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Complementation of the subspace of radial multipliers in the space of Fourier multipliers on \mathbb{R}^n

Cédric Arhancet and Christoph Kriegler

Abstract. In this short note, we prove that the subspace of radial multipliers is contractively complemented in the space of Fourier multipliers on the Bochner space $L^p(\mathbb{R}^n, X)$ where X is a Banach space and where $1 \leq p < \infty$. Moreover, if $X = \mathbb{C}$, then this complementation preserves the positivity of multipliers.

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If $|\cdot|$ denotes the euclidean norm on \mathbb{R}^n , recall that a complex function $\phi: \mathbb{R}^n \rightarrow \mathbb{C}$ is radial if we can write $\phi(x) = \dot{\phi}(|x|)$ for some function $\dot{\phi}: \mathbb{R}_+ \rightarrow \mathbb{C}$. If X is a Banach space and if $1 \leq p < \infty$ then Theorem 3 below says that the subspace $\mathfrak{M}_{\text{rad}}^p(\mathbb{R}^n, X)$ of radial Fourier multipliers on the Bochner space $L^p(\mathbb{R}^n, X)$ is contractively complemented in the space $\mathfrak{M}^p(\mathbb{R}^n, X)$ of (scalar-valued) Fourier multipliers on $L^p(\mathbb{R}^n, X)$.

We refer to [6, Definition 5.3.3] for the definition of $\mathfrak{M}^p(\mathbb{R}^n, X)$ ¹. We will use the notation $\mathfrak{M}_{\text{rad}}^p(\mathbb{R}^n, X) = \{\phi \in \mathfrak{M}^p(\mathbb{R}^n, X) : \phi \text{ is radial}\}$ equipped with the norm induced by the one of the Banach space $B(L^p(\mathbb{R}^n, X))$ of bounded operators on the Bochner space $L^p(\mathbb{R}^n, X)$. We say that a bounded linear operator $T: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is positive if $T(f) \geq 0$ for any $f \geq 0$.

Let E be a Hausdorff locally convex space and E' its topological dual. If we denote by E'' the strong bidual of E , that is the topological dual of E' where E' is equipped with the strong topology, we have an injective map $j_E: E \rightarrow E''$. This map allows us to identify E as a subspace of E'' . We say that E is semireflexive if this map is surjective, see [3, page 523]. In this case,

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¹In the book [6], the notation $\mathfrak{ML}^p(\mathbb{R}^n; X)$ is used.

we can identify E and E'' as *vector spaces*. By [3, Remarks 8.16.5], if E is barrelled (this condition is fulfilled whenever E is a Banach space) then E' equipped with the weak* topology is semireflexive.

Let Ω be a locally compact space which is countable at infinity² equipped with a Radon measure μ and let E be a Hausdorff locally convex space. We say that a function $f: \Omega \rightarrow E$ is scalarly μ -integrable if for any element φ of the topological dual E' the scalar-valued function $\varphi \circ f: \Omega \rightarrow \mathbb{C}$ is integrable, see [3, page 558]. In this case, we denote by $\int_{\Omega} f d\mu: E' \rightarrow \mathbb{C}$ the not necessarily continuous linear form on E' defined by

$$\left\langle \int_{\Omega} f d\mu, \varphi \right\rangle_{E'^*, E'} = \int_{\Omega} \varphi \circ f d\mu,$$

where E'^* is the algebraic dual of the topological dual E' . We say that a scalarly μ -integrable function $f: \Omega \rightarrow E$ is Gelfand integrable if the element $\int_{\Omega} f d\mu$ belongs to E'' [3, page 565]. In this case, $\int_{\Omega} f d\mu$ is called the Gelfand integral of f . If in addition, $\int_{\Omega} f d\mu$ belongs to E we have for any $\varphi \in E'$

$$\left\langle \int_{\Omega} f d\mu, \varphi \right\rangle_{E, E'} = \int_{\Omega} \varphi \circ f d\mu. \quad (1)$$

By [1, Corollary, VI.6] (see also [3, Corollary 8.14.10]), if E is semireflexive and if $f: \Omega \rightarrow E$ is a scalarly μ -integrable function such that, for every compact subset K of Ω , $f(K)$ is bounded then f is Gelfand integrable with $\int_{\Omega} f d\mu \in E$.

We will use the following lemma which is a straightforward consequence of [3, Proposition 8.14.5].

Lemma 1. *Let Ω be a locally compact space which is countable at infinity equipped with a Radon measure μ . Let $T: E \rightarrow F$ be a continuous linear map between Hausdorff locally convex spaces. If the function $f: \Omega \rightarrow E$ is Gelfand integrable with $\int_{\Omega} f d\mu \in E$ then the function $T \circ f: \Omega \rightarrow F$ is Gelfand integrable with $\int_{\Omega} T \circ f d\mu \in F$ and we have*

$$T\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} T \circ f d\mu. \quad (2)$$

Proof. Since Ω is countable at infinity, by [3, Proposition 8.14.5], the function $T \circ f: \Omega \rightarrow F$ is scalarly integrable. Moreover, the same reference says that if $T'^*: E'^* \rightarrow F'^*$ is the canonical *extension* of $T: E \rightarrow F$, we have

$$T'^*\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} T \circ f d\mu.$$

where we consider the integrals $\int_{\Omega} f d\mu$ and $\int_{\Omega} T \circ f d\mu$ as elements of E'^* and F'^* . Since $\int_{\Omega} f d\mu$ belongs to E , we deduce that $T'^*(\int_{\Omega} f d\mu)$ belongs to F and is equal to $T(\int_{\Omega} f d\mu)$. We infer that $\int_{\Omega} T \circ f d\mu$ also belongs to F and that the equality (2) is true. \square

²In this case, by [1, V.1], we can remove the words “essentially” and “essential” of the statements of [3] used in our paper.

We need the following variant of [6, Proposition 2.5.2].

Lemma 2. *Let X be a Banach space. If a function f of $L^1_{\text{loc}}(\mathbb{R}^n, X)$ satisfies*

$$\int_{\mathbb{R}^n} \langle f(s), g(s) \rangle_{X, X'} ds = 0$$

for any $g \in \mathcal{D}(\mathbb{R}^n, X')$ then $f = 0$ almost everywhere.

Proof. If $\varphi \in X'$ and if $h \in \mathcal{D}(\mathbb{R}^n)$, using the function $g = h \otimes \varphi$ of $\mathcal{D}(\mathbb{R}^n, X')$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} h(s) \langle f(s), \varphi \rangle_{X, X'} ds &= \int_{\mathbb{R}^n} \langle f(s), (h \otimes \varphi)(s) \rangle_{X, X'} ds \\ &= \int_{\mathbb{R}^n} \langle f(s), g(s) \rangle_{X, X'} ds = 0. \end{aligned}$$

By [6, Proposition 2.5.2], we deduce that the function $\langle f(\cdot), \varphi \rangle_{X, X'}$ is null almost everywhere. By [6, Corollary 1.1.25], we conclude that $f = 0$ almost everywhere. \square

Theorem 3. *Let X be a Banach space. Suppose $1 \leq p < \infty$. Then there exists a contractive projection $P_{p, X}: \mathfrak{M}^p(\mathbb{R}^n, X) \rightarrow \mathfrak{M}^p(\mathbb{R}^n, X)$ onto the subspace $\mathfrak{M}^p_{\text{rad}}(\mathbb{R}^n, X)$. Moreover, the map $M_{P_{p, c}(\phi)}: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ associated to the radial multiplier $P_{p, c}(\phi)$ is positive if the map $M_\phi: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is positive.*

Proof. Let $\text{SO}(\mathbb{R}^n)$ be the compact group of orthogonal mappings $R: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with determinant 1 equipped with its normalized left Haar measure μ . For $R \in \text{SO}(\mathbb{R}^n)$, consider the induced map $S_R: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ defined by $S_R(f) = f(R \cdot)$. Clearly, S_R and its inverse $S_{R^{-1}}$ are isometric and positive. By [6, Theorem 2.1.3], we obtain that $S_R \otimes \text{Id}_X: L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)$ is a well-defined isometric map. For any $f \in L^p(\mathbb{R}^n)$, by [2, Chap. VIII, §2, Section 5], the map $\text{SO}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $R \mapsto S_R(f)$ is continuous. More generally, it is easy to see that for any $f \in L^p(\mathbb{R}^n, X)$ the map $\text{SO}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n, X)$, $R \mapsto (S_R \otimes \text{Id}_X)(f)$ is also continuous. Since the composition of operators is strongly continuous on bounded sets by [4, Proposition C.19], for any $\phi \in \mathfrak{M}^p(\mathbb{R}^n, X)$, we deduce that the map $\text{SO}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n, X)$, $R \mapsto ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X)f$ is also continuous, hence Bochner integrable on the compact $\text{SO}(\mathbb{R}^n)$. Now, for any $\phi \in \mathfrak{M}^p(\mathbb{R}^n, X)$, put

$$Q_{p, X}(\phi)f = \int_{\text{SO}(\mathbb{R}^n)} ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X)f d\mu(R). \quad (3)$$

For any $f \in L^p(\mathbb{R}^n, X)$ and any symbol $\phi \in \mathfrak{M}^p(\mathbb{R}^n, X)$, we have

$$\begin{aligned} \|Q_{p, X}(\phi)f\|_{L^p(\mathbb{R}^n, X)} &= \left\| \int_{\text{SO}(\mathbb{R}^n)} ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X)f d\mu(R) \right\|_{L^p(\mathbb{R}^n, X)} \\ &\leq \int_{\text{SO}(\mathbb{R}^n)} \|((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X)f\|_{L^p(\mathbb{R}^n, X)} d\mu(R) \\ &\leq \|M_\phi \otimes \text{Id}_X\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)} \|f\|_{L^p(\mathbb{R}^n, X)}. \end{aligned}$$

Consequently, we have a well-defined contractive map $Q_{p,X}: \mathfrak{M}^p(\mathbb{R}^n, X) \rightarrow \mathbb{B}(L^p(\mathbb{R}^n, X))$.

In the sequel, we denote by $L^1(\mathbb{R}^n, X)_w$ the space $L^1(\mathbb{R}^n, X)$ equipped with the locally convex topology $\sigma(L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X'))$ obtained with the inclusion $L^\infty(\mathbb{R}^n, X') \subset (L^1(\mathbb{R}^n, X))'$ given by [6, Proposition 1.3.1]. Note that $L^\infty(\mathbb{R}^n, X')$ is norming for $L^1(\mathbb{R}^n, X)$ again by [6, Proposition 1.3.1]. So we have a well-defined dual pair and furthermore $L^1(\mathbb{R}^n, X)$ is Hausdorff for this topology. Similarly, we will use $L^\infty(\mathbb{R}^n, X)_{w^*}$ for the space $L^\infty(\mathbb{R}^n, X)$ equipped with the topology $\sigma(L^\infty(\mathbb{R}^n, X), L^1(\mathbb{R}^n, X'))$, which is locally convex, obtained with the inclusion $L^1(\mathbb{R}^n, X') \subset (L^\infty(\mathbb{R}^n, X))'$.

Note that for any $g \in L^1(\mathbb{R}^n)$, the complex function

$$R \mapsto \langle g, \phi(R^{-1} \cdot) \rangle_{L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)} = \langle S_R(g), \phi \rangle_{L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)}$$

is continuous on $\text{SO}(\mathbb{R}^n)$ and consequently measurable. Since $\text{SO}(\mathbb{R}^n)$ is compact, we deduce that this function is integrable. Consequently, the function $\text{SO}(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)_{w^*}$, $R \mapsto \phi(R^{-1} \cdot)$ is scalarly μ -integrable. Moreover, it is bounded since in $L^\infty(\mathbb{R}^n)$ weak* bounded subsets coincide with bounded subsets by [7, Theorem 2.6.7]. Since $L^\infty(\mathbb{R}^n)_{w^*}$ is semireflexive, we deduce that this bounded function is Gelfand integrable by [1, Corollary, VI.6] and that $\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) d\mu(R)$ belongs to $L^\infty(\mathbb{R}^n)$.

Let $f: \mathbb{R}^n \rightarrow X$ be a function of the vector-valued Schwartz space $\mathcal{S}(\mathbb{R}^n, X)$ such that the continuous function $\hat{f}: \mathbb{R}^n \rightarrow X$ has compact support. Note that the subset of such functions is dense in the Bochner space $L^p(\mathbb{R}^n, X)$ by [6, Proposition 2.4.23] and that \hat{f} belongs to $L^1(\mathbb{R}^n, X)$. By [1, IV.94, Corollary 1], the product $T: L^\infty(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n, X)$, $h \mapsto h\hat{f}$ is continuous. Moreover, T remains continuous when considered as a map $T: L^\infty(\mathbb{R}^n)_{w^*} \rightarrow L^1(\mathbb{R}^n, X)_w$. Indeed, suppose that the net (h_i) of $L^\infty(\mathbb{R}^n)$ converges to h in the weak* topology. Then for any $g \in L^\infty(\mathbb{R}^n, X')$, the function $x \mapsto \langle \hat{f}(x), g(x) \rangle_{X, X'}$ belongs to $L^1(\mathbb{R}^n)$ by [1, IV.94, Corollary 1], and consequently

$$\begin{aligned} \langle T(h_i), g \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')} &= \langle h_i \hat{f}, g \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')} \\ &= \int_{\mathbb{R}^n} h_i(x) \langle \hat{f}(x), g(x) \rangle_{X, X'} dx \xrightarrow{i} \int_{\mathbb{R}^n} h(x) \langle \hat{f}(x), g(x) \rangle_{X, X'} dx \\ &= \langle h \hat{f}, g \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')} = \langle T(h), g \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')}. \end{aligned}$$

Then Lemma 1 implies that the function $\text{SO}(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n, X)_w$, $R \mapsto \phi(R^{-1} \cdot) \hat{f}$ is Gelfand integrable with $\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \hat{f} d\mu(R)$ belonging to $L^1(\mathbb{R}^n, X)_w$ and that

$$\left(\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) d\mu(R) \right) \hat{f} \stackrel{(2)}{=} \int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \hat{f} d\mu(R), \quad (4)$$

where the integrals are Gelfand integrals and both sides belong to $L^1(\mathbb{R}^n, X)$. Now using an obvious vector-valued extension of [5, Proposition 1.3 (vii)] in

the third and in the fifth equality, we obtain a.e.

$$\begin{aligned}
((S_R^{-1}M_\phi S_R) \otimes \text{Id}_X) f &= ((S_R^{-1}M_\phi) \otimes \text{Id}_X)(f(R \cdot)) \\
&= (S_R^{-1} \otimes \text{Id}_X) \mathcal{F}^{-1}[\widehat{\phi(\cdot) f(R \cdot)}] \\
&= (S_R^{-1} \otimes \text{Id}_X) \mathcal{F}^{-1}[\widehat{\phi(\cdot) \hat{f}(R \cdot)}] \\
&= (S_R^{-1} \otimes \text{Id}_X)((\mathcal{F}^{-1}[\widehat{\phi(R^{-1} \cdot) \hat{f}}])(R \cdot)) \\
&= \mathcal{F}^{-1}(\phi(R^{-1} \cdot) \hat{f}).
\end{aligned} \tag{5}$$

According to [6, page 105], the inverse Fourier transform $\mathcal{F}^{-1}: L^1(\mathbb{R}^n, X) \rightarrow L^\infty(\mathbb{R}^n, X)$ is bounded. We will show that it remains continuous when considered as a map $\mathcal{F}^{-1}: L^1(\mathbb{R}^n, X)_w \rightarrow L^\infty(\mathbb{R}^n, X)_{w^*}$. Indeed, suppose that the net (h_i) of $L^1(\mathbb{R}^n, X)_w$ converges to h . For any $g \in L^1(\mathbb{R}^n, X')$, using two times an obvious vector-valued extension of [5, Theorem 1.12], we obtain

$$\begin{aligned}
\langle \mathcal{F}^{-1}(h_i), g \rangle_{L^\infty(\mathbb{R}^n, X), L^1(\mathbb{R}^n, X')} &= \langle h_i, \mathcal{F}^{-1}(g) \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')} \\
&\xrightarrow{i} \langle h, \mathcal{F}^{-1}(g) \rangle_{L^1(\mathbb{R}^n, X), L^\infty(\mathbb{R}^n, X')} \\
&= \langle \mathcal{F}^{-1}(h), g \rangle_{L^\infty(\mathbb{R}^n, X), L^1(\mathbb{R}^n, X')}.
\end{aligned}$$

Using again Lemma 1 with \mathcal{F}^{-1} instead of T , we obtain that the function

$$\text{SO}(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n, X)_{w^*}, \quad R \mapsto \mathcal{F}^{-1}(\phi(R^{-1} \cdot) \hat{f})$$

is Gelfand integrable with $\int_{\text{SO}(\mathbb{R}^n)} \mathcal{F}^{-1}(\phi(R^{-1} \cdot) \hat{f}) d\mu(R)$ belonging to the space $L^\infty(\mathbb{R}^n, X)$ and that a.e.

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\left(\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) d\mu(R)\right) \hat{f}\right) \\
&\stackrel{(4)}{=} \mathcal{F}^{-1}\left(\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \hat{f} d\mu(R)\right) \\
&\stackrel{(2)}{=} \int_{\text{SO}(\mathbb{R}^n)} \mathcal{F}^{-1}(\phi(R^{-1} \cdot) \hat{f}) d\mu(R) \\
&\stackrel{\text{def}}{=} \int_{\text{SO}(\mathbb{R}^n)}^{\text{Gel}} \xi_R d\mu(R),
\end{aligned} \tag{6}$$

where the last Gelfand integral defines an element of $L^\infty(\mathbb{R}, X)$. Now, we show that this Gelfand integral and the Bochner integral (3) in $L^p(\mathbb{R}^n, X)$ are equal almost everywhere. If g belongs to $\mathcal{D}(\mathbb{R}^n, X')$, using [6, (1.2) page 15] in the sixth equality with the bounded map $T: L^p(\mathbb{R}^n, X) \rightarrow \mathbb{C}$, $h \mapsto$

$\langle h, g \rangle_{L^p(\mathbb{R}^n, X), L^{p'}(\mathbb{R}^n, X')}$, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left\langle \left(\int_{\text{SO}(\mathbb{R}^n)}^{\text{Gel}} \xi_R \, d\mu(R) \right) (s), g(s) \right\rangle_{X, X'} \, ds \\
&= \left\langle \int_{\text{SO}(\mathbb{R}^n)}^{\text{Gel}} \xi_R \, d\mu(R), g \right\rangle_{L^\infty(\mathbb{R}^n, X), L^1(\mathbb{R}^n, X')} \\
&\stackrel{(1)}{=} \int_{\text{SO}(\mathbb{R}^n)} \langle \xi_R, g \rangle_{L^\infty(\mathbb{R}^n, X), L^1(\mathbb{R}^n, X')} \, d\mu(R) \\
&= \int_{\text{SO}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \langle \xi_R(s), g(s) \rangle_{X, X'} \, ds \right) \, d\mu(R) \\
&\stackrel{(5)}{=} \int_{\text{SO}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \langle (((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X) f)(s), g(s) \rangle_{X, X'} \, ds \right) \, d\mu(R) \\
&= \int_{\text{SO}(\mathbb{R}^n)} \langle ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X) f, g \rangle_{L^p(\mathbb{R}^n, X), L^{p'}(\mathbb{R}^n, X')} \, d\mu(R) \\
&= \left\langle \int_{\text{SO}(\mathbb{R}^n)}^{\text{Boc}} ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X) f \, d\mu(R), g \right\rangle_{L^p(\mathbb{R}^n, X), L^{p'}(\mathbb{R}^n, X')} \\
&= \int_{\mathbb{R}^n} \left\langle \left(\int_{\text{SO}(\mathbb{R}^n)}^{\text{Boc}} ((S_R^{-1} M_\phi S_R) \otimes \text{Id}_X) f \, d\mu(R) \right) (s), g(s) \right\rangle_{X, X'} \, ds \\
&\stackrel{(3)}{=} \int_{\mathbb{R}^n} \langle (Q_{p, X}(\phi) f)(s), g(s) \rangle_{X, X'} \, ds.
\end{aligned}$$

Using Lemma 2 in the first equality, we conclude that a.e.

$$Q_{p, X}(\phi) f = \int_{\text{SO}(\mathbb{R}^n)}^{\text{Gel}} \xi_R \, d\mu(R) \stackrel{(6)}{=} \mathcal{F}^{-1} \left(\left(\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \, d\mu(R) \right) \hat{f} \right).$$

Using the above mentioned density, we conclude that the bounded operator

$$Q_{p, X}(\phi): L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)$$

is induced by the Fourier multiplier

$$P_{p, X}(\phi) \stackrel{\text{def}}{=} \int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \, d\mu(R) \tag{7}$$

(Gelfand integral). Moreover, for any $R_0 \in \text{SO}(\mathbb{R}^n)$, using the obvious continuity of the linear map $S_{R_0}: L^\infty(\mathbb{R}^n)_{w^*} \rightarrow L^\infty(\mathbb{R}^n)_{w^*}$ together with Lemma 1, we see that

$$\begin{aligned}
S_{R_0}(P_{p, X}(\phi)) &\stackrel{(7)}{=} S_{R_0} \left(\int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \, d\mu(R) \right) \stackrel{(2)}{=} \int_{\text{SO}(\mathbb{R}^n)} S_{R_0} S_R^{-1}(\phi) \, d\mu(R) \\
&= \int_{\text{SO}(\mathbb{R}^n)} S_R^{-1}(\phi) \, d\mu(R) = \int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) \, d\mu(R) \\
&\stackrel{(7)}{=} P_{p, X}(\phi).
\end{aligned}$$

We deduce that the multiplier $P_{p,X}(\phi)$ is radial, i.e. $P_{p,X}(\phi)$ belongs to $\mathfrak{M}_{\text{rad}}^p(\mathbb{R}^n, X)$. In addition, if ϕ itself is already radial, then

$$P_{p,X}(\phi) \stackrel{(7)}{=} \int_{\text{SO}(\mathbb{R}^n)} \phi(R^{-1} \cdot) d\mu(R) = \int_{\text{SO}(\mathbb{R}^n)} \phi d\mu(R) = \phi.$$

Finally, if $X = \mathbb{C}$ and if the map $M_\phi: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is positive, then $S_R^{-1}M_\phi S_R: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is positive, so it is easy to prove that the map $M_{P_{p,c}(\phi)}: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is also positive by using [6, Proposition 1.2.25]. \square

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