

SADDLE HYPERBOLICITY IMPLIES HYPERBOLICITY FOR POLYNOMIAL AUTOMORPHISMS OF \mathbb{C}^2

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ABSTRACT. We prove that for a polynomial diffeomorphism of \mathbb{C}^2 , uniform hyperbolicity on the set of saddle periodic points implies that saddle points are dense in the Julia set. In particular f satisfies Smale's Axiom A on \mathbb{C}^2 .

1. INTRODUCTION

Let f be a polynomial automorphism of \mathbb{C}^2 with non-trivial dynamics. For such a dynamical system there are two natural definitions for the Julia set. The first one is in terms of normal families: $J = J^+ \cap J^-$ is the set of points at which neither $(f^n)_{n \geq 0}$ nor $(f^{-n})_{n \geq 0}$ is locally equicontinuous. The second one is the closure J^* of the set of saddle periodic orbits. The inclusion $J^* \subset J$ is obvious, and whether the reverse inclusion holds is one of the major open questions in higher dimensional holomorphic dynamics.

Following Bedford and Smillie [BS1], we say that f is *hyperbolic* if J is a hyperbolic set for f . Under this assumption we have a rather satisfactory understanding of the global dynamics of f . Indeed it was shown in [BS1] that under this assumption the forward and backward Julia sets J^+ and J^- (see §2.1 below for precise definitions) are laminated by stable and unstable manifolds, that the Fatou set is the union of finitely many cycles of attracting basins, that f satisfies Smale's Axiom A on \mathbb{C}^2 and finally that $J = J^*$. It was shown by Buzzard and Jenkins [BJ] that f is structurally stable on \mathbb{C}^2 . There are also tentative models for a description of the topological dynamics on J (see Ishii [I] for a survey).

On the other hand it is sometimes more natural to postulate that f is uniformly hyperbolic on J^* . One reason is that this information can be read off from the periodic points of f . This happens for instance in the study of the stability/bifurcation dichotomy for families of polynomial automorphisms [DL, BD]. The global consequences of hyperbolicity on J^* are then less easy to analyze, in particular it does not *a priori* imply a uniform laminar structure on J^\pm .

The main result of this paper is that these two notions actually coincide.

Main Theorem. *Let f be a polynomial automorphism of \mathbb{C}^2 with non-trivial dynamics. If f is hyperbolic on J^* , then $J = J^*$.*

In particular, if f is hyperbolic on J^* , then it is hyperbolic in the sense of [BS1].

Recall that the Jacobian $\text{Jac}(f)$ of a polynomial automorphism is a non-zero constant: thus f is dissipative when $|\text{Jac}(f)| < 1$ and conservative when $|\text{Jac}(f)| = 1$.

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This result was first announced in the dissipative case in [F], but the published proof is not correct¹, and it has remained an intriguing open problem since then. Recently, Guerini and Peters [GP] managed to establish the result under the more stringent assumption that f is *substantially dissipative*, that is $|\text{Jac}(f)| < d^{-2}$, where d is the dynamical degree (see §2.1 for this notion). Observe that only *quasi-hyperbolicity* on J^* is assumed in [GP] while our approach seems to require the full strength of hyperbolicity.

The proof of the main theorem starts with the dissipative case (Section 3). We assume by contradiction that f is dissipative, hyperbolic on J^* and that $J \neq J^*$. In a first stage we show that for some $p \in J^*$, J^- intersects $W^s(p)$ along a non-trivial relatively open subset, which is an unexpected property in the dissipative setting (for instance in the substantially dissipative case, the main point of [GP] is to show that $J^- \cap W^s(p)$ is totally disconnected). The main input here is the ergodic closing lemma that we obtained in a previous work [Du2]. In a second stage we use the results of [BS6] on the properties of stable slices of J^- together with some potential-theoretic ideas to actually derive a contradiction.

The conservative case is treated in Section 4 by a perturbative argument. If f is conservative and hyperbolic on J^* , we can find a holomorphic family (f_λ) with $f_0 = f$ containing dissipative parameters, on which J^* moves under a holomorphic motion. Again we assume that $J^*(f) \neq J(f)$, and use the extension properties of the holomorphic motion of J^* obtained in [DL] to derive a contradiction from the previously established dissipative case.

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2. PRELIMINARIES

In this section we recall some basic facts on the dynamics of polynomial automorphisms of \mathbb{C}^2 and hyperbolic dynamics, and establish a few preliminary results.

2.1. Vocabulary and basic facts. Let f be a polynomial diffeomorphism of \mathbb{C}^2 with non-trivial dynamics. This is the case exactly when the *dynamical degree* $\lim_{n \rightarrow \infty} (\deg(f^n))^{1/n}$ is larger than 1. By [FM] there exists a polynomial change of coordinates in which f is expressed as a composition of Hénon mappings $(z, w) \mapsto (p_i(z) + a_i w, a_i z)$. We fix such coordinates from now on. The degree of f is $d = \prod \deg(p_i) \geq 2$ and the relation $\deg(f^n) = d^n$ holds so that d coincides with the dynamical degree of the original map.

In these adapted coordinates, let

$$V_R^- = \{(z, w) \in \mathbb{C}^2, |w| \geq R, |z| < |w|\} \text{ and } V_R^+ = \{(z, w) \in \mathbb{C}^2, |z| \geq R, |w| < |z|\}.$$

We fix $R_0 > 0$ so large that for $R \geq R_0$ $f(V_R^+) \subset V_{2R}^+$ and $f^{-1}(V_R^-) \subset V_{2R}^-$. Hence the points of V_R^+ (resp. V_R^-) escape under forward (resp. backward) iteration. We denote by \mathbb{B} the bidisk $D(0, R_0)^2$. The non-wandering set of f is contained in \mathbb{B} .

An object (subset, current, or subvariety) in \mathbb{B} is said to be *vertical* (resp. *horizontal*) if its closure in $\overline{\mathbb{B}}$ is disjoint from $\{|z| = R_0\}$ (resp. $\{|w| = R_0\}$). A vertical subvariety has a *degree*, which is the number of intersection points with a generic horizontal line.

¹A first problem happens in the proof of [F, Thm 2], which corresponds to Step 1 in our proof. Indeed in the construction of the “queer” disk V , the sequence (y_n) is contained in $W^s(J^*)$ but not a priori in $W_{\text{loc}}^s(J^*)$, hence one cannot directly deduce that $G^+(y_n) \geq c$. Also, Lemma 6 is not correct: local product structure does not allow to transport whole components of $W^s(x) \cap J$ to components of $W^s(y) \cap J$ when x and y belong to the same global unstable manifold; in particular the boundedness of such a component is *not* an invariant property.

Here are some standard facts and notation (see e.g. [BS1, BS2, BLS]):

- K^\pm is the set of points with bounded forward orbits under $f^{\pm 1}$ and $K = K^+ \cap K^-$. Note that K^+ is vertical in \mathbb{B} and $f(\mathbb{B} \cap K^+) \subset K^+$. Similarly, K^- is horizontal and $f^{-1}(\mathbb{B} \cap K^-) \subset K^-$.
- The complement of K^+ is denoted by U^+ and the complement of K^- is U^- .
- $J^\pm = \partial K^\pm$ are the forward and backward Julia sets. If f is dissipative then $K^- = J^-$.
- $J = J^+ \cap J^-$ is the Julia set.
- $J^* \subset J$ is the closure of the set of saddle periodic points. It is also the support of the unique measure of maximal entropy $\log d$.

The *dynamical Green functions* G^\pm are defined by $G^\pm(z, w) = \lim d^{-n} \log^+ \|f^{\pm n}(z, w)\|$ (where $\log^+(x) = \max(\log(x), 0)$). These are non-negative continuous plurisubharmonic functions on \mathbb{C}^2 , such that $K^\pm = \{G^\pm = 0\}$ and G^\pm is pluriharmonic on $U^{+/-} := \{G^{+/-} > 0\}$. We let $T^\pm = dd^c G^\pm$. The maximum principle implies that $\text{Supp}(T^\pm) = J^\pm$.

The *restriction* $T^+|_D$ of T^+ to a complex submanifold D is a positive measure on D locally defined by $\Delta(G^+|_D)$ and since G^+ is continuous this measure coincides² with the wedge product $T^+ \wedge [D]$. A useful remark is that if x belongs to K^+ and $D \subset \mathbb{C}^2$ is a holomorphic disk through x along which G^+ is harmonic, then $D \subset K^+$ and $(f^n|_D)_{n \geq 1}$ is a normal family.

If p is a saddle periodic point or more generally if it belongs to a hyperbolic saddle set, it admits stable and unstable manifolds $W^{s/u}(p)$. Each of them is an immersed Riemann surface biholomorphic to \mathbb{C} and by [BS2, FS] $W^s(p)$ (resp. $W^u(p)$) is dense in J^+ (resp. J^-). A key point in the present paper is to analyse the topological properties of sets of the form $K^- \cap W^s(p)$ or $K^+ \cap W^u(p)$. Following [DL], we define the *intrinsic topology* to be the topology induced on a stable (resp. unstable) manifold by the biholomorphism $W^s \simeq \mathbb{C}$, and the corresponding concepts of boundary, interior, etc. will be labelled with the subscript i : ∂_i , Int_i , etc.

The following basic lemma will be used several times.

Lemma 2.1 ([DL, Lemma 5.1]). *Let p be a saddle periodic point. Then the boundary of $W^s(p) \cap J^-$ relative to the intrinsic topology in $W^s(p)$ is contained in J^* .*

We denote by $W_{\mathbb{B}}^s(p)$ the connected component of $W^s(p) \cap \mathbb{B}$ containing p (and accordingly for W^u). Likewise, $W_\delta^s(p)$ is the connected component of $W^s(p) \cap B(p, \delta)$ containing p , and $W_{\text{loc}}^s(p)$ denotes an unspecified open neighborhood of p in $W^s(p)$.

By [BLS], the currents T^\pm have geometric structure, related to the decomposition of J^\pm into stable and unstable manifolds. By *lamination* by Riemann surfaces we mean a closed subset \mathcal{L} of some open set $\Omega \subset \mathbb{C}^2$ such that every $p \in \mathcal{L}$ admits a neighborhood B biholomorphic to a bidisk, such that in the corresponding coordinates, a neighborhood of p in \mathcal{L} is a union of disjoint graphs (that is, a holomorphic motion) over the first coordinate in B . A positive current S is *uniformly laminar* if there is a lamination of $\text{Supp}(S)$ by Riemann surfaces and in the corresponding local coordinates S is locally expressed as $\int [\Delta_a] d\nu(a)$. These disks will be said *subordinate* to S .

A holomorphic disk D is *subordinate* to T^+ if there exists a non-zero uniformly laminar current $S \leq T^+$ such that D is subordinate to S . By [Du1, Prop. 2.3], if p is any saddle point, then any relatively compact disk $D \subset W^s(p)$ is subordinate to T^+ .

²It is standard to define $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$. Accordingly, Δ here is $1/2\pi$ times the ordinary Laplacian.

2.2. Stable (dis)connectivity. It was shown in [BS6] that the connectivity properties of sets of the form $K^+ \cap W^u(p)$ (resp. $K^- \cap W^s(p)$) carry deep information on the geometry of the Julia set. We say that f is *stably connected* if $U^+ \cap W^s(p)$ is simply connected for some (and then any) saddle point p , and *stably disconnected* otherwise. Equivalently, f is stably disconnected if for some saddle point p , $W^s(p) \cap K^-$ admits a compact component relative to the intrinsic topology. This actually implies the stronger property that most components of $W^s(p) \cap K^-$ are points (see the proof of Lemma 3.2 below for more details).

By [BS6, Cor. 7.4], a dissipative polynomial automorphism is always stably disconnected. It was observed in [Du1] that this implies a strong non-extremality property for the current $T^+|_{\mathbb{B}}$: there exists a decomposition $T^+|_{\mathbb{B}} = \sum_{k=1}^{\infty} T_k^+$ where T_k^+ is an average of integration currents over a family of disjoint vertical disks of degree k (see [Du1, Thm 2.4]).

Lemma 2.2. *Let f be dissipative and hyperbolic on J^* and let $q \in J^*$. Then q belongs to the support of $T^+|_{W^u(q)}$ and for $(T^+|_{W^u(q)})$ -a.e. q' near q , $W_{\mathbb{B}}^s(q')$ is a vertical manifold of finite degree in \mathbb{B} .*

Proof. The first assertion easily follows from the fact that $(f^n)_{n \geq 0}$ cannot be a normal family on $W_{\text{loc}}^u(q)$ (see [BLS, Lemma 2.8]). The second one is a consequence of [Du1, Thm 2.4]. Indeed as observed above $T^+|_{\mathbb{B}}$ admits a decomposition $T^+|_{\mathbb{B}} = \sum_{k=1}^{\infty} T_k^+$ where T_k^+ is made of vertical disks of degree k . Thus $T^+|_{W_{\text{loc}}^u(q)} = T^+ \wedge [W_{\text{loc}}^u(q)] = \sum_k T_k^+ \wedge [W_{\text{loc}}^u(q)]$. Now if Γ is a leaf of some T_k^+ intersecting $W_{\text{loc}}^u(q)$ at q' , then since $W_{\text{loc}}^u(q)$ is subordinate to T^- , q' belongs to J^* and Γ is a manifold through q' along which forward iterates are bounded, hence $\Gamma = W_{\mathbb{B}}^s(q')$. \square

2.3. Hyperbolicity and local product structure. Let us recall some generalities from hyperbolic dynamics, specialized to our situation. A *(saddle) hyperbolic set* for f is a compact invariant set $\Lambda \subset \mathbb{C}^2$ such that $T\mathbb{C}^2|_{\Lambda}$ admits a hyperbolic splitting, i.e. $T\mathbb{C}^2|_{\Lambda} = E^s \oplus E^u$, where E^s and E^u are continuous line bundles such that E^s (resp. E^u) is uniformly contracted (resp. expanded) by df . Then there exists $\delta_1 > 0$ such that $W_{\delta_1}^s(\Lambda) := \bigcup_{p \in \Lambda} W_{\delta_1}^s(p)$ and $W_{\delta_1}^u(\Lambda) := \bigcup_{p \in \Lambda} W_{\delta_1}^u(p)$ form laminations in the δ_1 -neighborhood of Λ .

A hyperbolic set is *locally maximal* if there exists an open neighborhood \mathcal{N} of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}(\mathcal{N})$. It has *local product structure* if there exists $0 < \delta_2 \leq \delta_1$ such that if $p, q \in \Lambda$ are such that $d(p, q) < \delta_2$ then $W_{\delta_1}^s(p) \cap W_{\delta_1}^u(q)$ consists of exactly one point belonging to Λ . It turns out that these two properties are equivalent (see [Y, §4.1]).

We will use the following consequence of the shadowing lemma.

Proposition 2.3. *Let Λ be a compact locally maximal hyperbolic set for a polynomial diffeomorphism f of \mathbb{C}^2 . Then there exist positive constants η, α and A such that for every $n \geq 0$; if x is such that $\{x, \dots, f^n(x)\} \subset \Lambda_{\eta}$ then there exists $y \in \Lambda$ such that x is $Ae^{-\alpha n}$ -close to the local stable manifold of y .*

A similar result holds for negative iterates: if $\{f^{-n}(x), \dots, x\} \subset \Lambda_{\eta}$ then there exists $z \in \Lambda$ such that x is $Ae^{-\alpha n}$ -close to the local unstable manifold of z .

The following corollary is well-known.

Corollary 2.4. *If Λ is a compact locally maximal hyperbolic set, then*

$$W^s(\Lambda) := \left\{ x \in \mathbb{C}^2, f^n(x) \xrightarrow{n \rightarrow \infty} \Lambda \right\} = \bigcup_{p \in \Lambda} W^s(p) \text{ and similarly } W^u(\Lambda) = \bigcup_{p \in \Lambda} W^u(p).$$

Note however that $W^s(\Lambda)$, being an increasing union of laminations, doesn't need to have a lamination structure (this is already false when Λ is a hyperbolic fixed point).

Proof of Proposition 2.3 (sketch). This is very classical. Given an orbit segment $\{x, \dots, f^n(x)\}$ as in the statement of the proposition, let $y^{(0)}$ (resp. $y^{(n)}$) be a point in Λ such that $d(x, y^{(0)}) < \eta$ (resp. $d(x, y^{(n)}) < \eta$). Then define a η -pseudo-orbit $(y^{(k)})_{k \in \mathbb{Z}}$ as follows

$$y^{(k)} = \begin{cases} f^k(y^{(0)}) & \text{for } k < 0; \\ f^k(x) & \text{for } 0 \leq k \leq n; \\ f^{k-n}(y^{(n)}) & \text{for } k > n. \end{cases}$$

Then if η is small enough by local maximality and the shadowing lemma there exists a unique $y \in \Lambda$ such that for every $k \in \mathbb{Z}$, $d(f^k(y), y^{(k)}) < C\eta$ (where C is some constant depending on (f, Λ) , see [Y, §4.1]). In particular for $0 \leq k \leq n$ we have $d(f^k(x), f^k(y)) < C\eta$ and it follows from standard graph transform estimates that $d(x, W_\delta^s(y)) \leq Ae^{-\alpha n}$. \square

The next result is a simple application of the techniques of [BLS].

Proposition 2.5. *If J^* is hyperbolic then it has local product structure. Furthermore global stable and unstable manifolds intersect only in J^* :*

$$W^s(J^*) \cap W^u(J^*) = J^*.$$

Proof. Hyperbolicity implies that for some $\delta > 0$, if p and q are close enough, $W_\delta^s(p) \cap W_\delta^u(q)$ consists of a single point r . We have to show that $r \in J^*$. Indeed, $W_\delta^s(p)$ (resp. $W_\delta^u(q)$) is a disk subordinate to T^+ (resp. T^-) so there exists a non-trivial uniformly laminar current $S^+ \leq T^+$ (resp. $S^- \leq T^-$) with $W_\delta^s(p)$ (resp. $W_\delta^u(q)$) as a leaf. By [BLS, Lem. 8.2], S^+ and S^- have continuous potentials, so the wedge product $S^+ \wedge S^-$ is well defined, and geometric intersection theory of uniformly laminar currents [BLS, Lem. 8.3] implies that $r \in \text{Supp}(S^+ \wedge S^-)$. Since $S^+ \wedge S^- \leq T^+ \wedge T^-$ we conclude that $r \in J^*$.

The proof of the second assertion is similar. By local product structure,

$$W^s(J^*) = \bigcup_{n \geq 0} f^{-n}(W_\delta^s(J^*)),$$

hence if $r \in W^s(J^*)$, there exists $p \in J^*$ such that $r \in W^s(p)$ so r belongs to a disk subordinate to T^+ , and likewise $r \in W^u(q)$ so it belongs to a disk subordinate to T^+ . Observe that these two disks are distinct: indeed otherwise we would have $W^s(p) = W^u(q)$ which is impossible because $W^s(p) \cap W^u(q)$ is contained in K which is bounded in \mathbb{C}^2 . So r is an isolated intersection between $W^s(p)$ and $W^u(q)$ for the leafwise topology. If this intersection is transverse, we argue as above to conclude that r belongs to J^* . If it is a tangency, then by [BLS, Lem. 6.4] for $q' \in J^*$ close to q , we get transverse intersections between $W^s(p)$ and $W^u(q')$ close to r and conclude as in the transverse case. \square

2.4. Stability. A theory of stability and bifurcations for polynomial automorphisms of \mathbb{C}^2 was developed in [DL], centered on the notion of *weak J^* -stability*.

A *branched holomorphic motion* over a complex manifold Λ in \mathbb{C}^2 is a family of holomorphic graphs over Λ in $\Lambda \times \mathbb{C}^2$. It is a holomorphic motion (i.e. an unbranched branched holomorphic motion!) when these graphs are disjoint. A holomorphic family $(f_\lambda)_{\lambda \in \Lambda}$ of polynomial automorphisms of dynamical degree d is *weakly J^* -stable* if the sets $J^*(f_\lambda)$ move under a branched holomorphic motion, and *J^* -stable* if this motion is unbranched. Note that if f_{λ_0}

is uniformly hyperbolic on $J^*(f_{\lambda_0})$, then it is J^* -stable near λ_0 in any holomorphic family containing f_{λ_0} .

A number of properties of weakly J^* -stable families are established in [DL], including extension properties of the branched holomorphic motion of J^* to K (and more generally to $J^+ \cup J^-$), that will be used in Section 4. These properties hold under the standing assumption that the family (f_λ) is *substantial*³: this means that either all members of the family are dissipative, or that no relation of a certain form between multipliers of periodic points persistently holds in the parameter space Λ . Without entering into the details, let us just note that by [BHI, Thm 1.4] any open subset of the family of all polynomial automorphisms of dynamical degree d is substantial.

3. PROOF OF THE MAIN THEOREM: THE DISSIPATIVE CASE

The proof is by contradiction so assume that f is a dissipative polynomial diffeomorphism of \mathbb{C}^2 , that is uniformly hyperbolic on J^* , and that $J^* \subsetneq J$.

Step 1. *There exists $p \in J^*$ and a holomorphic disk $\Delta \subset W^s(p)$ such that $G^-|_\Delta \equiv 0$.*

The purpose of the remaining steps 2 and 3 will be to show that such a “queer” component of $W^s(p) \cap J^-$ actually does not exist.

Proof. We first claim that it is enough to show that there exists $p \in J^*$ and $q \in W^s(p)$ such that $q \in J \setminus J^*$. Indeed observe that $W^s(p) \cap J = W^s(p) \cap J^- = W^s(p) \cap K^-$. By Lemma 2.1 we have that $\partial_i(W^s(p) \cap J^-) \subset J^*$. Hence if q belongs to $W^s(p) \cap (J \setminus J^*)$, it belongs to the intrinsic interior $\text{Int}_i(W^s(p) \cap J^-)$, hence $G^- \equiv 0$ in a neighborhood of q in $W^s(p)$.

Let now $x \in J \setminus J^*$. By Corollary 2.4, if $\omega(x) \subset J^*$ then $x \in W^s(J^*)$ and if $\alpha(x) \subset J^*$ then $x \in W^u(J^*)$. Since $W^u(J^*) \cap W^s(J^*) = J^*$, we infer that either $\omega(x) \not\subset J^*$ or $\alpha(x) \not\subset J^*$. In either case we will show that both $W^s(J^*) \cap (J \setminus J^*)$ and $W^u(J^*) \cap (J \setminus J^*)$ are non-empty. Thus by symmetry it is enough to deal with the case where $\omega(x) \not\subset J^*$.

Choose η so small that Proposition 2.3 holds for J^* and $\omega(x)$ is not contained in $\overline{\mathcal{N}}$, where $\mathcal{N} := (J^*)_\eta$ is the η -neighborhood of J^* .

Consider the sequence of Cesarò averages $\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$. By the ergodic closing lemma of [Du2], every cluster value of the sequence (ν_n) is supported on J^* . It follows that the asymptotic proportion of iterates of x belonging to \mathcal{N} tends to 1, i.e.

$$\frac{1}{n} \# \left\{ 0 \leq k \leq n-1, f^k(x) \in \mathcal{N} \right\} \xrightarrow{n \rightarrow \infty} 1.$$

Indeed if a positive proportion of iterates stayed outside \mathcal{N} , any cluster limit of ν_n would have to give positive mass to \mathcal{N}^c .

We thus infer that there are arbitrary long strings $\{x_i, \dots, x_{i+n}\}$ in the orbit of x that are entirely contained in \mathcal{N} . Indeed, if on the contrary the length of such a string were uniformly bounded by some n_0 , then the density of iterates outside \mathcal{N} would be bounded below by $1/(n_0 + 1)$. Therefore for every n there exists i_n such that $\{x_{i_n}, \dots, x_{i_n+n}\} \subset \mathcal{N}$. Choose i_n to be minimal with this property. Since $\omega(x) \not\subset \mathcal{N}$, there exists $j > i$ such that $x_j \notin \mathcal{N}$. So finally for every n we can find $i_n < j_n$ such that $j_n - i_n \geq n$, $\{x_{i_n}, \dots, x_{j_n}\} \subset \mathcal{N}$, $x_{i_n-1} \notin \mathcal{N}$ and $x_{j_n+1} \notin \mathcal{N}$.

³There is an unfortunate terminological conflict here: this should not be confused with the notion of substantial dissipativity mentioned in the introduction.

Let p (resp. p') be a cluster value of (x_{i_n-1}) (resp. (x_{j_n+1})). The points p and p' belong to J because x does, but not to J^* because they lie outside \mathcal{N} . It follows from Proposition 2.3 that $p \in W^s(q)$ for some $q \in J^*$ and $p' \in W^u(q')$ for some $q' \in J^*$. The proof is complete. \square

Remark 3.1. If f is substantially dissipative i.e. $|\text{Jac}(f)| < \deg(f)^{-2}$, then the contradiction readily follows from this first step. Indeed Wiman's theorem together with uniform hyperbolicity imply that the vertical degree of components of stable manifolds in some large bidisk \mathbb{B} is uniformly bounded (see [GP, Prop. 4.2] or [LP, Lem. 5.1]), and it follows that $J^- \cap W^s(x)$ is totally disconnected for every x (see [Du1, Thm 2.10] or [GP, Thm. 4.3]), which contradicts the conclusion of Step 1.

In the second and third steps we do not use the assumption that $J \setminus J^* \neq \emptyset$.

Step 2. *For every $p \in J^*$, if Ω is a component of $\text{Int}_i(W^s(p) \cap J^-)$, then Ω is unbounded for the leafwise topology.*

Proof. Note first that by the maximum principle, any component of $\text{Int}_i(W^s(p) \cap J^-) = \text{Int}_i(W^s(p) \cap K^-)$ is simply connected, so Ω is a topological disk. Assume by contradiction that Ω is bounded for the leafwise topology. Then iterating forward a few times if needed, we can suppose that Ω is entirely contained in a local product structure box.

More precisely for small $\delta > 0$, we can fix holomorphic local coordinates (z, w) near p in which $p = (0, 0)$, $W_{2\delta}^s(p) = \{z = 0\}$ and $W_{2\delta}^u(p) = \{w = 0\}$, and assume $\bar{\Omega}$ is contained in $W_\delta^s(p)$. Note that by Lemma 2.1, $\partial_i \Omega \subset J^*$. We can assume that for every $q \in W_\delta^s(p) \cap J^*$, $W_{2\delta}^u(q)$ contains a graph over the disk $D(0, \delta)$ in the first coordinate, with slope bounded by $1/2$. Then if $|z_0| \leq \delta$, the holonomy h_{0,z_0}^u along local unstable leaves is well defined on $W_\delta^s(p) \cap J^*$ and maps $W_\delta^s(p) \cap J^*$ into $\{z = z_0\} \cap J^-$. This holonomy is a holomorphic motion so by Ślodkowski's theorem [S] it extends to a holomorphic motion of $W_\delta^s(p)$. In particular the motion of $\partial_i \Omega$ extends to a motion of Ω and it makes sense to speak about $h_{0,z_0}^u(\Omega)$. This is an open subset of $\{z = z_0\}$, which is topologically a disk and whose boundary is contained in J^- . Thus for every $n \geq 0$, $f^{-n}(\partial(h_{0,z_0}^u(\Omega)))$ is contained in \mathbb{B} and by the maximum principle, the same holds for $f^{-n}(h_{0,z_0}^u(\Omega))$.

Finally, $\mathcal{O} := \bigcup_{|z_0| < \delta} h_{0,z_0}^u(\Omega)$ is an open set whose negative iterates remain in \mathbb{B} , hence it is contained in the Fatou set of f^{-1} . But since f is dissipative, this Fatou set is empty, which is the desired contradiction. \square

Step 3. *The unstable holonomy preserves the decomposition*

$$W^s(p) = (W^s(p) \cap J^-) \sqcup (W^s(p) \cap U^-).$$

To make this statement precise, observe that for every $p \in J^*$, the components of the complement of $\partial_i(W^s(p) \cap J^-)$ in $W^s(p)$ can be divided into two types: components of $\text{Int}_i(W^s(p) \cap J^-)$ and components of $W^s(p) \cap U^-$ (note that since U^- is open in \mathbb{C}^2 , $W^s(p) \cap U^-$ is open for the intrinsic topology as well). Consider as above local coordinates (z, w) near p in which $p = (0, 0)$, $W_{2\delta}^s(p) = \{z = 0\}$ and $W_{2\delta}^u(p) = \{w = 0\}$. The unstable holonomy h_{0,z_0}^u is initially only defined for points of $W_\delta^s(p) \cap J^* = \partial_i(W_\delta^s(p) \cap J^-)$, however by Ślodkowski's theorem it can be extended to $W_\delta^s(p)$. By Step 2, components of $\text{Int}_i(W^s(p) \cap J^-)$ are leafwise unbounded so they cannot be contained in $W_{2\delta}^s(p)$. Obviously, the same holds for components of $W^s(p) \cap U^-$.

If q belongs to $J^* \cap W_\delta^u(p)$, the extended holonomy $h_{p,q}^u$ defines a homeomorphism $W_\delta^s(p) \rightarrow h_{p,q}^u(W_\delta^s(p))$. By local product structure this homeomorphism preserves J^* so any component of $W_\delta^s(p) \setminus J^*$ is mapped onto a component of $h_{p,q}^u(W_\delta^s(p) \setminus J^*)$, which is itself contained in a component of $W^s(q) \setminus J^*$. The claim of Step 3 is that the extended holonomy $h_{p,q}^u$ preserves the type of components.

Since it doesn't make sense to transport a whole leafwise unbounded component by unstable holonomy, to prove this assertion we need to find a criterion that recognizes the type of a component just from local topological properties near a point of its boundary. As already said the maximum principle implies that any component of $\text{Int}_1(W^s(p) \cap J^-)$ is simply connected. Thus Step 3 follows from:

Lemma 3.2. *If Ω is a component of $W^s(p) \cap U^-$, then Ω is not simply connected near any point of $\partial\Omega$, more precisely: if $q \in \partial\Omega$ and N is any neighborhood of q , there is a loop in $N \cap \Omega$, homotopic to a point in N , and enclosing a component of $W^s(p) \cap J^-$*

Proof. Since f is dissipative by [BS6, Cor. 7.4] it is stably disconnected. It follows that almost every unstable component of K^+ is a point (see [BS6, Thm 7.1] and also [Du1, Thm 2.10]). More specifically, if μ is the unique measure of maximal entropy, then for μ -a.e. x , the measure $T^-|_{W^s(x)}$ (which is locally given by the wedge product $T^- \wedge [W^s(x)]$) gives full mass to the point components of $J^- \cap W^s(x)$. Obviously by Lemma 2.1 every such point component belongs to J^* so we can transport it to nearby stable manifolds by unstable holonomy. In addition, the measure $T^-|_{W^s(x)}$ is holonomy invariant (see [BS1, Thm 6.5] or [BLS, Thm 4.5]) so if x is such that $T^- \wedge [W_\delta^s(x)]$ gives full mass to point components, then the same holds for nearby x' . Thus we conclude that this property holds for every $p \in J^*$: $T^-|_{W^s(p)}$ gives full mass to the point components of $J^- \cap W^s(p)$.

Lemma 3.3. *Let $p \in J^*$ and Ω be a component of $W^s(p) \cap U^-$ such that Ω is locally simply connected near some $q \in \partial\Omega$. Then $\partial\Omega$ has positive $(T^-|_{W^s(p)})$ -measure.*

This proves Lemma 3.2. Indeed, assuming that Ω is locally simply connected near some $q \in \partial\Omega$, Lemma 3.3 asserts that $T^-|_{W^s(p)}$ carries positive mass on a non-trivial continuum so f cannot be stably disconnected. On the other hand f must be stably disconnected because it is dissipative, and we reach a contradiction. \square

The idea of Lemma 3.3 is as follows: every neighborhood of q in $\partial\Omega$ has positive harmonic measure when viewed from Ω . But the harmonic measure viewed from Ω is absolutely continuous with respect to $T^- \wedge [W^s(p)]$, hence the result. The formalization of this argument requires some elementary potential theory, for which we refer the reader to Doob's classical monograph⁴ [Do].

In particular we shall use the formalism of *sweeping* (or *balayage*). Let D be a smoothly bounded domain in \mathbb{C} , A a non-polar compact subset of D and ν a positive measure on D . The swept measure $\rho_{\nu,D,A}$ of ν on A is the distribution on A of the exit point of the Brownian motion in $D \setminus A$ whose starting point is distributed according to ν . In particular its mass is lower than that of ν since a positive proportion of Brownian paths escape from ∂D . If $G_{\nu,D}$ is the Green potential of ν in D , that is the unique negative subharmonic function on D such that $G_{\nu,D}|_{\partial D} = 0$ and $\Delta G_{\nu,D} = \nu$, then the swept measure of ν on A is $\Delta R_{\nu,D,A}$, where

$$(1) \quad R_{\nu,D,A}(z) = \sup \{u(z), u \leq 0 \text{ subharmonic on } D \text{ and } u \leq G_{\nu,D} \text{ on } A\}$$

⁴Note that Doob works with superharmonic functions so all inequalities have to be reversed.

(see Sections 1.III.4, 1.X and 2.IX.14 in [Do]). If ν and ν' have their supports disjoint from A , then the corresponding swept measures are mutually absolutely continuous (as follows from instance from Theorem 1.X.2 in [Do]).

Proof. We first claim that we can shift q slightly so that the assumptions of the lemma hold and in addition $W_{\mathbb{B}}^s(q')$ is of bounded vertical degree. Indeed q belongs to J^* and for $q' \in W_{\delta}^u(q) \cap J^*$, there is a component of $W_{\delta}^s(q') \setminus J^*$ corresponding to Ω under unstable holonomy, which is locally simply connected near q' . Since G^- is continuous, if q' is close enough to q , it takes positive values on that component, so we infer that the property that Ω is a component of U^- is open. Now by Lemma 2.2, for $(T^+|_{W_{\delta}^u})$ -a.e. q' , $W_{\mathbb{B}}^s(q')$ is a vertical manifold in \mathbb{B} of finite degree which establishes our claim. Without loss of generality rename q' into q . For every $g_0 < \min_{\partial\mathbb{B}} G^-$, the component of $\{G^- < g_0\}$ containing q in $W^s(q)$ is relatively compact for the intrinsic topology. We fix such a g_0 which is not a critical value of G^- and let D be the corresponding component, which is a smoothly bounded topological disk. From now on we work exclusively in D .

By assumption there is a neighborhood N of q in D and a component U of $\{G^- > 0\} \cap N$ that is simply connected. We have to show that $\partial U \cap N$ has positive mass relative to ΔG^- . We choose N to be closed so that $\partial U \cap N$ is compact. First, observe that for every $z_0 \in U \cap N$, the probability that the Brownian motion issued from z_0 hits $\partial U \cap N$ before leaving U is positive. Therefore the swept measure $\rho_{\delta_{z_0}, D, \partial U \cap N}$ has positive mass and to prove the lemma it is enough to show that is absolutely continuous with respect to ΔG^- . Recall that the measure class of the swept measure does not depend on the starting point so we can replace δ_{z_0} by an arbitrary positive measure on $D \setminus (D \cap K^-)$. Let $0 < g_1 < g_0$ and $\mu_{g_1} := \Delta(\max(G^-, g_1))$ be the natural measure induced by G^- on the level set $\{G^- = g_1\}$. We choose μ_{g_1} for the initial distribution of Brownian motion. Since $G^- \equiv g_0$ on ∂D , the Green function $G_{\Delta G^-, D}$ of the restriction of ΔG^- to D is equal to $G^- - g_0$, and likewise

$$G_{\mu_{g_1}, D} = \max(G^-, g_1) - g_0.$$

Thus from (1) we get that

$$\begin{aligned} R_{\mu_{g_1}, D, K^- \cap D} &= \sup \{u(z), u \text{ s.h.} \leq 0 \text{ on } D \text{ and } u \leq G_{\mu_{g_1}, D} \text{ on } K^- \cap D\} \\ &= \sup \{u(z), u \text{ s.h.} \leq 0 \text{ on } D \text{ and } u \leq g_1 - g_0 \text{ on } K^- \cap D\} \\ &= |g_1 - g_0| \frac{G^- - g_0}{g_0} \end{aligned}$$

and finally

$$\rho_{\mu_{g_1}, D, K^- \cap D} = \frac{g_0 - g_1}{g_0} \Delta G^-.$$

The proof is complete. \square

Step 4. Conclusion.

We just have to assemble the three previous steps. Assume as before by contradiction that f is dissipative, uniformly hyperbolic on J^* and $J^* \subsetneq J$. Then by Step 1 there exists $p \in J^*$ and a “queer” component Ω of $W^s(p) \setminus J^*$ along which $G^- \equiv 0$. Pick $q \in \partial\Omega$. By Lemma 2.1, $q \in J^*$ so we can follow $\Omega \cap W_{\delta}^s(q)$ using the holonomy along local unstable manifolds. Then for $q' \in W_{\text{loc}}^u(q)$ near q , the holonomy image $h_{q, q'}(\Omega \cap W_{\delta}^s(q))$ is contained in a queer component of $W^s(q') \setminus J^*$, which must be leafwise unbounded by Step 2. On the other hand

by Lemma 2.2, for generic q' in $W_{\text{loc}}^u(q)$ (relative to the transverse measure $T^+|_{W_{\text{loc}}^u(q)}$) $W_{\mathbb{B}}^s(q')$ is of bounded degree, in particular any component of $K^- \cap W_{\mathbb{B}}^s(q')$ is leafwise bounded. This contradiction finishes the proof. \square

4. PROOF OF THE MAIN THEOREM: THE CONSERVATIVE CASE

Again the proof is by contradiction, so assume that f is a conservative polynomial automorphism of \mathbb{C}^2 such that J^* is a hyperbolic set and $J^* \subsetneq J$. We will use a perturbative argument and the dissipative case of the theorem to reach a contradiction.

Assume that f is written as a product of Hénon mappings $f = h_1 \circ \dots \circ h_k$ and let $(f_\lambda)_{\lambda \in B}$ be a parameterization of a neighborhood of f in the space of such products, that is, the space of coefficients of the h_i , and such that $f_0 = f$. We can assume that B is a ball in \mathbb{C}^N for some N . Since $\lambda \mapsto \text{Jac}(f_\lambda)$ is an open map, there exist parameters arbitrary close to 0 for which f_λ is dissipative. As already said, by [BHI, Thm 1.4] there is no persistent relation between multipliers of periodic orbits so the family is substantial in the sense of [DL].

Since f_0 is hyperbolic on $J^*(f_0)$ the family (f_λ) is J^* -stable in a neighborhood of the origin, that is, $J^*(f_\lambda)$ moves under a holomorphic motion. Reducing the parameter space we can assume that J^* is hyperbolic throughout B . Pick a point $p = p(0) \in J(f_0) \setminus J^*(f_0)$. It was shown in [DL, Thm 5.12] that in a (weakly) J^* -stable family, the motion of J^* extends to a branched holomorphic motion of K . Thus there exists a holomorphic continuation $p(\lambda)$ of $p(0)$ such that for every $\lambda \in B$, $p(\lambda)$ belongs to $K(f_\lambda)$. Furthermore for every $\lambda \in B$, $p(\lambda)$ is disjoint from $J^*(f_\lambda)$. Indeed if for some $\lambda_0 \in B$ we had $p(\lambda_0) \in J^*(f_{\lambda_0})$, then by [DL, Lem 4.10] $p(\lambda)$ would have to coincide throughout the family (f_λ) with the natural continuation of $p(\lambda_0)$ as a point of the hyperbolic set J^* , which is not the case since $p(0) \notin J^*(f_0)$.

Let now $\lambda_1 \in B$ be such that f_{λ_1} is dissipative. Then by the first part of the proof $J(f_{\lambda_1}) = J^*(f_{\lambda_1})$, and $K(f_{\lambda_1}) \setminus J(f_{\lambda_1})$ is non-empty since it contains $p(\lambda_1)$. For a dissipative hyperbolic map

$$K \setminus J = (K^+ \cap J^-) \setminus (J^+ \cap J^-) = \text{Int}(K^+) \cap J^-,$$

so we deduce that $\text{Int}(K^+(f_{\lambda_1}))$ is non-empty. By [BS1], $\text{Int}(K^+(f_{\lambda_1}))$ is a finite union of attracting basins of periodic sinks, therefore f_{λ_1} admits an attracting periodic point. On the other hand by [DL, Thm 4.2], periodic points stay of constant type in a J^* -stable family (this holds even in the presence of conservative maps, provided the family is substantial), so f_0 must have an attracting orbit, which is contradictory since it is conservative. The proof is complete. \square

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