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# Invariant cone and synchronization state stability of the mean field models

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## Abstract

In this article we prove the stability of mean field systems as the Winfree model in the synchronized state. The model is governed by the coupling strength parameter  $\kappa$  and the natural frequency of each oscillator. The stability is proved independently of the number of oscillators and the distribution of the natural frequencies. In order to prove the main result, we introduce the positive invariant cone and we start by studying the linearized system. The method can be applied to others mean field models as the Kuramoto model.

**Keywords:** Stability, coupled oscillators, mean field, interconnected systems, synchronization, Winfree model.

## 1 Introduction and Main result

In 1967 Winfree [8] proposed a mean field model describing the synchronization of a population of organisms or *oscillators* that interact simultaneously [1, 7]. The mean field models are used, for example, in the Neurosciences to study of neuronal synchronization in the brain [2, 3]. The Winfree model is given by the following differential equation

$$\dot{x}_i = \omega_i - \kappa\sigma(x)R(x_i), \quad t \geq 0, \quad x = (x_1, \dots, x_n), \quad (1)$$

where  $n \geq 1$  is the number of oscillators, the mean field part  $\sigma$  is defined by

$$\sigma(z) := \frac{1}{n} \sum_{j=1}^n P(z_j), \quad \forall z = (z_1, \dots, z_n) \in \mathbb{R}^n,$$

$P$  and  $R$  are  $C^2$   $2\pi$ -periodic functions,  $x_i(t)$  is the phase of the  $i$ -th oscillator, and  $x(t) = (x_1(t), \dots, x_n(t))$  is the global state of the system. The natural frequencies  $\omega_i$  are chosen in  $(1 - \gamma, 1 + \gamma)$  with  $\gamma \in (0, 1)$ . The coupling strength  $\kappa$  is taken in the interval  $(0, 1)$ . The stability studied in this article takes the meaning of Lyapunov's stability. In the second Section, we start by introducing the ingredients to study the Winfree model's stability. In the last Section we prove the Main result. Now, we define the invariant set and the positive stability. Let  $n \in \mathbb{N}^*$  and let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  function and denote  $dF$  its Jacobian such that

$$\max\left\{\sup_{z \in \mathbb{R}^n} \|F(z)\|, \sup_{z \in \mathbb{R}^n} \|dF(z)\|\right\} = r < \infty.$$

where  $\|\cdot\|$  is the usual matrix norm. Let  $\phi^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the flow of the following autonomous system

$$\dot{x} = F(x), \quad t \geq 0. \quad (2)$$

**Definition 1.** Consider the system (2) and let  $C \subset \mathbb{R}^n$  be an open set. We say that  $C$  is  $\phi^t$ -positive invariant if  $\phi^t(C) \subset C$  for all  $t \geq 0$ .

**Definition 2.** [Stability] Consider the system (2) and let  $C \subset \mathbb{R}^n$  be an open set. We say that the system(2) is  $C$ -positive stable if  $C$  is  $\phi^t$ -positive invariant and

$$\begin{aligned} &\exists \lambda > 0, \forall x \in C, \exists \delta > 0, \forall y \in C : \|x - y\| < \delta \\ &\implies \left\| \phi^t(x) - \phi^t(y) \right\| \leq \lambda \|x - y\|, \quad \forall t \geq 0. \end{aligned}$$

Denote  $\Phi^t$  the flow of the Winfree model (1). In [5] the existence of the synchronization state of the Winfree model is proved independently of the number  $n$  and the choice of natural frequencies. More precisely, if the Winfree model (1) satisfies the synchronization hypothesis

$$\text{H} : \int_0^{2\pi} \frac{P(s)R'(s)}{1 - \kappa P(s)R(s)} ds > 0, \quad \forall \kappa \in (0, \kappa_*),$$

where the *locking bifurcation critical* parameter  $\kappa_*$  is defined by

$$\kappa_* := \sup \left\{ \kappa > 0 : 1 - \kappa P(z)R(z) > 0, \quad \forall z \in \mathbb{R} \right\},$$

then there exists an open set  $U$  in the space of parameters  $(\gamma, \kappa) \in (0, 1) \times (0, \kappa_*)$ , independent of  $n$ , containing in its closure  $\{0\} \times [0, \kappa_*]$ , such that for every  $n \geq 1$  and every parameter  $(\gamma, \kappa) \in U$  there exist  $D \in (0, 1)$  and a  $\Phi^t$ -positive invariant open set  $C_{\gamma, \kappa}^n$  independent of choice of natural frequencies  $(\omega_i)_{i=1}^n$  and of the form,

$$C_{\gamma, \kappa}^n := \left\{ X = (x_i)_{i=1}^n \in \mathbb{R}^n : \max_{i,j} |x_j - x_i| < \Delta_{\gamma, \kappa} \left( \frac{1}{n} \sum_{i=1}^n x_i \right) \right\} \quad (3)$$

where  $\Delta_{\gamma, \kappa} : \mathbb{R} \rightarrow (0, 1)$  is  $C^2$   $2\pi$ -periodic function and solution of

$$\begin{aligned} \frac{d}{ds} \Delta_{\gamma, \kappa}(s) &= \alpha(\gamma, \kappa, D) - \frac{\kappa P(s) R'(s)}{1 - \kappa P(s) R(s)} \Delta_{\gamma, \kappa}(s), \quad (4) \\ \alpha(\gamma, \kappa, D) &:= \frac{2\gamma + \tilde{C}\kappa D^2}{1 - \kappa/\kappa_*} + \frac{(2\gamma + \tilde{C}\kappa D^2 + \tilde{C}\kappa D)(\gamma + \tilde{C}\kappa D)}{(1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_*)(1 - \kappa/\kappa_*)}, \\ \tilde{C} &:= 2 \sum_{0 \leq i, j \leq 2} \left\| P^{(i)} \right\|_{\infty} \left\| R^{(j)} \right\|_{\infty}. \end{aligned}$$

The periodic function  $\Delta_{\gamma, \kappa}$  satisfies  $\max_{s \in [0, 2\pi]} \Delta_{\gamma, \kappa}(s) < D$ . Further, for all  $(\gamma, \kappa) \in U$ ,  $x(0) \in C_{\gamma, \kappa}^n$  and  $t \geq 0$  we have

$$1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_* > 0 \text{ and } |1 - \kappa P(\mu_x(t)) R(\mu_x(t)) - \dot{\mu}_x(t)| < \gamma + \tilde{C}\kappa D, \quad (5)$$

where  $\mu_x(t) := \frac{1}{n} \sum_{j=1}^n x_j(t)$ . In the following Main result, we show that the Winfree model is  $C_{\gamma, \kappa}^n$ -positive stable when the synchronization hypothesis H is satisfied and when  $(\gamma, \kappa) \in U$ .

**Main result.** *Consider the Winfree model (1) and assume that the hypothesis H is satisfied. Then for every parameter  $(\gamma, \kappa) \in U$ , for every  $n \geq 1$  and every choice of natural frequencies  $(\omega_i)_{i=1}^n$  the Winfree model (1) is  $C_{\gamma, \kappa}^n$ -positive stable, where the open set  $C_{\gamma, \kappa}^n$  is defined by equation (3).*

Note that, the periodic system's stability as Winfree model implies  $\phi^t(x) = \rho t + p_x(t)$  where  $\rho \in \mathbb{R}^n$  is the *rotation vector* and  $p_x(t)$  is an *almost periodic function*; for more details, see [4, 6]. The existence of a synchronized periodic orbit on the torus with initial condition in  $C_{\gamma, \kappa}^n$  of the Winfree model is proved in [5].

## 2 Invariant cone and stability

In this Section we study the stability by introducing the positive invariant cone. Under some hypothesis, we prove that the system (2) is stable if its linearized admits such an invariant cone. The next independent Propositions

are the main ingredients to study the linearized Winfree model's stability. In this Section we consider the linear system

$$\dot{y} = A(x, t)y, \quad t \geq t_0, \quad (6)$$

where  $t_0 \in \mathbb{R}$  and  $A(x, t) := \{a_{i,j}(x, t)\}_{1 \leq i, j \leq n}$  is a continuous  $n \times n$  matrix function of  $x \in C \subset \mathbb{R}^n$  and the time  $t \geq t_0$ . For all  $x \in C$  let  $\Psi_x^t$  the flow of the system (6). Now, we define the *positive invariant cone*

**Definition 3.** Denote  $V_+ := \{(z_1, \dots, z_n) \in \mathbb{R}^n : z_i \geq 0, \forall i = 1, \dots, n\}$  and consider the linear system (6). We say that the cone  $V_+$  is  $\Psi_C^{t-t_0}$ -positive invariant if there exists a *control time*  $\delta \geq 0$  such that

$$\forall x \in C, \exists t_x \in [t_0, t_0 + \delta] : \Psi_x^{t-t_x}(V_+) \subset V_+, \quad \forall t \geq t_x.$$

The stability of the linear system (6) in the following Definition is equivalent to the stability of Definition (2); just the linear system is not autonomous.

**Definition 4.** Consider the linear system (6) and let  $\Psi_x^t$  be its flow. We say that the system(2) is *linearly C-positive stable* if

$$\exists \lambda > 0, \forall x \in C : \left\| \Psi_x^t(z) \right\| \leq \lambda \|z\|, \quad \forall z \in \mathbb{R}^n, \forall t \geq 0.$$

In the next proposition, we state the stability of class of nonlinear systems using the invariant cone  $V_+$ .

**Proposition 5.** Consider the system (2) and put  $F := (f_1, \dots, f_n)$ . Suppose that there exists a  $\phi^t$ -positive invariant open set  $C \subset \mathbb{R}^n$  and there exists  $\alpha > 0$  such that

$$f_i(\phi^t(x)) \geq \alpha, \quad \forall x \in C, \forall t \geq 0, \forall i \in \{1, \dots, n\}.$$

Let  $x \in C$  and let  $\Psi_x^t$  be the flow of the linearized system

$$\dot{y} = dF(\phi^t(x))y, \quad t \geq 0. \quad (7)$$

Suppose that the cone  $V_+$  is  $\Psi_C^t$ -positive invariant, then the system (2) is  $C$ -positive stable.

To prove the Proposition 5 we use the next Lemma which gives a sufficient condition of the stability of the system (2) as defined in Definition 2.

**Lemma 6.** *We consider the system (2). Suppose that there exists an open  $\phi^t$ -positive invariant set  $C \subset \mathbb{R}^n$  such that the linear system*

$$\dot{y} = dF(\phi^t(x))y, \quad t \geq 0, \quad (8)$$

*is linearly  $C$ -positive stable. Then the system (2) is  $C$ -positive stable.*

*Proof.* The system (2) can be written as

$$\frac{d}{dt}d\phi^t(x) = dF(\phi^t(x))d\phi^t(x),$$

Since the linear system (8) is linearly  $C$ -positive stable. Then

$$\exists \lambda > 0, \forall x \in C : \left\| d\phi^t(x) \right\| \leq \lambda \left\| d\phi^0(x) \right\| = \lambda, \quad \forall t \geq 0. \quad (9)$$

Let  $(z_1, z_2) \in C \times C$  such that  $z(s) := (1-s)z_2 + sz_1 \in C$  for all  $s \in [0, 1]$ .

$$\begin{aligned} \implies \left\| \phi^t(z_1) - \phi^t(z_2) \right\| &= \left\| \int_0^1 \frac{d}{ds} \phi^t(z(s)) ds \right\| = \left\| \int_0^1 d\phi^t(z(s)) \frac{dz(s)}{ds} ds \right\|, \\ &= \left\| \int_0^1 d\phi^t(z(s))(z_1 - z_2) ds \right\| \leq \sup_{s \in [0,1]} \left\| d\phi^t(z(s))(z_1 - z_2) \right\|. \end{aligned}$$

Finally, use the fact  $z(s) \in C$  and use equation (9) to obtain

$$\left\| \phi^t(z_1) - \phi^t(z_2) \right\| \leq \lambda \|z_1 - z_2\|, \quad \forall t \geq 0,$$

which implies that the system (2) is  $C$ -positive stable.  $\square$

*Proof of Proposition 5.* By hypotheses  $V_+$  is  $\Psi_C^t$ -positive invariant. Then there exists a control time  $\delta \geq 0$  such that for all  $x \in C$  there exists  $t_x \geq [0, \delta]$  such that

$$z \in V_+ \implies \Psi_x^{t-t_x}(z) \in V_+, \quad \forall t \geq t_x.$$

Let  $\alpha > 0$  satisfying the hypotheses of the present Proposition and put  $\eta := \frac{1}{\alpha}$ . Let  $x \in C$ ,  $y \in \mathbb{R}^n$  and denote

$$z^+ := \eta \|y\| F(\phi^{t_x}(x)) + y \quad \text{and} \quad z^- := \eta \|y\| F(\phi^{t_x}(x)) - y.$$

On the one hand,  $z^+ := (z_1^+, \dots, z_n^+) \in V_+$  and  $z^- := (z_1^-, \dots, z_n^-) \in V_+$ . In fact,

$$\min_{1 \leq i \leq n} \{z_i^-, z_i^+\} \geq \eta \|y\| \min_{1 \leq i \leq n} \left\{ \inf_{x \in C} f_i(\phi^{t_x}(x)) \right\} - \|y\| \geq (\eta\alpha - 1) \|y\| = 0.$$

On the other hand,  $\frac{d}{dt}\phi^t(x)$  is solution of the linearized system (7) of initial condition  $\frac{d}{dt}\phi^{t_x}(x) = F(\phi^{t_x}(x))$  then  $\Psi_x^{t-t_x}(F(\phi^{t_x}(x))) = \frac{d}{dt}\phi^t(x) = F(\phi^t(x))$ . Since  $V_+$  is  $\Psi_C^t$ -positive invariant, we obtain

$$\begin{aligned}\eta\|y\| F(\phi^t(x)) + \Psi_x^{t-t_x}(y) &= \Psi_x^{t-t_x}(z^+) \in V_+, \quad \forall t \geq t_x, \\ \eta\|y\| F(\phi^t(x)) - \Psi_x^{t-t_x}(y) &= \Psi_x^{t-t_x}(z^-) \in V_+, \quad \forall t \geq t_x.\end{aligned}$$

Knowing that  $\max\{\|F(\phi^t(x))\|, \|dF(\phi^t(x))\|\} \leq r$  we obtain

$$\begin{aligned}\|\Psi_x^{t-t_x}(y)\| &\leq \eta r \|y\|, \quad \forall t \geq \delta \text{ and } \|\Psi_x^t(y)\| \leq \exp(r\delta)\|y\|, \quad \forall t \in [0, \delta]. \\ \implies \|\Psi_x^t(y)\| &\leq \lambda \|y\|, \quad \forall x \in C, \quad \forall t \geq 0,\end{aligned}$$

where  $\lambda := \eta r \exp(r\delta)$ . The linearized system (7) is linearly  $C$ -positive stable. Lemma 6 implies that (2) is  $C$ -positive stable.  $\square$

We have shown in the previous Proposition that the system is stable when the cone  $V_+$  is invariant under the flow of its linearized. We show in the next Proposition one case of a linear systems such that the cone  $V_+$  is invariant under the flow. The linear systems in the next Proposition play the role of the linear system (7) of Proposition 5.

**Proposition 7.** *Consider the linear system (6). Suppose that there exist a continuous 1-periodic functions  $p, \beta : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\beta$  is a positive function and such that for all  $t \geq t_0$*

$$\max_{1 \leq i, j, l, k \leq n} \sup_{x \in C} \left\{ p(t) - a_{i,j}(x, t), \left| a_{i,j}(x, t) - a_{k,l}(x, t) \right| \right\} < \beta(t). \quad (10)$$

*Suppose that there exist  $d > 0$  and a positive 1-periodic function  $\theta$  such that*

$$\dot{\theta} = -np(t)\theta + 2nd\beta(t) \quad \text{and} \quad \theta(t) < d, \quad \forall t \geq t_0. \quad (11)$$

*Then the cone  $V_+$  is  $\Psi_C^{t-t_0}$ -positive invariant of control time  $\delta = 1$ .*

*Proof.* Let be  $\theta$  be the positive 1-periodic solution of Equation (11). Let be  $t_* \in [t_0, t_0 + 1]$  such that

$$\max_{t \in [0, 1]} \theta(t) = \theta(t_*). \quad (12)$$

Let be  $x \in C$  and recall that  $\Psi_x^t = (\Psi_{1,x}^t, \dots, \Psi_{n,x}^t)$  is the flow of the system (6). Let  $\mathring{V}_+$  be the interior of the set  $V_+$  and let be  $y \in \mathring{V}_+$ . Put  $\tilde{\mu}(z) = \sum_j^n z_j$  for all  $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ . By continuity,

$$\exists t > t_* : \quad \Psi_x^{s-t_*}(y) \in \mathring{V}_+, \quad \forall s \in [t_*, t].$$

Let define

$$T := \sup \left\{ t > t_*, \Psi_x^{s-t_*}(y) \in \overset{\circ}{V}_+, \quad \forall s \in (t_*, t) \right\}.$$

By uniqueness of solution  $\tilde{\mu}(\Psi_x^{s-t_*}(y)) > 0$  for all  $s \in [t_*, T]$ . The proposition is proved if we prove that  $T = +\infty$ . By contradiction, suppose that  $T < +\infty$ , then

$$\Psi_x^{T-t_*}(y) \notin \overset{\circ}{V}_+. \quad (13)$$

Use Equation (6), we obtain for all  $1 \leq i \leq n$  and  $t \in [t_*, T)$  the following equations

$$\begin{aligned} \frac{d}{dt} \Psi_{i,x}^{t-t_*}(y) &= a_i(t) \tilde{\mu}(\Psi_x^{t-t_*}(y)), \quad \text{where } a_i(t) := \frac{\sum_{j=1}^n a_{i,j}(x, t) \Psi_{j,x}^{t-t_*}(y)}{\tilde{\mu}(\Psi_x^{t-t_*}(y))}. \\ \frac{d}{dt} \Psi_{i,x}^{t-t_*}(y) &= a_i(t) \exp \left( \int_{t_*}^t a(s) ds \right) \tilde{\mu}(y), \quad \text{where } a := \sum_{j=1}^n a_j. \end{aligned}$$

By hypothesis (10), we have  $|na_i(s) - a(s)| < n\beta(s)$  and  $\theta(t_*) < d$ , therefore

$$\frac{n\theta(t_*)}{\tilde{\mu}(y)} \exp \left( - \int_{t_*}^T a(\zeta) d\zeta \right) \left( \Psi_{i,x}^{T-t_*}(y) - y_i \right) \quad (14)$$

$$= \theta(t_*) \int_{t_*}^T [a(s) + (na_i(s) - a(s))] \exp \left( - \int_s^T a(\zeta) d\zeta \right) ds > \theta(t_*) - z(T)$$

$$z(t) := \exp \left( - \int_{t_*}^t a(\zeta) d\zeta \right) \theta(t_*) + \int_{t_*}^t nd\beta(s) \exp \left( - \int_s^t a(\zeta) d\zeta \right) ds, \quad t \in [t_*, T].$$

$$\implies \dot{z}(t) := -a(t)z(t) + nd\beta(t), \quad z(t_*) = \theta(t_*), \quad t \in [t_*, T].$$

To obtain  $\Psi_x^{T-t_*}(y) \in \overset{\circ}{V}_+$  and get a contradiction with (13), it is sufficient to prove that  $z(T) \leq \theta(t_*)$ , knowing that  $y_i > 0$ . For that, we use the comparison principle of differential equations. By Equation (12), the strategy is to prove that  $z(T) \leq \theta(T)$ . Since  $\theta(t_*)$  is a maximum of  $\theta$  then  $\dot{\theta}(t_*) = 0$ . By definition of the function  $a(t)$  and by hypothesis (10), we have  $-a(t) < -np(t) + n\beta(t)$  for all  $[t_*, T)$  and  $\theta(t_*) < d$ , therefore

$$\begin{aligned} \dot{z}(t_*) &:= -a(t_*)z(t_*) + n\beta(t_*) < -np(t_*)z(t_*) + n\beta(t_*)z(t_*) + dn\beta(t_*) \\ &= -np(t_*)\theta(t_*) + n\beta(t_*)\theta(t_*) + dn\beta(t_*) < -np(t_*)\theta(t_*) + 2dn\beta(t_*) \\ &= \dot{\theta}(t_*) = 0. \end{aligned}$$

Then there exists  $\epsilon > 0$  such that  $z(t) < \theta(t)$  for all  $t \in (t_*, t_* + \epsilon)$ . Let define

$$\tilde{T} := \sup \left\{ t > t_*, \quad z(s) < \theta(s), \quad \forall s \in (t_*, t) \right\}.$$



Proving that  $\tilde{T} \geq T$ . By contradiction, if  $\tilde{T} < T$ , then  $z(\tilde{T}) = \theta(\tilde{T})$ . That implies

$$\begin{aligned} \dot{z}(\tilde{T}) &= -a(\tilde{T})z(\tilde{T}) + nd\beta(\tilde{T}) = -a(\tilde{T})\theta(\tilde{T}) + nd\beta(\tilde{T}) \\ &< -np(\tilde{T})\theta(\tilde{T}) + 2nd\beta(\tilde{T}) = \dot{\theta}(\tilde{T}). \end{aligned}$$

There exists  $t < \tilde{T}$  close enough to  $\tilde{T}$  such that  $z(t) > \theta(t)$ . We have obtained a contradiction. Then  $\tilde{T} \geq T$  and  $z(\tilde{T}) < \theta(\tilde{T}) \leq \theta(t_*)$ . By Equation (14), we deduce that  $\Psi_x^{T-t_*}(y) \in \mathring{V}_+$ . Contradiction with the definition of the point  $T$ . We have  $\Psi_x^{t-t_*}(y) \in \mathring{V}_+$  for all  $t \geq t_*$  and all  $x \in C$ . By continuity of the flow, we have then proved that  $\Psi_x^{t-t_*}(y) \in V_+$  for all  $y \in V_+$ , all  $t \geq t_*$  and all  $x \in C$ . In the other word, we have proved that the cone  $V_+$  is  $\Psi_C^{t-t_0}$ -positive invariant.  $\square$

### 3 Proof of the Main result

We prove in this Section the Main result of Section 1. For this, we consider the Winfree model (1) and recall that  $\Phi^t$  is its flow. Without loss of generality we put  $x = (x_1, \dots, x_n) := \Phi^t(x)$ . As is stated in the Main result, we will focus on the case when the Winfree model satisfies the hypothesis H . To prove the Main result, for  $(\gamma, \kappa) \in U$  and  $x \in C_{\gamma, \kappa}^n$  (as defined in equation(3)) we consider the *linearized Winfree model*

$$\frac{d}{dt}y(t) = d\mathcal{W}(\Phi^t(x))y(t), \quad t \geq 0, \quad y = (y_1, \dots, y_n), \quad (15)$$

where  $\mathcal{W}(z) := -\kappa\sigma(z)(R(z_1), \dots, R(z_n))$  for all  $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ . The next Lemma shows that the linearized Winfree model (15) can be written as the linear system of Proposition 7. In the Lemma (9), we prove then the cone  $V_+$  is positive invariant under the flow of the linearized Winfree model. Finally, we use Proposition 5 to prove the Main result.

**Lemma 8.** *Let  $n \in \mathbb{N}^*$  and assume that the Winfree model (1) satisfies the hypothesis H. Then for all  $(\gamma, \kappa) \in U$  and all  $x \in C_{\gamma, \kappa}^n$  there exist an homeomorphism  $t \rightarrow s := \mu_x(t)$  and a change of variables of the form*

$$\tilde{y}_i(s) := \exp\left(\int_{s_0}^s \frac{\kappa P(\zeta)R'(\zeta)}{1 - \kappa P(\zeta)R(\zeta)} d\zeta\right) y_i(\tau_x(s)), \quad s \geq s_0 := \mu_x(0),$$

where  $s \rightarrow t := \tau_x(s)$  is the inverse function of  $\mu_x$  such that the linearized Winfree model (15) is equivalent to

$$\frac{d}{ds}\tilde{y}(s) = \tilde{A}(x, s)\tilde{y}(s), \quad s \geq s_0, \quad \tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n), \quad (16)$$

and such that the matrix function  $\tilde{A}(x, s) = \{\tilde{a}_{i,j}(x, s)\}_{1 \leq i, j \leq n}$  satisfies

$$\max_{1 \leq i, j, l, k \leq n} \sup_{x \in C} \left\{ p(s) - \tilde{a}_{i,j}(x, s), \left| \tilde{a}_{i,j}(x, s) - \tilde{a}_{k,l}(x, s) \right| \right\} < \beta(s), \quad s \geq s_0,$$

$$p(s) := -\frac{1}{n} \frac{\kappa P'(s)R(s)}{1 - \kappa P(s)R(s)}, \quad \beta(s) := \frac{1 - \kappa/\kappa_*}{1 - \kappa P(s)R(s)} \frac{\alpha(\gamma, \kappa, D)}{2nD}$$

where  $D$  and  $\alpha(\gamma, \kappa, D)$  are defined in Equation (4).

*Proof.* Let  $(\gamma, \kappa) \in U$ ,  $x \in C_{\gamma, \kappa}^n$  and consider the linearized Winfree model (15). Let  $A(z) := \{a_{i,j}(z)\}_{1 \leq i, j \leq n}$  be the matrix function defined by

$$a_{i,j}(z) := -\frac{1}{n} \kappa P'(z_j)R(z_i), \quad \forall z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

The Jacobian  $dW(z) = \{w_{i,j}(z)\}_{1 \leq i, j \leq n}$ ,  $z \in \mathbb{R}^n$ , of the field  $W$  is defined by

$$w_{ii}(z) := -\kappa \sigma(z)R'(z_i) + a_{i,i}(z) \text{ and } w_{ij}(z) := a_{i,j}(z), \quad i \neq j.$$

Denote  $\mu_x(t) := \frac{1}{n} \sum_{j=1}^n x_j(t)$ . Since  $(\gamma, \kappa) \in U$  and by equation (5)

$$0 < 1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_* < \dot{\mu}_x(t), \quad \forall t \geq 0.$$

Consider the change of variable  $t \rightarrow s := \mu_x(t)$  and put  $w(s) := y(\tau_x(s))$  where  $\tau_x$  is the inverse function of  $\mu_x$ . We have  $\frac{d}{dt}y(t) = \frac{d}{ds}w(s)\dot{\mu}_x$ . Put  $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^n$ . In terms of the new variables the linearized Winfree model (15) becomes

$$\frac{d}{ds}w(s) = \left[ \frac{dW(s\mathbb{1})}{1 - \kappa P(s)R(s)} + H(x, s) \right] w(s), \quad s \geq s_0 := \mu_x(0), \quad (17)$$

where the matrix function  $H(x, s) = \{h_{i,j}(x, s)\}_{1 \leq i, j \leq n}$  is defined by

$$H(x, s) := \frac{dW(\Phi^{\tau_x(s)}(x)) - dW(s\mathbb{1})}{1 - \kappa P(s)R(s)} - \frac{dW(\Phi^{\tau_x(s)}(x))}{\dot{\mu}_x} \left( \frac{\dot{\mu}_x}{1 - \kappa P(s)R(s)} - 1 \right).$$

Now consider the change of variables

$$\tilde{y}_i(s) := \exp \left( \int_{s_0}^s \frac{\kappa P(\zeta)R'(\zeta)}{1 - \kappa P(\zeta)R(\zeta)} d\zeta \right) w_i(s), \quad s \geq s_0.$$

In terms of the new variables the linear system (17) and so the linearized Winfree model (15) are equivalent to the linear system (16) where the matrix function  $\tilde{A}(x, s) = \{\tilde{a}_{i,j}(x, s)\}_{1 \leq i, j \leq n}$  is defined by

$$\tilde{A}(x, s) := \frac{A(s\mathbb{1})}{1 - \kappa P(s)R(s)} + H(x, s), \quad s \geq s_0, \quad 1 \leq i, j \leq n.$$

Recall that  $\max_{1 \leq i \leq N} |x_i - \mu_x| \leq \Delta_{\gamma, \kappa}(\mu_x) < D$ . Use Equation (5) and the Mean Value Theorem to get for all  $1 \leq i, j, k, l \leq n$  the following inequality

$$|\tilde{a}_{i,j}(x, s) - \tilde{a}_{k,l}(x, s)| < \frac{1}{n} \frac{\frac{\tilde{C}}{2} \kappa D}{1 - \kappa P(s)R(s)} + \frac{1}{n} \frac{\frac{\tilde{C}}{2} \kappa (\gamma + \frac{1}{2} \tilde{C} \kappa D)}{\dot{\mu}_x (1 - \kappa P(s)R(s))} < \beta(s),$$

$$|h_{i,j}(x, s)| < \frac{1}{n} \frac{\frac{\tilde{C}}{2} \kappa D}{1 - \kappa P(s)R(s)} + \frac{1}{n} \frac{\frac{\tilde{C}}{2} \kappa (\gamma + \frac{1}{2} \tilde{C} \kappa D)}{\dot{\mu}_x (1 - \kappa P(s)R(s))} < \beta(s).$$

where

$$\beta(s) := \frac{1 - \kappa/\kappa_*}{1 - \kappa P(s)R(s)} \frac{\alpha(\gamma, \kappa, D)}{2nD}.$$

□

**Lemma 9.** *Let  $n \in \mathbb{N}^*$  and assume that the Winfree model (1) satisfies the hypothesis H. Let  $(\gamma, \kappa) \in U$ ,  $x \in C_{\gamma, \kappa}^n$  and let  $\Psi_x^t$  be the flow of the linearized Winfree model (15). Then the cone  $V_+$  is  $\Psi_{C_{\gamma, \kappa}^n}^t$ -positive invariant.*

*Proof.* The strategy is to use Proposition 7. Let  $(\gamma, \kappa) \in U$  and  $x \in C_{\gamma, \kappa}^n$ . Lemma 8 implies that the linearized Winfree model (15) is equivalent to the linear system (16) such that the matrix function  $\tilde{A}(x, s) = \{\tilde{a}_{i,j}(x, s)\}_{1 \leq i, j \leq n}$  satisfies

$$\max_{1 \leq i, j, l, k \leq n} \sup_{x \in C} \left\{ p(t) - \tilde{a}_{i,j}(x, s), \left| \tilde{a}_{i,j}(x, s) - \tilde{a}_{k,l}(x, s) \right| \right\} < \beta(s), \quad s \geq s_0,$$

$$p(s) := -\frac{1}{n} \frac{\kappa P'(s)R(s)}{1 - \kappa P(s)R(s)}, \quad \beta(s) := \frac{1 - \kappa/\kappa_*}{1 - \kappa P(s)R(s)} \frac{\alpha(\gamma, \kappa, D)}{2nD}.$$

Let  $\Delta_{\gamma, \kappa}(s)$  the positive  $2\pi$ -periodic function as defined in Equation (4). By definition of the locking bifurcation critical parameter  $\kappa_*$ , we have

$$\min_{s \in [0, 2\pi]} 1 - \kappa P(s)R(s) = 1 - \kappa/\kappa_*. \quad (18)$$

Let  $\theta$  be the positive  $2\pi$ -periodic function defined by

$$\theta(s) := \frac{1 - \kappa/\kappa_*}{1 - \kappa P(s)R(s)} \Delta_{\gamma, \kappa}(s).$$

Knowing that  $\|\theta\|_\infty \leq \|\Delta_{\gamma, \kappa}\|_\infty < D$ , by Equations (4) and (18), we find

$$\dot{\theta} = -np(s)\theta + \frac{1 - \kappa/\kappa_*}{1 - \kappa P(s)R(s)} \alpha(\gamma, \kappa, D) = -np(s)\theta + 2nD\beta(s).$$

Since  $\theta$  is a positive  $2\pi$ -periodic function and satisfies  $\|\theta\|_\infty < D$ , by Proposition 7 the cone  $V_+$  is  $\tilde{\Psi}_{C_{\gamma,\kappa}^n}^{s-s_0}$ -positive invariant, where  $\tilde{\Psi}_x^s$  is the flow of the linear system (16) and where  $s_0 = \mu_x(0)$ . Further, the control time satisfies  $\delta = 2\pi$ . In the other word

$$\forall x \in C_{\gamma,\kappa}^n, \exists s_x \in [s_0, s_0 + 2\pi] : z \in V_+, \implies \tilde{\Psi}_x^{s-s_x}(z) \in V_+, \forall s \geq s_x.$$

Recall that  $\tau_x$  is the inverse function of  $\mu_x$ . We deduce that for all  $x \in C_{\gamma,\kappa}^n$  there exists  $\tau_x(s_x) \in [0, \tau_x(2\pi)]$  such that the flows family  $\{\Psi_x^t\}_{x \in C_{\gamma,\kappa}^n}$  of the linearized Winfree model (15) satisfies

$$z \in V_+, \implies \Psi_x^{t-\tau_x(s_x)}(z) \in V_+, \forall t \geq \tau_x(s_x).$$

Finally, we have

$$\tau_x(2\pi)(1-\gamma-\tilde{C}\kappa D-\kappa/\kappa_*) < \int_0^{\tau_x(2\pi)} \dot{\mu}_x(t) dt = 2\pi \implies \sup_{x \in C_{\gamma,\kappa}^n} \tau_x(2\pi) < +\infty.$$

□

Now, we prove the Main result of Section 1.

*Proof of the Main result.* We use in this proof the Proposition 5. On the one hand, by definition of the Winfree model (1)

$$\dot{x}_i = \omega_i - \kappa\sigma(x)R(x_i) = \omega_i - \kappa P(x_i)R(x_i) + \frac{\kappa}{n} \sum_{j=1}^n (P(x_i) - P(x_j))R(x_i).$$

Using the Mean Value Theorem, we obtain for all  $(\gamma, \kappa) \in U$  and  $1 \leq i \leq n$ ,

$$0 < 1 - \gamma - \tilde{C}\kappa D - \kappa/\kappa_* < \omega_i - \kappa\sigma(x)R(x_i) < 1 + \gamma + \kappa\tilde{C}, \forall x \in C_{\gamma,\kappa}^n, \forall t \geq 0.$$

On the other hand, by Lemma 9, for all  $(\gamma, \kappa) \in U$  the cone  $V_+$  is  $\Psi_{C_{\gamma,\kappa}^n}^t$ -positive invariant. Proposition 5 implies that the Winfree model (1) is  $C_{\gamma,\kappa}^n$ -positive stable for all  $(\gamma, \kappa) \in U$ . □

## 4 Conclusion

We have studied the stability of the Winfree model in its synchronized state. The method based on the study of the linearized system's stability. We have introduced the  $V_+$  invariant cone. The invariance of the cone  $V_+$  is a sufficient condition of the stability of the linear system (Proposition 5). To obtain the stability of the Winfree model we have proved that  $V_+$  is invariant under the flow of its linearized system.

## References

- [1] J.T. Ariaratnam and S.H. Strogatz, Phase diagram for the Winfree model of coupled nonlinear oscillators. *Phys. Rev. Lett.* 86, 4278 (2001).
- [2] D. Cumin and C.P. Unsworth, Generalising the Kuramoto model for the study of neuronal synchronization in the brain. *Physica D: Nonlinear Phenomena*, 2, 226, 181–196. (2007).
- [3] G. B. Ermentrout, M. Pascal and B. S. Gutkin, The effects of spike frequency adaptation and negative feedback on the synchronization of neural oscillators. *Neural Computation*, 6, 13, 1285-1310.(2001)
- [4] A.M. Fink, *Almost periodic functions*. L.N.M. Springer (1974)
- [5] W. Oukil, A. Kessi and Ph. Thieullen, Synchronization hypothesis in the Winfree model. *Dynamical Systems*, Taylor & Francis. V 32, N 3, 326-339 (2017).
- [6] T. Saito, On dynamical systems in n-dimensional torus. *Proceedings of the Symposium on Differential Equations and Dynamical Systems*, Lecture Notes in Mathematics, Vol. 206, 18–19. (1971)
- [7] D.D. Quinn, R. H. Rand and S.H. Strogatz, Singular unlocking transition in the Winfree model of coupled oscillators. *Physical Review E* 75, 036218 (2007)
- [8] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators. *J. Theor. Biol.* 16 15-42.(1967)