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HODGE DECOMPOSITION FOR SYMMETRIC MATRIX FIELDS AND THE ELASTICITY COMPLEX IN LIPSCHITZ DOMAINS

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Abstract. In 1999 M. Eastwood has used the general construction known as the Bernstein-Gelfand-Gelfand (BGG) resolution to prove, at least in smooth situation, the equivalence of the linear elasticity complex and of the de Rham complex in $\mathbf{R}^3$. The main objective of this paper is to study the linear elasticity complex for general Lipschitz domains in $\mathbf{R}^3$ and deduce a complete Hodge orthogonal decomposition for symmetric matrix fields in L^2, counterpart of the Hodge decomposition for vector fields. As a byproduct one obtains that the finite dimensional terms of this Hodge decomposition can be interpreted in homological terms as the corresponding terms for the de Rham complex if one takes the homology with value in RIG $\cong \mathbf{R}^6$ as in the (BGG) resolution.

Contents

1. Introduction

Let Ω be an open, connected and bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Relative to an orthonormal cartesian basis $\{\mathbf{e}^i\}$, (i=1,2,3), the coordinates of a generic point will be denoted by $\{x_1, x_2, x_3\}$, the components of a vector field $\mathbf{v}$ by v_i and the components of a square matrix field $\mathbf{S}$ of order three by S_{ij}. M^3 (resp. M^3_{sym}) denotes the vector space of second-order (resp. symmetric second-order) matrices. Latin indices range in the set $\{1, 2, 3\}$. The summation convention with respect to the repeated indices is used. The scalar product of the matrices $\mathbf{E}$ and $\mathbf{S}$ is denoted by $\mathbf{E}$:$\mathbf{S} := E_{ij}S_{ij}$.

Let $\mathbf{v}$ a smooth vector field defined on Ω; then the corresponding strain field $\nabla_s(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$ is a symmetric matrix field. The characterization of the smooth symmetric matrix fields $\mathbf{E}$ that are strain fields, i.e. that can be written as $\mathbf{E} = \nabla_s(\mathbf{v})$ for some $\mathbf{v}$, goes

back to the second half of the Nineteenth Century. Indeed, it was discovered by A. J.C. B. de Saint Venant (1864) the following result:

Theorem 1.1 (Saint Venant's necessary compatibility theorem)**.** *The strain field* $\boldsymbol{E}$ *corresponding to a class* $C^\infty(\Omega;\mathbf{R}^3)$ *displacement vector field* $\boldsymbol{v}$ *satisfies the compatibility equations.*

$$CURL\,CURL\,\boldsymbol{E}=0. \tag{1.1}$$

The components of the matrix $CURL\,\mathbf{E}$ are: $(CURL\,\mathbf{E})_{ij}=\epsilon_{ipk}E_{jk,p}$ where the commas stand for partial differentiations with respect to x and ϵ_{ipk} denotes the alternator:

$$\epsilon_{ipk}=\begin{cases}+1, & \text{(i,p,k) is an even permutation of (1,2,3);}\\ -1, & \text{(i,p,k) is an odd permutation of (1,2,3);}\\ 0, & \text{(i,p,k) is not a permutation of (1,2,3).}\end{cases}$$

The first [1] rigorous proof of sufficiency was given by E. Beltrami (1886) in the following form.

Theorem 1.2 (Beltrami's sufficiency compatibility theorem)**.** *If* Ω *is a simply-connected domain and if a symmetric matrix field* $\boldsymbol{E}\in C^\infty\left(\Omega;M^3_{sym}\right)$ *satisfies the compatibility equations (1.1), then there exists a vector field* $\boldsymbol{v}\in C^\infty(\Omega;\mathbf{R}^3)$ *satisfying the strain-displacement relations :*

$$\boldsymbol{E}=\frac{1}{2}(\nabla\boldsymbol{v}^T+\nabla\boldsymbol{v}) \tag{1.2}$$

A smooth symmetric matrix field $\mathbf{S}$ is said a stress field when:

$$Div\,\mathbf{S}=0 \tag{1.3}$$

where $(Div\mathbf{S})_i=S_{ij,j}$. In 1890, L. Donati has proved the following theorem :

Theorem 1.3 (Donati's theorem)**.** *Let* $\boldsymbol{E}$ *be a matrix field of class* $C^2\left(\Omega;M^3_{sym}\right)$*, such that*

$$\int_\Omega \boldsymbol{E}\!:\!\boldsymbol{S}\,d\Omega=0 \tag{1.4}$$

for every stress field $\boldsymbol{S}$ *in* $C^\infty\left(\overline{\Omega};M^3_{sym}\right)$ *that vanishes near the boundary* $\partial\Omega$*. Then* $\boldsymbol{E}$ *satisfies the equation of compatibility (1.1).*

When Ω is simply connected, thanks to the theorem 1.2, the Donati's theorem gives an orthogonal (in the L^2-sense) decomposition for symmetric matrix fields analogous to the Helmoltz decomposition of vector fields. The Donati's theorem has been extended in various directions: see e.g.[11], [3] and their bibliography and theorems 2.1 and 2.4.

[1] For more details and historical notes see Gurtin [16], Sect.14.

Beltrami (1892) has remarked that for a smooth symmetric matrix field $\mathbf{E}$ one has $Div\, CURL\, CURL\, \mathbf{E} = 0$ and, moreover, that with the representation (called Beltrami's solution):

$$\mathbf{S} = CURL\, CURL\, \mathbf{E} \tag{1.5}$$

it is possible to recover, with a suitable choice of $\mathbf{E}$, the Airy's, the Maxwell's and the Morera's solution, see Gurtin [16], Sect.17. There exists stress fields that do not admit a Beltrami representation. The characterization of the smooth stress fields $\mathbf{S}$ admitting the representation (1.5) for some symmetric matrix field $\mathbf{E}$ is given by the following theorem of Gurtin [15]:

Theorem 1.4 (Completeness of the Beltrami solution)**.** *A necessary and sufficient condition for the existence of a smooth solution of (1.3) such that $\boldsymbol{S} = CURL\, CURL\, \boldsymbol{E}$ for some smooth symmetric matrix field $\boldsymbol{E}$ is that on each closed smooth surface $\boldsymbol{\Sigma}$ in Ω the resultant force and the moment vanish, i.e.*

$$\int_{\boldsymbol{\Sigma}} \boldsymbol{S}\boldsymbol{n}\, d\,\mathrm{s} = \int_{\boldsymbol{\Sigma}} \boldsymbol{S}\boldsymbol{n} \wedge \boldsymbol{x}\, d\,\mathrm{s} = 0 \tag{1.6}$$

Let us remark that (1.3) implies that this condition is automatically satisfied when the boundary of Ω consists in a single closed surface.

The previous results can also be presented in the framework of differential forms and exterior calculus as a complex of linear operators (called the linear elasticity complex, see [8]):

$$\begin{aligned} 0 \to \text{RIG} \to C^\infty\left(\Omega; \mathbf{R}^3\right) &\xrightarrow{\nabla_s} C^\infty\left(\Omega; M^3_{sym}\right) \\ &\xrightarrow{CURL\, CURL} C^\infty\left(\Omega; M^3_{sym}\right) \xrightarrow{Div} C^\infty\left(\Omega; \mathbf{R}^3\right) \to 0 \end{aligned} \tag{1.7}$$

where the kernel of each linear operator contains the image of the previous one and:

$$\text{RIG} = \{\mathbf{v}; \nabla_s(\mathbf{v}) = 0\} = \{\mathbf{v}; \mathbf{v} = \mathbf{a} + \mathbf{b} \wedge \mathbf{id}_\Omega\} \cong \mathbf{R}^6 \tag{1.8}$$

For later use we denote by $\mathbf{e}^i$ and $\mathbf{p}^i = -\epsilon_{ijk} x_k \mathbf{e}^j$, $i = 1, 2, 3$, a basis of RIG. The Beltrami's sufficiency compatibility theorem and the Gurtin's theorem give sufficient conditions in order that the sequence is exact, i.e. that each operator provides the integrability conditions for the one which precedes it.

Volterra (1906) has given a characterization of $ker(CURL\, CURL)$ for a general non simply connected domain Ω, inspired by the results of Poincaré on the complex (also called the De Rham-Poincaré complex):

$$\begin{aligned} 0 \to \mathbb{R} \to C^\infty(\Omega) &\xrightarrow{grad} C^\infty\left(\Omega; \mathbf{R}^3\right) \\ &\xrightarrow{\mathbf{curl}} C^\infty\left(\Omega; \mathbf{R}^3\right) \xrightarrow{div} C^\infty(\Omega) \longrightarrow 0 \end{aligned} \tag{1.9}$$

where: $\mathbb{R} = \{\phi; grad\, \phi = 0\}$, $(\mathbf{curl}\, \mathbf{v})_i = \epsilon_{ijk} v_{k,j}$ and $div\, \mathbf{v} = v_{i,i}$.

The analogy between the two complexes can be further developed. Indeed M. Eastwood [8] has proved that, with a general construction known as the Bernstein-Gelfand-Gelfand (BGG) resolution, the two sequences are equivalent for smooth functions. He has also conjectured that "anything which is true of the de Rham complex should have a counterpart for linear elasticity". For example, a smooth matrix field $\mathbf{S}$ has a generalized Beltrami representation of the form (see [9]):

$$\mathbf{S} = CURL\,CURL\mathbf{A} + \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v}) \tag{1.10}$$

where $\mathbf{A}$ can be chosen such that $Div\mathbf{A} = 0$; this representation is the counterpart to the Helmholtz decomposition of vector fields, $\frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$ being the analogous of the irrotational term and $CURL\,CURL\mathbf{A}$ the analogous of the solenoidal term [2] .

The aim of this paper is to prove "a further possible instance of this": a general Hodge decomposition for symmetric matrix fields analogous to the classical Hodge decomposition for vector fields (see e.g. [4]); for this we study the linear elasticity complex for general Lipschitz domains in various Sobolev spaces settings. Let us define

$$H(Div;\Omega) = \{\mathbf{S} \in L^2\left(\Omega; M^3_{sym}\right); Div\,\mathbf{S} \in L^2\left(\Omega;\mathbf{R}^3\right)\} \tag{1.11}$$

and

$$\begin{aligned} &H(CURL\,CURL;\Omega) = \\ &\qquad \{\mathbf{E} \in L^2\left(\Omega; M^3_{sym}\right); CURL\,CURL\,\mathbf{E} \in L^2\left(\Omega; M^3_{sym}\right)\} \end{aligned} \tag{1.12}$$

where the operators Div and $CURL\,CURL$ are to be taken in the distribution sense. We will consider the following situations:

i: Beltrami completeness condition in L^2 setting:

$$H^2\left(\Omega; M^3_{sym}\right) \overset{CURL\,CURL}{\longrightarrow} H(Div;\Omega) \overset{Div}{\longrightarrow} L^2\left(\Omega;\mathbf{R}^3\right) \longrightarrow 0 \tag{1.13}$$

ii: Saint Venant compatibility condition in L^2 setting:

$$\text{RIG} \hookrightarrow H^1\left(\Omega;\mathbf{R}^3\right) \overset{\nabla_s}{\longrightarrow} L^2\left(\Omega; M^3_{sym}\right) \overset{CURL\,CURL}{\longrightarrow} H^{-2}\left(\Omega; M^3_{sym}\right) \tag{1.14}$$

Some of the results presented concerning the Beltrami completeness condition has been given with different proofs in [12] and some on the Saint Venant compatibility condition in L^2 setting have been announced in [6].

[2] Also in [3] it has been shown, that in a proper perspective, the operators ∇_s and $CURL\ CURL$ are the "matrix analogs" of the operators *grad* and *curl* and it has been remarked that the extension there given of Saint Venant's theorem is the matrix analog of a weak form of Poincare's lemma.

2. Beltrami completeness condition in L^2 setting

Let Ω be a general bounded Lipschitz connected but eventually multiply connected domain in $\mathbf{R}^3$ with boundary $\partial\Omega$. Let be γ_0 the exterior boundary of Ω, i. e. the boundary of the unbounded connected component of $\mathbf{R}^3\backslash\Omega$, and γ_q, $q = 1, \ldots, Q$ the others connected components of $\partial\Omega$. Let denote with $\mathfrak{B}$ an open ball such that $\overline{\Omega}$ is contained in $\mathfrak{B}$ and, for $q = 1, \ldots, Q$, let Ω_q be the connected component of $\mathfrak{B}\backslash\overline{\Omega}$ with boundary γ_q. At last let Ω_0 be the connected component of $\mathfrak{B}\backslash\overline{\Omega}$ with boundary $\gamma_0 \bigcup \partial\mathfrak{B}$. For future use let remark that like every multiply-connected domain, Ω can be reduced to be a simply-connected one, Ω^*, with a finite number N of planar, non-intersecting cuts, $\mathcal{C}_\alpha$, $\alpha = 1, \ldots, N$, linking the γ_q, $q = 0, \ldots, Q$, i.e. such that the boundary of $\mathcal{C}_\alpha$ is contained in $\partial\Omega$. Moreover the cuts are such that the simply-connected domain $\Omega^* = \Omega\backslash \bigcup_{\alpha=1}^{N} \mathcal{C}_\alpha$ verifies the cone condition. Hence the usual Sobolev properties are satisfied, [1], [7].

Following the well-known approach of Lions-Magenes [17], it has been proved in [13] that $\mathfrak{D}\left(\overline{\Omega}; M^3_{sym}\right)$ is dense in $H(Div;\Omega)$ and that the map

$$\mathbf{S} \mapsto \Gamma_{\mathbf{n}}(\mathbf{S}) = (\mathbf{S}.\mathbf{n})\,|_{\partial\Omega}\,,$$

well-defined for $\mathbf{S} \in \mathfrak{D}\left(\overline{\Omega}; M^3_{sym}\right)$, can be extended to a linear and continuous map, still denoted $\Gamma_{\mathbf{n}}$, from $H(Div;\Omega)$ to $H^{-1/2}\left(\partial\Omega;\mathbf{R}^3\right)$. Moreover for every $\mathbf{S} \in H(Div;\Omega)$ and every $\mathbf{v} \in H^1\left(\Omega;\mathbf{R}^3\right)$ the following Green's formula holds:

$$\int_\Omega \nabla_s(\mathbf{v}){:}\mathbf{S}\, d\Omega + \int_\Omega Div\,\mathbf{S}.\mathbf{v}\, d\Omega = \langle \Gamma_{\mathbf{n}}(\mathbf{S})\,, \mathbf{v}\rangle_{\partial\Omega} \tag{2.1}$$

where $\langle .\,, .\rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}\left(\partial\Omega;\mathbf{R}^3\right)$ and $H^{1/2}\left(\partial\Omega;\mathbf{R}^3\right)$. We also denote by $\langle .\,, .\rangle_{\gamma_q}$ the duality pairing between $H^{-1/2}\left(\gamma_q;\mathbf{R}^3\right)$ and $H^{1/2}\left(\gamma_q;\mathbf{R}^3\right)$ and we denote by $\Gamma^q_{\mathbf{n}}(\mathbf{S})$ the restriction of $\Gamma_{\mathbf{n}}(\mathbf{S})$ to γ_q. Hence:

$$\langle \Gamma_{\mathbf{n}}(\mathbf{S})\,, \mathbf{v}\rangle_{\partial\Omega} = \sum_{q=0}^{Q} \langle \Gamma^q_{\mathbf{n}}(\mathbf{S})\,, \mathbf{v}\rangle_{\gamma_q}$$

Let at first recall two extensions of the Donati's theorem. For this let us set $Ker(Div;L^2) := \left\{\mathbf{S} \in L^2\left(\Omega; M^3_{sym}\right);\ div\mathbf{S} = 0 \text{ in } \Omega\right\}$. The first result is essentially a reformulation of Theorem 4.2 of [3].

Theorem 2.1. *The following decomposition of $L^2\left(\Omega; M^3_{sym}\right)$ in mutually orthogonal closed subspaces holds true:*

$$L^2\left(\Omega; M^3_{sym}\right) = \nabla_s(H^1_0\left(\Omega;\mathbf{R}^3\right)) \overset{\perp}{\oplus} Ker(Div;L^2) \tag{2.2}$$

In [3] it has been proved that also $\nabla_s(H^1(\Omega;\mathbf{R}^3))$ is a closed subspace of $L^2\left(\Omega;M^3_{sym}\right)$.

Proposition 2.2. *One has the following orthogonal decomposition:*

$$\nabla_s(H^1\left(\Omega;\mathbf{R}^3\right)) = \nabla_s(H^1_0\left(\Omega;\mathbf{R}^3\right)) \overset{\perp}{\oplus} \mathbb{DFG} \tag{2.3}$$

where:

$$\mathbb{DFG} = \left\{ \boldsymbol{S} \in L^2\left(\Omega;M^3_{sym}\right);\quad div\,\boldsymbol{S}=0 \quad and \right. \tag{2.4}$$
$$\left. \boldsymbol{S} = \nabla_s \dot{\boldsymbol{u}} \text{ with } \dot{\boldsymbol{u}} \in H^1(\Omega;\mathbf{R}^3)/H^1_0(\Omega;\mathbf{R}^3) \right\}$$

and $\nabla_s \dot{\boldsymbol{u}} := \nabla_s \boldsymbol{w}$ *for any* $\boldsymbol{w} \in \dot{\boldsymbol{u}}$.

Proof. The orthogonality of the decomposition follows from the Green's formula (2.1). Let be given $\mathbf{u} \in H^1(\Omega;\mathbf{R}^3)$ and set $\mathbf{f} := div\,(\nabla_s \mathbf{u}) \in H^{-1}(\Omega;\mathbf{R}^3)$ and $\mathbf{g} := \mathbf{u}|_{\partial\Omega} \in H^{1/2}(\partial\Omega;\mathbf{R}^3)$. Then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where $\mathbf{w} \in H^1(\Omega;\mathbf{R}^3)$ verifies

$$\begin{cases} div\,(\nabla_s \mathbf{w}) = 0\,, & \text{in}\,\Omega; \\ \mathbf{w}|_{\partial\Omega} = \mathbf{g}\,, & \text{on}\,\partial\Omega. \end{cases} \tag{2.5}$$

and $\mathbf{v} \in H^1_0(\Omega;\mathbf{R}^3)$ verifies

$$\begin{cases} div\,(\nabla_s \mathbf{v}) = \mathbf{f}\,, & \text{in}\,\Omega; \\ \mathbf{v}|_{\partial\Omega} = 0\,, & \text{on}\,\partial\Omega. \end{cases} \tag{2.6}$$

It is worth note that when $\mathbf{u}|_{\partial\Omega} = \hat{\mathbf{u}}|_{\partial\Omega}$ then $\mathbf{S} = \nabla_s \mathbf{w} = \nabla_s \hat{\mathbf{w}}$. □

Let introduce the spaces:

$$\mathcal{W} := \left\{ \mathbf{S} \in H^1_0\left(\Omega;M^3_{sym}\right);\ div\mathbf{S} = 0 \text{ in } \Omega \right\} \tag{2.7}$$

(2.8)
$$\Sigma_{ad}(\Omega) := \left\{ \mathbf{S} \in L^2\left(\Omega;M^3_{sym}\right);\ div\mathbf{S} = 0 \text{ in } \Omega \text{ and } \Gamma_{\mathbf{n}}(\mathbf{S}) = 0 \text{ on } \partial\Omega \right\}$$

The next theorem provides an answer to a remark of [3]; a slightly different result has been given in a general L^p setting in [11].

Theorem 2.3. *The space* $\mathcal{W}$ *is dense in* $\Sigma_{ad}(\Omega)$.

Proof. Let $L(\mathbf{S})$ be a linear and continuous functional on $\Sigma_{ad}(\Omega)$; then there exist $\widehat{\mathbf{S}} \in \Sigma_{ad}(\Omega)$ such that :

$$L(\mathbf{S}) = \int_\Omega \mathbf{S} : \widehat{\mathbf{S}}\, d\Omega$$

Let us suppose that $L(\mathbf{T}) = \int_\Omega \mathbf{T} : \widehat{\mathbf{S}}\, d\Omega = 0$ for all $\mathbf{T} \in \mathcal{W}$. From Theorem 4.3 of [3] there exists $\mathbf{v} \in H^1(\Omega;\mathbf{R}^3)$ such that $\nabla_s(\mathbf{v}) = \widehat{\mathbf{S}}$. Since $\widehat{\mathbf{S}} \in \Sigma_{ad}(\Omega)$ it follows that $\mathbf{v} \in$ RIG and so $\widehat{\mathbf{S}} = 0$; hence $L(\mathbf{S}) = 0$ for all $\mathbf{S} \in \Sigma_{ad}(\Omega)$. □

The second extension of the Donati's theorem (for a proof in a general L^p setting see [11]) provides a complement to the Theorem 4.3 of [3].

Theorem 2.4. *The orthogonal of* $\nabla_s(H^1(\Omega;\mathbf{R}^3))$ *in* $L^2\left(\Omega;M^3_{sym}\right)$ *is* $\Sigma_{ad}(\Omega)$

Proof. Since obviously $\nabla_s(H^1(\Omega;\mathbf{R}^3)) \subset (\Sigma_{ad}(\Omega))^{\perp}$ it is enough to prove that $(\nabla_s(H^1(\Omega;\mathbf{R}^3)))^{\perp} \subset \Sigma_{ad}(\Omega)$. Hence let $\mathbf{S}$ be such that for all $\mathbf{v} \in H^1(\Omega;\mathbf{R}^3)$:

$$\int_\Omega \mathbf{S} : \nabla_s(\mathbf{v})\, d\Omega = 0$$

Taking $\mathbf{v} \in \mathfrak{D}(\Omega;\mathbf{R}^3)$ one finds that $div\mathbf{S} = 0$ in Ω; then the Green's formula (2.1) implies that $\langle \Gamma_{\mathbf{n}}(\mathbf{S}), \mathbf{v}\rangle_{\partial\Omega} = 0$ for all $\mathbf{v}$. Therefore $\mathbf{S} \in \Sigma_{ad}(\Omega)$. □

It also follows from this theorem, theorem 2.1 and (2.3) that:

$$Ker(Div;L^2) = \Sigma_{ad}(\Omega) \overset{\perp}{\oplus} \mathbb{DFG} \tag{2.9}$$

The image of the operator $CURL\,CURL$, linear and continuous from $H^2\left(\Omega;M^3_{sym}\right)$ into $L^2(\Omega;\mathbf{R}^3)$, is contained in $Ker(Div;L^2)$; indeed if $\mathbf{E} \in H^2\left(\Omega;M^3_{sym}\right)$, then $\mathbf{S} = CURL\,CURL\,\mathbf{E}$ satisfies $Div\,\mathbf{S} = 0$. A natural extension of the Gurtin's theorem 1.4 is the characterization of $CURL\,CURL$ $(H^2\left(\Omega;M^3_{sym}\right))$. For this let define *the Gurtin space of totally self-equilibrated stress fields*:

$$\mathbb{G} := \{\mathbf{S} \in L^2\left(\Omega;M^3_{sym}\right); (a)\ \ Div\,\mathbf{S} = 0 \quad and \quad \text{for } q = 0,\dots,Q,$$
$$i = 1,2,3\ \ (b)\left\langle \Gamma^q_{\mathbf{n}}(\mathbf{S}), \mathbf{e}^i\right\rangle_{\gamma_q} = 0 \quad and \quad (c)\left\langle \Gamma^q_{\mathbf{n}}(\mathbf{S}), \mathbf{p}^{\,i}\right\rangle_{\gamma_q} = 0\}$$

In the definition of $\mathbb{G}$ the conditions (a), (b) and (c) are not linearly independent since taking in (2.1) $\mathbf{v} = \mathbf{e}^i$ and $\mathbf{v} = \mathbf{p}^{\,i}$ one obtains for $i = 1,2,3$:

$$\int_\Omega Div\,\mathbf{S}.\mathbf{e}^i\, d\Omega = \left\langle \Gamma_{\mathbf{n}}(\mathbf{S}), \mathbf{e}^i\right\rangle_{\partial\Omega} \tag{2.10}$$

and

$$\int_\Omega Div\,\mathbf{S}.\mathbf{p}^{\,i}\, d\Omega = \left\langle \Gamma_{\mathbf{n}}(\mathbf{S}), \mathbf{p}^{\,i}\right\rangle_{\partial\Omega} \tag{2.11}$$

Since $\Sigma_{ad}(\Omega)$ is a closed subspace of $\mathbb{G}$, from theorem 2.4 one deduces the orthogonal decomposition:

$$\mathbb{G} = \Sigma_{ad}(\Omega) \overset{\perp}{\oplus} \mathbb{H} \tag{2.12}$$

where:

$$\mathbb{H} = \mathbb{G} \cap \nabla_s(H^1\left(\Omega;\mathbf{R}^3\right)) \tag{2.13}$$

The extension of the Gurtin's theorem 1.4 is the object of the following:

Theorem 2.5. (i) *For any* $\boldsymbol{A} \in H^2(\Omega; M^3_{sym})$ *the matrix field* $\boldsymbol{S} = CURL\, CURL\, \boldsymbol{A}$ *belongs to* $\mathbb{G}$.

(ii) *There exists a linear and continuous map* $\mathfrak{B} : \mathbb{G} \to H^2(\Omega; M^3_{sym})$ *such that* $\boldsymbol{A} = \mathfrak{B}(\boldsymbol{S})$ *satisfies:*

$$CURL\, CURL\, \boldsymbol{A} = \boldsymbol{S} \quad and \quad Div\, \boldsymbol{A} = Div\, CURL\, \boldsymbol{A} = 0 \tag{2.14}$$

The proof given here is inspired by a proof of a similar result for solenoidal vector fields given in Girault-Raviart ([14], Chapter I, Theorem 3.4, see also [2]). A different proof largely inspired to the proof of Gurtin [16], Sect. 17 and that also uses the results of [14] has been given in [12] (However, let us explicitly remark that the statement (ii) of Theorem 2.2 there is not correct). The case of the Airy's stress function in Lipschitz domain has been considered in [10].

Proof. (i) Let be $\mathbf{A} \in H^2(\Omega; M^3_{sym})$ then $\mathbf{S} = CURL\, CURL\, \mathbf{A} \in L^2\left(\Omega; M^3_{sym}\right)$. Since $Div\, \mathbf{S} = 0$ in order to prove that $\mathbf{S} \in \mathbb{G}$ one has only to verify the conditions (b) and (c) or else that for $q = 0, \ldots, Q$ and for every $\mathbf{v} \in \text{RIG}$ one has $\langle \Gamma^q_{\mathbf{n}}(\mathbf{S}), \mathbf{v} \rangle_{\gamma_q} = 0$. For this let $\chi_q \in \mathfrak{D}\left(\overline{\Omega}\right)$ with $\chi_q = 1$ in a neighborhood of γ_q and $\chi_q = 0$ in a neighborhood of every γ_r with $r \neq q$. Then one has:

$$\begin{aligned} 0 &= \int_\Omega Div\left(CURL\, CURL\left(\chi_q \mathbf{A}\right)\right).\mathbf{v}\, d\Omega \\ &= \langle \Gamma_{\mathbf{n}}(CURL\, CURL\left(\chi_q \mathbf{A}\right)), \mathbf{v} \rangle_{\partial\Omega} = \langle \Gamma^q_{\mathbf{n}}(\mathbf{S}), \mathbf{v} \rangle_{\gamma_q} \end{aligned}$$

(ii) Let $\mathbf{S} \in \mathbb{G}$ and for every $q = 0, \ldots, Q$ let $\mathbf{v}_q \in H^1\left(\Omega_q; \mathbf{R}^3\right)$ the solution (unique up to a rigid displacement) of the problem:

$$\begin{cases} -Div\, \nabla_s(\mathbf{v}_0) = 0 \text{ on } \Omega_0 \\ \nabla_s(\mathbf{v}_0).\mathbf{n} = \Gamma^0_{\mathbf{n}}(\mathbf{S}) \text{ on } \gamma_0 \\ \nabla_s(\mathbf{v}_0).\mathbf{n} = 0 \text{ on } \partial\mathfrak{B} \end{cases}$$

$$\begin{cases} -Div\, \nabla_s(\mathbf{v}_q) = 0 \text{ on } \Omega_q \\ \nabla_s(\mathbf{v}_q).\mathbf{n} = \Gamma^q_{\mathbf{n}}(\mathbf{S}) \text{ on } \gamma_q \end{cases}$$

These solutions exist thanks to the conditions (b) and (c) of the definition of $\mathbb{G}$. Let us define

$$\widetilde{\mathbf{S}} = \begin{cases} \mathbf{S} \text{ on } \Omega \\ \nabla_s(\mathbf{v}_q) \text{ on } \Omega_q, \quad q = 0, \ldots, Q \\ \mathbf{0} \text{ on } \mathbf{R}^3 \backslash \mathfrak{B} \end{cases}$$

Then $\widetilde{\mathbf{S}} \in L^2_{comp}\left(\mathbf{R}^3; M^3_{sym}\right)$ and satisfies $Div\, \widetilde{\mathbf{S}} = 0$. Its Fourier transform, denoted $\widehat{\widetilde{\mathbf{S}}} = (\widehat{S}_{ij})$, belongs to $L^2\left(\mathbf{R}^3; M^3_{sym}\right)$, $\widehat{S}_{ij}$ is holomorphic in $\mathbb{R}^3$ and satisfies

$$\widehat{Div\, \widetilde{\mathbf{S}}} = \widehat{Div}\, \widehat{\widetilde{\mathbf{S}}} = \xi \widehat{\mathbf{S}} = (\xi_j \widehat{S}_{ij}) = 0 \tag{2.15}$$

where $\xi = (\xi_j)$ is the dual variable of $\mathbf{x}$. Let now define $\widehat{\mathbf{A}} = (\widehat{A}_{ij})$ with:

$$\widehat{A}_{ij} = \frac{\varepsilon_{jnl}\varepsilon_{ikm}\xi_l\xi_m\widehat{S}_{nk}}{|\xi|^4} \tag{2.16}$$

From the symmetry of $\widehat{\mathbf{S}}$ it follows that also $\widehat{\mathbf{A}}$ is symmetric; from (2.15) it follows that

$$(\widehat{Curl}\widehat{\mathbf{A}})_{js} = \varepsilon_{jkm}\xi_k\widehat{A}_{sm} = \frac{\varepsilon_{skm}\xi_m\widehat{S}_{jk}}{|\xi|^2}$$

and hence

$$\widehat{Div}\,(\widehat{CURL}\widehat{\mathbf{A}}) = (\xi_s(\widehat{CURL}\widehat{\mathbf{A}})_{js}) = 0 \tag{2.17}$$

and, using once more (2.15),

$$(\widehat{CURL}\,(\widehat{CURL}\widehat{\mathbf{A}}))_{ij} = \widehat{S}_{ji} = \widehat{S}_{ij}. \tag{2.18}$$

At last one also has:

$$\widehat{Div}\,\widehat{\mathbf{A}} = (\xi_j\widehat{A}_{ij}) = 0. \tag{2.19}$$

If one defines $\mathbf{A} = \mathfrak{B}(\mathbf{S})$ as the restriction to Ω of the inverse Fourier transform of (2.16) it is immediately seen from (2.17), (2.18) and (2.19) that the conditions of (2.14) are all satisfied. In order to conclude it is enough to prove that $\mathbf{A} \in H^2(\Omega; M^3_{sym})$. One can at first remark that from (2.16) it follows that $\xi_r\xi_s\widehat{A}_{ij} \in L^2(\mathbb{R}^3)$ and hence the second order derivatives of A_{ij} are in $L^2(\Omega)$. It is enough to prove that also $A_{ij} \in L^2(\Omega)$. For this let $\omega(\xi) \in \mathfrak{D}(\mathbb{R}^3)$ with $\omega(\xi) = 1$ for $|\xi| \leq 1$. Since $\omega(\xi)\widehat{A}_{ij}$ has compact support its inverse Fourier transform is analytic and hence its restriction to Ω is in $L^2(\Omega)$. Since $(1-\omega(\xi))\widehat{A}_{ij} \in L^2(\mathbb{R}^3)$ it follows that $A_{ij} \in L^2(\Omega)$ and the proof is complete. □

Proposition 2.6. *Let be:* $\mathbb{Y} := \{\boldsymbol{T} \in Ker(Div; L^2);\ \boldsymbol{T} = \nabla_s(\boldsymbol{u}),\ \boldsymbol{u} \in H^1(\Omega; \mathbf{R}^3),\ \boldsymbol{u}_{|\gamma_0} = 0$ *and* $\boldsymbol{u}_{|\gamma_q} \in \text{RIG},\ q = 1, \ldots, Q\}$. $\mathbb{Y}$ *has dimension* $6Q$ *and:*

$$Ker(Div; L^2) = \mathbb{G} \overset{\perp}{\oplus} \mathbb{Y} = CURL\,CURL\,(H^2\left(\Omega; M^3_{sym}\right)) \overset{\perp}{\oplus} \mathbb{Y} \tag{2.20}$$

with orthogonality in $L^2\left(\Omega; M^3_{sym}\right)$.

Proof. Since the orthogonality of the decomposition follows from the Green's formula, one has only to prove that the sum in (2.20) is direct. For this let be $\mathbf{S} \in Ker(Div; L^2)$ and let us remark that there exists a unique $\mathbf{u} \in \mathfrak{Y} := \{\mathbf{v} \in H^1(\Omega; \mathbf{R}^3),\ \mathbf{v}_{|\gamma_0} = 0$ and $\mathbf{v}_{|\gamma_q} \in \text{RIG},\ q = 1, \ldots, Q\}$ such that for all $\mathbf{v} \in \mathfrak{Y}$:

$$\int_\Omega \nabla_s(\mathbf{u})\nabla_s(\mathbf{v})d\Omega = \sum_{q=1}^{Q} \langle \Gamma^q_{\mathbf{n}}(\mathbf{S}), \mathbf{v}\rangle_{\gamma_q}$$

Then $\mathbf{T} = \nabla_s(\mathbf{u}) \in \mathbb{Y}$ and $\mathbf{S} - \mathbf{T} \in \mathbb{G}$. □

Remark 2.7. One can easily give a variational characterization of a basis of $\mathbb{Y}$.

From this proposition and from theorem 2.1 one deduces the following decompositions of $L^2\left(\Omega; M^3_{sym}\right)$ in mutually orthogonal subspaces:
(2.21)

$$L^2\left(\Omega; M^3_{sym}\right) = \nabla_s(H^1_0\left(\Omega; \mathbf{R}^3\right)) \overset{\perp}{\oplus} CURL\, CURL\left(H^2\left(\Omega; M^3_{sym}\right)\right) \overset{\perp}{\oplus} \mathbb{Y}$$

and from (2.12)

$$L^2\left(\Omega; M^3_{sym}\right) = \nabla_s(H^1_0\left(\Omega; \mathbf{R}^3\right)) \overset{\perp}{\oplus} \Sigma_{ad}\left(\Omega\right) \overset{\perp}{\oplus} \mathbb{H} \overset{\perp}{\oplus} \mathbb{Y} \tag{2.22}$$

From (2.3) and theorem 2.4 one also has:

$$\mathbb{DFG} = \mathbb{H} \overset{\perp}{\oplus} \mathbb{Y} \tag{2.23}$$

Let us explicitly remark that these decompositions of $L^2\left(\Omega; M^3_{sym}\right)$ in mutually orthogonal subspaces are the counterpart of the decompositions of Helmoltz type for vector fields in $L^2\left(\Omega; \mathbf{R}^3\right)$. In order to obtain a complete decomposition of Hodge type we need the results of the next section.

3. Saint Venant compatibility condition in L^2 setting

$CURL\, CURL$ is a linear and continuous operator from $H^2_0\left(\Omega;\ M^3_{sym}\right)$ into $L^2(\Omega;\ M^3_{sym})$ and for any $\mathbf{E} \in L^2\left(\Omega;\ M^3_{sym}\right)$ and $\mathbf{S} \in \mathfrak{D}\left(\Omega; M^3_{sym}\right)$ one has :

$$\int_\Omega \mathbf{E} : CURL\, CURL\, \mathbf{S} d\Omega = {}_{\mathfrak{D}'\left(\Omega; M^3_{sym}\right)}\left\langle CURL\, CURL\, \mathbf{E}, \mathbf{S}\right\rangle_{\mathfrak{D}\left(\Omega; M^3_{sym}\right)}$$

where ${}_{\mathfrak{D}'\left(\Omega; M^3_{sym}\right)}\left\langle \bullet, \bullet\right\rangle_{\mathfrak{D}\left(\Omega; M^3_{sym}\right)}$ denotes the duality pairing between $\mathfrak{D}'\left(\Omega; M^3_{sym}\right)$ and $\mathfrak{D}\left(\Omega; M^3_{sym}\right)$. Hence its adjoint, linear and continuous from $L^2(\Omega; M^3_{sym})$ into $H^{-2}\left(\Omega; M^3_{sym}\right)$, is the operator $CURL\, CURL$ defined in the distribution sense. Let be $ker(CURL\ CURL\ ; L^2)$ its kernel satisfying:

$$ker(CURL\, CURL\, ; L^2) = (CURL\, CURL\, (H^2_0\left(\Omega; M^3_{sym}\right))^\perp$$

Since for every $\mathbf{v} \in H^1\left(\Omega; \mathbf{R}^3\right)$ one has $CURL\ CURL(\nabla_s(\mathbf{v})) = 0$ from the Green's formula (2.1) with $\mathbf{S} \in CURL\, CURL\, (H^2_0\left(\Omega; M^3_{sym}\right))$ it follows that $CURL\, CURL\, (H^2_0\left(\Omega; M^3_{sym}\right)) \subset \Sigma_{ad}\left(\Omega\right)$. Moreover one has :

$$\nabla_s(H^1\left(\Omega; \mathbf{R}^3\right)) \subset ker(CURL\, CURL\, ; L^2) \tag{3.1}$$

In order to completely characterize $ker(CURL\ CURL\ ; L^2)$ it suffices, thanks to theorem 2.4, to study its intersection with $\Sigma_{ad}\left(\Omega\right)$. From a direct inspection it appears that this space is:

$$\begin{aligned}\mathbb{K} = \big\{\mathbf{S} \in L^2(\Omega; M^3_{sym});\ & CURL\, CURL\, \mathbf{S} = 0 \\ & \text{and } Div\, \mathbf{S} = 0 \text{ in } \Omega,\ \Gamma_{\mathbf{n}}(\mathbf{S}) = 0 \text{ on } \partial\Omega\big\}.\end{aligned} \tag{3.2}$$

Proposition 3.1. *When Ω is simply connected then $\mathbb{K} = 0$.*

In [5] the proof that $\nabla_s(H^1(\Omega;\mathbf{R}^3)) = ker(CURL\ CURL\ ; L^2)$ when Ω is simply connected is reduced to a weak version of the classical Theorem of Poincaré. We give here a different proof.

Proof. One remarks at first that the following identities hold true:

$$CURL\ CURL\,\mathbf{S} = -\Delta\,\mathbf{S} - \nabla\nabla(tr\,\mathbf{S}) + 2\nabla_s\,Div\,\mathbf{S} + [\triangle(tr\,\mathbf{S}) - div\,Div\,\mathbf{S}]\mathbb{I}$$

$$tr[\Delta\,\mathbf{S} + \nabla\nabla(tr\,\mathbf{S}) - 2\nabla\,Div\,\mathbf{S}] = 2[\triangle(tr\,\mathbf{S}) - div\,Div\,\mathbf{S}]$$

Hence when $\mathbf{S} \in \mathbb{K}$ one has :

$$\Delta\,\mathbf{S} = -\nabla\nabla(tr\,\mathbf{S}) + [\triangle(tr\,\mathbf{S})]\mathbb{I} \tag{3.3}$$

and taking the trace of this equation one finds :

$$\triangle(tr\,\mathbf{S}) = 0 \tag{3.4}$$

From (3.4) and the hypoellipticity of the Laplacian it follows that $tr\,\mathbf{S} \in \mathcal{C}^\infty(\Omega)$ and then from (3.3) also that $\mathbf{S} \in \mathcal{C}^\infty(\Omega; M^3_{sym})$. Thanks to the Beltrami's sufficiency compatibility theorem 1.2 there exists a vector field $\mathbf{v} \in C^\infty(\Omega;\mathbf{R}^3)$ satisfying the strain-displacement relations : $\mathbf{S} = \frac{1}{2}(\nabla\mathbf{v}^T + \nabla\mathbf{v})$. The matrix version of the J.L. Lion's lemma (see [3]) implies that $\mathbf{v} \in H^1(\Omega;\mathbf{R}^3)$ and the conditions $Div\,\mathbf{S} = 0$ in Ω and $\Gamma_{\mathbf{n}}(\mathbf{S}) = 0$ on $\partial\Omega$ imply that $\mathbf{v} \in$ RIG and hence $\mathbf{S} = 0$. □

One can now prove the following result correcting the statement (ii) of Theorem 2.2 of [12].

Proposition 3.2. *$CURL\ CURL\ (H^2_0(\Omega; M^3_{sym}))$ is dense in $\Sigma_{ad}(\Omega)$ when Ω is simply connected.*

Proof. Let $L(\mathbf{S})$ be a linear and continuous functional on $\Sigma_{ad}(\Omega)$; then there exist $\widehat{\mathbf{S}} \in \Sigma_{ad}(\Omega)$ such that :

$$L(\mathbf{S}) = \int_\Omega \mathbf{S} : \widehat{\mathbf{S}}\,d\Omega \tag{3.5}$$

Let us suppose that $L(CURL\ CURL\ \mathbf{T}) = 0$ for all $\mathbf{T} \in H^2_0(\Omega; M^3_{sym})$. In order to prove the density we have to prove that then $L(\mathbf{S}) = 0$ for all $\mathbf{S} \in \Sigma_{ad}(\Omega)$. The assumption $L(CURL\ CURL\ \mathbf{T}) = 0$ for all $\mathbf{T} \in H^2_0(\Omega; M^3_{sym})$ means that:

$$0 = \int_\Omega CURL\ CURL\,\mathbf{T} : \widehat{\mathbf{S}}\,d\Omega$$

and hence $CURL\ CURL\,\widehat{\mathbf{S}} = 0$ in the distribution sense. From proposition 3.1 it then follows $\widehat{\mathbf{S}} = 0$ and so $L(\mathbf{S}) = 0$ for all $\mathbf{S} \in \Sigma_{ad}(\Omega)$. □

As it has been remarked for the first time by V. Volterra in 1906 [18], $\mathbb{K} \neq 0$ for a general non simply-connected domain. In order to study this general situation, let us define the following *space of Volterra's dislocations*:

$$\mathcal{VD} = \left\{ \mathbf{v} \in H^1(\Omega^*;\ \mathbf{R}^3); [[\mathbf{v}]]_{\mathcal{C}_\alpha} \in \text{RIG, for } \alpha = 1, \ldots, N \right\},$$

where $[[\mathbf{v}]]_{\mathcal{C}_\alpha}$ is the jump of $\mathbf{v}$ across the cut $\mathcal{C}_\alpha$ and $\Omega^* = \Omega \setminus \bigcup_{\alpha=1}^{N} \mathcal{C}_\alpha$. Let explicitly remark that when all these jumps vanish then $\mathbf{v} \in H^1(\Omega; \mathbf{R}^3)$. Hence $H^1(\Omega; \mathbf{R}^3) \subset \mathcal{VD}$.

Let us write the jump of $\mathbf{v}$ across the cut $\mathcal{C}_\alpha$ in the form $\mathbf{a}^\alpha(\mathbf{v}) + \mathbf{b}^\alpha(\mathbf{v}) \wedge \mathbf{id}_\Omega$ where $\mathbf{a}^\alpha(\mathbf{v}) = a_i^\alpha(\mathbf{v})\mathbf{e}^i$ and $\mathbf{b}^\alpha(\mathbf{v}) \wedge \mathbf{id}_\Omega = b_i^\alpha(\mathbf{v})\mathbf{p}^i$. Let us also define on $\mathcal{VD}$ the functionals:

$$\mathcal{I}^1_{i,\alpha}(\mathbf{v}) = \frac{1}{2}\int_{\Omega^*} \nabla_s\mathbf{v}{:}\nabla_s\mathbf{v}\, d\Omega - a_i^\alpha(\mathbf{v}) \tag{3.6}$$

$$\mathcal{I}^2_{i,\alpha}(\mathbf{v}) = \frac{1}{2}\int_{\Omega^*} \nabla_s\mathbf{v}{:}\nabla_s\mathbf{v}\, d\Omega - b_i^\alpha(\mathbf{v}) \tag{3.7}$$

Proposition 3.3. *For every $\alpha = 1, \ldots, N$ and $i = 1, 2, 3$ there exist $\boldsymbol{u}_i^\alpha \in \mathcal{VD}$ and $\boldsymbol{r}_i^\alpha \in \mathcal{VD}$ such that:*

$$\mathcal{I}^1_{i,\alpha}(\boldsymbol{u}_i^\alpha) \leq \mathcal{I}^1_{i,\alpha}(\boldsymbol{v}),\ \ \forall \boldsymbol{v} \in \mathcal{VD}\,; \tag{3.8}$$

$$\mathcal{I}^2_{i,\alpha}(\boldsymbol{r}_i^\alpha) \leq \mathcal{I}^2_{i,\alpha}(\boldsymbol{v}),\ \ \forall \boldsymbol{v} \in \mathcal{VD}. \tag{3.9}$$

Moreover, each vector field $\boldsymbol{u}_i^\alpha$ and $\boldsymbol{r}_i^\alpha$ is uniquely determined modulo a global infinitesimal rigid displacement on Ω.

The proof is obvious. For later use we write the corresponding Euler equations:

$$\int_{\Omega^*} \nabla_s\mathbf{u}_i^\alpha{:}\nabla_s\mathbf{v}\, d\Omega - a_i^\alpha(\mathbf{v}) = 0,\ \ \forall \mathbf{v} \in \mathcal{VD}\,; \tag{3.10}$$

$$\int_{\Omega^*} \nabla_s\mathbf{r}_i^\alpha{:}\nabla_s\mathbf{v}\, d\Omega - b_i^\alpha(\mathbf{v}) = 0,\ \ \forall \mathbf{v} \in \mathcal{VD}. \tag{3.11}$$

Since $meas\,(\Omega) = meas\,(\Omega^*)$, there is a canonical isomorphism of $L^2(\Omega^*; M^3_{sym})$ with $L^2(\Omega; M^3_{sym})$. Hence for a given $\mathbf{v} \in \mathcal{VD}$, one can associate with $\nabla_s\mathbf{v} \in L^2(\Omega^*; M^3_{sym})$ the corresponding element in $L^2(\Omega; M^3_{sym})$ denoted with $\widetilde{\nabla_s\mathbf{v}}$.

Theorem 3.4. *For every $\alpha = 1, \ldots, N$, and $i = 1, 2, 3$ the symmetric matrix fields $(\widetilde{\nabla_s\boldsymbol{u}_i^\alpha})$ and $(\widetilde{\nabla_s\boldsymbol{r}_i^\alpha})$ belong to the space $\mathbb{K}$.*

Proof. Let be α and i fixed. Since $\mathcal{D}(\Omega; M^3_{sym}) \subset \mathcal{VD}$ from (3.10) it follows :

$$\int_\Omega \widetilde{\nabla_s\mathbf{u}_i^\alpha}{:}\nabla_s\mathbf{v}\, d\Omega = \int_{\Omega^*} \nabla_s\mathbf{u}_i^\alpha{:}\nabla_s\mathbf{v}\, d\Omega = 0,\ \ \forall \mathbf{v} \in \mathcal{D}(\Omega; M^3_{sym})$$

Hence, in the distribution sense,

$$Div\left(\widetilde{\nabla_s \mathbf{u}_i^\alpha}\right) = 0 \text{ in } \Omega. \tag{3.12}$$

Taking $\mathbf{v} \in H^1(\Omega;\mathbf{R}^3)$ in (3.10) one then finds from (2.1):

$$\Gamma_{\mathbf{n}}\left(\widetilde{\nabla_s \mathbf{u}_i^\alpha}\right) = \left(\widetilde{\nabla_s \mathbf{u}_i^\alpha}\right).\mathbf{n}|_{\partial\Omega} = 0 \text{ in } H^{-1/2}(\partial\Omega;\mathbf{R}^3). \tag{3.13}$$

Hence we conclude that $\widetilde{\nabla_s \mathbf{u}_i^\alpha}$ belongs to $\Sigma_{ad}(\Omega)$. The same arguments can be repeated to obtain that $\widetilde{\nabla_s \mathbf{r}_i^\alpha}$ belongs to $\Sigma_{ad}(\Omega)$. With suitable choices of $\mathbf{v} \in \mathcal{VD}$ (for instance $\mathbf{v}$ with $[[\mathbf{v}]]_{\mathcal{C}_\beta} = 0$ for all $\beta \neq \alpha$, with $[[\mathbf{v}]]_{\mathcal{C}_\alpha} = \mathbf{e}^j$ for $j \neq i$, with $[[\mathbf{v}]]_{\mathcal{C}_\alpha} = \mathbf{p}^j$ for $j = 1, 2, 3$, etc.) one further finds that:

$$[[\mathbf{u}_i^\alpha]]_{\mathcal{C}_\beta} = 0 \quad [[\mathbf{u}_i^\alpha]]_{\mathcal{C}_\alpha} = \mathbf{e}^i \tag{3.14}$$

$$[[\mathbf{r}_i^\alpha]]_{\mathcal{C}_\beta} = 0 \quad [[\mathbf{r}_i^\alpha]]_{\mathcal{C}_\alpha} = \mathbf{p}^i \tag{3.15}$$

It remains to prove that $CURL\, CURL\,(\widetilde{\nabla_s \mathbf{u}_i^\alpha}) = 0$ in the distribution sense. This result is a consequence of the relation:

$$\mathfrak{D}'\left(\Omega;M^3_{sym}\right)\left\langle CURL\, CURL\,(\widetilde{\nabla_s \mathbf{u}_i^\alpha}), \mathbf{S}\right\rangle_{\mathfrak{D}\left(\Omega;M^3_{sym}\right)} =$$

$$\int_\Omega \widetilde{\nabla_s \mathbf{u}_i^\alpha} : CURL\, CURL\, \mathbf{S}\, d\Omega = \int_{\Omega^*} \nabla_s \mathbf{u}_i^\alpha : CURL\, CURL\, \mathbf{S}\, d\Omega =$$

$$\int_{\partial\Omega^*} \mathbf{u}_i^\alpha \cdot (CURL\, CURL\, \mathbf{S})\, \mathbf{n}\, d\Gamma =$$

$$\sum_\beta \int_{\mathcal{C}_\beta} [[\mathbf{u}_i^\alpha]] \cdot (CURL\, CURL\, \mathbf{S})\, \mathbf{n}\, d\mathcal{C} =$$

$$\int_{\mathcal{C}_\alpha} \mathbf{e}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n}\, d\mathcal{C}$$

Since the cuts $\mathcal{C}_\alpha$ are planar and each $\mathbf{u}_i^\alpha$ is unique modulo a global rigid displacement of Ω, one can take for simplicity $\mathcal{C}_\alpha$ contained in the plane spanned by $\mathbf{e}^1$ and $\mathbf{e}^2$ and so $\mathbf{n} = \mathbf{e}^3$ and $d\mathcal{C} = dx_1 dx_2$. A simple computation then gives:

$$(CURL\, CURL\ \ \mathbf{S})\, \mathbf{n} = (\varepsilon_{isr}\varepsilon_{3pq} S_{rq,ps}) \tag{3.16}$$

Since $S_{ij}|_{\mathcal{C}_\alpha} \in \mathcal{D}(\mathcal{C}_\alpha)$ it follows that

$$\int_{\mathcal{C}_\alpha} \mathbf{e}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n}\, d\mathcal{C} = \int_{\mathcal{C}_\alpha} \varepsilon_{isr}\varepsilon_{3pq} S_{rq,ps} dx_1 dx_2 = 0$$

and hence $(\widetilde{\nabla_s \mathbf{u}_i^\alpha}) \in \mathbb{K}$.

The proof that $(\widetilde{\nabla_s \mathbf{r}_i^\alpha}) \in \mathbb{K}$ is analogous. One obtains :

$$\mathfrak{D}'\left(\Omega;M^3_{sym}\right)\left\langle CURL\, CURL\,(\widetilde{\nabla_s \mathbf{r}_i^\alpha}), \mathbf{S}\right\rangle_{\mathfrak{D}\left(\Omega;M^3_{sym}\right)} =$$

$$\int_\Omega \widetilde{\nabla_s \mathbf{r}_i^\alpha} : CURL\, CURL\, \mathbf{S}\, d\Omega =$$

$$\int_{\mathcal{C}_\alpha} \mathbf{p}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n}\, d\mathcal{C}$$

From (3.16) one has

$$\mathbf{p}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n} = -\varepsilon_{ijk} x_k \varepsilon_{jsr} \varepsilon_{3pq} S_{rq,ps}$$

with $k \neq 3$. Recalling that $-\varepsilon_{ijk}\varepsilon_{jsr} = \delta_{is}\delta_{kr} - \delta_{ir}\delta_{ks}$ it follows that

$$\mathbf{p}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n} = x_k \varepsilon_{3pq} (S_{kq,pi} - S_{iq,pk})$$

Integrating by parts every term, one finds since $k \neq 3$ and $S_{ij}|_{\mathcal{C}_\alpha} \in \mathcal{D}(\mathcal{C}_\alpha)$:

$$\int_{\mathcal{C}_\alpha} \mathbf{p}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n}\, dx_1 dx_2 = \int_{\mathcal{C}_\alpha} (x_1 S_{12,1i} - x_2 S_{21,2i})\, dx_1 dx_2$$

Hence still integrating by parts one obtains, thanks to the symmetry of $\mathbf{S}$:

$$\int_{\mathcal{C}_\alpha} \mathbf{p}^i \cdot (CURL\, CURL\ \ \mathbf{S})\, \mathbf{n}\, dx_1 dx_2 = 0$$

This concludes the proof for $\widetilde{\nabla_s \mathbf{r}_i^\alpha}$. □

From (3.14),(3.15), (3.10) and (3.11) it follows:

Corollary 3.5. *The $6N$ matrix fields $\widetilde{\nabla_s \boldsymbol{u}_i^\alpha}$ and $\widetilde{\nabla_s \boldsymbol{r}_i^\alpha}$ are linearly independent in $L^2(\Omega; M^3_{sym})$.*

For later use one needs another form of the Green's formula. For this let remark at first that when $\mathbf{W} \in \Sigma_{ad}(\Omega)$, then the map $\mathbf{W} \mapsto \{(\ \mathbf{W}\mathbf{n})|_{\mathcal{C}_\alpha}\}_{\alpha=1}^{\alpha=N}$ well defined for $\mathbf{W} \in \mathcal{W}$, can be extended to a linear and continuous map from $\Sigma_{ad}(\Omega)$ into $\prod_{\alpha=1}^{\alpha=N} H^{-1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)$. Indeed, for any fixed α and any $\mathbf{g} \in H^{1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)$ one can find $\mathbf{u} \in H^1(\Omega^*; \mathbf{R}^3)$ such that $[[\mathbf{u}]]_{\mathcal{C}_\alpha} = \mathbf{g}$ and $[[\mathbf{u}]]_{\mathcal{C}_\beta} = 0$ when $\beta \neq \alpha$ and such that the map $\mathbf{g} \mapsto \mathbf{u}$ is linear and continuous. The Green's formula for $\mathbf{W} \in \mathcal{W}$ and $\mathbf{u} \in H^1(\Omega^*; \mathbf{R}^3)$ reads:

$$\int_{\Omega^*} \mathbf{W}{:}\nabla_s(\mathbf{u})\, d\Omega = \int_{\mathcal{C}_\alpha} (\ \mathbf{W}\mathbf{n}).\mathbf{g}\, d\mathcal{C}_\alpha$$

and hence

$$\left| \int_{\mathcal{C}_\alpha} (\ \mathbf{W}\mathbf{n}).\mathbf{g}\, d\mathcal{C}_\alpha \right| \leq c \|\mathbf{W}\|_{L^2(\Omega; M^3_{sym})} \|\mathbf{g}\|_{H^{1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)}.$$

The density of $\mathcal{W}$ in $\Sigma_{ad}(\Omega)$ (Theorem 2.3) allows to extend the map $\mathbf{W} \mapsto (\ \mathbf{W}\mathbf{n})|_{\mathcal{C}_\alpha}$ to a linear and continuous map from $\Sigma_{ad}(\Omega)$ into $H^{-1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)$. Moreover the following extended Green's formula holds true for $\mathbf{W} \in \Sigma_{ad}(\Omega)$ and $\mathbf{u} \in H^1(\Omega^*; \mathbf{R}^3)$:

$$\int_{\Omega^*} \mathbf{W}{:}\nabla_s(\mathbf{u})\, d\Omega = \sum_{\alpha=1}^{N} \left\langle (\ \mathbf{W}\mathbf{n})|_{\mathcal{C}_\alpha}\, ,\, [[\mathbf{u}]]_{\mathcal{C}_\alpha} \right\rangle_{\mathcal{C}_\alpha} \tag{3.17}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{C}_\alpha}$ denotes the duality pairing between $H^{-1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)$ and $H^{1/2}(\mathcal{C}_\alpha; \mathbf{R}^3)$. We can now give the announced characterization.

Theorem 3.6. *The space* $\mathbb{K}$ *is spanned by* $\widetilde{\nabla_s \boldsymbol{u}_i^\alpha}$ *and* $\widetilde{\nabla_s \boldsymbol{r}_i^\alpha}$, $\alpha = 1, \ldots, N$, $i = 1, 2, 3$.

Proof. Given $\mathbf{W} \in \mathbb{K}$, let $\mathbf{V} \in \mathbb{K}$ be defined by:

(3.18)

$$\mathbf{V} = \mathbf{W} - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} \widetilde{\nabla_s \mathbf{u}_i^\alpha} \right\} - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i \rangle_{\mathcal{C}_\alpha} \widetilde{\nabla_s \mathbf{r}_i^\alpha} \right\},$$

Let be $\mathbf{V}_*$, $\mathbf{W}_* \in L^2(\Omega^*; M^3_{sym})$ the restriction of $\mathbf{V}$ and $\mathbf{W}$ to Ω^*. From an inspection of (3.18) it appears that $CURL\, CURL\,(\mathbf{V}_*) = 0$. Because Ω^* is simply-connected, there exists (see proposition 3.1) $\widehat{\mathbf{u}} \in H^1(\Omega^*; \mathbf{R}^3)$ such that

(3.19) $\nabla_s \widehat{\mathbf{u}} = \mathbf{V}_* =$

$$\mathbf{W}_* - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} \nabla_s \mathbf{u}_i^\alpha \right\} - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i \rangle_{\mathcal{C}_\alpha} \nabla_s \mathbf{r}_i^\alpha \right\}.$$

Let now be $\mathbf{z}$ an arbitrary element of $\mathcal{VD}$. Using the Green's formula in Ω^* (3.17) and (3.10), (3.11) one finds:

$$\begin{aligned}
&\int_{\Omega^*} \nabla_s \widehat{\mathbf{u}} \mathbf{:} \nabla_s \mathbf{z}\, d\Omega = \int_{\Omega^*} \mathbf{W}_* \mathbf{:} \nabla_s \mathbf{z}\, d\Omega \\
&- \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} \int_{\Omega^*} \nabla_s \mathbf{u}_i^\alpha \mathbf{:} \nabla_s \mathbf{z}\, d\Omega \right\} \\
&- \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i \rangle_{\mathcal{C}_\alpha} \int_{\Omega^*} \nabla_s \mathbf{r}_i^\alpha \mathbf{:} \nabla_s \mathbf{z}\, d\Omega \right\} = \langle\, \mathbf{W}_* \mathbf{n}|_{\partial\Omega^*}, \mathbf{z} \rangle_{\partial\Omega^*} \\
&- \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} a_i^\alpha(\mathbf{z}) \right\} - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i \rangle_{\mathcal{C}_\alpha} b_i^\alpha(\mathbf{z}) \right\} \\
&= \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, [[\mathbf{z}]]_{\mathcal{C}_\alpha} \rangle_{\mathcal{C}_\alpha} \right\} - \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} a_i^\alpha(\mathbf{z}) \right\} \\
&- \sum_{\alpha=1}^{N} \left\{ \langle\, \mathbf{Wn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i \rangle_{\mathcal{C}_\alpha} b_i^\alpha(\mathbf{z}) \right\} = 0
\end{aligned}$$

It follows that $\nabla_s \widehat{\mathbf{u}} = \mathbf{V}_* = 0$ and so $\mathbf{V} = 0$. □

Since the matrix fields $(\widetilde{\nabla_s \mathbf{u}_i^\alpha})$ and $(\widetilde{\nabla_s \mathbf{r}_i^\alpha})$ are linearly independent in $L^2(\Omega; M^3_{sym})$, we also have

Corollary 3.7. *The space* $\mathbb{K}$ *is of dimension* $6N$.

Corollary 3.8. $\Sigma_{ad}(\Omega) = \mathbb{K} \overset{\perp}{\oplus} \mathbb{X}$ *with*

$$\mathbb{X} = \left\{ \mathbb{S} \in \Sigma_{ad}(\Omega)\ ;\ \langle\, \boldsymbol{Sn}|_{\mathcal{C}_\alpha}, \boldsymbol{e}^i \rangle_{\mathcal{C}_\alpha} = 0,\ \langle\, \boldsymbol{Sn}|_{\mathcal{C}_\alpha}, \boldsymbol{p}^i \rangle_{\mathcal{C}_\alpha} = 0,\ \alpha = 1, \ldots, N,\ i = 1, 2, 3 \right\}.$$

One may wonder how the results depend on the choice of the cuts. The following result gives the answer.

Proposition 3.9. *The definition of the space* $\mathbb{X}$ *is independent of the way the cuts are chosen.*

Proof. In order to prove this, let change the first cut $\mathcal{C}_1$ into another one $\widehat{\mathcal{C}_1}$ equivalent in the sense that (i) its boundary is contained in $\partial\Omega$, (ii) the set of planar non -intersecting cuts $\{\widehat{\mathcal{C}_1}, \mathcal{C}_2, \dots, \mathcal{C}_N\}$ is such that $\widehat{\Omega^*} = \Omega \backslash \{\widehat{\mathcal{C}_1} \bigcup_{\alpha=2}^{N} \mathcal{C}_\alpha\}$ is simply connected and satisfies the cone condition. Let O the subset of Ω whose boundary is $\mathcal{C}_1$, $\widehat{\mathcal{C}_1}$ and the part of $\partial\Omega$ connecting the boundaries of $\mathcal{C}_1$, $\widehat{\mathcal{C}_1}$. Then the Green's formula in O implies that for all $\mathbf{v} \in \text{RIG}$ and all $\mathbb{S} \in \Sigma_{ad}(\Omega)$ one has:

$$0 = \int_O Div\ \mathbf{S}.\mathbf{v}\, d\Omega = \langle\ \mathbf{Sn}|_{\mathcal{C}_1}, \mathbf{v}\rangle_{\mathcal{C}_\alpha} + \left\langle\ \mathbf{Sn}|_{\widehat{\mathcal{C}_1}}, \mathbf{v}\right\rangle_{\widehat{\mathcal{C}_1}}$$

and hence $\langle\ \mathbf{Sn}|_{\mathcal{C}_1}, \mathbf{v}\rangle_{\mathcal{C}_\alpha} = 0$ if and only if $\left\langle\ \mathbf{Sn}|_{\widehat{\mathcal{C}_1}}, \mathbf{v}\right\rangle_{\widehat{\mathcal{C}_1}} = 0$. □

Remark 3.10. From the previous proof it appears that the assumption that the cuts are planar is not necessary; it is enough that the non-intersecting cuts be Lipschitz.

Theorem 3.11. $\mathbb{X}$ *is the closure of* $CURL\, CURL\left(H_0^2(\Omega; M^3_{sym})\right)$

Proof. It is enough to remark that $ker(CURL\ CURL\ ; L^2) = \mathbb{K} \overset{\perp}{\oplus} \nabla_s(H^1(\Omega; \mathbf{R}^3))$. □

4. The complete Hodge decomposition

Collecting the different results we get the following general Hodge orthogonal decompositions of $L^2(\Omega; M^3_{sym})$:

$$L^2\left(\Omega; M^3_{sym}\right) = \mathbb{X} \overset{\perp}{\oplus} \mathbb{K} \overset{\perp}{\oplus} \mathbb{H} \overset{\perp}{\oplus} \mathbb{Y} \overset{\perp}{\oplus} \nabla_s(H_0^1\left(\Omega; \mathbf{R}^3\right)) \tag{4.1}$$

where:

$$\begin{aligned}\mathbb{X} = \ \big\{\mathbb{S} \in \Sigma_{ad}(\Omega)\ ;\ \langle\ \mathbf{Sn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i\rangle_{\mathcal{C}_\alpha} = 0,\ \langle\ \mathbf{Sn}|_{\mathcal{C}_\alpha}, \mathbf{p}^i\rangle_{\mathcal{C}_\alpha} = 0,\\ \alpha = 1, \dots, N,\ i = 1, 2, 3\big\}\,,\end{aligned}$$

$$\begin{aligned}\mathbb{K} = \big\{\mathbf{S} \in L^2(\Omega; M^3_{sym});\ CURL\, CURL\, \mathbf{S} = 0\\ \text{and } Div\, \mathbf{S} = 0 \text{ in } \Omega,\ \Gamma_{\mathbf{n}}(\mathbf{S}) = 0 \text{ on } \partial\Omega\big\}\,,\end{aligned}$$

$$\mathbb{H} = CURL\, CURL\left(H^2\left(\Omega; M^3_{sym}\right)\right) \cap \nabla_s(H^1\left(\Omega; \mathbf{R}^3\right)),$$

$$\begin{aligned}\mathbb{Y} = \{\mathbf{T} \in Ker(Div; L^2); \mathbf{T} = \nabla_s(\mathbf{u}),\ \mathbf{u} \in H^1(\Omega; \mathbf{R}^3),\ \mathbf{u}|_{\gamma_0} = 0\\ \text{and } \ \mathbf{u}|_{\gamma_q} \in \text{RIG},\ \ q = 1, \dots, Q\big\}\,.\end{aligned}$$

Let us stress that this orthogonal decomposition is the analogous of the following Hodge decomposition of $L^2\left(\Omega; \mathbf{R}^3\right)$ (see e.g. [2], [4]):

$$L^2\left(\Omega; \mathbf{R}^3\right) = FK \overset{\perp}{\oplus} HK \overset{\perp}{\oplus} CG \overset{\perp}{\oplus} HG \overset{\perp}{\oplus} grad(H_0^1(\Omega)) \tag{4.2}$$

where with clear notations:

$$FK = \quad \left\{ \mathbf{v} \in L^2(\Omega; \mathbf{R}^3)\,;\ div\mathbf{v} = 0 \ \text{ in } \ \Omega,\ \mathbf{vn}|_{\partial\Omega} = 0 \right.$$
$$\left. \text{and } \langle\ \mathbf{vn}|_{\mathcal{C}_\alpha}, \mathbf{e}^i \rangle_{\mathcal{C}_\alpha} = 0,\ \alpha = 1, \ldots, N,\ i = 1, 2, 3 \right\},$$

$$HK = \left\{ \mathbf{v} \in L^2(\Omega; \mathbf{R}^3);\ \mathbf{curlv} = 0 \,\text{and}\, div\,\mathbf{v} = 0 \,\text{in}\,\Omega,\ \mathbf{vn}|_{\partial\Omega} = 0 \right\},$$

$$CG = \{ \mathbf{v} \in L^2(\Omega; \mathbf{R}^3);\ div\,\mathbf{v} = 0 \,\text{in}\,\Omega,\ \mathbf{v} = grad\,\phi,\ \phi \in H^1(\Omega),$$
$$\left. \langle \mathbf{vn}\,, \mathbf{e}^i \rangle_{\gamma_q} = 0 \,\text{for}\, q = 1, \ldots, Q \right\}$$

$$HG = \{ \mathbf{v} \in L^2(\Omega; \mathbf{R}^3);\ div\,\mathbf{v} = 0 \,\text{in}\,\Omega,\ \mathbf{v} = grad\,\phi,\ \phi \in H^1(\Omega),$$
$$\left. \phi|_{\gamma_0} = 0 \ \text{ and } \phi|_{\gamma_q} \in \mathbb{R},\ \ q = 1, \ldots, Q \right\}$$

In [4] the spaces HK and HG are also characterized in homological terms :

$$HK \cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^N$$
$$HG \cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \cong \mathbf{R}^Q$$

In our situation it follows from proposition 2.6 that $\mathbb{K} \cong \mathbf{R}^{6N}$ and from corollary 3.7 that $\mathbb{Y} \cong \mathbf{R}^{6Q}$. This corresponds to homology with value in RIG $\cong \mathbf{R}^6$ and so it is coherent with the Bernstein-Gelfand-Gelfand resolution as suggested by Eastwood [8].

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