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Jean Jacques Moreau

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FACING THE PLURALITY OF SOLUTIONS IN NONSMOOTH MECHANICS

J. J. MOREAU

*Laboratoire de Mécanique et Génie Civil,
Université Montpellier II, 34095 Montpellier cedex 5, France
E-mail: moreau@lmgc.univ-montp2.fr*

1 SUMMARY

In the mechanics of systems involving dry frictional contacts it is known that the problem of evaluating at some instant the contact forces and the accelerations may admit a plurality of solutions. Starting from an elementary example of wedging one proposes, as a consistent way of handling information, to include the contact forces in the description of each state. This in fact is what popular time-stepping computation techniques do, as they make reference at each time-step to the constraint state calculated in the antecedent one (actually a way of treating Coulomb law incrementally). The Contact Dynamics technique optionally offers the alternative of discarding this information, exploring the consequent indeterminations and displaying the sets of solutions as clouds of dots. Examples arising from Granular Mechanics are presented. The so-called Painlevé paradox is commented. Finally, an explanation of the isostaticity of the equilibrium of a collection of frictionless round grains is proposed.

2 INTRODUCTION

However complex may be the phenomena which take place when two bodies touch each other, should sliding occur or not, the law of Coulomb provides an irreplaceable framework for approaching *dry friction*. In many industrial situations, the only quantitative information provided by technical documents about friction-affected devices is communicated in terms of Coulomb friction coefficients (with possible distinction between sticking and sliding values).

That the motions calculated from such a model frequently are mathematically nonsmooth reflects everyday observation: in the absence of lubrication, mechanisms may exhibit chattering motions, emit creaking noise, or jam.

Discussions have taken place for a long time about the fact that, in the dynamics of systems involving Coulomb friction, the initial value problem, i.e. the prediction of the motion consequent to an instant at which the positions and the velocities of the system elements are given, may have several solution or no solution at all. In the eyes of a 19th. century scientist like Painlevé, impressed by the theory of differential equations and Laplace's "determinism", this was an unacceptable defect, leading him to the point of rejecting the very concept of a contact force. The disquieting locution of "Painlevé's paradox" caused definite harm to mechanical science since engineers who had, from other parts, some reasons to be unsatisfied with the moderate quantitative precision of Coulomb law, tended to view this wording as a hint at some logical defect. The irrelevance of Painlevé's standpoint was already pointed at in his time [5].

In our view, a physical model is nothing but a *format*, in the sense of data processing, in which one decides to record the available information concerning a certain physical situation, to treat this information and finally to communicate conclusions. Fundamentally, the information one may collect about the investigated situation is always incomplete, so the model through which this information is coded cannot be expected to generate exhaustive predictions. Also, as the system evolves, the validity limits of the model may come to be overrun.

2 ELEMENTARY EXAMPLE

The system shown on Fig.1 is contained in a vertical plane. Two fixed walls slightly converge upward. A rigid rod, subject to gravity, is inserted between them, with Coulomb friction at the two contact points and abandoned at zero velocity. If the configuration angles have suitable values, compared with the friction angles at contact points, the problem of finding the consequent motion visibly has two solutions: either the rod loses contact and falls freely or it remains *wedged*.

The quadrilateral area filled with light gray is the intersection of the Coulomb cones drawn from the two contact points. The triangular part above AB intersects the vertical of the center of gravity G along a line segment. Any position of I on this segment corresponds to values of the contact forces \mathbf{R}_A and \mathbf{R}_B compatible with the no-sliding Coulomb condition and equilibrating the rod weight. But zero values for \mathbf{R}_A and \mathbf{R}_B are also compatible with the friction law, yielding a downward acceleration of G , hence the breaking of contact.

Intuitively, the outcome depends on the “intensity of wedging” of the rod. The natural measure of such a degree precisely consists of the values of contact forces and it turns out that, in the present case, knowing these forces allows one to predict the further motion unambiguously. For more complicated systems, the numerical investigations presented in the sequel show that treating contact forces as part of the description of each state reduces ambiguity considerably.

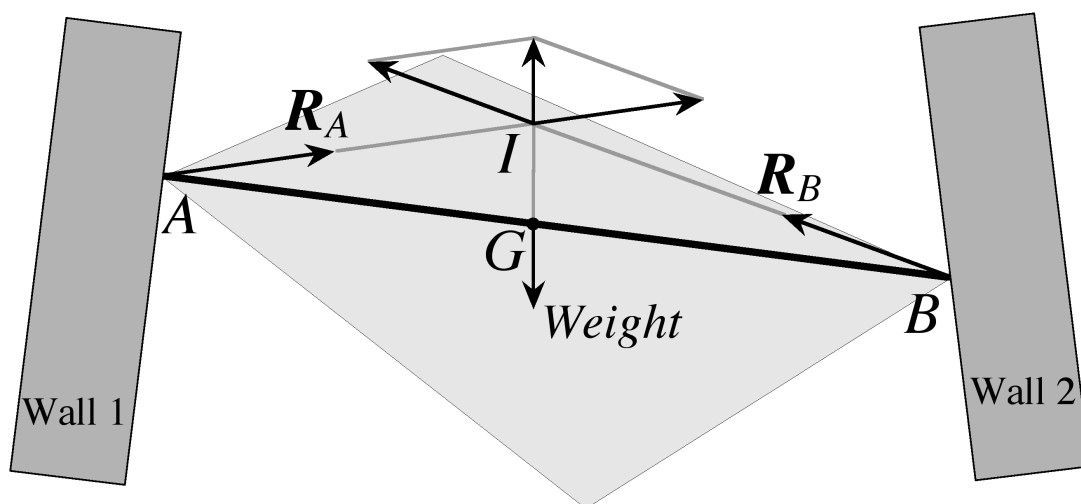


Figure 1: A rod inserted between convergent walls

A deeper insight into the present situation could be expected from the use of a *richer model* than above. To this end, one may take into account a certain *longitudinal elasticity* of the rod. This involves the introduction of a “small” variable s , the rod elongation counted from some reference state. As long as contacts are effective, the value of s is geometrically related to the displacements of points A and B on the respective walls, still treated as indeformable. Given a law of longitudinal elasticity of the rod, a differential equation with the longitudinal

components of \mathbf{R}_A and \mathbf{R}_B as endpoint data allows one to determine the stress distribution along AB . For simplicity, assume AB horizontal and all the applied forces vertical. Then this elastic analysis simply results in a monotone relationship between the variable s and the (opposite to each other) longitudinal components of \mathbf{R}_A and \mathbf{R}_B .

Such a model provides an understanding of *the scenario of rod insertion and extraction*. If, after contact is established, increasing upward forces are applied, the evolution (assumed quasi-static) consists in points A and B sliding upward. At every time, vectors \mathbf{R}_A and \mathbf{R}_B have lines of action imposed by Coulomb law in sliding regime. Thanks to the quasi-equilibrium equations of the rod, they may be calculated from the values of the applied forces.

If, at some instant, one decides to make the upward pushing forces decrease and eventually take downward direction, a new episode of motion begins, still assumed quasi-static. Points A and B remain fixed so that the elastic relationship, *whatever it is*, makes the longitudinal components of \mathbf{R}_A and \mathbf{R}_B keep the values they had at the instant of reversal. At every time, the values of \mathbf{R}_A and \mathbf{R}_B result from the values of the applied forces and the no-slide regime persists so long as these calculated vectors lie in the respective Coulomb cones. This allows one to determine the value that the resultant of the downward applied forces should eventually reach to produce detachment.

This model has the merit of making understand the processes of insertion and extraction but, to the lay observer, it conveys no quantitatively tractable information in the absence of a measuring device for microdisplacements, commonly not at hand. Since the proper elasticity law of the rod turns out to be immaterial, one may expect that some model merely ignoring this elasticity could be able to reproduce the above scenario. In fact, the Contact Dynamics computation method has been found to yield numerically a response of contact forces to the pushing history which agrees with what precedes.

3 TIME-STEPPING COMPUTATION

A widespread method for the numerical simulation of multibody systems is derived from the techniques used in *Molecular Dynamics*, hence referred to as MD. It consists of approximating the impenetrability constraints, when two bodies come close to each other, by repulsion forces steeply dependent on the mutual positions. Friction forces are similarly approximated, yielding an evolution problem governed by differential equations smooth enough for being handled through standard integration methods. In this way, the interaction forces are connected with fictitious deformations passed from step to step, so that the numerical scheme complies with the proposition made in the foregoing of *including these forces in the description of each state*. The drawback of the method is that, for the sake of precision, very stiff repulsion laws are needed so that very short step-lengths have to be used in the integration of the differential equations, usually with artificial damping introduced to prevent numerical instability. These added features may alter significantly the behaviour of the investigated system.

Another approach entitled *Contact Dynamics* (*abbr.* CD) will be preferred here in view of its ability to explore also the plurality of solutions. It faces the essential nonsmoothness of the impenetrability constraints and of dry friction without resorting to any regularizing alteration of the model [6] (for a recent exposition, see [7]). The dynamics of a system with n degrees of freedom is formulated in terms of two functions of a time-interval $[0, T]$ to \mathbb{R}^n , the *position function* $t \mapsto q$, and the *velocity function* $t \mapsto u$. In the simplest case, q equals the time-integral of u ; the kinematical relationship only becomes slightly more complicated when some members of u consist for instance in components of the spin vectors of rigid parts.

The geometric effect of non-interpenetrability is expressed by a finite set of inequalities $f_\alpha(q) \leq 0$, with equality corresponding to contact. The leading feature of the CD approach is that these inequalities are actually handled at the velocity level: by astraining $u(t)$ to belong for (almost) every t to a certain cone $\mathcal{K}(t, q)$, one secures inequalities to hold throughout $[0, T]$ provided they hold at initial instant.

In addition, some mechanical information about each contact should be available in terms of a *contact law*, i.e. a relationship connecting, for each α such that $f_\alpha(q) = 0$, the local kinematics with the contact force \mathcal{R}_α whose generalized components are noted $r_\alpha \in \mathbb{R}^n$. Coulomb law in particular may be written in such a form.

The equations of classical (smooth) Mechanics may be viewed as a differential equation, say $du/dt = \mathcal{E}(t, q, u, \Sigma r_\alpha)$, governing the function u and involving the other unknown q as a t -dependent parameter. But in the present context, the *nonsmoothness* arising from contact unilaterality and dry friction is liable to entail discontinuities for the function u . The suitable mathematical framework is that of u being an \mathbb{R}^n -valued function of t with bounded variation, so that the differential equation of the classical case is replaced by a *measure-differential equation* MDE. This means that the derivative du/dt should then be understood as an \mathbb{R}^n -valued measure on $[0, T]$, with punctual atoms at the instants of possible collisions. Mechanical actions, in particular the unknown contact forces or contact percussions are also described through such measures.

A time-stepping scheme is constructed by integrating both members of MDE on each interval of the time grid, say $[t_i, t_f]$, with $t_f = t_i + h$. Starting from the approximant u_i of $u(t_i)$ delivered by the antecedent step, the objective is to calculate an approximant u_f of $u(t_f)$. Concomitant unknown is the contact impulsion $s = \Sigma s_\alpha$, i.e. the integral of Σr_α over $[t_i, t_f]$. The discretized MDE thus has to be complemented by some impulsional form of the contact laws, hopefully valid to connect each s_α with some estimate \mathcal{U}_α of the local relative velocity of the contacting bodies. If one chooses as \mathcal{U}_α the value kinematically associated with the unknown u_f , the resulting discretization scheme turns out to be of the *implicit* type. Whatever is the procedure additionally used to update the continuous function q at the end of each step, such an implicit character secures numerical stability, allowing one to use considerably larger step-lengths than in MD methods. The price to pay is that the core of the calculation at each time-step consists of a highly nonlinear and nonsmooth problem.

Various techniques are available to solve the core problem [10], possibly parallelized [11]. The most commonly used is a *nonlinear Gauss-Seidel procedure* which suits well the present needs. It consists in reviewing the detected contacts cyclically again and again, solving for each of them a single-contact problem in which the other contact forces are treated as known and using the result to update interactions until some convergence criterium is met.

By connecting the contact impulsions s_α with u_f , the CD numerical scheme turns out to treat possible collisions as perfectly inelastic. An improvement, leading to a core problem of the same computational cost, consists in connecting each s_α with some *weighted mean of the initial and final values of \mathcal{U}_α* (possibly using different weights for normal and tangential components). In the special case of a binary collision this trick is found equivalent to the introduction of some Newton restitution coefficients (for spherical grains, this has been fairly well validated by experiments in [1]), but in general it offers the advantage of handling also multicontact systems with plausible results. Example: the rocking of a slender block on a horizontal ground [4].

The CD numerical strategy may also be applied to collections of deformable bodies, discretized

through finite elements [3].

Gauss-Seidel iterations have to be launched from some *initial guess* of the terms s_α . This guess may consist of zero values but, in the case of dense collections of rigid bodies, one considerably accelerates convergence by starting from the values found for these impulsions at the antecedent step, for the contacts which were already active. *If this is done, computation does comply with the proposition made in the foregoing of treating contact forces as state variables passed from step to step and very little indeterminacy is observed.*

Of course such a way of passing some information about the past makes sense only in time-stepping computation. At the level of theoretical formulation, what this numerical practice reflects is an *incremental* formulation of Coulomb law.

In contrast, given a time-step, if the information about antecedent contact forces is disregarded the Gauss-Seidel procedure allows one to explore the plurality of the solutions to the core problem. One possibility for this exploration is to execute a large number of runs of the cyclic iteration process, each time with initial guess drawn at random from some plausible range. Any solution then has a chance to be approached. Alternatively, one may launch all runs from zero initial guess while choosing at random, for each run, the ordering of the cyclic review of contacts. The limit of a run depends on this ordering. With κ contacts, the number of solutions attained in the latter way equals at most $(\kappa - 1)!$ which has been found sufficient to generate a significant outline of the solution set, as illustrated in Sect.4 below.

To determine whether a given position is an *equilibrium*, a single computation step is enough: starting with $u_i = 0$, one checks whether $u_f = 0$. The values found for the contact impulsions s_α yield the respective contact forces as $r_\alpha = s_\alpha/h$.

4 AN EQUILIBRIUM

Fig.2 presents a detail of a collection of $N = 28$ frictional circular bodies in equilibrium under gravity. The number of contact points being denoted by κ , the problem is to calculate the element r of $\mathbb{R}^{2\kappa}$ consisting of the components of the contact forces.

The $3N$ independent equations of equilibrium restrain r in an affine submanifold \mathcal{A} of $\mathbb{R}^{2\kappa}$ with dimension $2\kappa - 3N$. Coulomb law requires of each contact force, say r_α , to belong to an angular region \mathcal{C}_α of the corresponding two-dimensional factor subspace of $\mathbb{R}^{2\kappa}$. This is equivalent to r belonging to a polyhedral cone \mathcal{C} in $\mathbb{R}^{2\kappa}$, the Cartesian product of regions \mathcal{C}_α . The set of equilibrium solutions thus equals the intersection $\mathcal{A} \cap \mathcal{C}$, a convex hyperpolyhedron. Coming back to an individual contact, the set of the possible values of the associated contact force equals the Cartesian projection of $\mathcal{A} \cap \mathcal{C}$ onto the corresponding two-dimensional factor subspace. The two procedures described in the preceding Section are applied to their exploration.

For a specified contact point in the assembly, each run of the Gauss-Seidel algorithm allows one to draw a dot at the extremity of the computed contact force. By repetition, one obtains clouds outlining the investigated sets; of course, they are included in the angular regions imposed by Coulomb law.

For grain 1, simply supported by two underlying ones, clouds reduce to rectilinear segments parallel to the undetermined “wedging components” acting along AB and the situation is similar for grain 2. In contrast, grain 3 is involved in a more complicated pattern of interactions resulting in extended indetermination clouds.

A surprise is the appearance inside these clouds of some ghost images of edges and vertices of the high-dimensional polyhedron $\mathcal{A} \cap \mathcal{C}$, not the same for both procedures.

For a similar collection of discs in equilibrium, an investigation of force indeterminacy in *statistical* spirit is reported in [15].

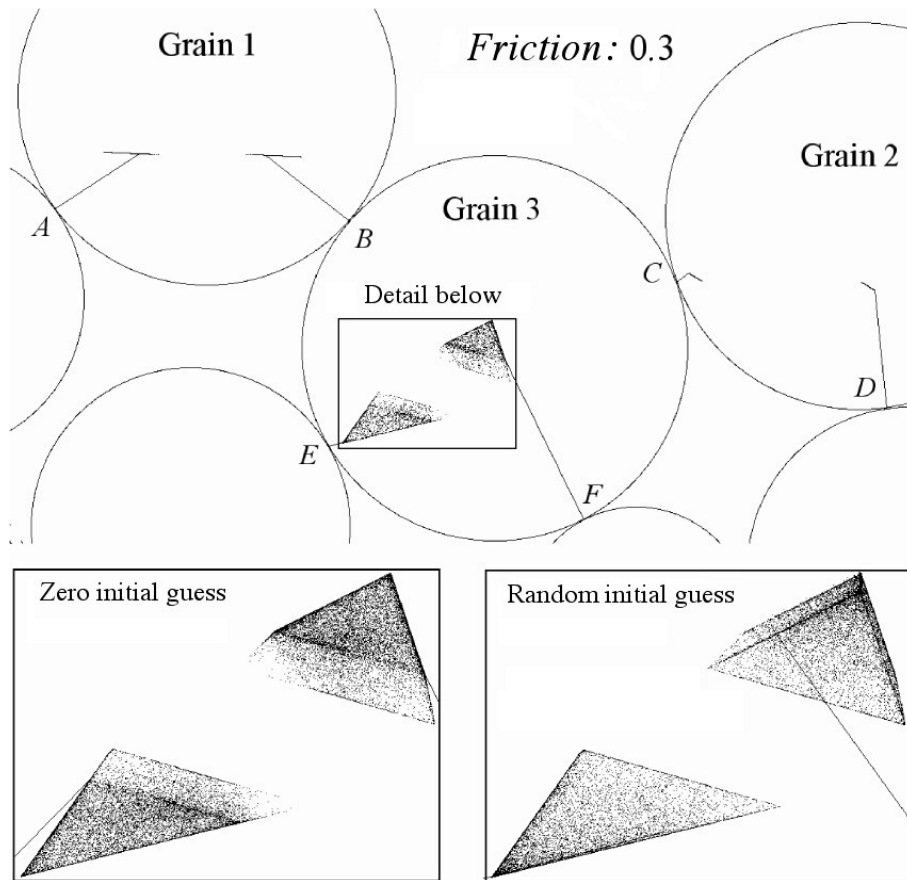


Figure 2: Contact force indeterminacy

5 GRANULAR FLOW ON A SLOPE

Fig.3 shows a detail of the two-dimensional flow on the ground $y = 0$ of a layer of $N = 355$ rigid disks, with gravity data corresponding to ground sloping down to the right by 24° .

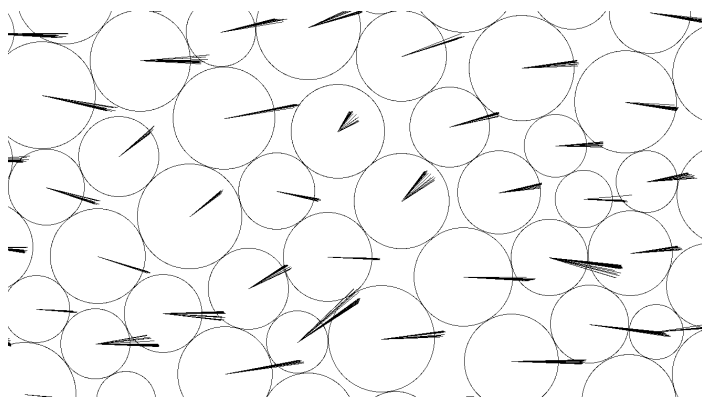


Figure 3: Dispersion of acceleration vectors at $t = .00044$

Periodic boundary conditions are applied, i.e. calculation is performed on disks with centers in the master band $-a \leq x < a$. Disks in the vicinity of $x = \pm a$ may interact through contact with the images of other ones under translations $(\pm 2a, 0)$; if they come to exit the master band they are removed from computation while some images of them are entered instead.

Grain-to-grain friction: 0.3; ground friction: 0.5; restitution of possible collisions: 0.

Flow is not in stationary regime but, starting from rest, it accelerates as the effect of gravity. Some results of computation are displayed on Fig.4 (11 computation steps after start) and Fig.5

(121 steps after start) both produced by repeated runs of the Gauss-Seidel procedure at zero initial guess, each run using a reviewing order selected at random.

An extended cloud at the first instant, the indeterminacy figure becomes a simple line and, as time grows, eventually shrinks to the size of numerical uncertainty. For instance, at $t = 0.0444$, the acceleration vector of some typical grain has a magnitude of about 520, while the indetermination range of its components is found to reduce to 0.007.

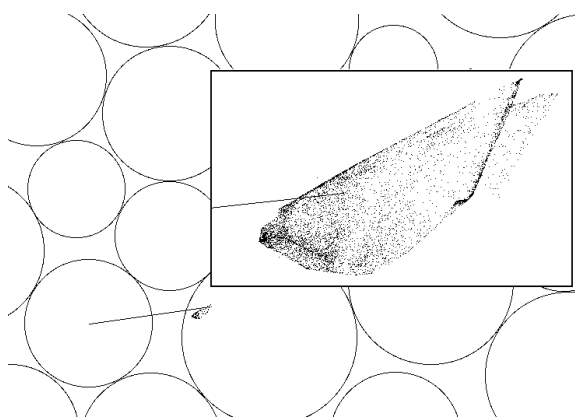


Figure 4: Acceleration vector indeterminacy at $t = .00044$

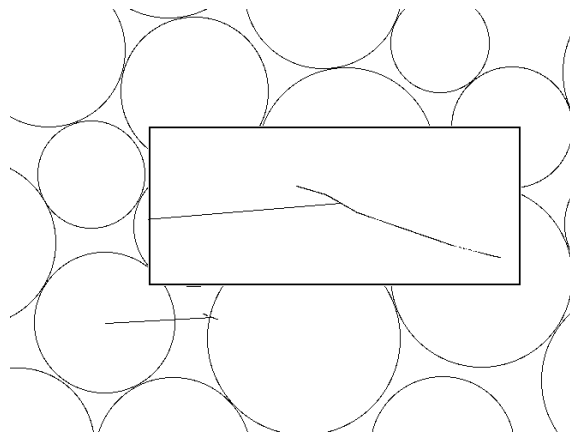


Figure 5: Acceleration vector indeterminacy at $t = .00484$

We have no explanation to propose for such a decrease of indeterminacy as the flow gains speed. One might think of the number of contacts involved in calculation and of the proportion of sticking ones as determinant parameters. Actually, at the instant of Fig.4, the number of contacts in the flow domain equals 556, among which 75% have zero sliding. At the instant of Fig.5, the number of contacts equals 544, with 70% of zero sliding ones. At the instant of negligible indetermination referred to above, the number of contacts is still as high as 506, with 57% of zero sliding ones.

6 PAINLEVE STATE AND FRICTIONAL PAROXYSM

In his criticism of the dry friction model, Painlevé invoked the very simple example of the motion of a rigid rod in a vertical plane, the extremity A of which slides with Coulomb friction on a fixed horizontal ground. Initial data consist of the angle θ of the rod with ground and of the velocity parameters (compatible with A remaining on ground). The problem of “incepting motion” is that of applying the equations of Dynamics in order to calculate the acceleration parameters at this instant. This is merely solving an algebraic system of linear equations and, if the friction coefficient is large enough, it turns out that, for a specific couple of values of θ and $\dot{\theta} = d\theta/dt$, no solution exists because a certain determinant vanishes (this may also lead to indeterminacy if some other quantity vanishes).

Today, one is not surprised to see a model refusing certain values of the parameters. A more interesting issue was recently addressed: may an episode of regular motion lead to such a *Painlevé state*? The answer is found “yes” through the elementary reduction of the system of second order differential equations to a single first order equation in the variables $\theta, \dot{\theta}$. The qualitative and numerical investigations of the integral lines reveal that the Painlevé state corresponds to some singular point in the $(\theta, \dot{\theta})$ -plane. All the integral lines emanating from a certain basin in this plane terminate at this very point and (with the exception of one of them) the corresponding angular acceleration $\ddot{\theta} = \dot{\theta}(d\dot{\theta}/d\theta)$ tends to $\pm\infty$. Consequently, the magnitude of the force exerted by the rod upon the ground tends to infinity; we propose to call that a

frictional paroxysm. A detailed mathematical analysis [2] confirms these graphical findings.

What could the motion be after the paroxystic instant depends on the mechanical assumptions one accepts about the friction phenomenon when contact forces tend to infinity. This is somewhat similar to a collision and the locution “tangential shock” is sometimes used. A model yielding a one-parameter indeterminacy in the after-paroxysm velocities is developed in [6]; it is automatically handled by CD time-stepping.

A definition of the concept of solution and an existence theorem (uniqueness not asserted) covering this case may be found in [14].

7 STATICS OF FRICTIONLESS ROUND RIGID BODIES

In contrast with the precedings we turn now to the statics of *frictionless* rigid body collections. Authors in [9][12] have asserted that, if a collection of *spherical bodies* (circular ones in 2D settings) contained in a fixed vessel is submitted to such external forces as gravity, then “generically” or “with probability 1”, the configurations taken by this system at equilibrium are *isostatic*. In the traditional context of frictionless systems involving only equality constraints, discussing isostaticity or more generally assessing the degree of hyperstaticity of an equilibrium amounts to evaluate the dimensions of the kernels or ranges of some linear mappings. In the present *unilateral* setting, equilibria are not characterized by equalities but by *complementarity conditions*, so that the discussion does not reduce to evaluating dimensions anymore.

Repeated numerical simulations have confirmed the assertion, but we are not fully convinced by the arguments of the authors, so we suggest below a novel approach to the question.

First observe that, in the statics of a frictionless collection of n spherical balls, the relevant position parameters merely are the coordinates of the respective centers. Use as abstract parameter the element q of \mathbb{R}^N , $N = 3n$, consisting of all these Cartesian coordinates.

The geometric effect of the non-interpenetrability of the balls and of their confinement by the vessel (one may assumed it polyhedral for simplicity) is expressed by inequalities

$$f_\alpha(q) \geq 0, \quad \alpha \in \{1, \dots, \nu\}, \quad (1)$$

with real functions f_α convex and smooth. This defines in \mathbb{R}^N the *feasible region* Φ , equivalently the complement of the union of the smooth convex sets defined by replacing the \geq symbols by $<$. The boundary Σ of Φ may thus be viewed as a “quilted” hypersurface constructed by piecing together some portions of the boundaries $S_\alpha, \alpha \in \{1, \dots, \nu\}$ of ν smooth convex sets with nonempty interiors. The S_α are smooth hypersurfaces which meet along hyperedges. In particular, hyperedges of dimension zero constitute *pits* in Σ .

For every $\alpha \in A(q) = \{\alpha : f_\alpha(q) = 0\}$, the configuration q involves a contact, which imparts on the system some contact forces with generalized components $r^\alpha \in \mathbb{R}^N$. Through the standard machinery of Analytical Mechanics [6], the assumption that contact is *frictionless* and *adhesionless* is equivalently transformed into

$$\exists \rho_\alpha \geq 0 : r^\alpha = \rho_\alpha \nabla f_\alpha(q) \quad (2)$$

(if all balls have nonzero radii, one may check that all $\nabla f_\alpha(q)$ are nonzero element of \mathbb{R}^N ; furthermore no pair of them have a common direction).

Let the applied forces consist of the action of gravity upon the respective balls; their generalized components make a vector $F \in \mathbb{R}^N$, independant of q . Provided gravity doesn’t lie in an

unboundedness directions of the vessel, the system possesses equilibrium positions belonging to Σ and, in view of the above description of the feasible region, it is only in pits that such equilibria may be stable.

Let q , located in some pit of Σ , denote an equilibrium configuration under force F ; in view of (2) this means that $-F$ equals a nonnegative combination of the elements $\nabla f_\alpha(q)$, $\alpha \in A$. We claim that “generically” these elements make a base in \mathbb{R}^N , hence the expected uniqueness.

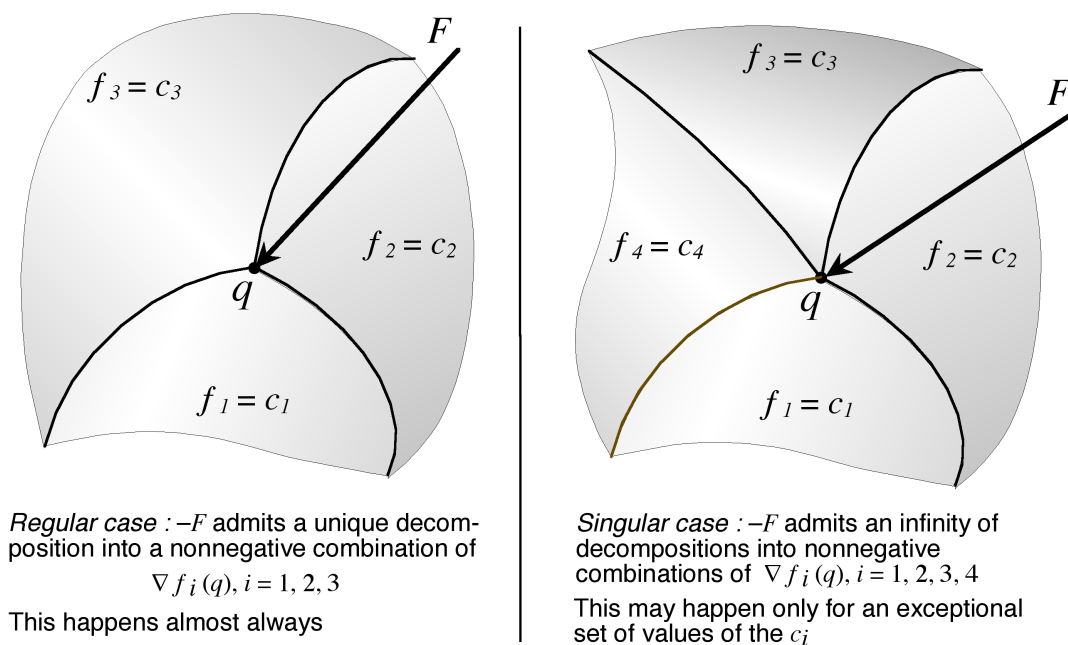


Figure 6: An illustration in \mathbb{R}^3

Figure 6 illustrates with $N = 3$ the proposed argumentation. After fixing ν real numbers c_1, \dots, c_ν , consider in \mathbb{R}^N the ν hypersurfaces $f_i = c_i$ (here $\nu \geq N$). We assert that generically no more than N of them can pass through a specified point and that the corresponding ∇f_i evaluated at this point are linearly independent. Here is an intuitive explanation.

Let N of the above hypersurfaces pass through some point p_0 . Effecting if necessary some arbitrarily small alterations (e.g. altering some of the c_i), one may obtain that two of them intersect *transversally* along a regular manifold M_{N-1} containing a point p_1 arbitrarily close to p_0 . Again after a possible alteration, a third surface may be made to intersect M_{N-1} transversally along a manifold M_{N-2} containing a point p_2 close to the preceding ones. The procedure has to be repeated, finally yielding N hypersurfaces which intersect in the expected regular way at some isolated point p_N close to p_0 . And it would be “exceptional” that another hypersurface of the collection pass through this point.

The essential issue is the meaning that one gives to the concept of *genericity*. A mathematical object depending on a parameter $p \in \mathbb{R}^m$ will be said to possess some property P generically if P holds for every p in a subset of \mathbb{R}^m whose complement is viewed as an exceptional set. A choice has to be made about the class of sets to be considered as exceptional, similarly to what is done in the *Theory of Transversality*.

- One may declare exceptional the subsets of \mathbb{R}^m with zero Lebesgue measure. This choice opens the way to *probabilistic* statements by viewing p as a random variable. If the probability distribution of p admits a density relative to Lebesgue measure, an exceptional set then has probability zero. The reader willing to develop the argument in this direction may take inspi-

ration from [13], a paper devoted to similar questions of genericity in nonlinear programming. The central analytic tool is *Sard's theorem*; it requires a high order of differentiability for the concerned functions, in fact secured in the present context.

- In a different approach, the property P will be declared generic if holds for every p in a *dense open subset* Ω of \mathbb{R}^m . In other words, if a point doesn't belong to Ω , an arbitrary small displacement is enough to bring it into Ω and, since Ω is open, property P is then secured in a "stable" way. Actually, results in this line need a slightly more complicated construction: the exceptional sets shall be the *Baire sets of first category* (also called meagre sets).

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