SADDLE HYPERBOLICITY IMPLIES HYPERBOLICITY FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

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Abstract. We prove that for a polynomial diffeomorphism of $\mathbb{C}^2$, uniform hyperbolicity on the set of saddle periodic points implies that saddle points are dense in the Julia set. In particular $f$ satisfies Smale’s Axiom A on $\mathbb{C}^2$.

1. Introduction

Let $f$ be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. For such a dynamical system there are two natural definitions for the Julia set. The first one is in terms of normal families: $J = J^+ \cap J^-$ is the set of points at which which neither $(f^n)_{n>0}$ nor $(f^{-n})_{n>0}$ is locally equicontinuous. The second one is the closure $J^*$ of the set of saddle periodic orbits. The inclusion $J^* \subset J$ is obvious, and whether the reverse inclusion holds is one of the major open questions in higher dimensional holomorphic dynamics.

Following Bedford and Smillie [BS1], we say that $f$ is hyperbolic if $J$ is a hyperbolic set for $f$. Under this assumption we have a rather satisfactory understanding of the global dynamics of $f$. Indeed it was shown in [BS1] that under this assumption the forward and backward Julia sets $J^+$ and $J^-$ (see §2.1 below for precise definitions) are laminated by stable and unstable manifolds, that the Fatou set is the union of finitely many cycles of attracting basins, that $f$ satisfies Smale’s Axiom A on $\mathbb{C}^2$ and finally that $J = J^*$. It was shown by Buzzard and Jenkins [BJ] that $f$ is structurally stable on $\mathbb{C}^2$. There are also tentative models for a description of the topological dynamics on $J$ (see Ishii [I] for a survey).

On the other hand it is sometimes more natural to postulate that $f$ is uniformly hyperbolic on $J^*$. One reason is that this information can be read off from the periodic points of $f$. This happens for instance in the study of the stability/bifurcation dichotomy for families of polynomial automorphisms [DL, BD]. The global consequences of hyperbolicity on $J^*$ are then less easy to analyze, in particular it does not a priori imply a uniform laminar structure on $J^\pm$.

The main result of this paper is that these two notions actually coincide.

Main Theorem. Let $f$ be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. If $f$ is hyperbolic on $J^*$, then $J = J^*$.

In particular, if $f$ is hyperbolic on $J^*$, then it is hyperbolic in the sense of [BS1].

Recall that the Jacobian $\text{Jac}(f)$ of a polynomial automorphism is a non-zero constant: thus $f$ is dissipative when $|\text{Jac}(f)| < 1$ and conservative when $|\text{Jac}(f)| = 1$.

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This result was first announced in the dissipative case in [F], but the published proof is not correct\(^1\), and it has remained an intriguing open problem since then. Recently, Guerini and Peters [GP] managed to establish the result under the more stringent assumption that \( f \) is substantial dissipative, that is \(|\text{Jac}(f)| < d^{-2}\), where \( d \) is the dynamical degree (see §2.1 for this notion). Observe that only quasi-hyperbolicity on \( J^* \) is assumed in [GP] while our approach seems to require the full strength of hyperbolicity.

The proof of the main theorem starts with the dissipative case (Section 3). We assume by contradiction that \( f \) is dissipative, hyperbolic on \( J^* \) and that \( J \neq J^* \). In a first stage we show that for some \( p \in J^* \), \( J^- \) intersects \( W^s(p) \) along a non-trivial relatively open subset, which is an unexpected property in the dissipative setting (for instance in the substantially dissipative case, the main point of [GP] is to show that \( J^- \cap W^s(p) \) is totally disconnected).

The main input here is the ergodic closing lemma that we obtained in a previous work [Du2].

The conservative case is treated in Section 4 by a perturbative argument. If \( f \) is conservative and hyperbolic on \( J^* \), we can find a holomorphic family \((f_\lambda)\) with \( f_0 = f \) containing dissipative parameters, on which \( J^* \) moves under a holomorphic motion. Again we assume that \( J^*(f) \neq J(f) \), and use the extension properties of the holomorphic motion of \( J^* \) obtained in [DL] to derive a contradiction from the previously established dissipative case.

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2. Preliminaries

In this section we recall some basic facts on the dynamics of polynomial automorphisms of \( \mathbb{C}^2 \) and hyperbolic dynamics, and establish a few preliminary results.

2.1. Vocabulary and basic facts. Let \( f \) be a polynomial diffeomorphism of \( \mathbb{C}^2 \) with non-trivial dynamics. This is the case exactly when the dynamical degree \( \lim_{n \to \infty} (\deg(f^n))^{1/n} \) is larger than 1. By [FM] there exists a polynomial change of coordinates in which \( f \) is expressed as a composition of Hénon mappings \((z, w) \mapsto (p_1(z) + a_i w, a_i z)\). We fix such coordinates from now on. The degree of \( f \) is \( d = \prod \deg(p_i) \geq 2 \) and the relation \( \deg(f^n) = d^n \) holds so that \( d \) coincides with the dynamical degree of the original map.

In these adapted coordinates, let

\[ V_R^- = \{(z, w) \in \mathbb{C}^2, \ |w| \geq R, \ |z| < |w|\} \quad \text{and} \quad V_R^+ = \{(z, w) \in \mathbb{C}^2, \ |z| \geq R, \ |w| < |z|\}. \]

We fix \( R_0 > 0 \) so large that for \( R \geq R_0 \) \( f(V_R^-) \subset V_{2R}^+ \) and \( f^{-1}(V_R^-) \subset V_{2R}^- \). Hence the points of \( V_R^+ \) (resp. \( V_R^- \)) escape under forward (resp. backward) iteration. We denote by \( \mathbb{D} \) the bidisk \( D(0, R_0)^2 \). The non-wandering set of \( f \) is contained in \( \mathbb{D} \).

An object (subset, current, or subvariety) in \( \mathbb{D} \) is said to be vertical (resp. horizontal) if its closure in \( \overline{\mathbb{D}} \) is disjoint from \( \{|z| = R_0\} \) (resp. \( \{|w| = R_0\} \)). A vertical subvariety has a degree, which is the number of intersection points with a generic horizontal line.

\(^1\)A first problem happens in the proof of [F, Thm 2], which corresponds to Step 1 in our proof. Indeed in the construction of the “queer” disk \( V \), the sequence \((y_n)\) is contained in \( W^s(J^*) \) but not a priori in \( W^s_{loc}(J^*) \), hence one cannot directly deduce that \( G^\ast(y_n) \geq c \). Also, Lemma 6 is not correct: local product structure does not allow to transport whole components of \( W^s(J^*) \cap J \) to components of \( W^s(y) \cap J \) when \( x \) and \( y \) belong to the same global unstable manifold; in particular the boundedness of such a component is not an invariant property.
Here are some standard facts and notation (see e.g. [BS1, BS2, BLS]):

- $K^\pm$ is the set of points with bounded forward orbits under $f^\pm_1$ and $K = K^+ \cap K^-$. Note that $K^+$ is vertical in $\mathbb{B}$ and $f(\mathbb{B} \cap K^+) \subset K^+$. Similarly, $K^-$ is horizontal and $f^{-1}(\mathbb{B} \cap K^-) \subset K^-$. 
- The complement of $K^+$ is denoted by $U^+$ and the complement of $K^-$ is $U^-$. 
- $J^\pm = \partial K^\pm$ are the forward and backward Julia sets. If $f$ is dissipative then $K^- = J^-$. 
- $J = J^+ \cap J^-$ is the Julia set. 
- $J^* \subset J$ is the closure of the set of saddle periodic points. It is also the support of the unique measure of maximal entropy $\log d$.

The dynamical Green functions $G^\pm$ are defined by $G^\pm(z, w) = \lim d^{-n} \log^+ \|f^{\pm n}(z, w)\|$ (where $\log^+(x) = \max(\log(x), 0)$). These are non-negative continuous plurisubharmonic functions on $\mathbb{C}^2$, such that $K^\pm = \{G^\pm = 0\}$ and $G^\pm$ is pluriharmonic on $U^+/-$ := $\{G^+/− > 0\}$. We let $T^\pm = d\nu G^\pm$. The maximum principle implies that $\text{Supp}(T^\pm) = J^\pm$.

The restriction $T^+|_D$ of $T^+$ to a complex submanifold $D$ is a positive measure on $D$ locally defined by $\Delta(G^+|_D)$ and since $G^+$ is continuous this measure coincides\(^2\) with the wedge product $T^+ \wedge [D]$. A useful remark is that if $x$ belongs to $K^+$ and $D \subset \mathbb{C}^2$ is a holomorphic disk through $x$ along which $G^+$ is harmonic, then $D \subset K^+$ and $(f^n|_D)_{n \geq 1}$ is a normal family.

If $p$ is a saddle periodic point or more generally if it belongs to a hyperbolic saddle set, it admits stable and unstable manifolds $W^{s/u}(p)$. Each of them is an immersed Riemann surface biholomorphic to $\mathbb{C}$ and by [BS2, FS] $W^s(p)$ (resp. $W^u(p)$) is dense in $J^+$ (resp. $J^-$). A key point in the present paper is to analyse the topological properties of sets of the form $K^- \cap W^s(p)$ or $K^+ \cap W^u(p)$. Following [DL], we define the intrinsic topology to be the topology induced on a stable (resp. unstable) manifold by the biholomorphism $W^s \simeq \mathbb{C}$, and the corresponding concepts of boundary, interior, etc. will be labelled with the subscript $i$: $\partial_i$, $\text{Int}_i$, etc.

The following basic lemma will be used several times.

**Lemma 2.1** ([DL, Lemma 5.1]). Let $p$ be a saddle periodic point. Then the boundary of $W^s(p) \cap J^-$ relative to the intrinsic topology in $W^s(p)$ is contained in $J^*$.

We denote by $W^s_{\mathbb{B}}(p)$ the connected component of $W^s(p) \cap \mathbb{B}$ containing $p$ (and accordingly for $W^u$). Likewise, $W^s_i(p)$ is the connected component of $W^s(p) \cap B(p, \delta)$ containing $p$, and $W^s_{\text{loc}}(p)$ denotes an unspecified open neighborhood of $p$ in $W^s(p)$.

By [BLS], the currents $T^\pm$ have geometric structure, related to the decomposition of $J^\pm$ into stable and unstable manifolds. By lamination by Riemann surfaces we mean a closed subset $\mathcal{L}$ of some open set $\Omega \subset \mathbb{C}^2$ such that every $p \in \mathcal{L}$ admits a neighborhood $B$ biholomorphic to a bidisk, such that in the corresponding coordinates, a neighborhood of $p$ in $\mathcal{L}$ is a union of disjoint graphs (that is, a holomorphic motion) over the first coordinate in $B$. A positive current $S$ is uniformly laminar if there is a lamination of $\text{Supp}(S)$ by Riemann surfaces and in the corresponding local coordinates $S$ is locally expressed as $\int [\Delta_a] \, dv(a)$. These disks will be said subordinate to $S$.

A holomorphic disk $D$ is subordinate to $T^+$ if there exists a non-zero uniformly laminar current $S \leq T^+$ such that $D$ is subordinate to $S$. By [Du1, Prop. 2.3], if $p$ is any saddle point, then any relatively compact disk $D \subset W^s(p)$ is subordinate to $T^+$.

\(^2\)It is standard to define $d^e = \frac{1}{2\pi} (\mathcal{J} - \partial)$. Accordingly, $\Delta$ here is $1/2\pi$ times the ordinary Laplacian.
2.2. Stable (dis)connectivity. It was shown in [BS6] that the connectivity properties of sets of the form $K^+ \cap W^u(p)$ (resp. $K^- \cap W^s(p)$) carry deep information on the geometry of the Julia set. We say that $f$ is **stably connected** if $U^+ \cap W^s(p)$ is simply connected for some (and then any) saddle point $p$, and **stably disconnected** otherwise. Equivalently, $f$ is stably disconnected if for some saddle point $p$, $W^s(p) \cap K^-$ admits a compact component relative to the intrinsic topology. This actually implies the stronger property that most components of $W^s(p) \cap K^-$ are points (see the proof of Lemma 3.2 below for more details).

By [BS6, Cor. 7.4], a dissipative polynomial automorphism is always stably disconnected. It was observed in [Du1] that this implies a strong non-extremality property for the current $T^+|_B$: there exists a decomposition $T^+|_B = \sum_{k=1}^{\infty} T_k^+$ where $T_k^+$ is an average of integration currents over a family of disjoint vertical disks of degree $k$ (see [Du1, Thm 2.4]).

**Lemma 2.2.** Let $f$ be dissipative and hyperbolic on $J^*$ and let $q \in J^*$. Then $q$ belongs to the support of $T^+|_{W^u(q)}$ and for $(T^+|_{W^u(q)})\cdot \text{a.e.} \ q' \in q$, $W^s_B(q')$ is a vertical manifold of finite degree in $\mathfrak{B}$.

**Proof.** The first assertion easily follows from the fact that $(f^n)_{n \geq 0}$ cannot be a normal family on $W^u_{\text{loc}}(q)$ (see [BLS, Lemma 2.8]). The second one is a consequence of [Du1, Thm 2.4]. Indeed as observed above $T^+|_B$ admits a decomposition $T^+|_B = \sum_{k=1}^{\infty} T_k^+$ where $T_k^+$ is made of vertical disks of degree $k$. Thus $T^+|_{W^u_{\text{loc}}(q)} = T^+ \cap [W^u_{\text{loc}}(q)] = \sum_{k=1}^{\infty} T_k^+ \cap [W^u_{\text{loc}}(q)]$. Now if $\Gamma$ is a leaf of some $T_k^+$ intersecting $W^u_{\text{loc}}(q)$ at $q'$, then $W^u_{\text{loc}}(q)$ is subordinate to $T^-$, $q'$ belongs to $J^*$ and $\Gamma$ is a manifold through $q'$ along which forward iterates are bounded, hence $\Gamma = W^s_B(q')$. □

2.3. Hyperbolicity and local product structure. Let us recall some generalities from hyperbolic dynamics, specialized to our situation. A (saddle) **hyperbolic set** for $f$ is a compact invariant set $\Lambda \subset \mathbb{C}^2$ such that $T\mathbb{C}^2|_{\Lambda}$ admits a hyperbolic splitting, i.e. $T\mathbb{C}^2|_{\Lambda} = E^s \oplus E^u$, where $E^s$ and $E^u$ are continuous line bundles such that $E^s$ (resp. $E^u$) is uniformly contracted (resp. expanded) by $df$. Then there exists $\delta_1 > 0$ such that $W^s_{\delta_1}(x) \subset \bigcup_{y \in \Lambda} W^u_\delta(x)$ and $W^s_\delta(x) \subset \bigcup_{y \in \Lambda} W^u_\delta(x)$ form laminations in the $\delta_1$-neighborhood of $\Lambda$.

A hyperbolic set is **locally maximal** if there exists an open neighborhood $\mathcal{N}$ of $\Lambda$ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}(\mathcal{N})$. It has **local product structure** if there exists $0 < \delta_2 \leq \delta_1$ such that if $p, q \in \Lambda$ are such that $d(p, q) < \delta_2$ then $W^s_\delta(p) \cap W^u_\delta(q)$ consists of exactly one point belonging to $\Lambda$. It turns out that these two properties are equivalent (see [Y, §4.1]).

We will use the following consequence of the shadowing lemma.

**Proposition 2.3.** Let $\Lambda$ be a compact locally maximal hyperbolic set for a polynomial diffeomorphism $f$ of $\mathbb{C}^2$. Then there exist positive constants $\eta$, $\alpha$ and $\Lambda$ such that for every $n \geq 0$; if $x$ is such that $\{x, \ldots, f^n(x)\} \subset \Lambda_{\eta}$ then there exists $y \in \Lambda$ such that $x$ is $Ae^{-\alpha n}$-close to the local stable manifold of $y$.

A similar result holds for negative iterates: if $\{f^{-n}(x), \ldots, x\} \subset \Lambda_{\eta}$ then there exists $z \in \Lambda$ such that $x$ is $Ae^{-\alpha n}$-close to the local unstable manifold of $z$.

The following corollary is well-known.

**Corollary 2.4.** If $\Lambda$ is a compact locally maximal hyperbolic set, then

$$W^s(\Lambda) := \left\{ x \in \mathbb{C}^2, f^n(x) \underset{n \to \infty}{\longrightarrow} \Lambda \right\} = \bigcup_{p \in \Lambda} W^s(p) \quad \text{and similarly} \quad W^u(\Lambda) = \bigcup_{p \in \Lambda} W^u(p).$$
Note however that $W^s(\Lambda)$, being an increasing union of laminations, doesn’t need to have a lamination structure (this is already false when $\Lambda$ is a hyperbolic fixed point).

**Proof of Proposition 2.3 (sketch).** This is very classical. Given an orbit segment $\{x, \ldots, f^n(x)\}$ as in the statement of the proposition, let $y^{(0)}$ (resp. $y^{(n)}$) be a point in $\Lambda$ such that $d(x, y^{(0)}) < \eta$ (resp. $d(x, y^{(n)}) < \eta$). Then define a $\eta$-pseudo-orbit $(y^{(k)})_{k \in \mathbb{Z}}$ as follows

$$y^{(k)} = \begin{cases} f^k(y^{(0)}) & \text{for } k < 0; \\ f^k(x) & \text{for } 0 \leq k \leq n; \\ f^{k-n}(y^{(n)}) & \text{for } k > n. \end{cases}$$

Then if $\eta$ is small enough by local maximality and the shadowing lemma there exists a unique $y \in \Lambda$ such that for every $k \in \mathbb{Z}$, $d(f^k(y), y^{(k)}) < C\eta$ (where $C$ is some constant depending on $(f, \Lambda)$, see [Y, §4.1]). In particular for $0 \leq k \leq n$ we have $d(f^k(x), f^k(y)) < C\eta$ and it follows from standard graph transform estimates that $d(x, W^s_\delta(y)) \leq Ae^{-an}$. \hfill $\square$

The next result is a simple application of the techniques of [BLS].

**Proposition 2.5.** If $J^*$ is hyperbolic then it has local product structure. Furthermore global stable and unstable manifolds intersect only in $J^*$:

$$W^s(J^*) \cap W^u(J^*) = \emptyset.$$

**Proof.** Hyperbolicity implies that for some $\delta > 0$, if $p$ and $q$ are close enough, $W^s_\delta(p) \cap W^u_\delta(q)$ consists of a single point $r$. We have to show that $r \in J^*$. Indeed, $W^s_\delta(p)$ (resp. $W^u_\delta(q)$) is a disk subordinate to $T^+$ (resp. $T^-$) so there exists a non-trivial uniformly laminar current $S^+ \leq T^+$ (resp. $S^- \leq T^-$) with $W^s_\delta(p)$ (resp $W^u_\delta(q)$) as a leaf. By [BLS, Lem. 8.2], $S^+$ and $S^-$ have continuous potentials, so the wedge product $S^+ \wedge S^-$ is well defined, and geometric intersection theory of uniformly laminar currents [BLS, Lem. 8.3] implies that $r \in \text{Supp}(S^+ \wedge S^-)$. Since $S^+ \wedge S^- \leq T^+ \wedge T^-$ we conclude that $r \in J^*$.

The proof of the second assertion is similar. By local product structure,

$$W^s(J^*) = \bigcup_{n \geq 0} f^{-n}(W^s_\delta(J^*)),$$

hence if $r \in W^s(J^*)$, there exists $p \in J^*$ such that $r \in W^s(p)$ so $r$ belongs to a disk subordinate to $T^+$, and likewise $r \in W^u(q)$ so it belongs to a disk subordinate to $T^+$. Observe that these two disks are distinct: indeed otherwise we would have $W^s(p) = W^u(q)$ which is impossible because $W^s(p) \cap W^u(q)$ is contained in $K$ which is bounded in $\mathbb{C}^2$. So $r$ is an isolated intersection between $W^s(p)$ and $W^u(q)$ for the leafwise topology. If this intersection is transverse, we argue as above to conclude that $r$ belongs to $J^*$. If it is a tangency, then by [BLS, Lem. 6.4] for $q' \in J^*$ close to $q$, we get transverse intersections between $W^s(p)$ and $W^u(q')$ close to $r$ and conclude as in the transverse case. \hfill $\square$

### 2.4. Stability.

A theory of stability and bifurcations for polynomial automorphisms of $\mathbb{C}^2$ was developed in [DL], centered on the notion of weak $J^*$-stability.

A branched holomorphic motion over a complex manifold $\Lambda$ in $\mathbb{C}^2$ is a family of holomorphic graphs over $\Lambda$ in $\mathbb{C}^2$. It is a holomorphic motion (i.e. an unbranched branched holomorphic motion!) when these graphs are disjoint. A holomorphic family $(f_\lambda)_{\lambda \in \Lambda}$ of polynomial automorphisms of dynamical degree $d$ is weakly $J^*$-stable if the sets $J^*(f_\lambda)$ move under a branched holomorphic motion, and $J^*$-stable if this motion is unbranched. Note that if $f_\lambda$
is uniformly hyperbolic on \(J^*(f_{\lambda_0})\), then it is \(J^*\)-stable near \(\lambda_0\) in any holomorphic family containing \(f_{\lambda_0}\).

A number of properties of weakly \(J^*\)-stable families are established in [DL], including extension properties of the branched holomorphic motion of \(J^*\) to \(K\) (and more generally to \(J^+ \cup J^-\)), that will be used in Section 4. These properties hold under the standing assumption that the family \((f_{\lambda})\) is *substantial*\(^3\): this means that either all members of the family are dissipative, or that no relation of a certain form between multipliers of periodic points persistently holds in the parameter space \(\Lambda\). Without entering into the details, let us just note that by [BHI, Thm 1.4] any open subset of the family of all polynomial automorphisms of dynamical degree \(d\) is substantial.

3. Proof of the main theorem: the dissipative case

The proof is by contradiction so assume that \(f\) is a dissipative polynomial diffeomorphism of \(\mathbb{C}^2\), that is uniformly hyperbolic on \(J^*\), and that \(J^* \subsetneq J\).

**Step 1.** There exists \(p \in J^*\) and a holomorphic disk \(\Delta \subset W^s(p)\) such that \(G^-|_\Delta \equiv 0\).

The purpose of the remaining steps 2 and 3 will be to show that such a "queer" component of \(W^s(p) \cap J^-\) actually does not exist.

*Proof.* We first claim that it is enough to show that there exists \(p \in J^*\) and \(q \in W^s(p)\) such that \(q \in J \setminus J^*\). Indeed observe that \(W^s(p) \cap J = W^s(p) \cap J^- = W^s(p) \cap K^-\). By Lemma 2.1 we have that \(\partial_i(W^s(p) \cap J^-) \subset J^*\). Hence if \(q\) belongs to \(W^s(p) \cap (J \setminus J^*)\), it belongs to the intrinsic interior \(\text{Int}_i(W^s(p) \cap J^-)\), hence \(G^- \equiv 0\) in a neighborhood of \(q\) in \(W^s(p)\).

Let now \(x \in J \setminus J^*\). By Corollary 2.4, if \(\omega(x) \subset J^*\) then \(x \in W^u(J^*)\) and if \(\alpha(x) \subset J^*\) then \(x \in W^u(J^*)\). Since \(W^u(J^*) \cap W^s(J^*) = J^*\), we infer that either \(\omega(x) \not\subset J^*\) or \(\alpha(x) \not\subset J^*\). In any case we will show that both \(W^u(J^*) \cap (J \setminus J^*)\) and \(W^u(J^*) \cap (J \setminus J^*)\) are non-empty. Thus by symmetry it is enough to deal with the case where \(\omega(x) \not\subset J^*\).

Choose \(\eta\) so small that Proposition 2.3 holds for \(J^*\) and \(\omega(x)\) is not contained in \(\overline{N}\), where \(N := (J^*)_\eta\) is the \(\eta\)-neighborhood of \(J^*\).

Consider the sequence of Cesàro averages \(\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}\). By the ergodic closing lemma of [D2], every cluster value of the sequence \((\nu_n)\) is supported on \(J^*\). It follows that the asymptotic proportion of iterates of \(x\) belonging to \(N\) tends to 1, i.e.

\[
\lim_{n \to \infty} \frac{1}{n} \# \left\{ 0 \leq k \leq n-1, f^k(x) \in \mathcal{N} \right\} = 1.
\]

Indeed if a positive proportion of iterates stayed outside \(\mathcal{N}\), any cluster limit of \(\nu_n\) would have to give positive mass to \(\mathcal{N}^c\).

We thus infer that there are arbitrary long strings \(\{x_i, \ldots, x_{i+n}\}\) in the orbit of \(x\) that are entirely contained in \(\mathcal{N}\). Indeed, if on the contrary the length of such a string were uniformly bounded by some \(n_0\), then the density of iterates outside \(\mathcal{N}\) would be bounded below by \(1/(n_0 + 1)\). Therefore for every \(n\) there exists \(i_n\) such that \(\{x_{i_n}, \ldots, x_{i_n+n}\} \subset \mathcal{N}\). Choose \(i_n\) to be minimal with this property. Since \(\omega(x) \not\subset \mathcal{N}\), there exists \(j > i\) such that \(x_j \not\in \mathcal{N}\). So finally for every \(n\) we can find \(i_n < j_n\) such that \(j_n - i_n \geq n\), \(\{x_{i_n}, \ldots, x_{j_n}\} \subset \mathcal{N}\), \(x_{i_n-1} \not\in \mathcal{N}\) and \(x_{j_n+1} \notin \mathcal{N}\).

\(^3\)There is an unfortunate terminological conflict here: this should not be confused with the notion of substantial dissipativity mentioned in the introduction.
Let $p$ (resp. $p'$) be a cluster value of $(x_{i_0-1})$ (resp. $(x_{j_0+1})$). The points $p$ and $p'$ belong to $J$ because $x$ does, but not to $J^*$ because they lie outside $N$. It follows from Proposition 2.3 that $p \in W^s(q)$ for some $q \in J^*$ and $p' \in W^u(q')$ for some $q' \in J^*$. The proof is complete.

**Remark 3.1.** If $f$ is substantially dissipative i.e. $|\text{Jac}(f)| < \deg(f)^{-2}$, then the contradiction readily follows from this first step. Indeed Wiman’s theorem together with uniform hyperbolicity imply that the vertical degree of components of stable manifolds in some large bidisk $B$ is uniformly bounded (see [GP, Prop. 4.2] or [LP, Lem. 5.1]), and it follows that $J^- \cap W^s(x)$ is totally disconnected for every $x$ (see [Du1, Thm 2.10] or [GP, Thm. 4.3]), which contradicts the conclusion of Step 1.

In the second and third steps we do not use the assumption that $J \setminus J^* \neq \emptyset$.

**Step 2.** For every $p \in J^*$, if $\Omega$ is a component of $\text{Int}_1(W^s(p) \cap J^-)$, then $\Omega$ is unbounded for the leafwise topology.

**Proof.** Note first that by the maximum principle, any component of $\text{Int}_1(W^s(p) \cap J^-) = \text{Int}_1(W^s(p) \cap K^-)$ is simply connected, so $\Omega$ is a topological disk. Assume by contradiction that $\Omega$ is bounded for the leafwise topology. Then iterating forward a few times if needed, we can suppose that $\Omega$ is entirely contained in a local product structure box.

More precisely for small $\delta > 0$, we can fix holomorphic local coordinates $(z, w)$ near $p$ in which $p = (0, 0)$, $W^s_\delta(p) = \{ z = 0 \}$ and $W^u_\delta(p) = \{ w = 0 \}$, and assume $\Omega$ is contained in $W^s_\delta(p)$. Note that by Lemma 2.1, $\partial_1 \Omega \subset J^*$. We can assume that for every $q \in W^s_\delta(p) \cap J^*$, $W^u_\delta(q)$ contains a graph over the disk $D(0, \delta)$ in the first coordinate, with slope bounded by $1/2$. Then if $|z_0| \leq \delta$, the holonomy $h^u_{0, z_0}$ along local unstable leaves is well defined on $W^s_\delta(p) \cap J^*$ and maps $W^s_\delta(p) \cap J^*$ into $\{ z = z_0 \} \cap J^-$. This holonomy is a holomorphic motion so by Slodkowski’s theorem [S] it extends to a holomorphic motion of $W^s_\delta(p)$. In particular the motion of $\partial_1 \Omega$ extends to a motion of $\Omega$ and it makes sense to speak about $h^u_{0, z_0}(\Omega)$. This is an open subset of $\{ z = z_0 \}$, which is topologically a disk and whose boundary is contained in $J^-$. Thus for every $n \geq 0$, $f^{-n}(\partial(h^u_{0, z_0}(\Omega)))$ in contained in $B$ and by the maximum principle, the same holds for $f^{-n}(h^u_{0, z_0}(\Omega))$.

Finally, $\mathcal{O} := \bigcup_{|z| < \delta} h^u_{0, z_0}(\Omega)$ is an open set whose negative iterates remain in $B$, hence it is contained in the Fatou set of $f^{-1}$. But since $f$ is dissipative, this Fatou set is empty, which is the desired contradiction. \[\Box\]

**Step 3.** The unstable holonomy preserves the decomposition

$$W^s(p) = (W^s(p) \cap J^-) \sqcup (W^s(p) \cap U^-).$$

To make this statement precise, observe that for every $p \in J^*$, the components of the complement of $\partial_1(W^s(p) \cap J^-)$ in $W^s(p)$ can be divided into two types: components of $\text{Int}_1(W^s(p) \cap J^-)$ and components of $W^s(p) \cap U^-$ (note that since $U^-$ is open in $\mathbb{C}^2$, $W^s(p) \cap U^-$ is open for the intrinsic topology as well). Consider as above local coordinates $(z, w)$ near $p$ in which $p = (0, 0)$, $W^s_\delta(p) = \{ z = 0 \}$ and $W^u_\delta(p) = \{ w = 0 \}$. The unstable holonomy $h^u_{0, z_0}$ is initially only defined for points of $W^s_\delta(p) \cap J^* = \partial_1(W^s_\delta(p) \cap J^-)$, however by Slodkowski’s theorem it can be extended to $W^s_\delta(p)$. By Step 2, components of $\text{Int}_1(W^s(p) \cap J^-)$ are leafwise unbounded so they cannot be contained in $W^s_\delta(p)$. Obviously, the same holds for components of $W^s(p) \cap U^-$. 

If $q$ belongs to $J^* \cap W^u(p)$, the extended holonomy $h^u_{p,q}$ defines a homeomorphism $W^u_\delta(p) \to h^u_{p,q}(W^u_\delta(p))$. By local product structure this homeomorphism preserves $J^*$ so any component of $W^u_\delta(p) \setminus J^*$ is mapped onto a component of $h^u_{p,q}(W^u_\delta(p)) \setminus J^*$, which is itself contained in a component of $W^u(q) \setminus J^*$. The claim of Step 3 is that the extended holonomy $h^u_{p,q}$ preserves the type of components.

Since it doesn’t make sense to transport a whole leafwise unbounded component by unstable holonomy, to prove this assertion we need to find a criterion that recognizes the type of a component just from local topological properties near a point of its boundary. As already said the maximum principle implies that any component of $\text{Int}(W^u(p) \cap J^-)$ is simply connected. Thus Step 3 follows from:

**Lemma 3.2.** If $\Omega$ is a component of $W^u(p) \cap U^-$, then $\Omega$ is not simply connected near any point of $\partial \Omega$, more precisely: if $q \in \partial \Omega$ and $N$ is any neighborhood of $q$, there is a loop in $N \cap \Omega$, homotopic to a point in $N$, and enclosing a component of $W^u(p) \cap J^-$

*Proof.* Since $f$ is dissipative by [BS6, Cor. 7.4] it is stably disconnected. It follows that almost every unstable component of $K^+$ is a point (see [BS6, Thm 7.1] and also [Du1, Thm 2.10]). More specifically, if $\mu$ is the unique measure of maximal entropy, then for $\mu$-a.e. $x$, the measure $T^*|_{W^u(x)}$ (which is locally given by the wedge product $T^* \wedge [W^u(x)]$) gives full mass to the point components of $J^* \cap W^u(x)$. Obviously by Lemma 2.1 every such point component belongs to $J^*$ so we can transport it to nearby stable manifolds by unstable holonomy. In addition, the measure $T^*|_{W^u(x)}$ is holonomy invariant (see [BS1, Thm 6.5] or [BLS, Thm 4.5]) so if $x$ is such that $T^* \wedge [W^u_\delta(x)]$ gives full mass to point components, then the same holds for nearby $x'$. Thus we conclude that this property holds for every $p \in J^*$: $T^*|_{W^u(p)}$ gives full mass to the point components of $J^- \cap W^u(p)$.

**Lemma 3.3.** Let $p \in J^*$ and $\Omega$ be a component of $W^u(p) \cap U^-$ such that $\Omega$ is locally simply connected near some $q \in \partial \Omega$. Then $\partial \Omega$ has positive $(T^*|_{W^u(p)})$-measure.

This proves Lemma 3.2. Indeed, assuming that $\Omega$ is locally simply connected near some $q \in \partial \Omega$, Lemma 3.3 asserts that $T^*|_{W^u(p)}$ carries positive mass on a non-trivial continuum so $f$ cannot be stably disconnected. On the other hand $f$ must be stably disconnected because it is dissipative, and we reach a contradiction. \hfill $\Box$

The idea of Lemma 3.3 is as follows: every neighborhood of $q$ in $\partial \Omega$ has positive harmonic measure when viewed from $\Omega$. But the harmonic measure viewed from $\Omega$ is absolutely continuous with respect to $T^* \wedge [W^u(p)]$, hence the result. The formalization of this argument requires some elementary potential theory, for which we refer the reader to Doob’s classical monograph\(^4\) [Do].

In particular we shall use the formalism of sweeping (or balayage). Let $D$ be a smoothly bounded domain in $\mathbb{C}$, $A$ a non-polar compact subset of $D$ and $\nu$ a positive measure on $D$. The swept measure $\rho_{\nu,D,A}$ of $\nu$ on $A$ is the distribution on $A$ of the exit point of the Brownian motion in $D \setminus A$ whose starting point is distributed according to $\nu$. In particular its mass is lower than that of $\nu$ since a positive proportion of Brownian paths escape from $\partial D$. If $G_{\nu,D}$ is the Green potential of $\nu$ in $D$, that is the unique negative subharmonic function on $D$ such that $G_{\nu,D}|_{\partial D} = 0$ and $\Delta G_{\nu,D} = \nu$, then the swept measure of $\nu$ on $A$ is $\Delta R_{\nu,D,A}$, where

$$R_{\nu,D,A}(z) = \sup \{u(z), \ u \leq 0 \text{ subharmonic on } D \text{ and } u \leq G_{\nu,D} \text{ on } A\}$$

\(^4\)Note that Doob works with superharmonic functions so all inequalities have to be reversed.
Then for $q^2.1$, Step 4. $f$ is dissipative, uniformly hyperbolic on $J^*$ and for $q' \in W^s_W(q') \setminus J^*$, there is a component of $W^s_W(q') \setminus J^*$ corresponding to $\Omega$ under unstable holonomy, which is locally simply connected near $q'$. Since $G^-$ is continuous, if $q'$ is close enough to $q$, it takes positive values on that component, so we infer that the property that $\Omega$ is a component of $U^-$ is open. Now by Lemma 2.2, for $(T^+|_{W^s_W})$-a.e. $q', W^s_W(q')$ is a vertical manifold in $B$ of finite degree which establishes our claim. Without loss of generality rename $q'$ into $q$. For every $g_0 = \min_B G^-$, the component of $\{G^- < g_0\}$ containing $q$ in $W^s_W(q)$ is relatively compact for the intrinsic topology. We fix such a $g_0$ which is not a critical value of $G^-$ and let $D$ be the corresponding component, which is a smoothly bounded topological disk. From now on we work exclusively in $D$.

By assumption there is a neighborhood $N$ of $q$ in $D$ and a component $U$ of $\{G^- > 0\} \cap N$ that is simply connected. We have to show that $\partial U \cap N$ has positive mass relative to $\Delta G^-$. We choose $N$ to be closed so that $\partial U \cap N$ is compact. First, observe that for every $z_0 \in U \cap N$, the probability that the Brownian motion issued from $z_0$ in $U \cap N$ before leaving $U$ is positive. Therefore the swept measure $\rho_\delta_{z_0},D,\partial U \cap N$ has positive mass and to prove the lemma it is enough to show that is absolutely continuous with respect to $\Delta G^-$. Recall that the measure class of the swept measure does not depend on the starting point so we can replace $\delta_{z_0}$ by an arbitrary positive measure on $D \setminus (D \cap K^-)$. Let $0 < g_1 < g_0$ and $\mu_{g_1} := \Delta(\max(G^- - G^-, g_1))$ be the natural measure induced by $G^-$ on the level set $\{G^- < g_1\}$. We choose $\mu_{g_1}$ for the initial distribution of Brownian motion. Since $G^- \equiv g_0$ on $\partial D$, the Green function $G_{\Delta G^-,D}$ of the restriction of $\Delta G^-$ to $D$ is equal to $G^- - g_0$, and likewise

$$G_{\mu_{g_1},D} = \max(G^-, g_1) - g_0.$$  

Thus from (1) we get that

$$R_{\mu_{g_1},D,K^- \cap D} = \sup \{u(z), \text{ u.s.h. } \leq 0 \text{ on } D \text{ and } u \leq G_{\mu_{g_1},D} \text{ on } K^- \cap D\}$$

$$= \sup \{u(z), \text{ u.s.h. } \leq 0 \text{ on } D \text{ and } u \leq g_1 - g_0 \text{ on } K^- \cap D\}$$

$$= |g_1 - g_0| \frac{G^- - g_0}{g_0}$$

and finally

$$\rho_{\mu_{g_1},D,K^- \cap D} = \frac{g_0 - g_1}{g_0} \Delta G^-.$$  

The proof is complete. \hfill \Box

**Step 4. Conclusion.**

We just have to assemble the three previous steps. Assume as before by contradiction that $f$ is dissipative, uniformly hyperbolic on $J^*$ and $J^* \subseteq J$. Then by Step 1 there exists $p \in J^*$ and a “queer” component $\Omega$ of $W^s_W(p) \setminus J^*$ along which $G^- \equiv 0$. Pick $q \in \partial \Omega$. By Lemma 2.1, $q \in J^*$ so we can follow $\Omega \cap W^s_W(q)$ using the holonomy along local unstable manifolds. Then for $q' \in W^u_W(q)$ near $q$, the holonomy image $h_{q,q'}(\Omega \cap W^s_W(q))$ is contained in a queer component of $W^s_W(q') \setminus J^*$, which must be leafwise unbounded by Step 2. On the other hand
by Lemma 2.2, for generic \( q' \) in \( W^u_{\text{loc}}(q) \) (relative to the transverse measure \( T^+|_{W^u_{\text{loc}}(q)} \) \( W^u_{\text{loc}}(q) \)) is of bounded degree, in particular any component of \( K^- \cap W^u_{\text{loc}}(q) \) is leafwise bounded. This contradiction finishes the proof. \( \square \)

4. PROOF OF THE MAIN THEOREM: THE CONSERVATIVE CASE

Again the proof is by contradiction, so assume that \( f \) is a conservative polynomial automorphism of \( \mathbb{C}^2 \) such that \( J^* \) is a hyperbolic set and \( J^* \subset J \). We will use a perturbative argument and the dissipative case of the theorem to reach a contradiction.

Assume that \( f \) is written as a product of Hénon mappings \( f = h_1 \circ \ldots \circ h_k \) and let \( (f_\lambda)_{\lambda \in B} \) be a parameterization of a neighborhood of \( f \) in the space of such products, that is, the space of coefficients of the \( h_i \), and such that \( f_0 = f \). We can assume that \( B \) is a ball in \( \mathbb{C}^N \) for some \( N \). Since \( \lambda \mapsto \text{Jac}(f_\lambda) \) is an open map, there exist parameters arbitrary close to 0 for which \( f_\lambda \) is dissipative. As already said, by [BHI, Thm 1.4] there is no persistent relation between multipliers of periodic orbits so the family is substantial in the sense of [DL].

Since \( f_0 \) is hyperbolic on \( J^*(f_0) \) the family \( (f_\lambda) \) is \( J^* \)-stable in a neighborhood of the origin, that is, \( J^*(f_\lambda) \) moves under a holomorphic motion. Reducing the parameter space we can assume that \( J^* \) is hyperbolic throughout \( B \). Pick a point \( p = p(0) \in J(f_0) \setminus J^*(f_0) \). It was shown in [DL, Thm 5.12] that in a (weakly) \( J^* \)-stable family, the motion of \( J^* \) extends to a branched holomorphic motion of \( K \). Thus there exists a holomorphic continuation \( p(\lambda) \) of \( p(0) \) such that for every \( \lambda \in B \), \( p(\lambda) \) belongs to \( K(f_\lambda) \). Furthermore for every \( \lambda \in B \), \( p(\lambda) \) is disjoint from \( J^*(f_\lambda) \). Indeed if for some \( \lambda_0 \in B \) we had \( p(\lambda_0) \in J^*(f_{\lambda_0}) \), then by [DL, Lem 4.10] \( p(\lambda) \) would have to coincide throughout the family \( (f_\lambda) \) with the natural continuation of \( p(\lambda_0) \) as a point of the hyperbolic set \( J^* \), which is not the case since \( p(0) \notin J^*(f_0) \).

Let now \( \lambda_1 \in B \) be such that \( f_{\lambda_1} \) is dissipative. Then by the first part of the proof \( J(f_{\lambda_1}) = J^*(f_{\lambda_1}) \), and \( K(f_{\lambda_1}) \setminus J(f_{\lambda_1}) \) is non-empty since it contains \( p(\lambda_1) \). For a dissipative hyperbolic map

\[
K \setminus J = (K^+ \cap J^-) \setminus (J^+ \cap J^-) = \text{Int}(K^+) \cap J^-,
\]

so we deduce that \( \text{Int}(K^+(f_{\lambda_1})) \) is non-empty. By [BS1], \( \text{Int}(K^+(f_{\lambda_1})) \) is a finite union of attracting basins of periodic sinks, therefore \( f_{\lambda_1} \) admits an attracting periodic point. On the other hand by [DL, Thm 4.2], periodic points stay of constant type in a \( J^* \)-stable family (this holds even in the presence of conservative maps, provided the family is substantial), so \( f_0 \) must have an attracting orbit, which is contradictory since it is conservative. The proof is complete. \( \square \)

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