h_j \alpha_{jk}^i$ therefore the same holds for the a_m , which are generators of I , hence $I = I'$. We obtain a *faithful* representation by replacing W with $W \oplus W'$, where W' is any faithful representation of G (which exists by Proposition 2.1 and Corollary 4.3).

We are left to prove that the same holds when $\mathcal{O}(H)$ or I has finite rank; the first case means that H is finite while the second case means that G/H is finite, because $\mathcal{O}(G/H) = R \oplus I^H$. In the first case consider $H \times \mathbb{G}_m \subset G \times \mathbb{G}_m$. Then the theorem holds for this closed embedding by what we have proven so far, hence there is a faithful finite free $(G \times \mathbb{G}_m)$ -representation $W = V \oplus V'$ with the desired properties, such that $H \times \mathbb{G}_m$ is the stabilizer of V . But then we can consider the restriction of W to $G \subset G \times \mathbb{G}_m$ and H will be the stabilizer of V . In the second case, consider $G \times \mathbb{G}_m$. Then there exists $W = V \oplus V'$ as before with H the stabilizer of V in the $(G \times \mathbb{G}_m)$ -module W . But then H is the stabilizer also once we restrict the representation W to G and we are done. \square

We deduce the sought-for extension of Raynaud's and Pappas–Zhu's results.

Theorem 5.2. *Let S be a scheme and $G, H \in \mathcal{C}_p(S)$ such that $H \subset G$ is a closed subgroup. Then the fppf quotient G/H is representable and Zariski-locally quasi-projective over S . More precisely, for every affine open $U \subset S$ the restriction $(G/H)_U \rightarrow U$ is quasi-projective.*

Proof. It is enough to assume S affine and prove that G/H is quasi-projective. We follow the ideas of the proof of [Con14, Thm. 2.3.1]. By Theorem 5.1, there exists a faithful free G -representation of finite rank $W = V \oplus V'$ with V, V' free sub-modules and V of rank 1, such that H is the stabilizer of V in W . The group G acts on the projectivization $\mathbb{P}(W)$, and H is the stabilizer of the point $P = [V]$. Now, $H \times_S G \rightarrow G \times_S G$ given by $(h, g) \mapsto (hg, g)$ is a closed embedding and defines an equivalence relation on G . As H is finitely presented and faithfully flat over S , it follows from Artin's theorem [Art74, Cor. 6.3] that the fppf quotient G/H is an algebraic space. But then the map $f : G/H \rightarrow \mathbb{P}(W)$ induced by $g \mapsto g(P)$ is well-defined and a monomorphism, as H is the stabilizer of P . Like any monomorphism the map f is separated ([SP18, Tag 042N]); being moreover quasi-finite it is quasi-affine ([SP18, Tag 05W4]). In particular G/H is a scheme. By Zariski's main theorem ([EGA4.4, Cor. 18.12.13]), f factors then into a quasi-compact immersion followed by a finite map. But anything finite over $\mathbb{P}(W)$ is projective over S , hence G/H is quasi-projective over S . \square

We now deduce a result of representability for those homogeneous spaces that have a section after a finite locally free base change (we might call such X *weakly isotrivial*), in the spirit of [Ray70, XI, Th. 3.1].

Proposition 5.3. *Let S be a scheme and $G \in \mathcal{C}_p(S)$. Let X be a separated homogeneous G -sheaf such that the morphism $j : G \times_S X \rightarrow X \times_S X$, $(g, x) \mapsto (gx, x)$ is faithfully flat, finitely presented and pure (the latter happens for example if the stabilizers of points are connected). Assume that X acquires a section after a finite locally free surjective base change $S' \rightarrow S$. Then X is representable and locally quasi-projective over S .*

Proof. Restricting to an open affine subscheme of S , it is enough to assume that S is affine and prove that X is representable and quasi-projective. After a finite locally free surjective extension $S' \rightarrow S$, the sheaf $X_{S'}$ has a section and there is an isomorphism $G_{S'}/H' \xrightarrow{\sim} X_{S'}$. Here $H' \rightarrow S'$ is the stabilizer, defined as the pullback of the universal stabilizer $\text{Stab} \rightarrow X$ so we have fibred products:

$$\begin{array}{ccc} H' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ \text{Stab} & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ G \times_S X & \xrightarrow{j} & X \times_S X. \end{array}$$

Since X is separated, the diagonal Δ is a closed immersion and so is $H' \rightarrow G_{S'}$. Given the assumptions on j , the morphism $H' \rightarrow S'$ makes H' an object of $\mathcal{C}_p(S')$. It follows from Proposition 5.2 that $X_{S'} = G_{S'}/H'$ is representable and quasi-projective over S' . By descent of quasi-projective schemes along finite locally free morphisms ([SGA1, Exp. VIII, Cor. 7.7] or [SP18, Tag 0CCH]), we obtain the desired result. \square

6. APPLICATIONS

In this section, we indicate various other applications.

6.1. Local isotriviality for groups of multiplicative type. We will follow here the exposition of [SGA3II, Exp. IX]. Let S be a scheme and G an affine group over S of multiplicative type, which by definition means that it is diagonalizable locally in the fpqc topology (we recall that a group over S is diagonalizable if it is of the form $D_S(M) := \underline{\text{Hom}}_{S\text{-gr}}(M_S, \mathbb{G}_m)$ for some commutative abstract group M). Then G is said to be *isotrivial* if there exists $S' \rightarrow S$ finite étale surjective such that $G_{S'}$ is diagonalizable and *locally isotrivial* if there exists a Zariski covering $\cup U_i$ of S such that each G_{U_i} is isotrivial.

Remark 6.1. Note that being locally isotrivial is stronger than being trivial in the étale topology: by [SGA3II, Exp. X, Cor. 4.5] every group scheme of multiplicative type and of finite type is automatically locally trivial in the étale topology. In particular we can always work in the étale topology rather than in the fpqc topology when we want to trivialize such a group.

We recall that a group scheme G is of multiplicative type if and only if it is of the form $\underline{\text{Hom}}_{S\text{-gr}}(M, \mathbb{G}_m)$ where M is a commutative group scheme over S which is locally constant in the fpqc topology; see [SGA3II, Exp. X, Th. 5.6, Cor. 5.7]. By [SGA3II, Exp. X, Prop. 5.3] we can again trivialize M with an étale covering, hence work in the étale topology rather than in the fpqc topology.

Proposition 6.2 ([SGA3II, Exp. XI, Rem. 4.6]). *Let S be a scheme and G a finitely presented group of multiplicative type over S . Then G is linear if and only if it is isotrivial.*

Proof. All references in this proof are to [SGA3II]. We are following verbatim Exp. XI, Rem. 4.6, simply giving a few more references for what is there claimed.

Assume that G is isotrivial, then there exists a finite étale cover $S' \rightarrow S$ such that $G_{S'} \simeq D_{S'}(M)$ for some abstract abelian group M , which is finitely generated as G is of finite type over S . The group $D_{S'}(M)$ obviously belongs to $\mathcal{C}_p(S')$, hence by descent $G \in \mathcal{C}_p(S)$. It follows that G is linear.

For the other direction, we start with some preliminary remarks. Let $M = \underline{\mathrm{Hom}}_{S\text{-gr}}(G, \mathbb{G}_m)$ be the group scheme over S locally constant in the fppf topology such that $G = \underline{\mathrm{Hom}}_{S\text{-gr}}(M, \mathbb{G}_m)$. By Exp. X, Prop. 5.11 the group G is isotrivial if and only if M is. Moreover M is isotrivial if and only if $M^\vee = \underline{\mathrm{Hom}}_{S\text{-gr}}(M, \mathbb{Z}_S)$ is isotrivial: of course if $S' \rightarrow S$ is finite étale trivializing M then it trivializes M^\vee as well. For the other direction, assume that $M_{S'}^\vee$ is trivial. As the objects under consideration are of finite type, M is fppf-locally isomorphic to a constant group scheme associated to an abstract group of finite type N . In particular $N = N_{\mathrm{tors}} \oplus \mathbb{Z}^m$ for some m , where the torsion subgroup N_{tors} is finite. By fppf descent (see Exp. X, Cor. 5.5), there exist a maximal torsion subgroup M_{tors} of M as well as the fppf quotient M/M_{tors} . As M is commutative, we can write it as the product of its torsion and free parts. The torsion part is finite and étale hence isotrivial, and the free part is isomorphic to $(M^\vee)^\vee$, which is isotrivial as M^\vee is.

By Exp. X, Cor. 5.7, if $M = \underline{\mathrm{Hom}}_{S\text{-gr}}(G, \mathbb{G}_m)$, then $M^\vee = \underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_m, G)$. By Exp. X, Prop. 5.11, the group G is isotrivial if and only if every connected component of M is finite over S , but as M is locally constant, it is the same to ask that M be affine over S . Putting together everything we have proven so far, we have that G is isotrivial if and only if M is isotrivial if and only if M^\vee is isotrivial, which is equivalent to ask that every connected component of M^\vee is affine over S .

Now assume that G is linear, that is $G \subset \mathrm{GL}(P)$ for some P locally free of finite rank over S . As the property of being affine is local on S , we can without loss of generality assume that $S = \mathrm{Spec} R$ is affine and that P is free, that is $G \subset \mathrm{GL}_{n,S}$ for some n . But then by Exp. VIII, Th. 6.4, the scheme $\underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_m, G)$ is a closed subscheme of $\underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_m, \mathrm{GL}_{n,S})$ hence it suffices to show that every connected component of the latter is affine (its representability is ensured by Exp. XI, Cor. 4.2). To conclude, it suffices to note that all cocharacters of GL_n are diagonalizable. Hence, writing $\underline{n} = (n_1, \dots, n_k)$ for the list of sizes of blocks in a diagonalization, we see that the scheme of cocharacters $X_*(\mathrm{GL}_n) = \underline{\mathrm{Hom}}_{S\text{-gr}}(\mathbb{G}_m, \mathrm{GL}_{n,S})$ is the disjoint sum over all \underline{n} of the schemes:

$$X_*(\mathrm{GL}_n)_{\underline{n}} = \mathrm{GL}_n / (\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}).$$

This homogeneous space is affine as $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$ is reductive. \square

Then the results of the previous sections imply:

Corollary 6.3. *If S is affine, every finitely presented group of multiplicative type is isotrivial. Hence on any scheme, every group of multiplicative type is locally isotrivial. In particular, if S is a scheme and $T \rightarrow S$ is a torus, there exists an open (affine) covering $\cup_i U_i$ of S and finite étale covers $V_i \rightarrow U_i$ such that $T \times_S V_i$ is trivial.*

Remark 6.4. In particular, in the example [SGA3II, Exp. X, Sec. 1.6] the constructed torus is not isotrivial as is proven in [SGA3II, Exp. XI, Rem. 4.6], but it is locally isotrivial, contrarily to what is claimed there.

6.2. Affineness for the functor of subgroups of multiplicative type. Let S be a scheme and $G \in \mathcal{C}_p(S)$ a pure S -group scheme. Then the functor F of subgroups of multiplicative type of G is representable by a disjoint sum of *affine* S -schemes, and it is smooth if G is. The proof of this is sketched in [SGA3II, Exp. XI, Rems. 4.3 and 4.6]; it proceeds by embedding G into GL_n and reducing to analogous statements for the functor F_{GL_n} . We do not repeat the details here, but note that a special case appeared in the proof of Prop. 6.2 above.

6.3. Existence of versal torsors over general rings. We refer to [Ser03, Def. 5.1] for the definition of a versal torsor. Let $S = \mathrm{Spec}(R)$ be an affine scheme and $G \in \mathcal{C}_p(S)$. It follows from Theorem 5.2 that there exists a versal torsor $E \rightarrow B$ with B quasi-projective over S . Indeed, take a closed group embedding $G \hookrightarrow \mathrm{GL}_{n,S}$ and let $B = \mathrm{GL}_{n,S}/G$. From [DG70, III, § 4, 4.6] we have the sequence of pointed sets:

$$\{1\} \rightarrow H^0(S, G) \rightarrow H^0(S, \mathrm{GL}_{n,S}) \rightarrow H^0(S, B) \rightarrow H^1(S, G) \rightarrow H^1(S, \mathrm{GL}_{n,S}).$$

Since $H^1(S, \mathrm{GL}_{n,S}) = \{1\}$ when R is a field, the versal property follows.

7. QUESTIONS

Finally we want to add some comments and ask a few questions.

- (1) The first comment is that $\mathcal{C}_p \neq \mathcal{C}_{\mathrm{lin}}$ in general. Let R be a discrete valuation ring with fraction field K . Let G be the affine étale R -group scheme obtained by removing the nontrivial closed point of the special fibre of the finite constant group $(\mathbb{Z}/2\mathbb{Z})_R$. Then G is not pure, however it is linear: given a uniformizer $\pi \in R$, the map taking the nontrivial element of G_K to the matrix $\begin{pmatrix} 1/\pi & 1/\pi^2 \\ \pi^2 - 1 & -1/\pi \end{pmatrix}$ extends to a closed immersion $G \hookrightarrow \mathrm{GL}_{2,R}$.
- (2) Let $\mathcal{C}_{\mathrm{qf}} \subset \mathcal{C}$ be the subcategory of quasi-finite group schemes, so we have $\mathcal{C}_{\mathrm{qf}} \cap \mathcal{C}_p = \mathcal{C}_{\mathrm{fin}}$. Does $\mathcal{C}_{\mathrm{qf}} \subset \mathcal{C}_{\mathrm{resol}}$? Do at least affine, étale group schemes over a regular ring of dimension 3 have a classifying space enjoying the resolution property?
- (3) Are there other inclusions among $\mathcal{C}_{\mathrm{lin}}$, $\mathcal{C}_{\mathrm{mono}}$ and $\mathcal{C}_{\mathrm{recons}}$? Assume that R is regular of dimension 2. Do we have $\mathcal{C}_{\mathrm{lin}}(R) \subset \mathcal{C}_{\mathrm{resol}}(R)$? Since in this case we have $\mathcal{C}_p(R) = \mathcal{C}_{\mathrm{resol}}(R)$ and $\mathcal{C}_{\mathrm{lin}}(R) = \mathcal{C}(R)$, it would follow that also $\mathcal{C}_{\mathrm{mono}}(R) = \mathcal{C}_{\mathrm{recons}}(R) = \mathcal{C}(R)$.
- (4) Does any of the categories $\mathcal{C}_{\mathrm{resol}}$, $\mathcal{C}_{\mathrm{lin}}$, $\mathcal{C}_{\mathrm{mono}}$, $\mathcal{C}_{\mathrm{recons}}$ satisfy (weak versions of) the properties (α) , (β) or (γ) ? It should be noted that apart from $\mathcal{C}_{\mathrm{lin}}$ these categories are not obviously fibred. Also, already the stack property for the Zariski topology may fail. For example, one can construct a 2-dimensional torus T of finite type over a singular proper curve S which is not linear (see [SGA3II, Exp. XI, Rem. 4.6] together with [SGA3II, Exp. X, Sec. 1.6]). This shows that $\mathcal{C}_{\mathrm{lin}}(S)$ does not satisfy fpqc descent as we have $T_{S'} \simeq \mathbb{G}_{m,S'}^2$ on some fpqc cover $S' \rightarrow S$.
- (5) Let $G \in \mathcal{C}_p(R)$ and let X be an affine, flat, finitely presented, pure R -scheme with an action of G . Does (G, S, X) satisfy the resolution property as in [Tho87, Def. 2.1]?
- (6) What can be said about the same questions for pure, flat, finitely presented, possibly non-affine group schemes?
- (7) The connectedness problem studies which groups are extensions of an étale group by a fibrewise connected group, see [Jaf93], [Rom11]. In this article, we saw that for our purposes the extensions of a finite locally free group

by a fibrewise connected one behave very nicely; for them, the variation of the number of connected components of the fibres is controlled by a finite scheme. We can enlarge the problem to the search of flat subgroups of G with fibres supported on the neutral component, and ask: which are the group schemes that can be written as extensions of a flat, quasi-finite group scheme Q by a flat, fibrewise connected group scheme C ? For such extensions, is the finiteness of Q an intrinsic property, that is: if G can be written in two ways as an extension $1 \rightarrow C_i \rightarrow G \rightarrow Q_i \rightarrow 1$ ($i = 1, 2$) with $C_i \in \mathcal{C}_{\text{conn}}$ and $Q_i \in \mathcal{C}_{\text{qf}}$, then is Q_1 finite if and only if Q_2 is finite?

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