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IDENTIFICATION OF AN ALGEBRAIC DOMAIN IN TWO DIMENSIONS FROM A FINITE NUMBER OF ITS GENERALIZED POLARIZATION TENSORS

HABIB AMMARI, MIHAI PUTINAR, ANDRIES STEENKAMP, AND FAOUZI TRIKI

Abstract. This paper aims at studying how finitely many generalized polarization tensors of an algebraic domain can be used to determine its shape. Precisely, given a planar set with real algebraic boundary, it is shown that the minimal polynomial with real coefficients vanishing on the boundary can be identified as the generator of a one dimensional kernel of a matrix whose entries are obtained from a finite number of generalized polarization tensors. The size of the matrix depends polynomially on the degree of the boundary of the algebraic domain. The density with respect to Hausdorff distance of algebraic domains among all bounded domains invites to extend via approximation our reconstruction procedure beyond its natural context. Based on this, a new algorithm for shape recognition/classification is proposed with some strong hints about its efficiency.

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1. Introduction

Given a conductivity contrast, the generalized polarization tensors (GPTs) of a bounded Lipschitz domain are an infinite sequence of tensors. The GPTs form the basic building blocks for the far-field behavior of the electric potential. Recently, many works have shown that the GPTs can be used to efficiently recover geometrical properties of the underlying shape. In fact the knowledge of the full set of GPTs determines uniquely the shape of the domain as proved in [5]. When the domain is transformed by a rigid motion or a dilation, the corresponding GPTs change according to certain rules. It is possible to construct as combinations of GPTs invariants under these transformations. This property makes GPTs suitable for the dictionary matching problem [1, 22]. The GPTs have also been used in various areas of applications such as imaging, cloaking, and plasmonics. We refer the reader to [4, 6, 7, 9, 23, 28, 31] and references therein for further information about these applications.

Since the GPTs appear naturally in imaging a small conductivity inclusion from boundary potential measurements, the amount of information about the shape of the inclusion encoded in the first tensors is richer than any other geometrical quantities. Recent numerical studies [3, 8] show that by using only the

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first few terms of GPTs reasonable approximation of the true shape can be recovered. Complete geometric identification of a conductivity inclusion from the knowledge of its first GPT is known to be possible for ellipsoid shapes. In fact if the contrast is given, only the first polarization tensor is needed to retrieve the major and minor axis of an ellipse [6]. For arbitrary shapes, it is proposed to approach them using ellipse-equivalent identification. This consists simply of determining the shape of an ellipse with the same first polarization tensor as that of the targeted inclusion [6]. The results of this approach are quite surprising since the recovered equivalent ellipse seems to hold much more information on the shape than anticipated. For example, the equivalent ellipse contains more knowledge than the first two or three Dirichlet Laplacian eigenvalues of the inclusion. An interesting question is whether one could recover other shapes of an inclusion from the knowledge of a finite number of its GPTs. In view of [10], inclusions with algebraic shapes represent good candidates for such an identification problem. To specify our terminology, an inclusion has algebraic shape if it is a bounded open subset of Euclidean space whose boundary is real algebraic, i.e., contained in the zero set of finitely many polynomials.

In this paper, we are interested in the inverse problem of recovering the shape of an algebraic inclusion given a finite number of its GPTs. We consider shapes unique up to rigid motions, that is orthogonal transformations.

The paper is organized as follows. In Section 2 we introduce the notion of GPTs of an inclusion and their relation to far-field expansion of the fields associated to a piecewise constant conductivity. The inverse problem in question is stated in Section 3, where a review of recent results in the recovery of the shape of an inclusion from the knowledge of all the GPTs is also given. Section 4 is dedicated to an introduction to real algebraic domains. Some basic notions of real algebraic geometry are recalled here. Our main identification result is stated in Theorem 5.1. The detailed proof of the main result as well as a uniqueness result are provided in Section 5. Section 6 is devoted to the generalization of the concept of ellipse-equivalent approach to higher-order GPTs. This generalization takes advantage of the density of algebraic domains in the set of smooth inclusions. We apply the main result in Section 6 by constructing a shape recognition algorithm and demonstrating its optimality by means of a few well chosen examples. The paper is concluded with some discussions in Section 7.

2. Generalized polarization tensors

Let D be a bounded Lipschitz domain in $\mathbb{R}^2$, of size of order one. Assume that its boundary ∂D contains the origin. Throughout this paper, we use standard notation concerning Sobolev spaces. For a density $\phi \in H^{-1/2}(\partial D)$, define the Neumann-Poincaré operator (NPO): $\mathcal{K}^*_{\partial D} : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$, by

$$\mathcal{K}^*_{\partial D}[\phi](x) = \frac{1}{2\pi}\text{p.v.} \int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2}\phi(y)\, d\sigma(y), \quad x \in \partial D,$$

where p.v. denotes the principal value, $\nu(x)$ is the outward unit normal to ∂D at $x \in \partial D$, and $\langle\, ,\, \rangle$ denotes the scalar product in $\mathbb{R}^2$.

The spectral properties of the Neumann-Poincaré operator have proven interesting in several contexts [2, 11,13–15]. Due to Plemelj-Calderón identity and energy estimates, the spectrum of $\mathcal{K}^*_{\partial D}$ is real [6,26]. When D is smooth (with $\mathcal{C}^{1,\alpha}$ boundary), $\mathcal{K}^*_{\partial D}$ is compact, hence its spectrum consists of a sequence of eigenvalues that accumulates to 0 [26]. When D is Lipschitz, the following proposition characterizes the resolvent set $\rho(\mathcal{K}^*_{\partial D})$ of the NPO [6,18].

Proposition 2.1. *We have $\mathbb{C}\setminus(-1/2, 1/2] \subset \rho(\mathcal{K}^*_{\partial D})$. Moreover, if $|\lambda| \geq 1/2$, then $(\lambda I - \mathcal{K}^*_{\partial D})$ is invertible on $H^{-1/2}_0(\partial D) := \{f \in H^{-1/2}(\partial D) : \langle f, 1\rangle_{-1/2,1/2} = 0\}$. Here, $\langle\ ,\ \rangle_{-1/2,1/2}$ denotes the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$.*

For $|\lambda| > 1/2$ and a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, where $\mathbb{N}$ is the set of all positive integers, define ϕ_α by

$$\phi_\alpha(y) := (\lambda I - \mathcal{K}^*_{\partial D})^{-1}\left[\nu(x)\cdot\nabla x^\alpha\right](y), \quad y \in \partial D.$$

Here and throughout this paper, we use the conventional notation: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, and $|\alpha| = \alpha_1 + \alpha_2$. We also use the graded lexicographic order: α, $\beta \in \mathbb{N}^2$ verifies $\alpha \leq \beta$ if $|\alpha| < |\beta|$, or, if $|\alpha| = |\beta|$, then $\alpha_1 \leq \beta_1$ or $\alpha_1 = \beta_1$ and $\alpha_2 \leq \beta_2$.

The GPTs $M_{\alpha\beta}$ for $\alpha, \beta \in \mathbb{N}^2$ ($|\alpha|, |\beta| \geq 1$), associated with the parameter λ and the domain D are defined by

$$M_{\alpha\beta}(\lambda, D) := \int_{\partial D} y^\beta \phi_\alpha(y)\, d\sigma(y). \tag{1}$$

As we said before, the GPTs are tensors that appear naturally in the asymptotic expansion of the electrical potential in the presence of a small inclusion D of conductivity contrast $0 \leq k \neq 1 \leq +\infty$. The parameter λ is related to the conductivity k via the relation

$$\lambda = \frac{k+1}{2(k-1)}.$$

The fact that $0 \leq k \leq +\infty$ implies that $|\lambda| \geq 1/2$, and hence $(\lambda I - \mathcal{K}^*_{\partial D})$ is invertible on $H_0^{-1/2}(\partial D)$. Assume now that the distribution of the conductivity in $\mathbb{R}^2$ is given by

$$\Upsilon = k\chi(D) + \chi(\mathbb{R}^2 \setminus \overline{D}),$$

where χ denotes the indicator function. For a given harmonic function h in $\mathbb{R}^2$, we consider the following transmission problem:

$$\begin{cases} \nabla \cdot \Upsilon \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x) - h(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

The electric potential u has the following integral representation (see, for instance, [6])

$$u(x) = h(x) + \mathcal{S}_{\partial D}(\lambda I - \mathcal{K}^*_{\partial D})^{-1} \left[\partial_\nu h(x)\right], \quad x \in \mathbb{R}^2,$$

where $\mathcal{S}_{\partial D} : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is the single layer potential given by

$$\mathcal{S}_{\partial D}[\phi](x) = \frac{1}{2\pi} \int_{\partial D} \Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^2.$$

Here, Γ is the fundamental solution of the Laplacian,

$$\Gamma(x) := \frac{1}{2\pi} \ln |x|.$$

It possesses the following Taylor expansion

$$\Gamma(x-y) = \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha \Gamma(x) y^\alpha, \quad y \in \overline{D}, \quad |x| \to +\infty,$$

where $\partial_\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$, and $\alpha! = \alpha_1! \alpha_2!$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$.

Then, the far-field perturbation of the voltage potential created by D is given by [6]

$$u(x) - h(x) = \sum_{|\alpha|,|\beta|=1}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial^\alpha \Gamma(x) M_{\alpha\beta} \partial^\beta h(0) \qquad \text{as } |x| \to +\infty. \tag{2}$$

From the asymptotic expansion (2), we deduce that the knowledge of $M_{\alpha\beta}$ for $\alpha, \beta \in \mathbb{N}^2$ ($|\alpha|, |\beta| \geq 1$) is equivalent to knowing the far-field responses of the inclusion for all harmonic excitations.

3. Shape reconstruction problem

In this section, a brief review of recent results in GPT based inclusion shape recovery is given. We first introduce the harmonic combinations of the GPTs. Positivity and symmetry properties of the GPTs are proved using their harmonic combinations [6]. A harmonic combination of the GPTs $M_{\alpha\beta}$ is

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta},$$

where $\sum_\alpha a_\alpha x^\alpha$ and $\sum_\beta b_\beta x^\beta$ are real harmonic polynomials. We further call such a_α and b_β harmonic coefficients. For example, if a_α and b_β are any two harmonic coefficients, we have the following symmetry property:

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta} = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\beta\alpha}.$$

The following uniqueness result has been proved in [5].

Theorem 3.1. *If all harmonic combinations of GPTs of two domains D and $\widetilde{D}$ with parameters λ and $\widetilde{\lambda}$, are identical, that is*

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(\lambda, D) = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(\widetilde{\lambda}, \widetilde{D}),$$

for all pairs a_α and b_β of harmonic coefficients, then $D = \widetilde{D}$ and $\lambda = \widetilde{\lambda}$.

Theorem 3.1 says that the full knowledge of (harmonic combinations of) GPTs determines uniquely the domain D and the parameter λ.

Recall that the first-order polarization tensor, $M_{\alpha\beta}$ with $|\alpha| = |\beta| = 1$, of any given inclusion is a 2×2 real valued and symmetric matrix. Remarking that the polarization tensors produced by rotating ellipses with size coincide with the set of 2×2 real valued symmetric matrices, it is known that the first-order polarization tensor yields the equivalent ellipse (see for instance [6] and references therein). The equivalent ellipse of D is the ellipse with the same first-order polarization tensor as D. However, it is not known explicitly what kind of information on D and λ the higher-order GPTs carry. The purpose of this work is to study the possibility of using higher-order GPTs for shape description. Our idea is based on first deriving a set of dense domains that can be identified from finitely many GPTs. We show that good candidates for such a dictionary is the class $\mathcal{G}$ of algebraic domains of size one. Then, by approximating the target domain using a sequence of algebraic domains in $\mathcal{G}$ we obtain a powerful tool for describing the shape of inclusions if only a finite number of GPTs is available. Next, we introduce the concept of real algebraic domains.

4. Real algebraic domains

In the present section we consider the class of bounded open subsets of Euclidean space $\mathbf{R}^2$ with real algebraic boundary. We adopt the following definition.

Definition 4.1. *An open set G in $\mathbf{R}^2$ is called real algebraic if there exists a finite number of real coefficient polynomials $g_i(x), i = 1, \cdots, m$ such that*

$$\partial G \subset V := \{x \in \mathbb{R}^2 \ : g_1(x) = \cdots = g_m(x) = 0\}.$$

The ellipse is a simple example of an algebraic domain, since its general boundary coincides with the zero set of the quadratic polynomial function

$$g(x) = \sum_{|\alpha|\leq 2} g_\alpha x^\alpha$$

for given real coefficients $(g_\alpha)_{|\alpha|\leq 2}$ and proper signs in the top degree part.

We further denote by $\mathcal{G}$ the collection of bounded algebraic domains and *of size of order one*. The differential structure of the boundary ∂G is then well known: it consists of algebraic arcs joining finitely many singular points, see for instance [29].

It is tempting to also impose connectedness of the respective sets, but this constraint is not accessible by the elementary linear algebra tools we develop in the present note, so we drop it. However, we call "domains" all elements $G \in \mathcal{G}$.

Following [27] we focus on a particular class of algebraic domains which are better adapted to the uniqueness and stability of the shape inverse quest. Let

$$\mathcal{G}^* := \left\{ G \in \mathcal{G} : G = \operatorname{int} \overline{G} \right\}. \tag{3}$$

An element of G^* is called an *admissible domain*, although it may not be connected.

The assumption that $G = \operatorname{int} \overline{G}$ implies that G contains no slits or ∂G does not have isolated points. If $G \in \mathcal{G}^*$, the algebraic dimension of ∂G is one, and the ideal associated to it is principal. To be more precise, ∂G is a finite union of irreducible algebraic sets $X_j,\ j \in J$, of dimension one each. The reduced ideal associated to every X_j is principal:

$$I(X_j) = (P_j), \quad j \in J$$

for instance see [12, Theorem 4.5.1]. We assume that each P_j is indefinite, i.e. it changes sign when crossing X_j. Therefore one can consider the polynomial $g = \prod_{j \in J} P_j$, vanishing of the first-order on ∂G, that is $|\nabla g| \neq 0$ on the regular locus of ∂G. According to the real version of Study's lemma (cf. Theorem 12 in [29]) every polynomial vanishing on ∂G is a multiple of g, that is $I(\partial G) = (g)$. We define the degree of ∂G as the degree of the generator g of the ideal $I(\partial G)$. For a thorough discussion of the reduced ideal of a real algebraic surface in $\mathbf{R}^d$, we refer to [17].

In the sequel, we denote by $g(x)$ the single polynomial vanishing on ∂G which is the generator of $I(\partial G)$ and satisfying the following normalization condition $g_{\alpha^*} = 1$, where $\alpha^* = \max_{g_\alpha \neq 0} \alpha$.

Assume $G \in \mathcal{G}^*$. If the degree d of ∂G and moments (up to order 3d) of the Lebesgue measure on G are known, then as shown in [27], the coefficients of g of degree d that vanishes on ∂G are uniquely determined (up to a constant). More precisely, it is shown in [27] that g is the generator of a one-dimensional kernel of a matrix whose entries are obtained from moments of the Lebesgue supported measure by G. That is, only finitely many moments (up to order 3d) are needed to recover the minimal degree polynomial vanishing on G. It turns out that computing g reduces to a solving a system of linear equations.

We stress that the main result contained in the present article identifies the minimal degree polynomial g, and not the exact boundary of the admissible domain G. To give a simple example, consider the defining equation of the boundary

$$g(x_1, x_2) = (x_1^3 - x_1)(x_2^3 - x_2). \tag{4}$$

The following algebraic domains

$$\begin{aligned} G_1 &= \left\{ (x_1, x_2), x_1 > 0 \text{ or } x_2 > 0 \right\}, \\ G_2 &= \left\{ \{(x_1, x_2), x_1 > 0 \text{ or } x_2 < 0 \right\}, \\ G_3 &= \left\{ \{(x_1, x_2), x_1 < 0 \text{ or } x_2 < 0 \right\}, \\ G_4 &= \left\{ \{(x_1, y_2), x_1 < 0 \text{ or } x_2 > 0 \right\}, \end{aligned} \tag{5}$$

are all admissible domains, sharing the same minimal degree defining function of the boundary.

Even when restricting the class of domains to those possessing an irreducible boundary, we may encounter pathologies. Without recalling cumbersome details, Example 28 in [29] produces a series of polynomials whose zero sets may contain curves and isolated points, but some of the isolated points are not irreducible components. The connectedness of G is also tricky, as for instance the intricate nature of the topology of the zero set of a lemniscate reveals: the curve

$$\prod_{j=1}^{k} \left((x_1 - a_j)^2 + (x_2 - b_j)^2 \right) = r, \tag{6}$$

has k distinct connected components for small values of $r > 0$, where the poles (a_j, b_j) are mutually distinct.

The main objective of our article, comparable to that in [27], is to isolate a finite pool of domains (we may call them bounded "chambers") from which we can select the shape of G and further on determine the parameter λ, both inferred from the knowledge of finitely many GPTs. It would be extremely interesting to unveil how the additional information encoded in the GPTs allows to select the correct chamber among the many potential candidates.

5. Main results

Let $\mathbb{R}[x]$ be the ring of polynomials in the variables $x = (x_1, x_2)$ and let $\mathbb{R}_N[x]$ be the vector space of polynomials of degree at most N (whose dimension is $r_N = (N+1)(N+2)/2$). For a polynomial function $p(x) \in \mathbb{R}_N[x]$, it has a unique expansion in the canonical basis $x^\alpha, |\alpha| \leq N$ of $\mathbb{R}_N[x]$, that is,

$$p(x) = \sum_{|\alpha| \leq N} p_\alpha x^\alpha$$

for some vector coefficients $\mathbf{p} = (p_\alpha) \in \mathbb{R}_{r_N}$. The following is the main result of the paper.

Theorem 5.1. *Let $G \in \mathcal{G}^*$ with ∂G Lipschitz of degree d, and let $g(x) = \sum_{|\alpha| \leq d} g_\alpha x^\alpha$, be a polynomial function that vanishes of the first-order on ∂G, satisfying $I(\partial G) = (g)$, $g_{\alpha^*} = 1$, and $g(0) = 0$, where $\alpha^* = \max_{g_\alpha \neq 0} \alpha$. Then, there exists a discrete set $\Sigma \subset \mathbb{C}_0 := \mathbb{C} \setminus [-1/2, 1, 2]$, such that $\mathbf{g} = (g_\alpha) \in \mathbb{R}_{r_d}$ is the unique solution to the following normalized linear system:*

$$\mathbf{p} = (p_\alpha) \in \mathbb{R}_{r_d}; \quad \sum_{|\beta| \leq d} M_{\alpha\beta}(\lambda, G) p_\beta = 0 \ \text{ for } |\alpha| \leq 2d; \ \ p_{\alpha^*} = 1, \ \ \alpha^* = \max_{p_\alpha \neq 0} \alpha \ \text{ for } \lambda \in \mathbb{C}_0 \setminus \Sigma. \tag{7}$$

Proof. The proof of the theorem has two main steps. In the first step, we show that $\mathbf{g}$ satisfies the normalized linear system (7). The second step consists in proving that it is indeed the unique solution to that system.

Step 1. For $|\lambda| > \frac{1}{2}$, recall from (1) the general expression of the GPTs

$$M_{\alpha\beta}(\lambda, G) := \int_{\partial G} (\lambda I - \mathcal{K}^*_{\partial G})^{-1} \left[\nu(x) \cdot \nabla x^\alpha\right] y^\beta \, d\sigma(y). \tag{8}$$

Since $g(x)$ vanishes on ∂G, we infer

$$\int_{\partial G} (\lambda I - \mathcal{K}^*_{\partial G})^{-1} \left[\nu(x) \cdot \nabla x^\alpha\right] g(y) \, d\sigma(y) = 0, \qquad \forall \alpha \in \mathbb{N}^2, \tag{9}$$

which combined with (8) leads to the desired system:

$$\sum_{|\beta| \leq d} M_{\alpha\beta}(\lambda, G) g_\beta = 0, \qquad \forall \alpha \in \mathbb{N}^2. \tag{10}$$

Step 2. Assume that $\mathbf{p} \in \mathbb{R}_{r_d}[x]$ satisfies the system (10). Our objective is to prove that $\mathbf{p}$ coincides with $\mathbf{g}$.

Denote by $\mathbb{C}_\star := \mathbb{C} \setminus (-\infty, -2] \cup [2, +\infty)$, and let $\mu = \lambda^{-1} \in \mathbb{C}_\star$. Define $\mathbb{M}(\mu)$ to be the rectangular matrix with coefficients: $(\mu M_{\alpha\beta})_{|\alpha| \leq 2d, |\beta| \leq d}$, that is

$$\mathbb{M}_{\alpha\beta}(\mu) := \int_{\partial G} (I - \mu \mathcal{K}^*_{\partial G})^{-1} \left[\nu(x) \cdot \nabla x^\alpha\right] y^\beta \, d\sigma(y). \tag{11}$$

Obviously, the following equality holds

$$\mathbb{M}(\mu)\mathbf{p} = 0.$$

Lemma 5.1. *The function $\mu \to \mathbb{M}(\mu) \in \mathcal{L}(\mathbb{R}^{r_d}, \mathbb{R}^{r_{2d}})$ is a holomorphic matrix-valued function on $\mathbb{C}_\star$, and $ker(\mathbb{M}(0)) = \{c\mathbf{g}; \ \ c \in \mathbb{R}\}$.*

Proof. Considering the properties of the resolvent set in Proposition 2.1, $\mu \mathcal{K}^*_{\partial G}$ is a contraction operator for μ small enough. In fact it can be easily verified that $\|\mathcal{K}^*_{\partial G}\| \leq 1/2$ where the norm of $\mathcal{K}^*_{\partial G}$ is taken in the energy space [26]. Moreover, the Neumann series

$$(I - \mu \mathcal{K}^*_{\partial G})^{-1} = \sum_{j=0}^{\infty} \mu^j (\mathcal{K}^*_{\partial G})^j,$$

has a holomorphic extension in the resolvent set $\mathbb{C}_\star$. Next, we investigate the kernel of $\mathbb{M}(0)$. More precisely, $\mathbb{M}(0)\mathbf{p} = 0$ is equivalent to

$$\int_{\partial G} \nu(y) \cdot \nabla q(y) p(y)\, d\sigma(y) = 0, \quad \forall q \in \mathbb{R}_d[x]. \tag{12}$$

Since g generates the ideal associated to ∂G and ∂G is Lipschitz, we have $\nu(x) = \frac{\nabla g}{\|\nabla g\|}$ on the regular part of the curve ∂G. Then (12) becomes

$$\int_{\partial G} \nabla g(y) \cdot \nabla q(y) p(y) \frac{1}{\|\nabla g\|}\, d\sigma = 0, \quad \forall q \in \mathbb{R}_d[x].$$

By taking $q(x) = x^\alpha g(x)$ for $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \leq d$, and considering the fact that $g(x)$ vanishes on ∂G, one finds

$$\int_{\partial G} \|\nabla g\| y^\alpha p(y)\, d\sigma = 0, \quad \forall \alpha \in \mathbb{N}^2, \ \ |\alpha| \leq d,$$

which in turn implies that

$$\int_{\partial G} \|\nabla g\| q(y) p(y)\, d\sigma = 0, \quad \forall q \in \mathbb{R}_d[x].$$

Then, taking $q = p$ in the last inequality gives $p(y) = 0$ on ∂G. Consequently, $p = cg$ for some real constant c, which is the desired result. □

At this point we return to the proof of the main theorem. The analytic family of matrices $M(\mu)$ annihilates g for all values of the parameter μ. Moreover, for $\mu = 0$ we saw that $M(\mu)$ has maximal rank, that is its kernel is spanned by g. Since maximal rank is constant on a Zariski open subset of the parameter domain, we infer that

$$\dim \ker M(\mu) = 1$$

for all μ in $\mathbb{C}_\star$, except a discrete subset $\widetilde{\Sigma}$.

We provide some details of the proof for the convenience of the general readership. Let $\mathfrak{H} = \{c\mathbf{g};\ \ c \in \mathbb{R}\}^\perp$ be the sub-vector space in $\mathbb{R}^{r_d}$ orthogonal to $\ker(\mathbb{M}(0))$. Denote the restriction of $\mathbb{M}(\mu)$ to $\mathfrak{H}$ by $\widetilde{\mathbb{M}}(\mu) \in \mathcal{L}(\mathfrak{H}, \mathbb{R}_{r_{2d}})$. Since the Hilbert space $\mathfrak{H}$ is μ independent, the matrix-valued function $\mu \to \widetilde{\mathbb{M}}(\mu)$ inherits the same regularity as the function $\mu \to \mathbb{M}(\mu)$, i.e., it is holomorphic on $\mathbb{C}_\star$. Then, as a direct consequence of Lemma 5.1, the restriction of $\mathbb{M}(0)$ to $\mathfrak{H}$, denoted by $\widetilde{\mathbb{M}}(0) \in \mathcal{L}(\mathfrak{H}, \mathbb{R}_{r_{2d}})$, is injective. Recall that a linear bounded operator is injective if and only if it has a left inverse [16]. Then, there exists a left inverse denoted by $L_0 \in \mathcal{L}(\mathbb{R}_{r_{2d}}, \mathfrak{H})$, that only depends on ∂G satisfying

$$L_0 \widetilde{\mathbb{M}}(0) = I_d,$$

where $I_{\mathfrak{H}}$ is the identity matrix acting on $\mathfrak{H}$. Now, let

$$\begin{aligned} &\mathfrak{H} \longrightarrow \mathfrak{H} \\ &\mathbb{T}(\mu) = L_0 \widetilde{\mathbb{M}}(\mu). \end{aligned}$$

Then, $\mu \to \mathbb{T}(\mu)$ is holomorphic on $\mathbb{C}_\star$, and by construction it verifies $\mathbb{T}(0) = I_{\mathfrak{H}}$. Thus, we deduce from Steinberg Theorem that $\mathbb{T}(\mu)$ is invertible everywhere on $\mathbb{C}_\star$ except at a discrete set of values $\widetilde{\Sigma}$ [24]. Hence, L_0 becomes the left inverse of the matrix $\widetilde{\mathbb{M}}(\mu)$ for all $\mu \in \mathbb{C}_\star \setminus \widetilde{\Sigma}$. Consequently, $\widetilde{\mathbb{M}}(\mu)$ is injective for all $\mu \in \mathbb{C}_\star \setminus \widetilde{\Sigma}$. Since $\widetilde{\mathbb{M}}(\mu)$ is the restriction of $\mathbb{M}(\mu)$ to $\mathfrak{H}$ which is the orthogonal space to the vector g in $\mathbb{R}^{r_d}$, we obtain that $\ker(\mathbb{M}(\mu)) = \{c\mathbf{g};\ \ c \in \mathbb{R}\}$ for all $\mu \in \mathbb{C}_\star \setminus \widetilde{\Sigma}$. Then, for $\mu \in \mathbb{C}_\star \setminus \widetilde{\Sigma}$, (12) implies $\mathbf{p} = c\mathbf{g}$ for some real constant c. Using the normalization condition, we obtain $\mathbf{p} = \mathbf{g}$, and hence $p = g$ for all $\lambda \in \mathbb{C} \setminus [-1/2, 1, 2]$ outside the set $\Sigma := \{\lambda \in \mathbb{C};\ \lambda^{-1} \in \widetilde{\Sigma}\}$.

□

Remark 5.1. *We propose here a direct proof of the uniqueness result in Theorem 3.1 in the case where all the GPTs of the algebraic domain G are known. In [5], the proof of uniqueness for general shapes is based on the relation between the far-field expansion and the Dirichlet-to-Neumann operator.*

Let $\mathbf{p}^{\mathbf{e}}$ be the extension of the vector $\mathbf{p} \in \mathbb{R}_{r_d}[x]$ by zero in the canonical basis $x^\alpha, |\alpha| > d$. Assume that $\mathbf{p} \in \mathbb{R}_{r_d}[x]$ satisfies the system (7), then

$$\sum_\beta M_{\alpha\beta}(\lambda, G) p^e_\beta = 0, \qquad \forall \alpha \in \mathbb{N}^2,$$

and consequently,

$$\int_{\partial G} (\lambda I - \mathcal{K}^*_{\partial G})^{-1} \left[\nu(x) \cdot \nabla x^\alpha\right] p(y)\, d\sigma(y) = 0, \qquad \forall \alpha \in \mathbb{N}^2.$$

Similarly, we have

$$\int_{\partial G} \nu(y) \cdot \nabla y^\alpha (\lambda I - \mathcal{K}_{\partial G})^{-1} \left[p(x)\right] d\sigma(y) = 0, \qquad \forall \alpha \in \mathbb{N}^2,$$

where $\mathcal{K}_{\partial G} : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ denotes the adjoint of $\mathcal{K}^*_{\partial G}$. Since $\Gamma(z-y)$ has the following expansion

$$\Gamma(z-y) = \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha \Gamma(z) y^\alpha, \quad y \in \overline{G}, \quad |z| \to +\infty,$$

we deduce that

$$\int_{\partial G} \nu(y) \cdot \nabla \Gamma(z-y) (\lambda I - \mathcal{K}_{\partial G})^{-1} \left[p(x)\right] d\sigma(y) = 0, \qquad \forall z \in \mathbb{R}^2 \setminus \overline{G},$$

which implies that

$$(\lambda I - \mathcal{K}_{\partial G})^{-1} \left[p(x)\right] = 0 \qquad \text{on } \partial D.$$

Since $|\lambda| > \frac{1}{2}$, $\lambda I - \mathcal{K}_{\partial G}$ is invertible and so p vanishes completely on ∂G. Consequently, $p = cg$ for some real constant c. Using the normalization condition, we obtain $p = g$.

Following the discussion in Section 4, we need to add a supplementary criteria to be able to identify uniquely the domain G from its minimal polynomial g. Let Ω_r be a bounded domain in $\mathbb{R}^2$, containing a ball of center zero and radius $r > 0$ large enough. Let $\mathcal{G}^*_0$ be the set of polynomial functions $g \in \mathcal{G}^*$ such that there exists a unique Lipshitz algebraic domain G containing zero and with size one satisfying $\partial G \subset \{g = 0\} \cap \Omega$.

Corollary 5.1. *Let $G \in \mathcal{G}^*_0, \widetilde{G} \in \mathcal{G}^*$ with respectively ∂G and $\partial \widetilde{G}$ of degree d. Let $g(x) = \sum_{|\alpha| \le d} g_\alpha x^\alpha$ and $\widetilde{g}(x) = \sum_{|\alpha| \le d} \widetilde{g}_\alpha x^\alpha$ be respectively polynomial functions that vanish respectively of the first-order on ∂G, and $\partial \widetilde{G}$ satisfying $I(\partial G) = (g)$, $g_{\alpha^*} = 1$, $g(0) = 0$, and $I(\partial \widetilde{G}) = (\widetilde{g})$, $\widetilde{g}_{\alpha^*} = 1$, $\widetilde{g}(0) = 0$. Let λ be fixed in $\mathbb{C}_0$ such that $\lambda \notin \Sigma$, where the set $\Sigma(\partial G)$ is as defined in Theorem 5.1. Then, the following uniqueness result holds:*

$$(M_{\alpha\beta}(G, \lambda))_{|\alpha| \le 2d, 0 < |\beta| \le d} = (M_{\alpha\beta}(\widetilde{G}, \lambda))_{|\alpha| \le 2d, 0 < |\beta| \le d} \quad \text{iff} \quad G = \widetilde{G}. \tag{13}$$

Proof. The result is a direct consequence of Theorem 5.1. Since the generalized polarization tensors coincide, and $\lambda \notin \Sigma$, we can deduce from Theorem 5.1 that $g = \widetilde{g}$. Since $g \in \mathcal{G}^*_0$ we finally obtain $G = \widetilde{G}$. □

Remark 5.2. *From applications point of view, the assumption $G \in \mathcal{G}^*_0$ in Corollary 5.1, is somehow related to the fact that in the inverse problem of identifying small inclusions from boundary voltage measurements, the location and the convex hull are well determined [7]. Our method will allow the recovery of the shape up to a certain precision fixed by the highest order of the considered GPTs. The approach can be seen as an extension of the equivalent ellipse approach [6]. The assumption $G \in \mathcal{G}^*_0$ can be dropped when the regular locus of ∂G coincides with ∂G. For example, when G is a lemniscate with a large enough level set constant such that ∂G contains all the complex roots of its associated complex polynomial (for r large enough in (6)).*

Remark 5.3. *In Theorem 5.1, λ is used implicitly. However, if the domain G is sufficiently well approximated by $\widetilde{G}$ then one can also attempt to recover λ by solving the following minimization problem:*

$$\widetilde{\lambda} = argmin \|M_{\alpha\beta}(\lambda, G) - M_{\alpha\beta}(\widetilde{\lambda}, \widetilde{G})\| \ ; \ \|M_{\alpha\beta}\| \neq 0.$$

6. Approximation by algebraic domains

For possibly non algebraic boundaries ∂G, we describe a simple procedure to compute a polynomial g whose level set $\{x : g(x) = 0\}$ approximates ∂D. We expect better approximation to be found among higher degree polynomials. Domains enclosed by real algebraic curves (henceforth simply called algebraic domains) are dense, in Hausdorff metric among all planar domains. A very particular case is offered by domains surrounded by a smooth curve. They can be approximated by a sequence of algebraic domains. This observation turns algebraic curves into an efficient tool for describing shapes [19, 25, 30]. Note that an algebraic domain which in addition *is the sub level set of a polynomial of degree d* can be determined from its set of two-dimensional moments of order less than or equal to $3d$ [27]. On a related topics, an exact reconstruction of quadrature domains for harmonic functions was proposed in [20] with the advantage of providing a potential type function (similar to the ubiquitous barrier method in global optimization), which detects the boundary of any complicated shape without having to make a choice among different potential chambers. More details about the approximation theory concepts related to this framework can be found in [21].

Theorem 5.1 suggests a strategy to approximately recover information on the boundary ∂D when the latter is not algebraic. We will follow the approach developed in [27] with interior moments. By considering the GPTs $(M_{\alpha\beta})_{|\alpha|\leq 2d, |\beta|\leq d}$, one may compute the polynomial $g \in \mathbb{R}_d[x]$ with coefficients $\mathbf{g} \in \mathbb{R}_{r_d}$ such that $\mathbf{g}$ is the most suitable singular vector corresponding to the smallest in absolute value singular value of $(M_{\alpha\beta})_{|\alpha|\leq 2d, |\beta|\leq d}$.

Building on this, we also suggest an algorithm for shape recognition. The algorithm has three steps: (i) recovering $\mathbf{g}$ for some degree d polynomial; (ii) checking to see if the recovered polynomial has bounded level set, and (iii) accounting for scaling and rotations via minimization problem.

We stress that not all simple real algebraic sets in $\mathbf{R}^2$, such as a triangle, are the sublevel set of a single polynomial. The reader should be aware that the algorithms below identify the minimal polynomial g vanishing on the boundary of an admissible planar domain G, but by no means this implies

$$G = \{x \in \mathbf{R}^2; \ g(x) > 0\}.$$

Even worse, our numerical procedure of plotting the potential boundary of G is based on a curve selection process, which may not detect in a single shot all irreducible components of ∂G. The intricate details of amending these weak points of our numerical schemes and experiments will be addressed in a forthcoming article.

We start by showing how orthogonal transformations on the domain relate to linear operation on the underlying coefficients. We use this relation to define a minimization scheme for finding the similarity between reference and target polynomials. Start by writing the algebraic domain as follows:

$$G := \{x \in \mathbb{R}^2 \ : \ g(x) = \mathbf{x}_{[k]}^t \mathbf{g}_{[k,k]}^2 \mathbf{x}_{[k]} + \sum_{j=0}^{2k-1} \mathbf{g}_{[j]} \mathbf{x}_j = \sum_{j=0}^{2k} \mathbf{g}_{[j]} \mathbf{x}_{[j]} = 0\},$$

where g is a polynomial of even degree $d = 2k$ (necessary) written as a sum of polynomial forms (homogeneous) and the superscript t denotes the transpose. With the notation explained by

Definition 6.1. *Let $\boldsymbol{x}_k^t := [x_1^k, x_1^{k-1}x_2^1, ..., x_1^1 x_2^{k-1}, x_2^k]$, $\boldsymbol{g}_{[k,k]}$ be a $(k+1) - by - (k+1)$ matrix, and $\boldsymbol{g}_{[j]}$ be a row vector of $j+1$ real coefficients. Note that $\boldsymbol{g}_{[2k]}$ represents the coefficients of the leading form and $\boldsymbol{g}_{[k,k]}$ is its quadratic form [30].*

The ability to write any polynomial in this form is a consequence of Euler's theorem [30]. This formulation is used in [30] to prove the following results.

Lemma 6.1. *If $\boldsymbol{g}_{[k,k]}$ is non-singular, then G is bounded and non-empty.*

Lemma 6.2. *All odd degree forms have unbounded level sets.*

In order to recover shapes, we must first define what it means for two shapes to be the same. We avoid the difficulty of defining a similarity measure and simply state that a shape should be invariant under rotations and scaling. This invariance takes the form of a matrix operation on the underlying coefficients. Let $\mathcal{O}_s(2) := \{A = sR \text{ with } R \text{ being a rotation matrix and } s > 0\}$. The following result holds.

Lemma 6.3. *Consider a transformation $A = (a_{i,j})_{i,j\in\{1,2\}} \in \mathcal{O}_s(2)$. Let $x' = Ax$. Then*

$$\boldsymbol{x}'_{[d]} = A_{[d]}\boldsymbol{x}_{[d]}$$

and

$$A(G) = \{x \in \mathbb{R}^2 \ : \ g(x) = \sum_{j=0}^{2k} \boldsymbol{g}_{[j]} A_{[j]}^{-1} \boldsymbol{x}_{[j]} = 0\}$$

with the $(d+1)\times(d+1)$ matrix $A_{[d]}$ having entries given by

$$(A_{[d]})_{h,j\leq d} = \sum_{p=0}^{j} c_p(h) d_{j-p}(h) \ ; \ c_j(h) = \binom{d-h}{j} a_{1,1}^{d-h-j} a_{1,2}^{j} \ : \ d_j(h) = \binom{h}{j} a_{2,1}^{h-j} a_{2,2}^{j}.$$

Here, c_j and d_j are extended by zeros to be defined up to $j \leq d$.

Proof. We write

$$\mathbf{x}'^{t}_{[d]} = [(a_{1,1}x_1 + a_{1,2}x_2)^d, (a_{1,1}x_1 + a_{1,2}x_2)^{d-1}(a_{2,1}x_1 + a_{2,2}x_2), ..., (a_{2,1}x_1 + a_{2,2}x_2)^d].$$

For the h-th entry, we have the following:

$$(a_{1,1}x_1 + a_{1,2}x_2)^{d-h}(a_{2,1}x_1 + a_{2,2}x_2)^h = \Big(\sum_{j=0}^{d-h} \binom{d-h}{j} a_{1,1}^{d-h-j} a_{1,2}^{j} x_1^{d-h-j} x_2^{j}\Big)\Big(\sum_{j=0}^{h} \binom{h}{j} a_{2,1}^{h-j} a_{2,2}^{j} x_1^{h-j} x_2^{j}\Big)$$

$$= \Big(x_1^{d-h}\sum_{j=0}^{d-h} c_j y^j\Big)\Big(x_1^{h}\sum_{j=0}^{h} d_j y^j\Big) = \sum_{j=0}^{d}\Big(\sum_{p=0}^{j} c_p(h) d_{j-p}(h)\Big) x_1^{d-j} x_2^{j} =: A_{[d],h}\mathbf{x}_{[d]},$$

where $y^j := x_1^{-j}x_2^j$, $c_j(h) = \binom{d-h}{j} a_{1,1}^{d-h-j} a_{1,2}^{j}$ and $d_j(h) = \binom{h}{j} a_{2,1}^{h-j} a_{2,2}^{j}$ are extended by zeros to be defined up to $j \leq d$. □

With this result, we can first suggest a simple algorithm for matching GPTs to a predefined real algebraic shape. The goal is to recover the matrix A as it gives the relation between reference and observed shapes.

Algorithm 6.1 Shape matching

Input: The GPTs $(M_{\alpha\beta})_{|\alpha|\leq 2d, |\beta|\leq d}$ and the coefficients $\mathbf{g}$ of the reference shape.
Procedure:

- Solve $\widetilde{\mathbf{g}}$ using Theorem 5.1;
- Check if Lemma 6.1 holds for $\widetilde{\mathbf{g}}$;
- Find $\widetilde{A} := argmin_{A\in\mathcal{O}_s(2)} \| \sum_{j=0}^{2k} (\widetilde{\mathbf{g}}_{[j]} - \mathbf{g}_{[j]} A_{[j]}) \|$ and $\epsilon_{match} := \| \sum_{j=0}^{2k} (\widetilde{\mathbf{g}}_{[j]} - \mathbf{g}_{[j]} \widetilde{A}_{[j]}) \|$.

Output: $\widetilde{A}$ and ϵ_{match}.

Example 6.1. *(**Shape search real algebraic domains**). The following 6 plots show real algebraic domains with degrees ranging from 6 to 2. The GPTs were calculated from algebraic domains and then the algebraic domains were recovered again via Theorem 5.1. The first shape is a reference shape, see Figure 6.1.*

The coefficients are minimized under rotations and scaling. The algorithm correctly matches the second shape.

In the general case, we propose a shape reconstruction algorithm for not necessary algebraic domains.

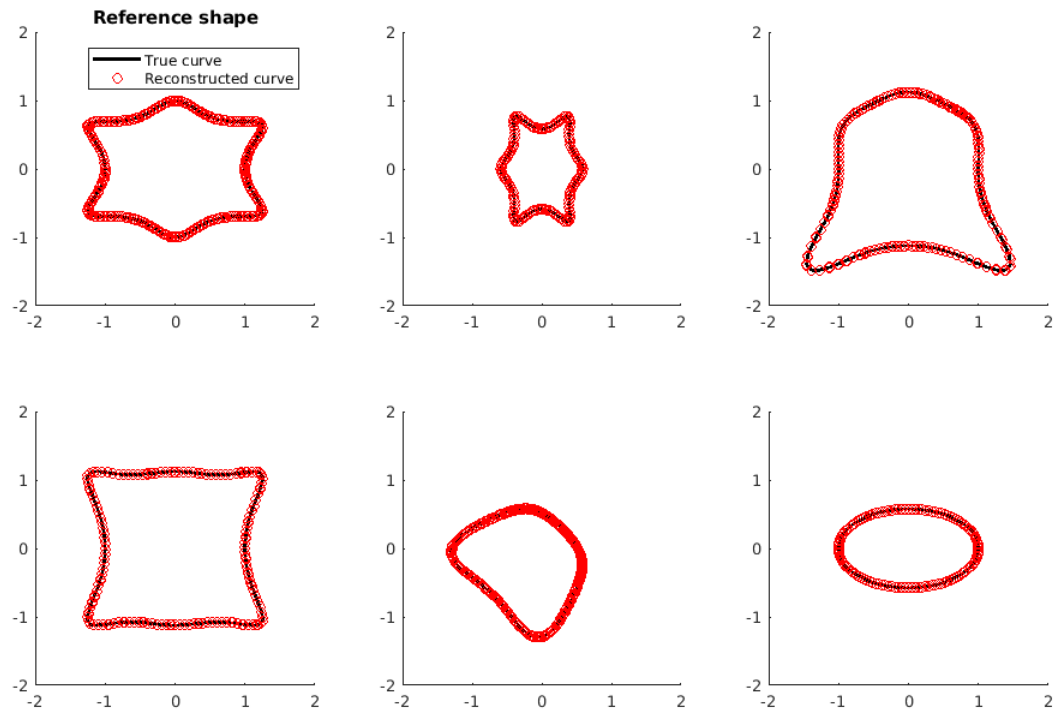

FIGURE 6.1. Reconstruction of real algebraic shapes.

Algorithm 6.2 Approximating general domains with algebraic ones

Input: The GPTs $(M_{\alpha\beta})_{|\alpha|\leq 2d,|\beta|\leq d}$ obtained from a non-algebraic domain.
Procedure:

- Let $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(r_d)}$ be an enumeration of the β's;
- Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r_{2d})}$ be an enumeration of the α's;
- Construct a matrix $[L]_{1\leq i\leq r_{2d},1\leq j\leq r_d} := M_{\alpha^{(i)}\beta^{(j)}}$;
- Find the singular vector $[\widetilde{g}^{(1)}, \ldots, \widetilde{g}^{(r_d)}]^t$ corresponding to the smallest in absolute value singular value of L;
- Set $\widetilde{g}^{(j)} = \widetilde{g}_{\beta^{(j)}}$.

Output: The coefficients $\widetilde{\mathbf{g}}$ of the reconstructed real algebraic shape.

Example 6.2. *(**Approximating non-algebraic domains**) As stated earlier, algebraic domains can approximate any planar domain. However, some shapes may require an infinite series of polynomials to be described. We approximate the shapes of a triangle, a diamond and a flower with one petal missing. The GPTs were calculated from these diametrically defined shapes which in turn was used to recover equivalent-polynomials, see Figure 6.2.*

Remark 6.1. *Note that a higher degree polynomial may not always yield a better approximation.*

Example 6.3. *(**Recovering non-connected real algebraic domains**) The proposed reconstruction method is powerful enough to reconstruct non-connected domains. Consider the polynomial given in (4). Below, we plot the real level sets of (5) for values* $-0.1,\ 0$ *and* 0.1. *On the* 0.1*-level two components of the domain are selected and their GPTs are computed. From these GPTs, a polynomial is recovered via Algorithm 6.2. The*

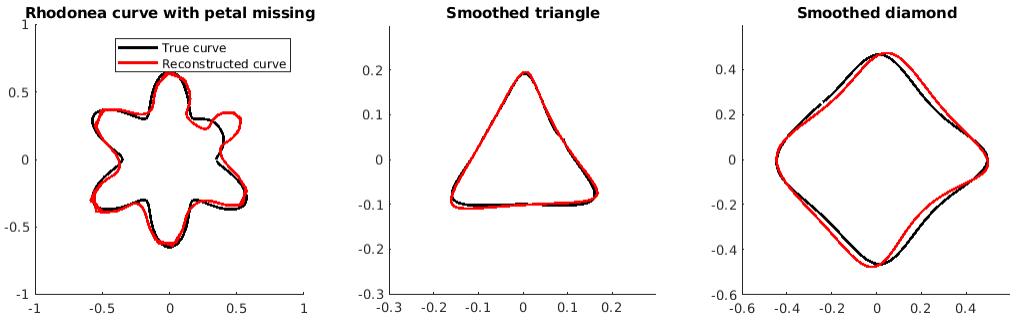

FIGURE 6.2. Reconstruction of non-algebraic shapes.

real 0*-level set of the recovered polynomial contains curves similar to the ones used to compute the GPTs but also unbounded components, see Figure 6.3.*

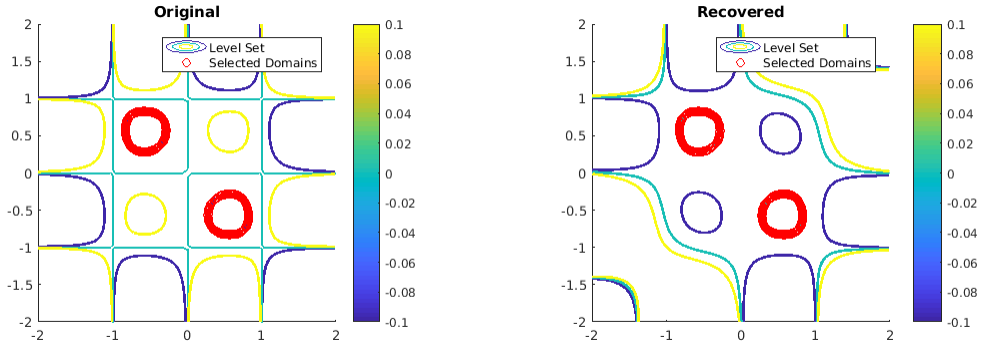

FIGURE 6.3. Reconstruction of non-connected domains.

7. CONCLUSION

In this paper, we have introduced a new tool for identifying shapes from finite numbers of their associated GPTs. We have shown in Theorem 5.1 and Corollary 5.1 that the knowledge of the $r_d \times r_{2d}$ GPTs of a real algebraic domain having a boundary of degree d is sufficient to identify it. This is a very promising path since it is now possible to recover intricate shapes with the knowledge of only a finite number of GPTs. These results confirm that GPTs are good shape descriptors and can be harnessed to great effect by even simple algorithms. We believe that the number $r_d \times r_{2d}$ of required GPTs can be dramatically reduced, and this will be the subject of a future work.

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Habib Ammari, Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: habib.ammari@math.ethz.ch

Mihai Putinar, Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA 93106-3080, USA, and School of Mathematics & Statistics, Newcastle University Newcastle upon Tyne, NE1 7RU, United Kingdom
E-mail address: mputinar@math.ucsb.edu

Andries Steenkamp, Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: andriess@student.ethz.ch

Faouzi Triki, Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d'Hères, France
E-mail address: faouzi.triki@univ-grenoble-alpes.fr