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► **To cite this version:**

Sophie Tarbouriech, Alexandre Seuret, Christophe Prieur, Luca Zaccarian. Insights on event-triggered control for linear systems subject to norm-bounded uncertainty. S.Tarbouriech, A. Girard, and L. Hetel Control subject to computational and communication constraints, Springer, pp.181-196, 2018, 978-3-319-78449-6. hal-01827110

**HAL Id: hal-01827110**

**<https://hal.science/hal-01827110>**

Submitted on 1 Jul 2018

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# Insights on event-triggered control for linear systems subject to norm-bounded uncertainty

S. Tarbouriech, A. Seuret, C. Prieur, L. Zaccarian

**Abstract** The chapter deals with the design of event-triggered rules to stabilize a class of uncertain linear control systems. The uncertainty affecting the plant is norm-bounded. The event-triggering rule uses only local information, namely it uses only the output signals available to the controller. The approach proposed combines a hybrid framework to describe the closed-loop system with techniques based on looped functionals. The proposed design conditions are formulated in terms of linear matrix inequalities (LMIs), ensuring global robust asymptotic stability of the closed-loop system. A tunable parameter allows guaranteeing an adjustable dwell-time property of the solutions. The effectiveness of the approach is evaluated on an example taken from the literature.

## 1 Introduction

In recent years, sampled-data control designs for linear or nonlinear plants have been studied in several works. In particular, robust stability analysis with respect to aperiodic sampling has been widely studied (see, for example, [7, 17, 23] and references therein), where variations on the sampling intervals are seen as a disturbance, an undesired perturbation of the periodic case. The objective is then to

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provide an analysis of such systems using the discrete-time approach [17, 8], the input delay approach [12, 31], or the impulsive systems approach [22]. Furthermore, an alternative and interesting vision of sampled-data systems has been proposed in [3, 5], suggesting to adapt the sampling sequence to certain events related to the state evolution (see, for example, [4, 15, 18, 20, 34, 38]). This is called “event-triggered sampling”, which naturally mixes continuous and discrete-time dynamics. Thus, the event-triggered algorithm design can be rewritten as the stability study of a hybrid dynamical system, which has been carried out in different contexts in [13, 14, 27, 29].

In the context of event-triggered control, two objectives can be pursued: 1) the controller is a priori designed and only the event-triggered rules have to be designed, or 2) the joint design of the control law and the event-triggering conditions has to be performed. The first case is called the emulation approach, whereas the second one corresponds to the co-design problem. A large part of the existing works is dedicated to the design of efficient event-triggering rules, that is the designs done by emulation: see, for example, [16, 37, 26, 35, 2] and references therein. Moreover, most of the results on event-triggered control consider that the full state is available, which can be unrealistic from a practical point of view. Hence, it is interesting to address the design of event-triggered controllers by using only measured signals. Some works have addressed this challenge as, for example, in [36] where the dynamic controller is an observer-based one, [1], where the co-design of the output feedback law and the event-triggering conditions is addressed by using the hybrid framework.

The results proposed in the current chapter take place in the context of the emulation approach, when the predesigned controller is issued from a hybrid dynamic output feedback controller, with the aim at using only the available signals. The controller under consideration is a continuous controller possibly including some reset loop as in [28] (see also [10, 11, 32] for more details on reset control systems). Actually, the approach proposed combines a hybrid framework to describe the sampled-data system with Lyapunov-based techniques. Constructive conditions, in the sense that linear matrix inequality (LMI) conditions are associated to a convex optimization scheme, are proposed to design the event-triggered rule ensuring asymptotic stability of the closed-loop system. Differently from [1], a condition involving the allowable maximal sampling period  $T$  can be deduced by solving a set of LMIs proposed using a similar approach to the one in [21]. Furthermore, complementary to most of the results in the literature, uncertainty affecting the continuous plant is considered in our approach. The results of this chapter are complementary to those in [33] where polytopic uncertainties (rather than norm bounded ones) are considered with similar design approaches.

The chapter is organized as follows. In Section 2, the system under consideration is defined, together with the sampled-data architecture. The problem that we intend to solve is also formally stated in Section 2, describing the associated hybrid formulation. Section 3 is dedicated to presenting the main conditions, allowing to design the event-triggering rules in both the nominal and the uncertain cases. The condition to design the associated dwell-time is also derived. We point out that the

contribution is twofold. On the first hand, we provide a new event-triggered algorithm yielding robust stability controllers for the closed-loop system. Secondly, the stability conditions depend on a dwell-time  $T$ , which appears as an explicit tuning parameter for the selection of the control law. Section 4 illustrates the results and compares them with some existing approach. Finally, in Section 5, some concluding remarks end the chapter.

**Notation.** The sets  $\mathbb{N}$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times n}$  and  $\mathbb{S}^n$  denote respectively the sets of positive integers, positive scalars,  $n$ -dimensional vectors,  $n \times n$  matrices and symmetric matrices in  $\mathbb{R}^{n \times n}$ . If a matrix  $P$  in  $\mathbb{S}_+^n$ , it means that  $P$  is symmetric positive definite. The superscript “ $\top$ ” stands for matrix transposition, and the notation  $\text{He}(P)$  stands for  $P + P^\top$ . The Euclidean norm is denoted  $|\cdot|$ . Given a compact set  $\mathcal{A}$ , the notation  $|x|_{\mathcal{A}} := \min\{|x - y|, y \in \mathcal{A}\}$  indicates the distance of the vector  $x$  from the set  $\mathcal{A}$ . The symbols  $I$  and  $0$  represent the identity and the zero matrices of appropriate dimensions.

## 2 Problem formulation

The chapter deals with linear systems fed by an output feedback sampled-data control described by the following hybrid dynamical system

$$\begin{cases} \dot{x} = Ax + Bu, \\ \dot{u} = 0, \\ \dot{\sigma} \in g_T(\sigma), \end{cases} \quad (x, u, \sigma) \in \mathcal{C}, \quad (1)$$

$$\begin{cases} x^+ = x, \\ u^+ = KCx, \\ \sigma^+ = 0, \end{cases} \quad (x, u, \sigma) \in \mathcal{D},$$

where  $x \in \mathbb{R}^n$  represents the state of the system and  $u \in \mathbb{R}^m$  represents the zero order holder of the system input since the last sampling time. The output  $y$  of the system is given by

$$y = Cx \in \mathbb{R}^p. \quad (2)$$

System (1)-(2) can appear when connecting, for instance, a linear continuous-time plant with a dynamic output feedback controller.

*Remark 1.* In [28], a reset controller is considered, which corresponds to modifying system (1)-(2) by considering  $x^+ = Jx$ , where  $J$  is a matrix of appropriate dimensions.

To study stability properties for (1), the hybrid formalism of [13, 27, 30] can be used. Matrices  $A, B, C$  characterize the system dynamics and matrix  $K$  corresponds to the controller gain. While  $C$  is assumed to be constant and known, let us assume that matrices  $A$  and  $B$  are constant but uncertain (see, for example, [24], [39]) and expressed by

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \end{bmatrix} + DF \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad (3)$$

with  $A_0$ ,  $B_0$ ,  $D$ ,  $E_1$  and  $E_2$  constant and known matrices. These matrices define the structure of the uncertainty.  $A_0$  and  $B_0$  define the nominal case and the uncertainty parameter is  $F$ , which is supposed to be constant and belongs to the set:

$$\mathbb{F} = \left\{ F \in \mathbb{R}^{f \times f}; F^\top F \leq I \right\}. \quad (4)$$

Timer  $\sigma \in [0, 2T]$  flows by keeping track of the elapsed time since the last sample (where it was reset to zero) according to the following set-valued dynamics:

$$g_T(\sigma) := \begin{cases} 1 & \sigma \leq 2T \\ [0, 1] & \sigma = 2T, \end{cases} \quad (5)$$

whose rationale is that whenever  $\sigma < 2T$ , its value exactly represents the elapsed time since the last sample, moreover  $\sigma \in [T, 2T]$  implies that at least  $T$  seconds have elapsed since the last sample.

*Remark 2.* The use of a set-valued map for the right hand side  $g_T$  of the flow equation for  $\sigma$  enables us to confine the timer  $\sigma$  to a compact set  $[0, 2T]$ . Note also that with the selection in (5), the regularity conditions in [14, As. 6.5] and the desirable robustness properties of stability of compact attractors established in [14, Ch. 7] are satisfied.

In (1), the so-called flow and jump sets  $\mathcal{C}$  and  $\mathcal{D}$  must be suitably selected to induce a desirable behavior of the sampled-data system, and are the available degrees of freedom in the design of the event-triggered algorithm addressed here. In particular, the problem that we intend to solve in this chapter can be summarized as follows.

**Problem 1.** Given an uncertain linear plant and a hybrid controller defined by (1), (2), (3), (4), (5), design an event-triggering rule, with a prescribed dwell-time  $T$  that makes the closed loop globally asymptotically stable to a compact set wherein  $x = 0$  and  $u = 0$ .

Problem 1 corresponds to an emulation problem (see, for example [16, 37, 26, 35] and the references therein) since we assume that the controller gain  $K$  is given.

### 3 Event-triggered design

In order to address Problem 1, we focus on hybrid dynamics (1) for suitably selecting the flow and jump sets  $\mathcal{C}$  and  $\mathcal{D}$ , whose role is precisely to rule when a sampling should happen, based on the available signals to the controller, namely output  $y = Cx$ , the last sampled input  $u$  and timer  $\sigma$ . Then, we select the following sets  $\mathcal{C}$  and  $\mathcal{D}$ :

$$\mathcal{C} := (\mathcal{F} \times [0, 2T]) \cup (\mathbb{R}^{n+m} \times [0, T]) \quad (6a)$$

$$\mathcal{D} := (\mathcal{J} \times [0, 2T]) \cap (\mathbb{R}^{n+m} \times [T, 2T]), \quad (6b)$$

where sets  $\mathcal{F}$  and  $\mathcal{J}$  are selected as

$$\mathcal{F} := \left\{ (x, u); \begin{bmatrix} y \\ s - Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s - Ky \end{bmatrix} \leq 0 \right\}, \quad (6c)$$

$$\mathcal{J} := \left\{ (x, u); \begin{bmatrix} y \\ s - Ky \end{bmatrix}^\top M \begin{bmatrix} y \\ s - Ky \end{bmatrix} \geq 0 \right\}, \quad (6d)$$

where matrix  $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{R}^{(p+m) \times (p+m)}$  has to be designed, and  $y$  is defined in (2). Solution (6) to the considered event-triggered problem is parametrized by  $M$  and  $T$ .

Note that the jump set selection in (6b) ensures that all solutions satisfy a dwell-time constraint corresponding to  $T$ . Moreover the definition of the flow and jump sets provided in (6) meets the one provided in the recent paper [25]. The novelty of this definition, which is also used in [33], relies on the consideration of a general matrix  $M$ . For example, selecting  $M_2 = 0$  leads to the definition of the flow and jump sets usually employed in the literature, issued from an Input-to-State (or Input-to-Output) analysis. See [25] for more details.

### 3.1 Nominal case

In this section, the design is addressed for the nominal case, namely  $A = A_0$  and  $B = B_0$  (which corresponds to  $F = 0$  in (3)).

**Theorem 1.** *Assume that, for a given  $T > 0$ , there exist matrices  $P \in \mathbb{S}_+^n$ ,  $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$  satisfying*

$$\begin{aligned} \Psi_{\mathcal{C}}(A_0, B_0) &:= \begin{bmatrix} \text{He}(P(A_0 + B_0 K C)) - C^\top M_1 C & \star \\ B_0^\top P - M_2^\top C & -M_3 \end{bmatrix} < 0, \\ \Psi_{\mathcal{D}}(A_0, B_0, T) &:= \begin{bmatrix} I \\ KC \end{bmatrix}^\top \Lambda^\top(A_0, B_0, T)^\top P \Lambda(A_0, B_0, T) \begin{bmatrix} I \\ KC \end{bmatrix} - P < 0, \end{aligned} \quad (7)$$

with

$$\Lambda(A_0, B_0, T) := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\begin{bmatrix} A_0 & B_0 \\ 0 & 0 \end{bmatrix} T} \in \mathbb{R}^{n \times (n+m)}. \quad (8)$$

Then the compact attractor

$$\mathcal{A} := \{(x, u, \sigma) : x = 0, u = 0, \sigma \in [0, 2T]\} \quad (9)$$

is globally asymptotically stable for the nominal closed-loop dynamics (1), (6) with  $\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \end{bmatrix}$ .

*Proof.* To prove this result one uses a non-smooth Lyapunov function and the hybrid invariance principle given in [33]. In particular, the following function is considered, with the standard notation  $|z|_P^2 := z^\top Pz$ :

$$V(x, u, \sigma) := \underbrace{e^{-\rho \min\{\sigma, T\}} \left| \Lambda(T - \min\{\sigma, T\}) \begin{bmatrix} x \\ u \end{bmatrix} \right|_P^2}_{=: V_0(x, u, \sigma)} + \underbrace{\eta |u|^2}_{=: V_u(u)}, \quad (10)$$

with  $\Lambda$  given in (8), and where  $\rho$  and  $\eta$  are sufficiently small positive scalars.

Let us first denote  $\xi := (x, u, \sigma)$ . We can also notice that in (6a) the flow set is the union of two sets, and then one can split the analysis in three cases:

- Case 1:  $\sigma \in [0, T)$ ;
- Case 2:  $\sigma = T$ ;
- Case 3:  $(x, u) \in \mathcal{F}$  and  $\sigma \in [T, 2T]$ .

Then along flowing solutions one gets:

- In Case 1, after some simplifications (as done in [33]) we get:

$$\dot{V}(\xi) = -\rho e^{-\rho \min\{\sigma, T\}} \left| \Lambda(T - \min\{\sigma, T\}) \begin{bmatrix} x \\ u \end{bmatrix} \right|_P^2 = -\rho V_0(\xi) \leq 0,$$

- In Case 3 (which also addresses Case 2, because no flowing is possible for a solution from  $\sigma = T$ , unless  $(x, u) \in \mathcal{F}$ ), from inequality  $\Psi_{\mathcal{E}}(A_0, B_0) < 0$  in (7), there exists a sufficiently small  $\varepsilon > 0$  such that  $\Psi_{\mathcal{E}}(A_0, B_0) < -\varepsilon I$ , and then one obtains the following strict decrease property:

$$\dot{V}(\xi) \leq -\varepsilon \left\| \begin{bmatrix} x \\ u - Ky \end{bmatrix} \right\|^2, \text{ if } (x, u) \in \mathcal{F} \text{ and } \sigma \geq T. \quad (11)$$

Therefore, the Lyapunov function  $V$  is strictly decreasing along flows.

Along jumps, it is easy to verify that, for all  $\xi \in \mathcal{D}$ , after some calculations (see again [33]), the condition  $\Psi_{\mathcal{D}}(A_0, B_0, T) < 0$  ensures that

$$V^+(\xi) = e^{-\rho T} V_0(\xi) \leq e^{-\rho T} V(\xi), \quad (12)$$

which proves the strict decrease of the Lyapunov function, across any jump from a point outside  $\mathcal{A}$ .

One can finally show that no “bad” complete solution exists, which keeps  $V$  constant and nonzero. If any such “bad” complete solution existed, then it would start outside  $\mathcal{A}$  (where  $V(\xi) \neq 0$ ) and it could not jump, because otherwise from (12), a decrease of  $V$  would be experienced across the jump. However, any solution flowing forever outside  $\mathcal{A}$  would eventually reach a point where  $\sigma > T$ , and (11)

would imply again some decrease of  $V$ . The proof is then completed by applying the invariance principle in [33].  $\square$

*Remark 3.* Let us provide some comments on the conditions of Theorem 1.

- The satisfaction of  $\Psi_{\mathcal{E}}(A_0, B_0) < 0$  imposes that the Lyapunov function  $V$  in (10) is decreasing while flowing with  $\sigma \geq T$  (which requires  $(x, u) \in \mathcal{F}$ ).
- The satisfaction of  $\Psi_{\mathcal{D}}(A_0, B_0, T) < 0$  can be interpreted as an asymptotic stability criterion for system (1) when the control updates are performed periodically with a period  $T$ , which motivates the union and intersection in (6a) and (6b).

*Remark 4.* The interest of the proposed approach with respect to the literature, where the dwell time is computed a posteriori, resides in the fact that Theorem 1 includes a guaranteed dwell-time  $T$  as a tuning parameter. In particular, if one can find a solution to the LMI conditions (7) for a given parameter  $T$ , then this same  $T$  can be employed in the definition of the flow and jumps sets (6) and is a guaranteed dwell time for all solutions of (1), (6). This method can be compared to [35] or [1] where a similar triggering rule includes a dwell time constraint, but in the current case, the dwell time  $T$  is a parameter for the design of event triggering algorithm.

*Remark 5.* Theorem 1 can be stated when an additional reset control component, as mentioned in Remark 1, is included in the jump dynamics of system (1)-(6). Preliminary results in this direction are provided in [28].

### 3.2 Uncertain case

Let us address now the case where matrices  $A$  and  $B$  are uncertain as defined in (3)-(4). In this case, it is difficult to verify the inequality  $\Psi_{\mathcal{D}}(A, B, T) < 0$ , which will depend nonlinearly on the uncertain parts. Nevertheless, it is possible to adapt results developed in [31, Thm 1], which are based on the recent developments arising from stability analysis of periodic sampled-data systems (see also [19]). Then, the following lemma can be proven by using the looped-functional approach developed in [31, 6].

**Lemma 1.** *For a given positive scalar  $T$  and matrices  $A, B, K, C$  as defined in (3)-(4), if there exist  $P, Z \in \mathbb{S}_+^n$ ,  $Q, X \in \mathbb{S}^n$ ,  $R \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{2n \times n}$  such that the inequalities*

$$\begin{aligned} \Psi_{\mathcal{D}1}(A, B, T) &:= F_0(A, B, T) + TF_1(A, B) < 0, \\ \Psi_{\mathcal{D}2}(A, B, T) &:= \begin{bmatrix} F_0(A, B, T) & TY \\ \star & -TZ \end{bmatrix} < 0, \end{aligned} \quad (13)$$

hold for all pairs  $(A, B)$  satisfying (3)–(4), where

$$\begin{aligned} F_0(A, B, T) &:= \text{He}(e_S^\top P e_1 - e_{12}^\top R e_2 - Y e_{12}) - e_{12}^\top Q e_{12} - T e_2^\top X e_2, \\ F_1(A, B) &:= \text{He}(e_S^\top Q e_{12} + e_S^\top R e_2) + e_S^\top Z e_S + 2e_2^\top X e_2, \end{aligned}$$

and  $e_S := [A \ BKC]$ ,  $e_1 := [I_n \ 0]$ ,  $e_2 := [0 \ I_n]$  and  $e_{12} := [I_n \ -I_n]$ , then inequality  $\Psi_{\mathcal{D}}(A, B, T) < 0$  in (7) holds for any pair  $(A, B)$  satisfying (3)-(4).

By adapting Lemma 1 to the case of norm-bounded uncertainty, we can derive the following theorem solving Problem 1. It is based on the non-smooth hybrid Lyapunov function introduced in (10), which is weak in the sense that it does not provide a strict decrease both during flow and across jumps (samplings) of the proposed event-triggered sampled-data system. The proof then relies on the non-smooth invariance principle presented in [33].

**Theorem 2.** Given positive scalar  $T > 0$  and matrices  $A_0, B_0, D, E_1, E_2$  as defined in (3)-(4). Assume that there exist matrices  $P \in \mathbb{S}_+^n$ ,  $M = \begin{bmatrix} M_1 & M_2 \\ M_2^\top & M_3 \end{bmatrix} \in \mathbb{S}^{p+m}$ , and matrices  $Z \in \mathbb{S}_+^n$ ,  $Q, U \in \mathbb{S}^n$ ,  $R \in \mathbb{R}^{n \times n}$  and  $Y \in \mathbb{R}^{2n \times n}$  and positive scalars  $\varepsilon_i$ ,  $i = 0, 1, 2, 3$  satisfying conditions

$$\Theta_{\mathcal{E}} := \begin{bmatrix} \Psi_{\mathcal{E}}(A_0, B_0) + \varepsilon_0 e_{\mathcal{E}}^\top e_{\mathcal{E}} & \star \\ [D^\top P \ 0] & -\varepsilon_0 I \end{bmatrix} < 0, \quad (14)$$

$$\Theta_{\mathcal{D}1} := \begin{bmatrix} F_0(A_0, B_0, T) + TF_1(A_0, B_0) + (\varepsilon_1 + T\varepsilon_3) e_{\mathcal{D}}^\top e_{\mathcal{D}} & \star \\ D^\top (Pe_1 + T(Qe_{12} + Re_2 + Ze_0)) & -\varepsilon_1 I \end{bmatrix} < 0, \quad (15)$$

$$\Theta_{\mathcal{D}2} := \begin{bmatrix} F_0(A_0, B_0, T) + \varepsilon_2 e_{\mathcal{D}}^\top e_{\mathcal{D}} & \star & \star \\ TY^\top & -TZ & \star \\ D^\top Pe_1 & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \quad (16)$$

$$\Theta_{\mathcal{D}3} := \varepsilon_3 I - D^\top Z D > 0, \quad (17)$$

with

$$\begin{aligned} F_0(A_0, B_0, T) &:= \text{He}(e_0^\top Pe_1 - e_{12}^\top Re_2 - Ye_{12}) - e_{12}^\top Qe_{12} - Te_2^\top Xe_2 \\ F_1(A_0, B_0) &:= \text{He}(e_0^\top Qe_{12} + e_0^\top Re_2) + e_0^\top Ze_0 + 2e_2^\top Xe_2 \end{aligned}$$

and  $e_0 := [A_0 \ B_0KC]$ ,  $e_{\mathcal{E}} = [E_1 + E_2KC \ E_2]$ ,  $e_{\mathcal{D}} = [E_1 \ E_2KC]$ ,  $e_1 := [I_n \ 0]$ ,  $e_2 := [0 \ I_n]$  and  $e_{12} := [I_n \ -I_n]$ . Then the compact attractor  $\mathcal{A}$  in (9) is globally asymptotically stable for the uncertain closed-loop dynamics (1)-(6) that is for each pair  $(A, B)$  satisfying (3)-(4).

*Proof.* The proof of the theorem follows by showing that (14)–(17) imply the two conditions in (7) (with  $[A_0 \ B_0]$  replaced by  $[A \ B]$ ), and then the proof can be completed by following the same exact steps as those in the proof of Theorem 1.

*First condition in (7).* Condition  $\Psi_{\mathcal{E}}(A_0, B_0) < 0$  of (7) has to be replaced by  $\Psi_{\mathcal{E}}(A, B) < 0$  where  $(A, B)$  have the expression in (3). After some calculations, this substitution gives:

$$\Psi_{\mathcal{E}}(A, B) = \Psi_{\mathcal{E}}(A_0, B_0) + \text{He} \left( \begin{bmatrix} PD \\ 0 \end{bmatrix} F \underbrace{[E_1 + E_2KC \ E_2]}_{=e_{\mathcal{E}}} \right) < 0 \quad (18)$$

Using the fact that  $F^\top F \leq I$ , this expression can be upper-bounded as follows:

$$\Psi_{\mathcal{C}}(A, B) \leq \Psi_{\mathcal{C}}(A_0, B_0) + \varepsilon_0 e_{\mathcal{C}}^\top e_{\mathcal{C}} + \varepsilon_0^{-1} \begin{bmatrix} PD \\ 0 \end{bmatrix} \begin{bmatrix} PD \\ 0 \end{bmatrix}^\top \quad (19)$$

for any positive scalar  $\varepsilon_0$ . By using a Schur complement on the rightmost term of (19), one obtains the matrix  $\Theta_{\mathcal{C}}$  in (14). Then it follows that inequality (14) implies that  $\Psi_{\mathcal{C}}(A, B) < 0$ .

*Second condition in (7).* To prove this second condition, we use Lemma 1 and show that (15)–(17) imply (13). In particular, the expressions of  $F_0(A, B, T)$  and  $F_1(A, B)$  in Lemma 1 are developed by substituting  $e_S = e_0 + DFe_{\mathcal{D}}$ , as defined in Theorem 2. Indeed, we note that

$$\begin{aligned} \Psi_{\mathcal{D}1}(A, B, T) &= \Psi_{\mathcal{D}1}(A_0, B_0, T) + \text{He} \left( e_{\mathcal{D}}^\top F^\top D^\top (Pe_1 + TQe_{12} + TRe_2 + TZe_0) \right. \\ &\quad \left. + Te_{\mathcal{D}}^\top F^\top D^\top ZDFe_{\mathcal{D}}, \right. \\ \Psi_{\mathcal{D}2}(A, B, T) &= \Psi_{\mathcal{D}2}(A_0, B_0, T) + \begin{bmatrix} \text{He} \left( e_{\mathcal{D}}^\top F^\top D^\top Pe_1 \right) & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where we recall that  $e_{\mathcal{D}} = [E_1 \ E_2 KC]$ . First we note that, since  $\Theta_{\mathcal{D}3} > 0$  and  $F^\top F \leq I$ , the last term of  $\Psi_{\mathcal{D}1}(A, B, T)$  can be upper-bounded by  $\varepsilon_3 Te_{\mathcal{D}} e_{\mathcal{D}}^\top$ . Following the same procedure as for  $\Theta_{\mathcal{C}}$  in (19), and using  $F^\top F \leq I$ , for any positive selection of  $\varepsilon_1, \varepsilon_2$  we have,

$$\begin{aligned} \Psi_{\mathcal{D}1}(A, B, T) &\leq \Psi_{\mathcal{D}1}(A_0, B_0, T) + (\varepsilon_1 + T\varepsilon_3) e_{\mathcal{D}}^\top e_{\mathcal{D}} \\ &\quad + \varepsilon_1^{-1} (Pe_1 + T(Qe_{12} + Re_2 + Ze_0))^\top DD^\top (Pe_1 + T(Qe_{12} + Re_2 + Ze_0)) \\ \Psi_{\mathcal{D}2}(A, B, T) &\leq \Psi_{\mathcal{D}2}(A_0, B_0, T) + \begin{bmatrix} \varepsilon_2 e_{\mathcal{D}}^\top e_{\mathcal{D}} + \varepsilon_2^{-1} e_1^\top PDD^\top Pe_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Finally the expression of  $\Theta_{\mathcal{D}1}$  and  $\Theta_{\mathcal{D}2}$  are retrieved by application of the Schur complement. This shows that (15)–(17) imply the two conditions in (13), and also the second inequality in (7) is proven to hold, which completes the proof.  $\square$

### 3.3 Optimization and computational issues

Note first that conditions of Theorems 1 and 2 are linear in all the decision variables provided that  $K$  and  $T$  are given, as classically in an emulation problem. It is important to note that if matrix  $A_0 + B_0 KC$  is Hurwitz, there always exists a small enough positive scalar  $T$  such that the conditions of Theorem 1 or 2 are feasible. Indeed, the event-triggered rule is defined through two design parameters, which are matrix  $M$  and dwell-time  $T$ . The implicit objective is to reduce the number of control updates.

Let us observe that in the LMI  $\Theta_{\mathcal{C}} < 0$  in (14) (see also (7)), the blocks  $\text{He}(P(A_0 + B_0 KC)) - C^\top M_1 C$  and  $-M_3$  are required to be negative definite. A natural optimization procedure could then consist in minimizing the effect of the

off-diagonal term  $PB_0 - C^\top M_2$ , which could be performed by minimizing the size of the positive definite matrix  $M_3$  appearing on the diagonal. Obtaining small values of the diagonal term  $-M_3$  will indeed reduce also the off-diagonal term in (7). This optimization problem can be formulated in terms of an LMI optimization as follows

$$\begin{aligned} & \min_{P, M, Z, Q, U, R, Y, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3} \text{Trace}(M_3), \\ & \text{subject to:} \\ & \Theta_{\mathcal{E}} < 0, \Theta_{\mathcal{D}1} < 0, \Theta_{\mathcal{D}2} < 0, \Theta_{\mathcal{D}3} > 0 \\ & P > I, M_1 < 0, \end{aligned} \quad (20)$$

In the optimization problem (20), the additional constraint  $P > I$  is imposed for well conditioning the LMI constraints. In addition, constraint  $M_1 < 0$  has been included in order to obtain negative definiteness of  $\text{He}(P(A_0 + B_0KC))$  in (7), which avoids exponentially unstable continuous dynamics, thereby giving more graceful inter-sample transients. Furthermore, the fact of minimizing the trace of  $M_3$  aims at increasing the negativity of matrix  $M_3$ , which leads to larger flow sets (see equation (6)). Since the jump set is the closed complement of the flow set, it is expected that solutions will flow longer and jump less in light of larger flow sets. Moreover, the dwell-time  $T$  being also a design parameter, whose role is connected to the expected average sampling rate of the event triggered implementation, one can seek for maximizing its value through problem (20) by iteratively increasing  $T$  and testing the conditions.

## 4 Illustrative example

In this example, we consider that system (1) is issued from the connection of a linear plant with a dynamical output feedback controller, inspired from [9, 1]. Furthermore, this example can be viewed as a complementary example to that one tackled in [33], where polytopic uncertainties were considered. Hence, we consider the following plant:

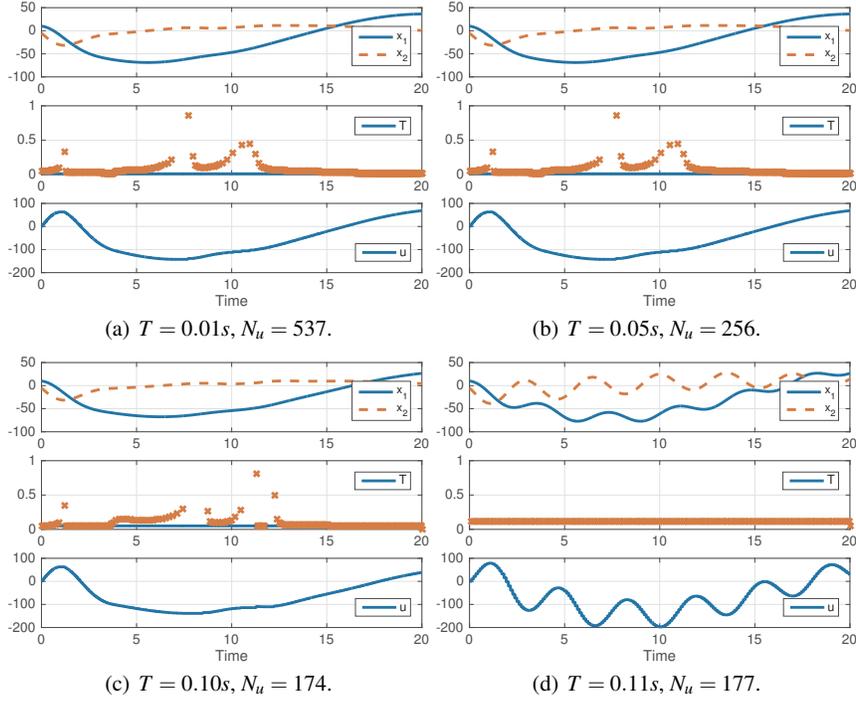
$$\begin{cases} \dot{x}_p = A_p x_p + B_p u_p = (A_{p0} + D_p F E_{p1}) x_p + (B_{p0} + D_p F E_{p2}) u_p, \\ y_p = C_p x_p, \end{cases} \quad (21)$$

for which the nominal data is given by

$$A_{p0} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, B_{p0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_p = [-1 \ 4] \quad (22)$$

and the matrices describing the norm-bounded uncertainty are defined by:

$$D_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{p1} = [0 \ \omega_0], E_{p2} = 0.1 \omega_0 \quad (23)$$



**Fig. 1** Figure representing the state of the plant  $x_p$ , the inter-sampling times (with the dwell time  $T$ ) and the control input  $u$  issued from Theorem 2 for several values of  $T$  for the same initial condition  $x_p(0) = [10 \ -5]^\top, x_c(0) = [0 \ 0]^\top$ .

with a positive constant  $\omega_0$ . The following controller obtained using an optimization process provided in [1] is also considered:

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y_p, \\ u_p = C_c x_c + D_c y_p, \end{cases} \quad (24)$$

With plant (21) interconnected to the controller (24), denote the overall state by  $x := \begin{bmatrix} x_p \\ x_c \end{bmatrix}$ . The complete closed-loop system under consideration in this chapter can be reformulated as system (1) with

$$\begin{bmatrix} A & B \\ K & C \end{bmatrix} = \begin{bmatrix} A_p & 0 & B_p & 0 \\ 0 & A_c & 0 & B_c \\ D_c & C_c & C_p & 0 \\ I & 0 & 0 & I \end{bmatrix} \quad (25)$$

Similarly, from (3) one can define the nominal and uncertain parts as follows:

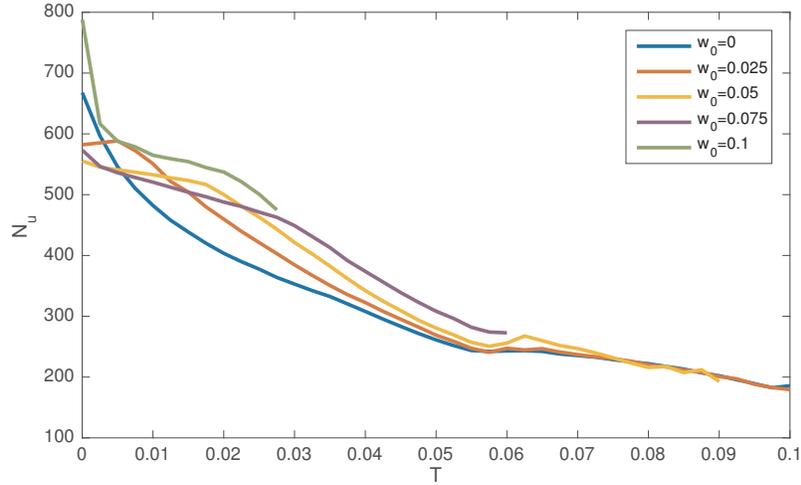
$$A_0 = \begin{bmatrix} A_{p0} & 0 \\ 0 & A_c \end{bmatrix}; B_0 = \begin{bmatrix} B_{p0} & 0 \\ 0 & B_c \end{bmatrix}; D = \begin{bmatrix} D_p \\ 0 \end{bmatrix}; E_1 = [E_{p1} \ 0]; E_2 = [E_{p2} \ 0] \quad (26)$$

### 4.1 Nominal case

The nominal case corresponds to setting  $\omega_0 = 0$ . Let us note that in [1], some improvements with respect to the literature (for instance with respect to [9]) have been reported in the nominal case. More precisely, the authors of [1] obtained a dwell-time  $T = 0.0114s$ , whereas by using the conditions of Theorem 1, one can verify that there exist values of the design parameter  $T$  up to  $0.113s$  providing feasible designs. This corresponds to a parameter  $T$  ten times larger than the solution provided in [1], which well illustrates the potential of the proposed method. Moreover, it is worth pointing out that the numerical results obtained from the application of Theorem 1 (which is only applicable to the nominal case) are very similar to the maximal guaranteed dwell-time obtained by Theorem 2 specialized to the nominal case by setting  $E_1 = 0$  and  $E_2 = 0$ . This means that the conservatism introduced by Theorem 2, to be able to provide an event-triggered algorithm for the system, is quite limited compared to Theorem 1.

Figure 1 shows several simulation results of the nominal system obtained for four dwell-time parameters  $T$  selected in Theorem 1. The caption of the figure also shows the number of control updates ( $N_u$ ) that have been required by each event-triggered simulation. While increasing  $T$  leads apparently to a notable reduction of the number of control updates, it can also be seen in Figure 1(d) that the selection of a too large guaranteed dwell-time has several drawbacks. First of all, a similar number of control updates  $N_u$  are required for simulations (c) and (d). The sampling algorithm in (c) is still able to often trigger the sampling well after  $T = 0.1$  times after the previous update, and the inter-sampling time may reach up to  $0.8s$ . The sampling algorithm employed in (d) results in a periodic implementation of the control law. As another consequence, the simulation provided in (d) shows some undesirable oscillatory behavior that makes this emulation rule not effective with respect to some performance index.

Apart from that, the three simulations depicted in Figure 1(a,b,c) are quite similar if one only regards the  $x$  state response. The main difference can be seen in the triggering rule and in the number of control updates  $N_u$ . Indeed, a trend can be seen in these simulations, which consists in noting that increasing the dwell-time parameter  $T$  allows to notably reduce the number of control updates, while obtaining similar responses in the  $x$  variable. Of course, regarding the previous remarks on Figure 1(d), increasing too much  $T$  up to the maximal feasibility value of the LMI conditions of Theorem 2 (or Theorem 1), is not a good option to obtain effective emulation algorithms.



**Fig. 2** Evolution of the average number of control updates  $N_u$  with respect to the dwell time parameter  $T$  for several values of  $\omega_0$ .

## 4.2 Uncertain case

In order to illustrate the uncertain case, corresponding to the situation where  $\omega_0$  is not equal to 0 anymore, we have conducted the following test. For five values of  $\omega_0$  taken in the interval  $[0, 0.1]$ , the average number of control updates  $N_u$  obtained over 60 different initial conditions is computed for several values of  $T$  in  $[0, 0.1]$ . Figure 2 shows these simulations. One can first note that the maximal value of  $\omega_0$ , for which a solution to the conditions of Theorem 2 can be found, depends on the dwell-time parameter  $T$ . More precisely, increasing  $T$  reduces the maximal allowable uncertainty range  $\omega_0$ . For  $T = 0.01$ , solutions to the conditions of Theorem 2 can be found up to  $\omega_0 = 0.1$  while for  $T = 0.1$ , solutions can be found up to  $\omega_0 = 0.025$ .

In addition, the decreasing trends shown in the figure reveal that the expected control updates, suitably averaged over the 60 initial conditions, are a decreasing function of the dwell-time parameter  $T$ .

## 5 Conclusion

In this chapter, we have presented a method to provide efficient output-feedback event-triggered controls for linear systems subject to norm-bounded uncertainties. Based on an existing control law, which ensures, a priori, the stability of the associated continuous-time closed-loop system, the chapter presents several constructive theorems providing an efficient event-triggered sampling algorithm dedicated to the

nominal and the uncertain cases. The conditions are expressed in terms of LMIs where a guaranteed dwell-time appears as a tunable parameter. The method is then evaluated on an example taken from the literature, which demonstrates the potential of the proposed solutions.

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