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NUMERICAL TREATMENT OF CONTACT AND FRICTION: THE CONTACT DYNAMICS METHOD

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ABSTRACT

Difficulties occur in the numerical treatment of mechanical systems when the unilateral constraints of mutual impenetrability of members are taken into account. At every time-step, algorithms have to be ready to handle the possible breaking of some contacts or, on the contrary, their sudden introduction by collisions. Furthermore, the modeling of dry friction, generally present in the case of contact, rests on mathematically irregular relationships. It is shown in this paper how the numerical approach named 'Contact Dynamics' allows one to calculate evolutions in dynamic or quasi-static regimes and also to analyze the possible equilibria. This method faces the essential nonsmoothness of the concerned problems without ressorting to the regularizing approximations applied by other commonly used techniques. It rests on time-discretization schemes of the implicit type, allowing for coarser time steps, at the price of applying nonsmooth iteration procedures.

1. INTRODUCTION

Various engineering situations call for the mechanical analysis of collections of bodies which, instead of the articulations considered in the traditional mechanics of systems, are only submitted to the unilateral constraints of not interpenetrating each other. At a priori unknown instants, such bodies may enter into contact or separate out, but they can never overlap.

For example, *granulatematerials*, studied from the static or the dynamic standpoints, are systems of this sort. Problems raised by the behaviour of a *work of masonry submitted to an earthquake* let themselves be formulated in a similar way, as well as various others in Geomechanics (Jean, 1995). In these instances, the concerned bodies are primarily treated as perfectly rigid, but the numerical investigation of the *forming of materials* (drawing, rolling, etc.) show the same type of constraint imposed to portions of deformable media (see e.g. Jean, 1993, Raous et al., 1995). Also in Robotics, some authors are currently taking into account, for simulation or control, the unilateral constraints of impenetrability (Brogliato et al.,1994)(Pierrot et al., 1994).

In terms of the configuration parameters of the system (in numerical computation, there is always a finite number of them, even if the considered bodies are deformable), impenetrability constraints are expressed by *inequality conditions*. Furthermore, when contact occurs, solving statical or dynamical problems requires some phenomenological information about the *contact forces* that the bodies exert on each other.

Even in the ideal case of frictionless contact, due to the *unilateral* character of the impenetrability constraints, the appraisal of contact forces in Dynamics when several contacts are simultaneously effective has long been known to need careful discussion. Additional difficulties come from *friction* at the contact locus, which is rarely negligible. Tribology describes it as a complex phenomenon so that the data needed for a quite accurate prediction of its effects are, in most engineering situations, unavailable.

In practice the Coulomb law of *dry friction* is very commonly invoked, for it retains the most salient features of frictional contact. But even this fairly simple law raises analytical and numerical difficulties, since the relationship it formulates between the contact forces and the relative velocities of the bodies at the respective contact points is mathematically irregular. In the event of zero relative velocity, the law only requires of the contact force to belong to a certain cone (the cone of static friction). For nonzero relative velocity, the law connects the tangential component of the contact force with its normal component and with the direction of the velocity vector. This direction may vary sharply when the velocity approaches zero, while the normal component itself is not usually given, but depends on the whole dynamical analysis. At equilibrium, familiar examples show that situations involving dry friction are frequently hyperstatic. For evolution problems, one should no more expect the uniqueness of solution.

Furthermore, in dynamical evolutions subject to impenetrability constraints, *collisions* implying velocity jumps are likely to occur. This constitutes another sort of analytical irregularity while, from the phenomenological standpoint, collisions may be affected by many aspects of the material behaviour which make their outcome difficult to predict with great precision.

In Operation Research, when problems of Optimization or Control in the presence of inequality conditions are treated, mathe-

matical difficulties of the same nature are met, though somewhat milder. The analytical and numerical tools developed to solve the latter problems constitute the mathematical domain called *Nonsmooth Analysis*. By analogy, questions of the sort considered here, as well as various others arising from plasticity, shape optimization, etc. have been said to belong to *Nonsmooth Mechanics* (Moreau and Panagiotopoulos, 1988; Moreau, Panagiotopoulos and Strang, 1988).

A natural approximation procedure, for the numerical treatment of such problems consists in replacing the impenetrability constraints by some short range repulsion laws which enter into action when two bodies come close to each other. This is a mechanical analog to the *penalization technique*, classical in Constrained Optimization. The dynamics of the approximant mechanical system is governed by proper differential equations to which standard numerical schemes, usually of the *explicit* type, are applied. Similarly, the nonsmooth character of dry friction may be numerically overcome by replacing Coulomb law by some *regularized* approximant.

After the pioneering work of P. Cundall (1971), many authors have developed numerical methods of this sort and used them to produce significant results. Depending on the respective authors environment and on technical specificities, such methods are named 'Distinct Element method', 'Discrete Element method' or 'Molecular Dynamics methods'. Some commercial programs rest on this numerical strategy. The main drawback is that the sake of precision in approximating impenetrability constraints requires of the repulsion laws to be stiff, so that the numerical integration of the corresponding differential equations needs fine time-steps. Furthermore, it may be difficult to distinguish the possible artefacts generated by such artificial elasticity from the effects of physical deformability. As a recent paper where these technical aspects are discussed, one may consult Drake and Walton (1995).

In contrast, the *Contact Dynamics* method, presented in this communication, faces nonsmoothness without resorting to any regularizing approximation. The key feature is that the impenetrability inequalities are treated at the level of velocities. Clearly, if in some state of the system a certain contact is in effect, the corresponding impenetrability inequality requires of the consequent velocities (i. e. the time-derivatives of the configuration parameters on the right of the concerned instant) to verify a certain linear inequality. Thus, for each possible configuration of the system, the set V of the 'right-admissible' velocity vectors is a convex polyhedral cone in $\mathbf{R}^n$. It turns out that the information to be entered, concerning contact forces is also connected with the same cone. For instance, in the academic case of frictionless contact, the condition imposed to the element of $\mathbf{R}^n$ which represents the contact forces is that of belonging to the 'polar cone' of V. If friction is taken into account, contact forces have to be related to velocity by more complicated relationships but fitting very well into such a Convex Analysis framework. In Sec.3 below we introduce the concept of a *complete contact law*, a formalism which secures the mechanical and numerical consistency of the model while leaving widely open the possibility of introducing phenomenological data.

The problem to be solved at each step of the time-discretization is to determine velocities and contact forces jointly, in order for the contact laws and the balance of momentum to be satisfied. The numerical schemes so constructed naturally turn out to be of the *implicit* type. This implicit character makes the algorithms ready, at each time-step, to face collisions, should one of them be detected as occurring during the time-interval of the step, a point developed in Sec.5 below.

The paper ends with a few examples of applications developed by the authors.

2. ANALYTICAL SETTING

Let the configurations of the investigated system be parametrized, at least locally, through generalized coordinates, say $q = (q^1, q^2, \ldots, q^n)$. If the system consists only of perfectly rigid bodies, this reduction to finite freedom usually results from some (bilateral) ideal constraints, such as the possible action of internal or external frictionless linkages. Also when deformable continuous bodies are involved, the numerical treatment rests on finite freedom approximations (modal representation or use of finite elements) so that the formalism presented in what follows applies as well.

After constructing the parametrization, one takes into account the constraints of impenetrability; their geometric effect is assumed expressed by a finite set of inequalities

$$f_\alpha(q) \le 0\,, \quad \alpha \in \{1, 2, \ldots, k\} \tag{1}$$

where $f_1, f_2, \ldots, f_k$ are given functions. Equality $f_\alpha=0$ corresponds to the occurrence of a contact between two members of the system or between one of these members and some external obstacle. For brevity, the latter will be assumed fixed in the reference frame in use. The case of obstacles with prescribed motion would call for the time t being also a variable in f_α and bring only notational complications to the sequel (see Moreau, 1994).

For every imagined motion $t \to q(t)$ and for t such that the derivative $\dot q(t) \in \mathbf{R}^n$ exists, the kinetic energy has a quadratic expression in $\dot q$, say

$$\mathcal{E}_k(q, \dot q) = \tfrac{1}{2} A_{ij}(q)\, \dot q^i\, \dot q^j \tag{2}$$

where A denotes a symmetric positive definite n×n-matrix.

As far as *smooth*, i.e. twice differentiable, motions are concerned, the system Dynamics is governed by Lagrange's equations, written below as an equality in $\mathbf{R}^n$

$$A(q)\, \ddot q = F(t, q, \dot q) + \sum_\alpha r^\alpha \tag{3}$$

The expression F comprises the $\dot q$-dependent terms of Lagrange's equations and the covariant components, relative to the parametrization (q), of some applied forces supposed given as functions of time, position and velocity. The element r^α of $\mathbf{R}^n$ is made of the covariant components of the a priori unknown *contact forces* experienced by the system in case the contact $f_\alpha=0$ holds. Their definition rests on the standard construction of the covariant components of forces located in physical space, connected as follows with the system kinematics.

Suppose that inequality $f_\alpha \le 0$ expresses the mutual impenetrability of some pair of members of the system, say $\mathcal{B}$ and $\mathcal{B}'$, so that equality $f_\alpha=0$ corresponds to these two bodies touching each other at some point of space denoted by M_α. One assumes it an isolated contact point, but other contacts, corresponding to different values of α, may also be in effect between the same bodies at the same instant. For every imagined motion $t \to q(t)$ bringing the system through the considered contact position for some value of t, the velocities $\mathcal{V}_\alpha$ and $\mathcal{V}'_\alpha$ of the respective particles of $\mathcal{B}$ and $\mathcal{B}'$ passing at point M_α let themselves be expressed as affine functions of the va-

lue u of the derivative $\dot{q}$. The same is thus true for the *relative velocity* $\mathcal{U}_\alpha=\mathcal{V}_\alpha-\mathcal{V}'_\alpha$ of $\mathcal{B}$ with respect to $\mathcal{B}'$ at this point, say

$$\mathcal{U}_\alpha = G_\alpha u \tag{4}$$

where G_α:$\mathbf{R}^n \to \mathbf{R}^3$ denotes a linear mapping, depending on t and q. No attention is paid at this stage to the imagined motion preserving contact or not.

Let $\mathcal{R}^\alpha$ denote the contact force that body $\mathcal{B}$ experiences at point M_α from body $\mathcal{B}'$; then $\mathcal{B}'$ experiences from $\mathcal{B}$ the force $-\mathcal{R}^\alpha$. Classically, the covariant components of this pair of forces are expressed by

$$r^\alpha = G^*_\alpha \mathcal{R}^\alpha \tag{5}$$

with G^*_α: $\mathbf{R}^3 \to \mathbf{R}^n$ denoting the transpose of G_α (the convention of implicit summation will never be applied to Greek indices).

Similar formulas hold if inequality $f_\alpha \le 0$ represents the confinement of a part $\mathcal{B}$ of the system by some external boundary. Then r^α is made of the covariant components of $\mathcal{R}^\alpha$ alone. In both cases, the following relationship is found (Moreau, 1988) to hold between the gradient $\partial f_\alpha/\partial q$ in $\mathbf{R}^n$ and the normal unit vecteur $\mathbf{n}^\alpha$ at point M_α to the two contacting bodies, *directed toward* $\mathcal{B}$

$$\exists \lambda_\alpha \ge 0 \text{ such that } \quad G^*_\alpha \mathbf{n}^\alpha = -\lambda_\alpha \, \partial f_\alpha/\partial q \tag{6}$$

In all the sequel, we shall assume that the mapping G_α is *onto* from $\mathbf{R}^n$ to $\mathbf{R}^3$; equivalently, its transpose G^*_α is *one-to-one* from $\mathbf{R}^3$ *into* $\mathbf{R}^n$. Only some special positions of certain linkages may give rise to 'wedging' effects which break this assumption.

3. CONTACT LAWS

To Eqs. (1), (3), (4), (5) one necessarily has to adjoin some phenomenological information regarding the contact forces. For instance, the ideal situation of *frictionless contact* is described by asserting that the contact force $\mathcal{R}^\alpha$ (vanishing if contact α is not effective, i.e. if $f_\alpha(q)<0$) is a vector belonging to the half-line generated in $\mathbf{R}^3$ by $\mathbf{n}^\alpha$.

As an elementary description of *dry friction*, the law of Coulomb will be invoked in the sequel. Generally, let us call a *contact law* a relationship of the form

$$\text{law}_\alpha(t, q, \mathcal{U}_\alpha, \mathcal{R}^\alpha) = \text{true} \tag{7}$$

imposed to the elements defined in the preceding Section.

An adequately designed formulation of contact laws greatly helps to face the difficulties raised by the unilateral character of the impenetrability constraints. The event of collisions, i.e. the sudden occurence of new contacts, which is expected to generate velocity jumps, is left for Sec.5. At the present stage, where only smooth motions are considered, one still has to face the possibility of the *breaking of some contacts*. The traditional approach consists in tentatively calculating the motion under the assumption that all contacts present at a considered instant remain effective in the sequel. If the evaluation of contact forces in the course of such a motion yields, at a further instant, an unfeasible direction for some of them, one concludes that some contacts should break at this instant, so the continued motion has to be calculated differently. But easy examples show that the contacts which break are not necessarily those for which unfeasible contact forces were just found. In the ideal case of frictionless contacts, Delassus (1917) has elucidated the determination of instant accelerations under such circumstances and the same question was more recently investigated in terms of Convex Analysis (Moreau, 1966).

Strictly speaking, the matrix G_α and the vector $\mathbf{n}^\alpha$ defined in the preceding Section make sense only in the case of contact, i.e. for a configuration q satisfying $f_\alpha(q)=0$. In computation, as well as in existential studies, the definition of these elements has to be extended, in a smooth arbitrary way, to configurations q laying in a neighbourhood of the hypersurface $f_\alpha=0$ of $\mathbf{R}^n$. In particular, one should be ready to face a certain amount of violation of the impenetrability inequalities (one manages to keep it small).

Put

$$\mathcal{K}_\alpha(q) = \begin{cases} \{\mathcal{U} \in \mathbf{R}^3 : \ \mathbf{n}^\alpha . \mathcal{U} \ge 0\} \text{ if } f_\alpha(q) \ge 0 \\ \mathbf{R}^3 \text{ if } f_\alpha(q) < 0 \end{cases}$$

which may be called the set of the *right-admissible* values for the relative velocity of the two concerned bodies at the contact point.

Let us agree to say that a contact law such as (7) *is complete if it implies* $\mathcal{U}_\alpha \in \mathcal{K}_\alpha$ *in all circumstances and* $\mathcal{R}_\alpha=0$ *if* $\mathcal{U}_\alpha \in$ interior $\mathcal{K}_\alpha$.

Through Eq. (4), the fact that a contact law is complete is translated into a property of the element u of $\mathbf{R}^n$. One source of interest of the concept lies in the following kinematical result (Moreau, 1988).

If, at some initial instant, all impenetrability inequalities $f_\alpha \le 0$ *are fullfilled and if all contacts possibly occurring in the sequel are governed by complete contact laws, the function* $t \to q(t)$ *constructed by integration of the velocity function* $t \to u(t)$, *fullfills the impenetrability inequalities in all the subsequent motion.*

The Contact Dynamics method rests on time-discretization. The essential stage is to compute a discrete approximant of the velocity function $t \to u(t)$, to which the position function $t \to q(t)$ is simply connected, step by step, through integration. Starting from the value obtained for u at the beginning of an interval of the discretization, one has to assess the value at the endpoint. Discretizing the dynamical equation (3) amounts to express the momentum balance of the system over the interval, without explicitely referring to accelerations. If the contact forces which appear in this balance of momentum verify complete contact laws, the above kinematical result makes that the impenetrability inequalities (1) are automatically taken care of.

Furthermore, the use of complete contact laws secures the correct handling of the possibility of some contact to get loose. Such is in fact the case for contact α if computation yields $\mathcal{U}_\alpha \in$ interior $\mathcal{K}_\alpha$ (i.e. $\mathbf{n}_\alpha . \mathcal{U} > 0$) hence $\mathcal{R}_\alpha=0$. Clearly also the complete contact law assumption implies that $\mathcal{R}_\alpha=0$ when $f_\alpha(q)<0$, since $\mathcal{K}_\alpha(q)$ equals the whole of $\mathbf{R}^3$ in that case.

4. BASIC CONTACT DYNAMICS ALGORITHM

Let $[t_I, t_F]$, $t_F = t_I + h$, denote an interval of the time discretization. Starting with q_I, u_I, the approximate values of q and u at time t_I, the objective is to calculate q_F, u_F, the approximate values at the endpoint t_F of this interval.

Identification of contacts

By using $q_M = q_I + h\, u_I /2$ as test position, the set of the contacts to be treated as active in the considered step is estimated to be

$$J = \{\alpha : f_\alpha(q_M) \ge 0\}$$

Discretization of the equation of Dynamics

Equation (3) may be discretized in the form

$$A(q_M)(u_F - u_I) = h\,F(t_M, q_M, u_I) + \sum_{\beta\in J} G_\beta^*(q_M)\mathcal{S}\beta$$

where $\mathcal{S}\beta \in \mathbf{R}^3$ denotes the impulsion at contact β, i.e. the integral of $\mathcal{R}\beta$ over the time interval. In short

$$u_F = u_I + hA^{-1}F + A^{-1}\sum_{\beta\in J} G_\beta^*\mathcal{S}\beta \qquad (8)$$

Contact laws

In contact laws such as (7), which are assumed satisfied all over the time interval, the time-dependent quantities $\mathcal{U}_\alpha$, $\mathcal{R}\alpha$ have to be replaced by some values estimated to be typical of the interval. We choose here to have (7) hold between the impulsion $\mathcal{S}\alpha$ and the *final* value

$$\mathcal{U}_{\alpha F} = G_\alpha u_F \qquad .(9)$$

Such a choice makes the present time-discretization scheme belong to the *implicit* sort, at least in what concerns the velocity function u. As a justification of entering impulsions instead of forces in (7) one may observe that Coulomb law states a *positively homogeneous* relationship between its two vector arguments.

Final position

$$q_F = q_M + \tfrac{1}{2} h\,u_F$$

Iterative procedure

By combining Eqs. (7) and (9), one obtains

$$\forall \alpha\in J: \quad law_\alpha(q_M, G_\alpha u_F, \mathcal{S}\alpha) = true \qquad (10)$$

Solving the system of conditions (8) to (10) constitutes the heaviest part of the computation. Here is a relaxation technique, amounting to treat a succession of single-contact problems.

Let an estimated solution u_F^{esti}, $\mathcal{S}_{esti}^{\beta}$, β running through J, be obtained, with (8) satisfied (the starting guess may consist in taking zero values for the $\mathcal{S}\beta$ or in using values found at the preceding step). One attempts to construct a corrected estimate, say $u_F^{corr}, \mathcal{S}_{corr}^{\beta}$, by altering only $\mathcal{S}^{\alpha}$, i.e. $\mathcal{S}_{corr}^{\beta} = \mathcal{S}_{esti}^{\beta}$ for $\beta \neq \alpha$. The new estimate is astrained to satisfy (8), i.e. since the old estimate satisfies the same,

$$u_F^{corr} = u_F^{esti} + A^{-1}G_\alpha^*(\mathcal{S}_{corr}^{\alpha} - \mathcal{S}_{esti}^{\alpha}) \qquad (11)$$

and to satisfy (10), hence, after applying G_α to both members of (11),

$$\forall \alpha\in J: \; law_\alpha(G_\alpha u_F^{esti} + H_\alpha(\mathcal{S}_{corr}^{\alpha} - \mathcal{S}_{esti}^{\alpha}), \mathcal{S}_{corr}^{\alpha}) = true \qquad (12)$$

where

$$H_\alpha = G_\alpha A^{-1} G_\alpha^*$$

can be proved to be a symmetric positive definite 3×3 matrix.

Solving (12) with regard to the unknown $\mathcal{S}_{corr}^{\alpha}$ may, in some usual cases, be reduced to finding the zeros of a piecewise affine mapping. The above computation will then be iterated, with α ranging cyclically through J. The decision of stopping iterations may be taken on observing the magnitude of $\mathcal{S}_{corr}^{\alpha} - \mathcal{S}_{esti}^{\alpha}$ and this turns out to be equivalent to checking the precision at which each pair $\mathcal{U}_\alpha, \mathcal{S}\alpha$ satisfies the corresponding contact law. Observe that, provided this precision check is made, the operator H_α in (12) may be replaced by any other mapping with zero limit at the origin: this may be used in tricks for accelerating convergence.

Clearly, this algorithm tolerates a certain amount of violation of the impenetrability inequalities. By adjusting the step-length and the stopping criterium, one may keep these errors arbitrarily small and prevent their accumulation.

The iterated calculation is very simple, but needs to be repeated many times in case of numerous contacts. Since the balance of momentum (8) is only preserved from one iteration to the next through the conservation condition (11), one should think of the possible accumulation of arithmetic errors. For safety, one may refresh u_F^{esti} from time to time, by returning to (8) while keeping the approximate values obtained for $\mathcal{S}\beta$. This proves useful only for motions involving thousands of contacts.

Technically, let us also observe that in many usual applications, the n×n matrix A is constant and diagonal. G^α is a 3×n matrix, but only the elements corresponding to the two bodies involved in contact α are nonzero. So the treatment of large collections of bodies does not require the handling of large matrices.

The convergence of this algorithm and the existence of a solution to the problem it addresses has only been proved in special cases (Monteiro Marques, 1993). Uniqueness cannot be expected to hold in general, since the mechanical problem of determining the reactions in a closely packed collection of rigid bodies (for instance a wall made of rectangular blocks) is usually hyperstatic.

The algorithm still works in analysing *equilibrium* situations. The balance of applied forces and contact forces simply yields $u_F = u_I = 0$ in such a case.

5. COLLISIONS

The sudden occurrence of new contacts produces velocity changes in some parts. The less deformable the colliding bodies are, the more brutal the phenomenon should be, involving large values of the contact forces at the impact locus. If the deformability of the bodies is fully taken into account, the problem is however not fundamentally different from others in the Dynamics of Continua.

But if it has been decided to treat bodies as perfectly rigid, one has to face strict discontinuities in velocities, so that the smooth dynamic framework of Sec.2 cannot apply. In the majority of the papers devoted to the topic of 'Rigid Body Collisions' (Brach, 1991), it is attempted to formulate some *collision equations*, connecting the values of velocities after the collision to the values they had before it. Traditionally, the intense effects which take place during a collision are assumed localized in the vicinity of the impact locus. Then, a multiple scaling analysis of what happens in an 'infinitely small' domain, during the 'infinitely short' collisional episode allows one to take into account in more or less detail the material behaviour of the involved bodies.

Situations in which such a treatment is justified certainly exist, but in general the effect of a collision should not be local. For instance, material dissipation in the vicinity of the impact is not the only cause of the energy loss detected at the macroscopic level of observation. Even if the concerned bodies are assumed perfectly elastic, energy conservation cannot be expected. Disturbances are to propagate from the collision locus to the whole system and also, if the latter is linked with some external support, to the outside world.

After contact recedes, vibrations are likely to persist somewhere. At the macroscopic observation level, this does not contradict the rigidity assertion, but the energy involved in microscopic agitation may not be negligible.

Also as a consequence of global deformation, a collision may, at the microscopic time-scale, split into several separate contact episodes: an example of such a double bounce is calculated in closed form in (Timoshenko, 1948, Chap. 12). Finite element computation of the collision of two elastic bodies performed in our laboratory has shown the same. So the conception of a collision as consisting of a compression phase followed by a so-called restitution phase cannot be considered as general.

All this makes the outcome of a collision depend on many factors, in particular on the shape of the concerned bodies.

Still more severe difficulties arise from collisions being frequently *multiple*, i.e. several contact loci are involved at the same time. Such is the case if a colliding body is part of a cluster of objects already in contact. The propagation of disturbances into a cluster is a problem similar to that of sound in granular media.

One thus has to accept that any given model of collision can only have a limited scope. Every occasion of comparing its results with calibrated experiments should be seized, in order to estimate this scope as precisely as possible.

The algorithm described in Sec.4 is found to work consistently in the face of collisions, i.e. when the test position q_M reveals some contacts which were not in effect at the preceding step. There only happens that the impulsive term in Eq.(8) is no more the same order of magnitude as h. The elements u_I and u_F of $\mathbf{R}^n$ in this case are naturally interpreted as representing u^- and u^+, the values of the system velocity before and after the collision. Now, the decision of constructing a time-discretization algorithm of the implicit type has led us to connect by Eq.(10) each contact impulsion S^α with the *final* value u_F. Contact laws are assumed *complete* in the sense precised in Sec.3. Clearly then S^α can be nonzero only if $\mathbf{n}_\alpha . u_{\alpha F}=0$. Since $u_{\alpha F}$ represents the after-collision value of the relative velocity at contact α, this means that all the contacts which take an active part in the collision process satisfy the traditional condition of *zero restitution*. Using standard vocabulary, one may say that the algorithm, as we just described it, treat all contacts as *inelastic*.

There remains to explain how situations of nonzero restitution can be handled in this framework. As a pragmatic way of overcoming the difficulties mentioned in the foregoing, one may decide to admit that a contact law of the form (7) holds between the contact impulsion S^α and some formal velocity $u_{\alpha a}$ constructed as a *weighted average* of the (known) vector $u_{\alpha I}$ and the (unknown) vector $u_{\alpha F}$. The averaging may be effected separately for normal and tangential components, using different weights, say

$$u_{\alpha a}^N = \frac{\rho_\alpha}{1+\rho_\alpha} u_{\alpha I}^N + \frac{1}{1+\rho_\alpha} u_{\alpha F}^N \in \mathbf{R}$$

$$u_{\alpha a}^T = \frac{\tau_\alpha}{1+\tau_\alpha} u_{\alpha I}^T + \frac{1}{1+\tau_\alpha} u_{\alpha F}^T \in \mathbf{T}$$

The numbers ρ_α and τ_α, with values in the interval [0, 1], may be interpreted as the normal and the tangential *restitution coefficients*. It turns out that, in the special case of the collision of two spherical objects, the velocity jumps deduced from this averaging trick coincides with what has been proposed by several authors on the basis of a microscopic analysis and found in fairly good agreement with experimental data (Foerster et al., 1994). The case $\rho_\alpha=\tau_\alpha=1$, with friction made equal to zero corresponds to *energy preserving collisions* (Moreau, 1988).

In the case of a two-body collision, ρ_α coincides with the traditional restitution coefficient of Newton. But it should be stressed that the above formalism still yields plausible results in multiple collisions for which the Newton rule fails. For instance, what precedes applies to the problem, familiar in earthquake engineering literature, of the *rocking* of a slender rectangular block supported by a fixed horizontal plane. For simplicity, assume the lower edge slightly concave, so that contact can only occur through corners. Let the left corner remain in contact during an episode where the block rotates to the right, until the right corner collides. If at this time Newton's rule was applied to both contact points, this would yield zero normal velocity for the left corner, so no rocking could be found. On the contrary, our assumption of complete contact laws leaves the possibility for this normal velocity to be nonzero while the normal contact impulsion vanishes. The result of the calculation depends on the aspect ratio of the block, in conformance with common observation.

For large collections of bodies such as samples of granular materials, one may have to handle many collisions in a single interval of the time-discretization even though they theorically are not simultaneous. This is the key of the efficiency of the method in the simulation of granular materials. Comparison with physical experiments on convection currents and size-segregation in shaken granulates have produced quite satisfactory results (Moreau, 1995). By disclosing the values of experimentally inaccessible variables, the numerical simulation has permitted to understand the undelying mechanisms.

6. EXAMPLES OF APPLICATION

Ancient column submitted to ground shake

Figure 1 is part of a feasibility test of the Contact Dynamics method in the simulation of the *behaviour of ancient monuments under earthquakes*. The column is made of 10 cylindrical blocks superimposed without mortar an of a cylindrical capitel. Instead of a proper earthquake, the ground is only affected by an oblique elliptic oscillation. (Development supported by the Commission of the European Communities, Environment programme, Contract N° EV5V CT93 0300)

Force transmission in a sand pile

Some experimentalists have found that the pressure exerted upon the ground by a sand pile may present a local minimum near the center of the pile (Smid and Novosad, 1981). The explanation of this effect is currently a matter of discussion among theoreticians. Figure 2 was produced by a numerical simulation intended, as part of these investigations, to explore the transmission of forces inside a pile. For clarity, the model is two-dimensional (corresponding to experiments on a pile of 'Schneebeli material', i.e. a collection of cylindrical objects). The pile construction has been computed as the successive addition of 3000 circular grains of dispersed sizes, falling down at a velocity of 25 cm/s, with slight lateral random dispersion. After that, the pile has been left to relax under gravity (actually, the observation described below is practically the same if made during the pile construction).

Two vertical rectilinear cuts are shown. For each of them, one draws the resultant of the contact forces exerted by the grains with centers on one side upon those with centers on the other side. This

demonstrates that the section comprised between the two cuts has part of its weight counter-balanced by the action of the rest of the pile. However, with the present data, the effect is not large enough to produce a visible local minimum of ground pressure.

Two-dimensional stress-strain experiment

A two-dimensional model of granulate (or Schneebeli material), composed of 1024 randomly located disks as shown in Fig.3, is being deformed between four walls. Horizontal walls are vertically loaded with opposite constant forces (average pressure $\sigma 2$=100 KPa). Vertical walls are moved inward with constant horizontal velocity v=5 cm/s. The granulate reacts on them by a pressure $\sigma 1$. Gravity is neglected in the computation.

The evolution investigated here makes the deformation ε of the sample, with regard to its initial configuration, go from 2.625% to 5.250% .

On Fig.3 are shown the displacement fields of the disk centers during two subintervals. The field on the left corresponds to some initial episode, ε going from 2.625% to 3.280%, and the field on the right to a final episode, ε going from 4.595% to 5:250%. In the initial episode, a band of quasi-stagnation in the 45° direction is visible, while in the final episode a similar feature appears with -45° direction. *So a significant change in the deformation pattern occurs in the course of the investigated experiment.* Such *crises* affecting the slow deformation of compact granular materials have been reported by experimentalists (Meftah et al., 1993) and make laboratory tests harder to interpret.

The subsequent figures show how the crisis, which takes place near the middle of the evolution, reflects in various parameters.

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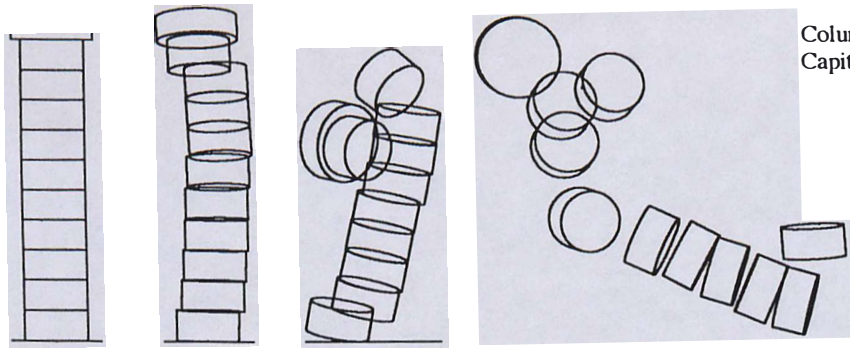
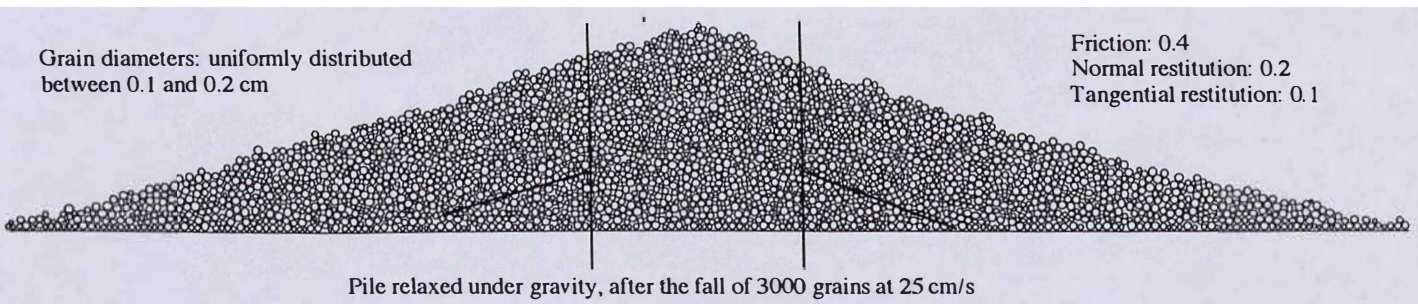


Column : 10 drums with diameter 2 m, height 1 m
Capitel: diameter 2.6 m, height 0.8 m

Friction between blocs : 0.4
Friction with ground : 0.7
Restitutions : 0

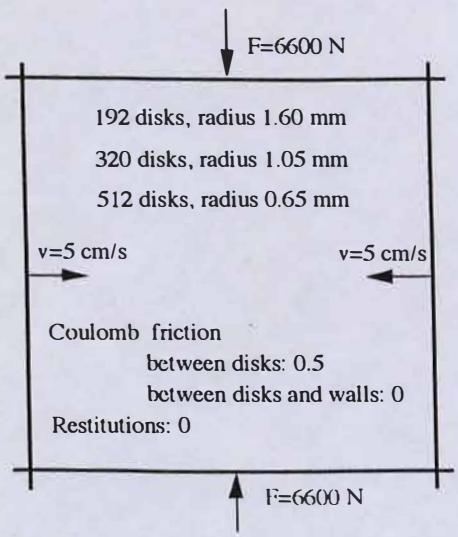
Ground shake : elliptic motion, period 1 s
Amplitude in x : 0.5 m (phase 60°)
in y : 0.15 m (phase 90°)
in z : 0.1 m (phase 0°)

Figure 1: Column on shaken ground



Pile relaxed under gravity, after the fall of 3000 grains at 25 cm/s

Figure 2: Internal forces in sand pile



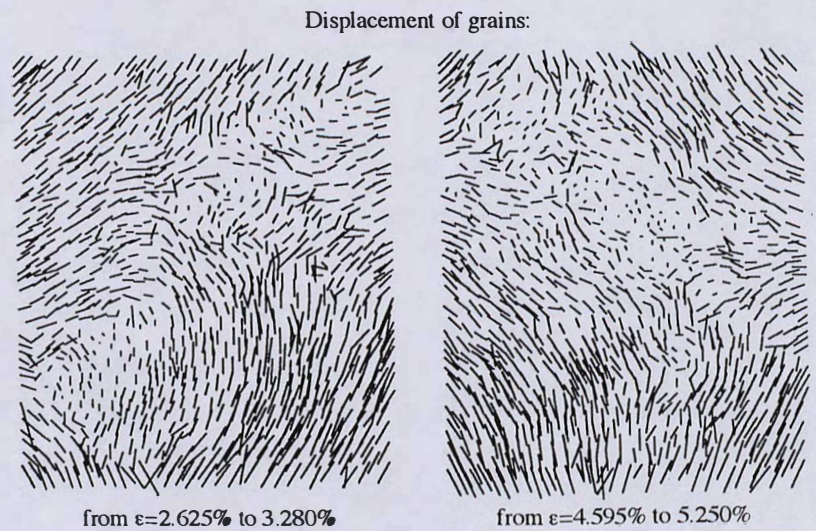


Figure 3: Stress-strain experiment

Figure 4: σ1/σ2 versus deformation

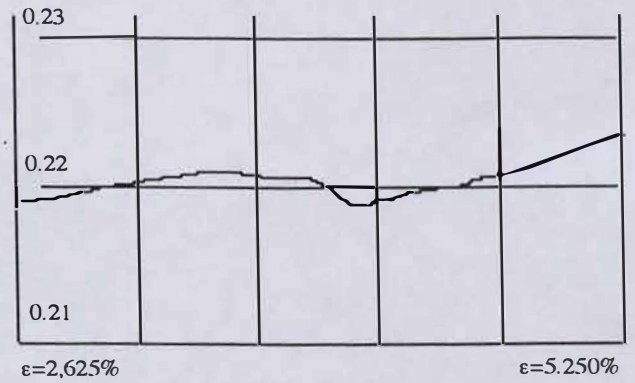


Figure 5: Void ratio versus deformation

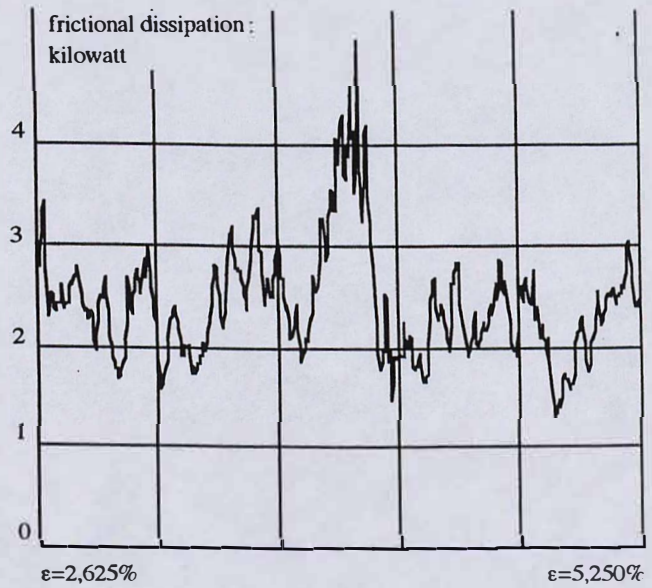


Figure 6: Frictional dissipation versus deformation

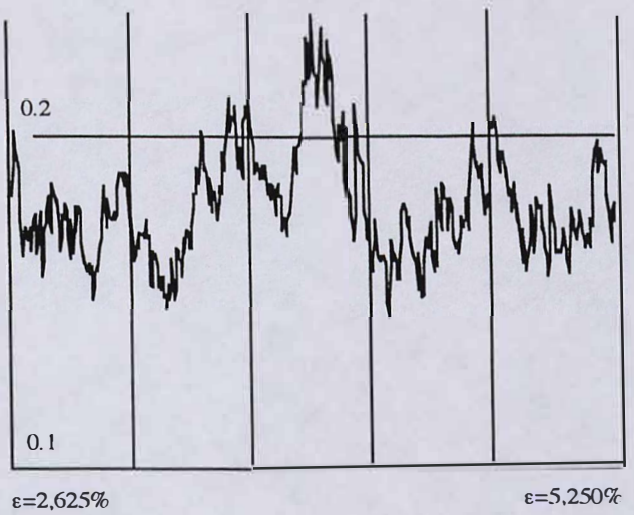


Figure 7: Ratio $\frac{\text{sliding contacts}}{\text{number of contacts}}$ versus deformation