# Proof-of-work certificates that can be efficiently computed in the cloud

Jean-Guillaume Dumas

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**Abstract**

In an emerging computing paradigm, computational capabilities, from processing power to storage capacities, are offered to users over communication networks as a cloud-based service. There, demanding computations are outsourced in order to limit infrastructure costs.

The idea of verifiable computing is to associate a data structure, a *proof-of-work certificate*, to the result of the outsourced computation. This allows a verification algorithm to prove the validity of the result, faster than by recomputing it. We talk about a Prover (the server performing the computations) and a Verifier.

Goldwasser, Kalai and Rothblum gave in 2008 a generic method to verify any parallelizable computation, in almost linear time in the size of the, potentially structured, inputs and the result. However, the extra cost of the computations for the Prover (and therefore the extra cost to the customer), although only almost a constant factor of the overall work, is nonetheless prohibitive in practice.

Differently, we will here present problem-specific procedures in computer algebra, e.g. for exact linear algebra computations, that are Prover-optimal, that is that have much less financial overhead.

## 1 Introduction

In an emerging computing paradigm, computational capabilities, from processing power to storage capacities, are offered to users over communication networks as a service.

Many such outsourcing platforms are now well established, as Amazon web services (through the Elastic Compute Cloud), Microsoft Azure, IBM Platform Computing or Google cloud platform (via Google Compute Engine), as shown in Figure 1. None of these platforms, however, offer any guarantee whatsoever on the calculation: no guarantee that the result is correct, nor even that the computation has even effectively been done.

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Figure 1: Some outsourced computing services

## 1.1 Verifiable computing

This new paradigm holds enormous promise for increasing the utility of computationally weak devices. A natural approach is for weak devices to delegate expensive tasks, such as storing a large file or running a complex computation, to more powerful entities (say servers) connected to the same network. While the delegation approach seems promising, it raises an immediate concern: when and how can a weak device verify that a computational task was completed correctly? This practically motivated question touches on foundational questions in cryptography, coding theory, complexity theory, proofs and algorithms.

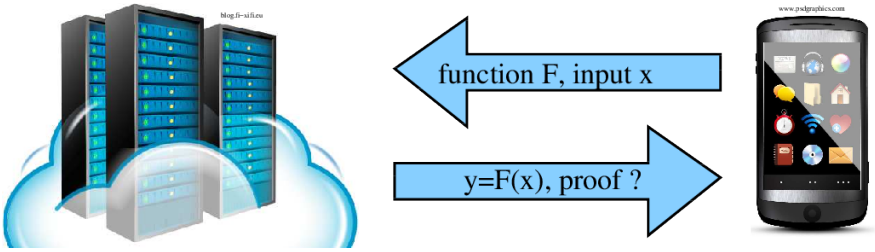



Figure 2: Verifying the computation should take less time than computing it

More generally, the question of verifying a result at a lower cost (time, memory) than that of recomputing it, as shown on Figure 2, is of paramount importance. Another example of application is for the extension of the trust about results computed via probabilistic or approximate algorithms. There the idea is to gain confidence into the correctness, but only at a cost negligible when compared to that of the computation.

## 1.2 Linear algebra, global optimization

For instance, GL7d19 is an $1\,911\,130 \times 1\,955\,309$ matrix whose rank $1\,033\,568$ was computed once in 2007 with a Monte-Carlo randomized algorithm [19].

This required 1050 CPU days of computation. We thus need publicly verifiable certificates to improve our confidence in computational results.

In linear algebra our original motivation is also related to sum-of-squares. By Artin's solution to Hilbert 17th Problem, any polynomial inequality $\forall \xi_1, \ldots, \xi_n \in \mathbb{R}, f(\xi_1, \ldots, \xi_n) \geq g(\xi_1, \ldots, \xi_n)$ can be proved by a fraction of sum-of-squares:

$$\exists u_i, v_j \in \mathbb{R}[x_1, \ldots, x_n], f - g = \left( \sum_{i=1}^{\ell} u_i^2 \right) / \left( \sum_{j=1}^{m} v_j^2 \right) \tag{1}$$

Such proofs can be used to establish global minimality for

$g = \inf_{\xi_v \in \mathbb{R}} f(\xi_1, \ldots, \xi_n)$ and constitute certificates in non-linear global optimization. A symmetric integer matrix $W \in \mathbb{SZ}^{n \times n}$ is positive semidefinite, denoted by $W \succeq 0$, if all its eigenvalues, which then must be real numbers, are non-negative. Then, a certificate for positive semidefiniteness of rational matrices constitutes, by its Cholesky factorizability, the final computer algebra step in an exact rational sum-of-squares proof, namely

$$\begin{aligned} \exists e \geq 0,\ W^{[1]} \succeq 0,\ W^{[2]} \succeq 0,\ W^{[2]} \neq \mathbf{0}: \\ (f-g)(x_1, \ldots, x_n) \cdot (m_e(x_1, \ldots, x_n)^T W^{[2]} m_e(x_1, \ldots, x_n)) = \\ m_d(x_1, \ldots, x_n)^T W^{[1]} m_d(x_1, \ldots, x_n), \end{aligned} \tag{2}$$

where the entries in the vectors $m_d, m_e$ are the terms occurring in $u_i, v_j$ in (1). In fact, (2) is the semidefinite program that one solves [43]. Then, the client can verify the positiveness by checking Descartes' rule of sign on the *certified* characteristic polynomial of $W^{[1]}$ and $W^{[2]}$. Thus arose the question how to give possibly probabilistically checkable certificates for linear algebra problems.

## 1.3 Techniques

The tools used to provide such efficient *proof-of-work certificates* stem from programs that check their work [12], to proof of knowledge protocols [7], via error-correcting codes [42, 35] complexity theory [1] or secure multiparty protocols [17], and the interaction of these different methodologies is crucial.

Here we will thus follow this road-map:

- We recalled that global optimization can be reduced to linear algebra. Thereupon we will focus on certificates for linear algebra problems [43] in computer algebra, which extend the randomized algorithms of Freivalds [32].
- We combine those with probabilistic interactive proofs of Babai [5] and Goldwasser et al. [39],
- as well as Fiat-Shamir heuristic [29, 9] turning interactive certificates into non-interactive heuristics subject to computational hardness.
- Overall, we obtain problem-specific efficient certificates for dense, sparse, structured matrices with coefficients in fields or integral domains.

# 2 Interactive protocols, the PCP theorem and homomorphic encryption

## 2.1 Arthur-Merlin interactive proof systems

A proof system usually has two parts, a theorem $T$ and a proof $\Pi$, and the validity of the proof can be checked by a verifier $V$. Now, an *interactive proof*, or a $\sum$*-protocol*, is a dialogue between a prover $P$ (or *Peggy* in the following) and a verifier $V$ (or *Victor* in the following), where $V$ can ask a series of questions, or challenges, $q_1$, $q_2$, ... and $P$ can respond alternatively, in successive *rounds*, with a series of strings $\pi_1$, $\pi_2$, ..., the responses, in order to prove the theorem $T$. The theorem is sometimes decomposed into two parts, the hypothesis, or input, $H$, and the commitment, $C$. Then the verifier can accept or reject the proof: $V(H, C, q_1, \pi_1, q_2, \pi_2, \ldots) \in \{\text{accept}, \text{reject}\}$.

To be useful, such proof systems should satisfy **completeness** (the prover can convince the verifier that a true statement is indeed true) and **soundness** (the prover cannot convince the verifier that a false statement is true). More precisely, the protocol is *complete* if the probability that a true statement is rejected by the verifier can be made arbitrarily small. Similarly, the protocol is *sound* if the probability that a false statement is accepted by the verifier can be made arbitrarily small. The completeness (resp. soundness) is *perfect* if accepted (resp. rejected) statement are always true (resp. false).

It turns out that interactive proofs with perfect completeness are as powerful as interactive proofs [33]. Thus in the following, as we want to prove correctness of a result more than proving knowledge of it, we will mainly show interactive proofs with perfect completeness.

The class of problems solvable by an interactive proof system (IP) is equal to the class PSPACE [55] and a probabilistically checkable proof, $\text{PCP}[r(n), \pi(n)]$, for an input of length $n$, is a type of proof that can be checked by a randomized algorithm using a bounded amount of randomness $r(n)$ and reading a bounded number of bits of the proof $\pi(n)$. For instance, $\text{PCP}[O(\log n), O(1)]$=NP [6, 3].

In general, interactive protocols encompass many kinds of proofs and Prover and Verifier settings. One can think of the difficulty of integer factorization versus that of re-multiplying found factors. Another example could be satisfiability checking, where the solver has to explore the state space, while verifying a variable assignment or a conflict clause could be much simpler [2]. In computer algebra, the Prover can be a probabilistic algorithm or a symbolic-numeric program, where the Verifier would perform the checks exactly or symbolically; further, computer algebra systems could perform a complex calculations where an interactive theorem prover (or proof assistant like Isabel-HOL or Coq) only has to a posteriori formally verify the certificate [16, 15].

Table 1 gives more examples of such settings.

| Prover | Verifier |
|---|---|
| Computer Scientist | Mathematician |
| Computer Algebra system | Formal proof assistant |
| Cloud | User |
| Server | Client |
| Cellphone | Trusted platform module |

Table 1: Examples of Prover/Verifier settings

## 2.2 Goldwasser et al. prover efficient interactive certificates

Now, efficient protocols (interactive proofs between a *Prover*, responsible for the computation, and a *Verifier*, to be convinced) can be designed for delegating computational tasks.

Recently, generic protocols, mixing zero-sum checks [45] and probabilistically checkable proofs, have been designed by teams around Shafi Goldwasser at the MIT or Harvard, for circuits with polylogarithmic depth [38, 57], namely for problems that can be efficiently solved on a parallel computer (in the NC or AC complexity class). These results have also been extended to any structured inputs (any polynomial-time-uniform polylog-depth Boolean circuits in the sense of Beame's et al, [8], division circuits) [23].

The resulting protocols are interactive and there is a trade-off between the number of interactive rounds, the volume of communication and the computational cost [50]; the cost for the verifier being usually only roughly proportional to the input size.

These protocols can, e.g., certify that two supersparse polynomials are relatively prime in verifier cost which is polylog time (and rounds) in the degree.

The produced certificates, in analogy to processor-efficient parallel algorithms, are Prover-efficient: if the cost to compute the result by the best known algorithm is $T(n)$ for a size $n$ problem, then the cost to produce the result together with the verifiable certificate is $T(n)^{1+o(1)}$.

Those techniques can however produce a non negligible practical overhead for the Prover and are restricted to certain classes of circuits.

## 2.3 Parno et al. homomorphic solutions

Another approach as been developed by Gentry et al., at Microsoft and IBM research, it is Pinocchio. It solves a broader range of problems, to the cost of using relatively inefficient homomorphic routines [48] in an amortized way.

The idea is that the Prover should run the program (or at least part of the program twice), once normally on the input, and once homomorphically on an encrypted version of the input. The Verifier will then verify the consistency between the normal output and the encrypted one. Usually the Verifier is required to run the algorithm at least once for a given size or structure of the input but

can reuse this for multiple inputs.

This generic procedure can be applied on specific linear algebra or polynomial problems [31, 60, 28, 25], or on generic quadratic arithmetic programs [48]. There, fully homomorphic encryption can be used [36] or sometimes just pairings [48] and/or cryptographic hashes [30].

Here also the Prover can be efficient, but subject in practice to the overhead of homomorphic computations.

## 2.4 Public verification, delegatability and zero-knowledge

Interactive certificates require some exchanges between the Prover and the Verifier. With such a protocol, the Verifier can be *privately* convinced that the computation of the Prover produced the correct answer. This does not mean that other people would be convinced be the transcript of their exchange: the Prover and Verifier could be in cahoots and the supposedly random challenges carefully crafted.

Fiat-Shamir heuristic [29, 9] can thus turn interactive certificates into non-interactive heuristics subject to computational hardness: the random challenges are replaced by cryptographic hashes of all previous data and exchanges. Crafting such values would then reduce to being able to forge cryptographic fingerprints [20, § 4.5].

Further, more properties could be sought for such protocols, such *privacy* of data and/or computations. In this setting, a publicly verifiable computation scheme can also be four algorithms (*KeyGen*, *ProbGen*, *Compute*, *Verify*), where *KeyGen* is some (amortized) preparation of the data, *ProbGen* is the preparation of the input, *Compute* is the work of the *Prover* and *Verify* is the work of the *Verifier* [49]. Usually the Verifier also executes *KeyGen* and *ProbGen* but in a more general setting these can be performed by different entities (respectively called a *Preparator* and a *Trustee*).

This allows to define several adversary models but usually the protocols are secure against a *malicious Prover only* (that is the Client must trust both the Preparator and the Trustee).

One can also further impose that there is no interaction between the Client and the Trustee after the Client has sent his input to the Server. Publicly verifiable protocols with this property are said to be *publicly delegatable* [60, 28, 25].

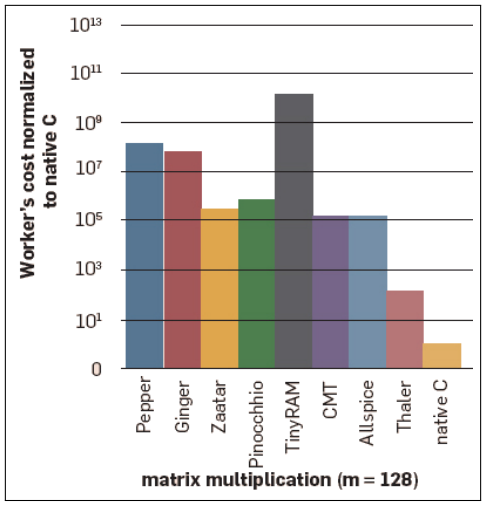
Then, some different properties of the protocol could be desirable, such as not disclosing the result but instead just providing a *proof-of-work*. This results in general in *zero-knowledge* protocols over confidential data, such as cryptocurrency transactions, as in, e.g., [39], with recent efficient implementations [13, 10, 11, 14].

## 2.5 Problem-specific efficient certificates

Differently, dedicated certificates (data structures and algorithms that are verifiable a posteriori, without interaction) have also been developed, e.g., in com-

puter algebra for exact linear algebra [32, 43, 20, 22, 24], even for problems that are not structured. There the certificate constitute a proof of correctness of a result, not of a computation, and can thus also **detect bugs** in the implementations.

Moreover, problem-specific certificates can gain crucial logarithmic factors for the verifier and allow for optimal prover computational time, see Figure 3.




| $2048 \times 2048$ | Thaler[57] | Ad-hoc [32] |
|---|---|---|
| Server time | 18.23s | 0.65s |
| Certificate overhead | 0.13s | 0.00s |
| Client time | 2.89s | 0.01s |

Figure 3: Generic protocols [58] versus dedicated protocols for matrix multiplication

For this, the main difficulty is to be able to design verification algorithms for a problem that are completely orthogonal to the computational algorithms solving it, while remaining checkable in time and space not much larger than the input.

# 3 Prover-optimal certificates in linear algebra

We show in this section, that such problem-specific certificates are attainable in linear algebra, where we allow certificates that are validated by Monte Carlo randomized algorithms.

### 3.1 Freivalds zero equivalence of matrix expressions

The seminal certificate in linear algebra is due to Rūsiņš Freivalds [32]: quadratic time is feasible because a matrix multiplication $AB$ can be certified by the resulting product matrix $C$ via the probabilistic projection to matrix-vector products (see also [44] who reduced the requirements to only $O(\log(n))$ random bits), shown in Protocol 1.

| *Prover* | *Communication* | *Verifier* |
|---|---|---|
| | $\mathbf{A} \in \mathbb{F}^{m\times k}$, $\mathbf{B} \in \mathbb{F}^{k\times n}$ | |
| Compute $\mathbf{C} = \mathbf{A}\cdot\mathbf{B}$ | $\xrightarrow{\mathbf{C}}$ | $r \xleftarrow{\$} S \subseteq \mathbb{F}$<br>Form $\vec{v} = [1, r, r^2, \ldots, r^{n-1}]^T$<br>$\mathbf{A}\,(\mathbf{B}\vec{v}) - \mathbf{C}\vec{v} \stackrel{?}{=} \vec{0}$ |

**Protocol 1.** Matrix multiplication certificate [44].

In Protocol 1, we give the variant of [44] that requires $\log(n)$ random bits, but works over sufficiently large coefficient domains, as its soundness is $1 - \frac{|S|}{n}$ by the DeMillo-Lipton/Schwartz/Zippel lemma [18, 61, 53]. Freivalds original version randomly selects a zero-one vector instead. This requires $n$ random bits instead but applies to any ring and gives a soundness larger than $\frac{1}{2}$.

In both cases it is sufficient to repeat the test several times to achieve any desired probability.

### 3.2 Reductions to matrix multiplication

With a certificate for matrix expressions, then one can certify **any algorithm that reduces to matrix multiplication**: the Prover records all the intermediate matrix products and sends them to the Verifier who reruns the same algorithm but checks the matrix products instead of computing them [43], as shown in Protocol 2.

| *Prover* | *Communication* | *Verifier* |
|---|---|---|
| Runs the algorithm | All intermediate matrix products $\longrightarrow$ | Runs the algorithm but replace each matrix products by Freivalds' checks |

**Protocol 2.** Certificates with reduction to matrix multiplication [43, § 5].

Overall, the communications and Verifier computational cost are given by taking $\omega = 2$ in the Prover's complexity bounds (with potential additional logarithmic factors due to summations). Further, *t*he production of the certificate

has no computational overhead for the Prover: it only adds the communication of the intermediate matrix products.

For instance, Storjohann's Las Vegas rank algorithm of integer matrices [56] becomes a non-interactive/non-cryptographic Monte Carlo checkable proof-of-work certificate that has soft-linear time communication and verifier bit complexity in the number of input bits!

### 3.3 Sparse or structured matrices

When the matrices are sparse or present some structure, quadratic run time and/or quadratic communications might be overkill for the Verifier. There it is better if his communications and computational cost is of the form $\mu(A)+n^{1+o(1)}$ where $\mu(A)$ is the number of operations to perform a matrix-vector products. This scheme is thus also interesting if the considered matrix is only given as a blackbox [40].

In that vein, we now have certificates for :

- **non-singularity**, Protocol 3;
- **an upper bound to the rank**, Protocol 4 (if elimination on the input matrix is possible for the Prover then a variant without preconditioners can be used [26, 24]);
- the **rank**, combining Protocols 3 and 4;
- the **minimal polynomial**, using Protocol 5 (where $f_u^{A,v}$ is the monic scalar minimal generating polynomial of the sequence $u^Tv, \ldots, u^TA^iv$, $\rho_u^{A,v}$ is such that $\rho_u^{A,v} = f_u^{A,v} \cdot G$ with $G$ the generating function of the latter sequence, for random vectors $u$ and $v$, chosen by the Verifier [41, Theorem 5]);
- the **determinant**, Protocol 6, which randomness could be reduced from $O(n)$ to a constant number of field elements [21, § 7].

Additionally, properties of the given matrices can also sometimes be discovered at low cost: whether the blackbox is a **band matrix**, has a **low displacement rank**, has a few or many **nilpotent blocks** or **invariant factors** [27]. Similarly, the existence of a **triangular one sided equivalence**, as well as the **rank profiles** can also be certified without sending an explicit factorization to the Verifier [24]. For the latter, the price to pay is to require a linear number of rounds.

### 3.4 Integer or polynomial matrices

Over an integral domain, the verification procedure can be performed via a randomly chosen modular projection. If there are sufficiently many *small* maximal ideals, then one can uniformly chose one at random and then ask for a certification of the result in the associated quotient field as shown in Protocol 7.

| *Prover* | | | *Verifier* |
|---|---|---|---|
| *Input* | | $\mathbf{A} \in \mathbb{F}^{n\times n}$ | |
| *Commitment* | | $1:$ non-singular $\longrightarrow$ | |
| *Challenge* | | $\longleftarrow 2:\ \vec{b}$ | $\vec{b} \xleftarrow{\$} S^n \subset \mathbb{F}^n$ |
| *Response* | $\vec{w} \in \mathbb{F}^n$ | $3:\ \vec{w} \longrightarrow$ | $\mathbf{A}\vec{w} \stackrel{?}{=} \vec{b}$ |

**Protocol 3.** Blackbox interactive certificate of non-singularity [20]

| *Prover* | | *Verifier* |
|---|---|---|
| | $A \in \mathbb{F}^{m\times n}$ | $S \subset \mathbb{F}$ |
| $rank(A) \leq r$ | $1:\ r \longrightarrow$ | $r \stackrel{?}{<} \min\{m,n\}$ |
| | $\longleftarrow 2:\ U,V$ | $U \in \mathbb{B}_S^{m\times m}, V \in \mathbb{B}_S^{n\times n}$ |
| | | preconditioners of size $n^{1+o(1)}$ |
| $w \in \mathbb{F}^{r+1} \neq 0$ | $3:\ w \longrightarrow$ | $w \stackrel{?}{\neq} 0$ |
| | | $[I_{r+1}\|0]\, UAV \begin{bmatrix} I_{r+1} \\ 0 \end{bmatrix} w \stackrel{?}{=} 0$ |

**Protocol 4.** Blackbox upper bound to the rank certificate [20]

| *Prover* | *Communication* | *Verifier* |
|---|---|---|
| $H(\lambda) = f_u^{A,v}(\lambda)$, | | |
| $h(\lambda) = \rho_u^{A,v}(\lambda)$. | $\xrightarrow{H,h}$ | |
| $\phi, \psi \in \mathbb{F}[\lambda]$ with | | |
| $\phi f_u^{A,v} + \psi \rho_u^{A,v} = 1$, | $\xrightarrow{\phi,\psi}$ | $\deg(\phi) \stackrel{?}{\leq} \deg(h) - 1$, |
| | | $\deg(\psi) \stackrel{?}{\leq} \deg(H) - 1$. |
| | | Random $r_0 \in S \subseteq \mathbb{F}$. |
| | | Checks $\mathrm{GCD}(H(\lambda), h(\lambda)) = 1$ by $\phi(r_0)H(r_0) + \psi(r_0)h(r_0) \stackrel{?}{=} 1$. |
| Computes $w$ such that | $\xleftarrow{r_1}$ | Random $r_1 \in S \subseteq \mathbb{F}$. |
| $(r_1 I_n - A)w = v$. | $\xrightarrow{w}$ | Checks $(r_1 I_n - A)w \stackrel{?}{=} v$ and $(u^T w)H(r_1) \stackrel{?}{=} h(r_1)$. |
| | | Returns $f_u^{A,v}(\lambda) = H(\lambda)$. |

**Protocol 5.** Certificate for $f_u^{A,v}$ [22]

| *Prover* | *Communication* | *Verifier* |
|---|---|---|
| 1. Form $B = DA$ with $D \in S^n \subseteq \mathbb{F}^{*n}$ and $u, v \in S^n$, s.t. $\deg(f_u^{B,v}) = n$. | $\xrightarrow{D,u,v}$ | |
| | Protocol 5 | |
| 2. | $\xrightarrow{H,h,\phi,\psi}$ | Checks: |
| 3. | $\xleftarrow{r_1}$ | $\deg(H) \stackrel{?}{=} n$, |
| 4. | $\xrightarrow{w}$ | $H \stackrel{?}{=} f_u^{B,v}$, w.h.p. |
| 5. | | Returns $\dfrac{f_u^{B,v}(0)}{\det(D)}$. |

**Protocol 6.** Determinant certificate for a non-singular blackbox [22]

| | *Prover* | *Communication* | *Verifier* |
|---|---|---|---|
| *Commitment* | Result $r \in \mathbb{R}$ | $\xrightarrow{r}$ | |
| *Challenge* | | $\xleftarrow{\mathcal{I}}$ | $\mathcal{I} \xleftarrow{\$}$ maximal ideals |
| *Response* | Result $x \in \mathbb{R}/\mathcal{I}$ with field certificate $\mathcal{C}_{\mathbb{R}/I}$ | $\xrightarrow{x, \mathcal{C}_{\mathbb{R}/I}}$ | $x \stackrel{?}{\equiv} r \mod \mathcal{I}$ and $\mathcal{C}_{\mathbb{R}/I}(x) \stackrel{?}{=}$ valid |

**Protocol 7.** Certification in a quotient field [20, § 3.2 and § 4.4].

For instance this gives very efficient certificates for polynomial or integer/rational matrices, provided that one has a bound on the degree or the magnitude of the coefficients:

- For **integral matrices**, if the true result $v$ is bounded in magnitude, then only a finite number of prime numbers will divide the difference between the commitment $r$ and the result. Therefore the result can be checked over a *small* prime field [20, Theorem 5].
- For **polynomial matrices**, if the true $v(X)$ result's degree is bounded, then only a finite number of evaluation points can be roots of the difference polynomial between the committed one $r(X)$ and the result. Therefore the result can be checked in the ground field at a *small* evaluation point [20, Theorem 2].

The latter results allows, for instance, to certify the global optimization problems of Section 1.2.

This is illustrated in Figure 4, where many of the reductions presented here are recalled.

## 3.5 Non-interactive certificates

The certificates in Sections 3.1 and 3.2 are non-interactive: all the communications can be recorded and publicly verified later.

On the contrary the certificates of Sections 3.3, 3.4 are interactive: the Verifier chooses some random bits during the computation of the certificate. Non-interactivity can be recovered via Fiat-Shamir scheme: any random bits are generated by cryptographic hashes of the inputs and all the previous intermediate commitments. Soundness is then subject to standard cryptographic assumptions.

For sparse or structured problems fewer results exists without this assumption, or with worse complexity bounds:

- For the minimal polynomial (scalar or matrix) or the determinant, non-interactive certificates exists, but with communications and computational cost $O(n\sqrt{\mu(A)})$ instead of $\mu(A) + n^{1+o(1)}$ [21].

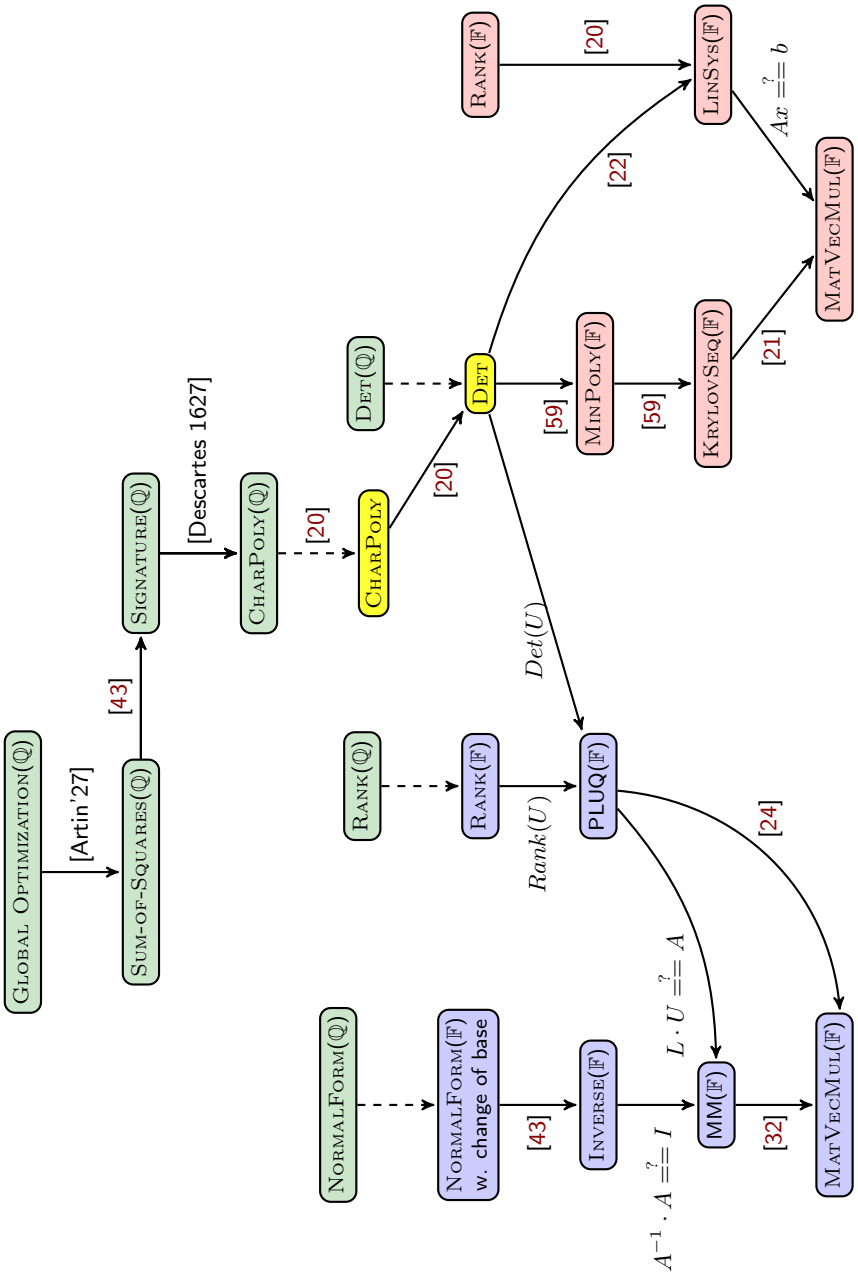



Figure 4: Global optimization via problem-specific interactive certificates: dense (purple) or sparse (red) algebraic problems, as well as over the reals (green) or oblivious (yellow).

- Non-interactive certificates can also verify polynomial minimal approximant bases in $O(mD + m^\omega)$, where $D$ is the sum of the column degrees of the output [37].

# 4 Some open problems

We conclude this survey with some open problems in the area of problem specific linear algebra certificates:

- **Sparse Smith form**: for dense matrices, one can interactively certify any normal form via a Freivalds certificate on a randomly chosen modular factorization. With sparse matrices, even the modular projection of the change of base can be too large. In that setting extending protocols for the rank or the determinant to deal with the Smith form should be possible.
- **Non integral domains certificates**: more generally, how to efficiently certify some properties when there is no quotients or if those properties do not carry over (e.g., Smith form)?
- We have defined certificates resisting a malicious server with unbounded power. This is error detection with unbounded number of errors. Thus the question of the complexity of **problem specific unbounded error correction** also arises. This path again was first taken for matrix multiplication [35] and was recently extended to the matrix inverse [51].

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