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SEDIMENTATION OF PARTICLES IN STOKES FLOW

AMINA MECHERBET

ABSTRACT. In this paper, we consider N identical spherical particles sedimenting in a uniform gravitational field. Particle rotation is included in the model while inertia is neglected. Using the method of reflections, we extend the investigation of [10] by discussing the optimal particle distance which is conserved in finite time. We also prove that the particles interact with a singular interaction force given by the Oseen tensor and justify the mean field approximation of Vlasov-Stokes equations in the spirit of [7] and [8].

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1. INTRODUCTION

In this paper, we consider a system of N spherical particles $(B_i)_{1 \leq i \leq N}$ with identical radii R immersed in a viscous fluid satisfying the following Stokes equation:

$$(1) \quad \begin{cases} -\Delta u^N + \nabla p^N &= 0, \\ \operatorname{div} u^N &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed with the no-slip boundary conditions:

$$(2) \quad \begin{cases} u^N &= V_i + \Omega_i \times (x - x_i), \text{ on } \partial B_i, \\ \lim_{|x| \rightarrow \infty} |u^N(x)| &= 0, \end{cases}$$

where $(V_i, \Omega_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $1 \leq i \leq N$ represent the linear and angular velocities,

$$B_i := B(x_i, R).$$

We describe the inertialess motion of the rigid spheres $(B_i)_{1 \leq i \leq N}$ by adding to the instantaneous Stokes equation the classical Newton dynamics for the particles $(x_i)_{1 \leq i \leq N}$:

$$(3) \quad \begin{cases} \dot{x}_i &= V_i, \\ F_i + mg &= 0, \\ T_i &= 0, \end{cases}$$

where m denotes the mass of the identical particles adjusted for buoyancy, g the gravitational acceleration, F_i (resp. T_i) the drag force (resp. the torque) applied by the fluid

on the i^{th} particle B_i defined as

$$\begin{aligned} F_i &:= \int_{\partial B_i} \sigma(u^N, p^N) n, \\ T_i &= \int_{\partial B_i} (x - x_i) \times [\sigma(u^N, p^N) n], \end{aligned}$$

with n the unit outer normal to ∂B_i and $\sigma(u^N, p^N) = 2D(u^N) - p^N \mathbb{I}$, the stress tensor where $2D(u^N) = \nabla u^N + \nabla u^{N\top}$.

Equations (1) – (3) are a model for suspensions sedimenting in a uniform gravitational field. We refer to [5, Chapter 1 section 1] for a physical explanation of the Stokes approximation in terms of the Reynolds number, we refer also to [1] for an introduction to the model. The constant velocities (V_i, Ω_i) of each particle are unknown and are determined by the prescribed force and torque $F_i = mg$ and $T_i = 0$. In [15], the author shows that the linear mapping on \mathbb{R}^{6N} :

$$(V_i, \Omega_i)_{1 \leq i \leq N} \mapsto (F_i, T_i)_{1 \leq i \leq N},$$

is bijective for all $N \in \mathbb{N}^*$. This ensures the existence and uniqueness of the velocities.

Given initial particle positions $x_i(0) := x_i^0$, $1 \leq i \leq N$, we are interested in the asymptotics of the solution when the number of particles N tends to infinity. The main motivation is to justify the representation of the motion of a dispersed phase inside a fluid using Vlasov-Stokes equations in spray theory [6], [2].

The analysis of the dynamics is done in [12] in the dilute case i.e when the minimal distance between particles is at least of order $1/N^{1/3}$. The authors prove that the particles do not get closer in finite time. Moreover, in the case where the minimal distance between particles is much larger than $1/N^{1/3}$ the result in [12] shows that particles do not interact and sink like single particles. We refer finally to [10] where the author considers a particle system with minimal distance of order $1/N^{1/3}$ and proves that, under a relevant time scale, the spatial density of the cloud converges in a certain averaged sense to the solution of the Vlasov-Stokes equation (58).

In this paper, we continue the investigation of [10] by looking for a more general set of particle configurations that is conserved in time and prove the convergence to the Vlasov-Stokes equation (58). Also, we include particle rotation in the modeling.

1.1. Description of initial configurations. We recall that the particles B_i are spherical with identical radii R :

$$B_i = B(x_i, R), \quad 1 \leq i \leq N,$$

where

$$R = \frac{r_0}{N}, \quad r_0 > 0.$$

We define ρ^N the spatial density of the cloud:

$$\rho^N(t, x) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}(x),$$

and set

$$\rho_0^N := \rho^N(0, x).$$

We assume that the cloud occupy initially a bounded domain:

$$(4) \quad \exists \bar{R} > 0, \quad x_i(0) \in B(0, \bar{R}), \quad \forall 1 \leq i \leq N, \quad \forall N \geq 1.$$

To describe the dilution regime we define d_{\min} the minimal particle distance:

$$d_{\min}(t) := \min_{i \neq j} \{ d_{ij}(t) = |x_i(t) - x_j(t)| \}.$$

We define also the particle concentration M^N as follows:

$$(5) \quad M^N(t) := \sup_{x \in \mathbb{R}^3} \{ \# \{ i \in \{1, \dots, N\} \text{ such that } |x_i(t) - x|_\infty \leq \lambda^N \} \},$$

where $|\cdot|_\infty$ stands for the l^∞ norm. $\lambda^N > 0$ is a positive quantity depending only on N . We assume that there exists two positive constants \bar{M} , \mathcal{E}_1 independent of N such that the minimal distance d_{\min} and the particle concentration M^N satisfy initially:

$$(6) \quad \frac{M^N(0)}{N|\lambda^N|^3} \leq \bar{M} \quad , \quad \frac{|\lambda^N|}{N^{2/3}d_{\min}^2(0)} \leq \mathcal{E}_1 \quad , \quad \frac{|\lambda^N|^2}{d_{\min}(0)} \ll 1.$$

We finally assume that the constant r_0 is small enough in the sense that:

$$(7) \quad \bar{M}r_0 \ll 1.$$

1.2. Main result. The main results of this paper are the two following theorems.

Theorem 1.1. *Assume that the particle configuration satisfies initially Assumptions 1.1. There exists $T > 0$ independent of N such that for all $t \in [0, T]$:*

$$d_{\min}(t) \geq \frac{1}{2}d_{\min}(0),$$

$$M^N(t) \leq 8^4 M^N(0),$$

for N large enough and depending on $r_0, \bar{M}, \mathcal{E}_1, \mathcal{E}_2$.

The second part of the result is the justification of the convergence of ρ^N when N tends to infinity.

Theorem 1.2. *Consider the maximal time $T > 0$ introduced in Theorem 1.1. Given an initial regular density ρ_0 , we denote by ρ the unique solution to the Vlasov-Stokes equation (58). There exists some positive constants $C_1, C_2 = C(\bar{M}, \bar{R}, \|\rho_0\|_{L^\infty \cap L^1})$ independent of N such that for all $t \in [0, T]$*

$$W_1(\rho^N(t, \cdot), \rho(t, \cdot)) \leq C_1 (\lambda^N + d_{\min}(0)t + W_1(\rho_0, \rho_0^N)) e^{C_2 t},$$

for sufficiently large N .

This shows that if the initial particle distribution ρ_0^N converges to ρ_0 then the particle distribution ρ^N converges toward the unique solution ρ of the Vlasov-Stokes equation (58) for all time $0 \leq t \leq T$. Moreover, Theorem 1.2 provides a quantitative convergence rate in terms of the initial Wasserstein distance $W_1(\rho_0, \rho_0^N)$.

Remark 1.1. *The assumption on the initial density ρ_0 is the one introduced by Höfer in [10] which is $\rho_0, \nabla \rho_0 \in X_\beta$, for some $\beta > 2$. See Section 5.1 for the definition of X_β . In particular, the assumption is satisfied if ρ_0 is compactly supported and \mathcal{C}^1 .*

The idea of proof of Theorem 1.2 is to formulate the problem considered as a mean-field problem. The mean-field theory consists in approaching equations of motion of large particles systems when the number of particles tends to infinity. The ODE governing the particle motion satisfies:

$$(8) \quad \begin{cases} \dot{x}_i &= \frac{1}{N} \sum_{j=1}^N F(x_i - x_j), \\ x_i(0) &= x_i^0, \end{cases}$$

where the kernel F is the interaction force of the particles. The limit model describing the time evolution for the spatial density $\rho(t, x)$ is given by

$$(9) \quad \begin{cases} \partial_t \rho + \mathcal{K} \rho \cdot \nabla \rho = 0, \\ \mathcal{K} \rho(x) := \int_{\mathbb{R}^3} F(x - y) \rho(t, y) dy, \end{cases}$$

In our case, the first difficulty is to extract a system similar to (8) for the particle motion and to identify the interaction force F . A key step is then a sharp expansion of the velocities for large N . We obtain for each $1 \leq i \leq N$:

$$V_i = \kappa g + 6\pi \frac{r_0}{N} \sum_{j \neq i} \Phi(x_i - x_j) \kappa g + O(d_{\min}) , \quad 1 \leq i \leq N,$$

where Φ is the Green's function for the Stokes equations, also called the Oseen tensor (see formula (18) for a definition), κ is such that

$$\kappa g = \frac{m}{6\pi R} g,$$

represents the fall speed of a sedimenting single particle under gravitational force. This shows that the particle system satisfies (8) with $F = \kappa g + 6\pi r_0 \Phi \kappa g$.

The proof of Theorem 1.2 is based on the two papers [8], [7] where, in the first one, the authors justify the mean field approximation and prove the propagation of chaos for a system of particles interacting with a singular interaction force and where the ODE governing the particle motion is second order. In [7] the author considers a different mean-field equation where the particle dynamics is a first order ODE. The results obtained hold true for a family of singular kernels and applies to the case of vortex system converging towards equations similar to the 2D Euler equation in vorticity formulation. The associated kernel in this case is the Biot-Savard kernel.

In order to extract the first order terms for the velocities (V_i, Ω_i) we apply the method of reflections. This method is introduced by Smoluchowski [16] in 1911. The main idea is to express the solution u^N of N separated particles as superposition of fields produced by the isolated N particle solutions. We refer to [13, Chapter 8] and [5, Section 4] for an introduction to the method. A convergence proof based on orthogonal projection operators is introduced by Luke [15] in 1989. We refer also to the method of reflections developed

in [11] which is used by Höfer in [10].

In this paper, we design a modified method of reflections that takes into account the particle rotation and relies on explicit solutions of Stokes flow generated by a translating, rotating and straining sphere. To obtain the convergence of the method of reflections we need to identify particle configuration that can be propagated in time. The particle configuration considered herein is the one introduced in [9] to study the homogenization of the Stokes problem in perforated domain. The novelty is that the author considers the minimal distance d_{\min} together with the particle concentration M^N as parameters to describe the cloud. The result in [9] extends in particular the validity of the homogenization problem for configurations having minimal distance lower than $1/N^{1/3}$. Note that the notion of particle concentration appears also in [8] to describe the cloud. Theorem 1.1 ensures that the particle configurations considered herein are preserved in time. We recover the result of [12] in the case where the minimal distance is at least of order $1/N^{1/3}$ when $\lambda^N = \frac{1}{N^{1/3}}$. In particular if d_{\min} is much larger than $1/N^{1/3}$, the explicit formula for the velocities implies

$$|V_i - \kappa g| \lesssim \frac{6\pi r_0}{N} \sum_{j \neq i} \frac{1}{|x_i - x_j|} |\kappa g| \lesssim \frac{1}{N} \frac{N^{2/3}}{d_{\min}} \ll 1$$

which is in accordance with the "non-interacting scenario" explained in [12]. More precisely, for $\lambda^N = \frac{1}{N^{1/3}}$, Theorem 1.1 extends the previous known results to configurations having minimal distance at least of order $\frac{1}{\sqrt{N}}$. This lower bound for the minimal distance appears naturally in our analysis and is closely related to the properties of the Green's function for the Stokes equations. We emphasize that this critical minimal distance appears also in the mean-field analysis due to [7, Theorem 1].

1.3. Outline of the paper. The remaining sections of this paper are organized as follows. In section 2 we recall the classical results for the existence and uniqueness of the Stokes solution u^N . We recall also the definition of the drag force, torque and strain and present in Section 2.1 the particular solutions to a Stokes flow generated by a translating, a rotating or a straining sphere. In section 3 we present and prove the convergence of the method of reflections in order to compute the first order terms for the velocities $(V_i, \Omega_i)_{1 \leq i \leq N}$. Section 4 is devoted to the proof of Theorem 1.1. In section 5 we recall some definitions associated to the Wasserstein distance. We present then the strong existence, uniqueness and stability theory for the Vlasov Stokes equations. In the second part of section 5 we show that the discrete density ρ^N satisfies weakly a Vlasov-Stokes equation. Section 6 is devoted to the proof of the second Theorem 1.2. Finally, some technical lemmas are presented in the appendix.

1.4. Notations. Given an exterior domain Ω with smooth boundaries, we set

$$\mathcal{C}^\infty(\overline{\Omega}) := \{v|_\Omega, v \in \mathcal{C}_c^\infty(\mathbb{R}^3)\},$$

and the following norm for all $u \in \mathcal{C}^\infty(\bar{\Omega})$

$$\|u\|_{1,2} := \|\nabla u\|_{L^2(\Omega)},$$

we define then the homogeneous Sobolev space $D(\Omega)$ as the closure of $\mathcal{C}^\infty(\bar{\Omega})$ for the norm $\|\cdot\|_{1,2}$ (see [4, Theorem II.7.2]). We also use the notation $D_\sigma(\Omega)$ for the subset of divergence-free $D(\Omega)$ fields

$$D_\sigma(\Omega) := \{u \in D(\Omega), \operatorname{div} u = 0\}.$$

Which is also the closure of the subset of divergence-free $\mathcal{C}^\infty(\bar{\Omega})$ fields for the $\|\cdot\|_{1,2}$ norm. Analogously, if $\Omega = \mathbb{R}^3$ we use the notation

$$\dot{H}_\sigma^1(\mathbb{R}^3) = D_\sigma(\mathbb{R}^3).$$

For all 3×3 matrix M , we define $\operatorname{sym}(M)$ (resp. $\operatorname{asym}(M)$) as the symmetric part of M (resp. the skew-symmetric part of M)

$$\begin{aligned} \operatorname{sym}(M) &= \frac{1}{2}(M + M^\top), \\ \operatorname{asym}(M) &= \frac{1}{2}(M - M^\top). \end{aligned}$$

We denote by \times the cross product on \mathbb{R}^3 and by \otimes the tensor product on $\mathbb{R}^3 \times \mathbb{R}^3$ which associates to each couple $(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ the 3×3 matrix defined as

$$(u \otimes v)_{ij} = u_i v_j, \quad 1 \leq i, j \leq 3.$$

In \mathbb{R}^3 , $|\cdot|$ stands for the euclidian norm while $|\cdot|_\infty$ represents the l^∞ norm. We use the notation $B_\infty(x, r)$ for the ball with center x and radius r for the l^∞ norm.

Finally, in the whole paper we use the symbol \lesssim to express an inequality with a multiplicative constant independent of N . We will also denote by $C > 0$ all the positive constants appearing in the estimates. These constants do not depend of N but can depend on the datas $r_0, \bar{R}, \bar{M}, \mathcal{E}_1, \mathcal{E}_2$.

2. REMINDER ON THE STOKES PROBLEM

In this section we recall some results concerning the Stokes equations. We remind that for all $N \in \mathbb{N}$ we denote by (u^N, p^N) the solution to (1) – (2). The classical theory for the Stokes equations yields:

Proposition 2.1. *For all $N \in \mathbb{N}$, there exists a unique pair $(u^N, p^N) \in D_\sigma(\mathbb{R}^3 \setminus \bigcup_i \bar{B}_i) \times L^2(\mathbb{R}^3 \setminus \bigcup_i \bar{B}_i)$ which realizes*

$$(10) \quad \inf \left\{ \int_{\mathbb{R}^3 \setminus \bigcup_i \bar{B}_i} |\nabla v|^2, v \in D_\sigma(\mathbb{R}^3 \setminus \bigcup_i \bar{B}_i), v = V_i + \Omega_i \times (x - x_i) \text{ on } \partial B_i, 1 \leq i \leq N \right\}.$$

The velocity field u^N can be extended to $V_i + \Omega_i \times (x - x_i)$ on each particle B_i . This extension denoted also u^N is in $\dot{H}_\sigma^1(\mathbb{R}^3)$.

We recall the definition of the force $F_i \in \mathbb{R}^3$, torque $T_i \in \mathbb{R}^3$ and strain $S_i \in \mathcal{M}_3(\mathbb{R})$ applied by the particle B_i on the fluid (see [5, Section 1.3])

$$(11) \quad \begin{aligned} F_i &= \int_{\partial B_i} \sigma(u^N, p^N) n. \\ M_i &= \int_{\partial B_i} (x - x_i) \otimes [\sigma(u^N, p^N) n]. \end{aligned}$$

The matrix M_i represents the first momentum which is decomposed into a symmetric and skew-symmetric part:

$$M_i = T_i + S_i,$$

the symmetric part is the strain S_i . Since the skew-symmetric part of a 3×3 matrix M has only three independent components, it can be associated to a unique vector T such that

$$\text{asym}(M) x = T \times x, \forall x \in \mathbb{R}^3.$$

In this paper, we allow the confusion between the skew-symmetric matrix $\text{asym}(M)$ and the vector T . Hence, we define the torque $T_i \in \mathbb{R}^3$ as being the skew-symmetric part of the first momentum M_i which satisfies:

$$(12) \quad \begin{aligned} T_i &= \text{asym}(M_i) = \int_{\partial B_i} (x - x_i) \times [\sigma(u^N, p^N) n], \\ S_i &= \text{sym}(M_i). \end{aligned}$$

2.1. Particular Stokes solutions. The linearity of the Stokes problem allows us to develop powerful tools that will be used in the method of reflections. In particular, we investigate in what follows the analytical solution to a Stokes flow generated by a translating, a rotating or a straining sphere. The motivation in considering these cases is that the fluid motion near a point x_0 may be approximated by

$$u(x) \sim u(x_0) + \nabla u(x_0) \cdot (x - x_0),$$

hence, if we replace the boundary condition on each particle by its Taylor series of order one, we can use these special solutions to approximate the flow u . The results and formulas of this section are detailed in [5, Section 2] and [13, Section 2.4.1]. In what follows $B := B(a, r)$ is a ball centered in $a \in \mathbb{R}^3$ with radius $r > 0$.

2.1.1. Case of translation: Stokeslet. Let $V \in \mathbb{R}^3$. We consider the unique solution $(U_{a,R}[V], P_{a,R}[V])$ to the following Stokes problem:

$$(13) \quad \begin{cases} -\Delta U_{a,R}[V] + \nabla P_{a,R}[V] &= 0, \\ \text{div } U_{a,R}[V] &= 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \overline{B},$$

completed by the boundary condition:

$$(14) \quad \begin{cases} U_{a,R}[V] &= V, \text{ on } \partial B, \\ \lim_{|x| \rightarrow \infty} |U_{a,R}[V](x)| &= 0. \end{cases}$$

$U_{a,R}[V]$ is the flow generated by a unique sphere immersed in a fluid moving at V and is called *Stokeslet*. The explicit formula for $(U_{a,R}[V], P_{a,R}[V])$ is derived in [13, section 3.3.1] and also in [5, Formula (2.12) and (2.13)]. These explicit formulas imply that for all $x \in \mathbb{R}^3 \setminus B(a, R)$:

$$(15) \quad |U_{a,R}[V](x)| \lesssim R \frac{|V|}{|x-a|}, \quad |\nabla U_{a,R}[V](x)| + |P_{a,R}[V](x)| \lesssim R \frac{|V|}{|x-a|^2}.$$

On the other hand, the force F , torque T and strain S exerted by the Stokeslet on the particle B as defined in (11) read:

$$(16) \quad F = -6\pi R V, \quad T = 0, \quad S = 0.$$

We recall now an important formula that links the Stokeslet to the Green's function of the Stokes problem. For all $x \in \mathbb{R}^3 \setminus B(a, R)$ we have:

$$(17) \quad U_{a,R}[V](x) = - \left(\Phi(x-a) - \frac{R^2}{6} \Delta \Phi(x-a) \right) F.$$

Where Φ is the Green's function for Stokes flow also called Oseen-tensor:

$$(18) \quad \Phi(x) = \frac{1}{8\pi} \left(\frac{1}{|x|} \mathbb{I}_3 + \frac{1}{|x|^3} x \otimes x \right),$$

the 3×3 matrix $\Delta \Phi$ represents the Laplacian of Φ and is given by:

$$\Delta \Phi(x) = \frac{1}{8\pi} \left(\frac{2}{|x|^3} \mathbb{I}_3 - \frac{6}{|x|^5} x \otimes x \right),$$

Remark 2.1. *Formula (17) is closely related to the Faxén law which represents the relations between the force F , torque T , stresslet S and the velocity V . We refer to [5, Section 2.3] and [13, section 3.5] for more details on the topic.*

Remark also that in (17) the first part retains the most slowly decaying portion, which is of order $\frac{R}{|x|}$. This property is useful in order to extract the first order terms for the velocities $(V_i)_{1 \leq i \leq N}$, see Lemma 3.8.

Moreover, we recall a Lipschitz-like inequality satisfied by the Oseen tensor:

$$(19) \quad |\Phi(x) - \Phi(y)| \lesssim \frac{|x-y|}{\min(|y|^2, |x|^2)}, \quad \forall x \neq y \neq 0.$$

Finally, in this paper, the velocity field $U_{a,R}[V]$ is extended by V on $B(a, R)$.

2.1.2. Case of rotation. Let $\omega \in \mathbb{R}^3$. Denote by $(A_{a,R}^{(1)}[\omega], P_{a,R}^{(1)}[\omega])$ the unique solution to

$$(20) \quad \begin{cases} -\Delta A_{a,R}^{(1)}[\omega] + \nabla P_{a,R}^{(1)}[\omega] &= 0, \\ \operatorname{div} A_{a,R}^{(1)}[\omega] &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \overline{B(a, R)},$$

completed with the boundary conditions

$$(21) \quad \begin{cases} A_{a,R}^{(1)}[\omega] &= \omega \times (x-a), \text{ on } \partial B(a, R), \\ \lim_{|x| \rightarrow \infty} |A_{a,R}^{(1)}[\omega]| &= 0. \end{cases}$$

$A_{a,R}^{(1)}[\omega]$ represents the flow generated by a sphere rotating with angular velocity ω . In particular we have $P_{a,R}^{(1)}[\omega] = 0$ due to symmetries. The drag force F and strain S also vanish:

$$F = 0 \quad , \quad S = 0.$$

On the other hand, the hydrodynamic torque resulting from the fluid traction on the surface defined in (12) is given by

$$(22) \quad T = -8\pi R^3 \omega.$$

Finally, for all $x \in \mathbb{R}^3 \setminus B(a, R)$

$$|A_{a,R}^{(1)}[\omega]| \lesssim R^3 \frac{|\omega|}{|x-a|^2} \quad , \quad |\nabla A_{a,R}^{(1)}[\omega]| + |P_{a,R}^{(1)}[\omega]| \lesssim R^3 \frac{|\omega|}{|x-a|^3}.$$

2.1.3. *Case of strain.* Let E be a trace-free 3×3 symmetric matrix.

Denote by $(A_{a,R}^{(2)}[E], P_{a,R}^{(2)}[E])$ the unique solution to

$$(23) \quad \begin{cases} -\Delta A_{a,R}^{(2)}[E] + \nabla P_{a,R}^{(2)}[E] &= 0, \\ \operatorname{div} A_{a,R}^{(2)}[E] &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \overline{B(a, R)},$$

completed with the boundary conditions:

$$(24) \quad \begin{cases} A_{a,R}^{(2)}[E] &= E(x-a), \text{ on } \partial B(a, R), \\ \lim_{|x| \rightarrow \infty} |A_{a,R}^{(2)}[E]| &= 0. \end{cases}$$

The velocity field $A_{a,R}^{(2)}[E]$ is the flow generated by a sphere submitted to the strain $E(x-a)$. In this case, the drag force and torque vanishes:

$$F = 0 \quad , \quad T = 0.$$

On the other hand, the *Stresslet* S as defined in (12) is given by

$$(25) \quad S = -\frac{20}{3}\pi R^3 E,$$

in the case of a straining flow.

Finally, for all $x \in \mathbb{R}^3 \setminus B(a, R)$ we have

$$(26) \quad |A_{a,R}^{(2)}[E]| \lesssim R^3 \frac{|E|}{|x|^2} \quad , \quad |\nabla A_{a,R}^{(2)}[E]| + |P_{a,R}^{(2)}[E](x)| \lesssim R^3 \frac{|E|}{|x|^3}.$$

2.1.4. *Final notations.* Now, assume that D is a trace-free 3×3 matrix. We denote by $(A_{a,R}[D], P_{a,R}[D])$ the unique solution to

$$(27) \quad \begin{cases} -\Delta A_{a,R}[D] + \nabla P_{a,R}[D] &= 0, \\ \operatorname{div} A_{a,R}[D] &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \overline{B(a, R)},$$

completed by the boundary conditions:

$$(28) \quad \begin{cases} A_{a,R}[D] &= D(x-a), \text{ on } \partial B(a, R), \\ \lim_{|x| \rightarrow \infty} |A_{a,R}[D]| &= 0. \end{cases}$$

We set then, $D = E + \omega$ with $E = \text{sym}(D)$ and $\omega = \text{asym}(D)$. As stated in the definition (12), ω represents also a 3D vector. Hence, the boundary condition (28) reads

$$A_{a,R}[D] = D(x-a) = E(x-a) + \omega \times (x-a), \quad \text{for all } x \in \partial B(a, R).$$

We have, thanks to the linearity of the Stokes equation, that

$$(A_{a,R}[D], P_{a,R}[D]) = (A_{a,R}^{(1)}[\omega], P_{a,R}^{(1)}[\omega]) + (A_{a,R}^{(2)}[E], P_{a,R}^{(2)}[E]).$$

Since the two solutions have the same decay-rate this yields for all $x \in \mathbb{R}^3 \setminus \overline{B(a, R)}$

$$(29) \quad |A_{a,R}[D]| \lesssim R^3 \frac{|D|}{|x|^2}, \quad |\nabla A_{a,R}[D]| + |P_{a,R}[D](x)| \lesssim R^3 \frac{|D|}{|x|^3}.$$

2.2. Approximation result. In this part we consider the unique solution (v, p) of the following Stokes problem:

$$(30) \quad \begin{cases} -\Delta v + \nabla p &= 0, \\ \text{div } v &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed with the boundary conditions:

$$(31) \quad \begin{cases} v &= V + D(x-x_1), \text{ on } \partial B_1, \\ v &= 0, \text{ on } \partial B_i, i \neq 1, \\ \lim_{|x| \rightarrow \infty} |v(x)| &= 0, \end{cases}$$

with $V \in \mathbb{R}^3$ and D a trace-free 3×3 matrix. We set

$$v_1 := U_{x_1, R}[V] + A_{x_1, R}[D].$$

We aim to show that the velocity field v_1 is a good approximation of the unique solution v .

Lemma 2.2. *We have the following error bound*

$$\|\nabla v - \nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 \lesssim \frac{R}{\sqrt{d_{\min}}} |V| + \frac{R^3}{d_{\min}^{3/2}} |D|.$$

Proof. We have

$$\|\nabla v - \nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 = \|\nabla v\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 - 2 \int_{\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}} \nabla v : \nabla v_1 + \|\nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2,$$

as v and v_1 satisfy the same boundary condition on ∂B_1 this yields

$$\int_{\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}} \nabla v : \nabla v_1 = - \int_{\partial B_1} (\partial_n v_1 - p_1 n) \cdot v = - \int_{\partial B_1} (\partial_n v_1 - p_1 n) \cdot v_1 = \|\nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2,$$

hence

$$\|\nabla v - \nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 = \|\nabla v\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 - \|\nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2.$$

In order to bound the first term we construct an extension \tilde{v} of the boundary conditions of v and apply the variational principle. We define:

$$\tilde{v} := \chi \left(\frac{\cdot - x_1}{d_{\min}/4} \right) v_1 - \mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}[\bar{f}],$$

where χ is a truncation function such that $\chi = 1$ on $B(0, 1)$ and $\chi = 0$ out of $B(0, 2)$, \bar{f} is defined as follow

$$\bar{f}(x) := v_1(x) \cdot \nabla \left[x \mapsto \chi \left(\frac{x - x_1}{d_{\min}/4} \right) \right],$$

and $\mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}$ denotes the Bogovskii operator satisfying for all $f \in L_0^q(B(x_1, d_{\min}/2) \setminus \overline{B(x_1, d_{\min}/4)})$, $q \in (0, \infty)$:

$$\operatorname{div} \mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}[f] = f,$$

we refer to [4, Theorem III.3.1] for a complete definition of the Bogovskii operator. In particular, from [9, Lemma 16], there exists a constant $C > 0$ independent of d_{\min} such that:

$$(32) \quad \|\nabla \mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}[\bar{f}]\|_{L^2(A_1)} \leq C \|\bar{f}\|_{L^2(A_1)},$$

where $A_1 := B(x_1, d_{\min}/2) \setminus \overline{B(x_1, d_{\min}/4)}$. With this construction \tilde{u} is a divergence-free field satisfying the same boundary conditions as u . Moreover, applying formula (32) yields

$$\begin{aligned} \|\nabla \tilde{v}\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 &= \int_{\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}} \left| \nabla \left[x \mapsto \chi \left(\frac{x - x_1}{d_{\min}/4} \right) v_1(x) \right] \right|^2 dx \\ &\quad + \int_{\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}} |\nabla \mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}[\bar{f}](x)|^2 dx \\ &\quad - 2 \int_{\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}} \nabla \left[x \mapsto \chi \left(\frac{x - x_1}{d_{\min}/4} \right) v_1(x) \right] : \nabla \mathcal{B}_{x_1, d_{\min}/4, d_{\min}/2}[\bar{f}](x) dx \\ &\leq \int_{\mathbb{R}^3 \setminus B_1} \left| \chi \left(\frac{x - x_1}{d_{\min}/4} \right) \nabla v_1(x) \right|^2 dx \\ &\quad + K \left(\int_{A_1} |\nabla v_1(x)|^2 + \frac{1}{d_{\min}^2} \left| x \mapsto \nabla \chi \left(\frac{x - x_1}{d_{\min}/4} \right) \right|^2 |v_1(x)|^2 \right) dx. \end{aligned}$$

As $\chi\left(\frac{\cdot - x_1}{d_{\min}/4}\right) = 1$ on $B(x_1, d_{\min}/4)$ we get

$$\begin{aligned} \|\nabla v - \nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 &\leq \|\nabla \tilde{v}\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 - \|\nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})}^2 \\ &\lesssim \int_{\mathbb{R}^3 \setminus B(x_1, d_{\min}/4)} |\nabla v_1(x)|^2 dx \\ &\quad + \int_{A_1} \frac{1}{d_{\min}^2} \left| x \mapsto \nabla \chi\left(\frac{x - x_1}{d_{\min}/4}\right) \right| |v_1|^2 dx, \end{aligned}$$

Thanks to (15) and (29) we have:

$$\begin{aligned} \int_{A_1} \frac{1}{d_{\min}^2} \left| \nabla \chi\left(\frac{x - x_1}{d_{\min}/4}\right) \otimes v_1 \right|^2 &\lesssim \|\nabla \chi\|_{\infty} \int_{A_1} \frac{1}{d_{\min}^2} \left(R^2 \frac{|V|^2}{|x - x_1|^2} + R^6 \frac{|D|^2}{|x - x_1|^4} \right) \\ &\lesssim \frac{1}{d_{\min}^2} \int_{d_{\min}/4}^{d_{\min}/2} \left(R^2 |V|^2 + R^6 \frac{|D|^2}{r^2} \right) dr \\ &\lesssim \frac{1}{d_{\min}^2} \left(R^2 |V|^2 d_{\min} + R^6 \frac{|D|^2}{d_{\min}} \right). \end{aligned}$$

Reproducing an analogous computation for the first term we obtain finally:

$$\|\nabla v - \nabla v_1\|_{L^2(\mathbb{R}^3 \setminus \bigcup_i B_i)}^2 \lesssim \frac{R^2}{d_{\min}} |V|^2 + \frac{R^6}{d_{\min}^3} |D|^2.$$

This yields the expected result. \square

3. ANALYSIS OF THE STATIONARY STOKES EQUATION

This section is devoted to the analysis of the method of reflections and computation of the unknown velocities $(V_i, \Omega_i)_{1 \leq i \leq N}$. We remind that, for fixed time, u^N is the unique solution to the stationary Stokes problem

$$\begin{cases} -\Delta u^N + \nabla p^N &= 0, \\ \operatorname{div} u^N &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed with the no-slip boundary conditions:

$$\begin{cases} u^N &= V_i + \Omega_i \times (x - x_i), \text{ on } \partial B_i, \\ \lim_{|x| \rightarrow \infty} |u^N(x)| &= 0, \end{cases}$$

where (V_i, Ω_i) are the unique velocities satisfying

$$(33) \quad F_i + mg = 0 \quad , \quad T_i = 0 \quad , \quad \forall 1 \leq i \leq N.$$

Note that the velocity (V_i, Ω_i) of each particle is only determined by equation (33) and the configuration $X^N := (x_1, \dots, x_N)$. We make precise now the set of configurations that we consider and which is propagated in finite time.

Definition 3.1 (Definition of the set of particle configuration). *Let $N \in \mathbb{N}$ and $\lambda^N > 0$ be a positive constant depending only on N . Given X^N a configuration of N particles we define the minimal distance:*

$$d[X^N] := \min_{i \neq j} \{|x_i - x_j|\},$$

and the particle concentration

$$M[X^N, \lambda^N] := \sup_{x \in \mathbb{R}^3} \{\#\{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N)}\}\}.$$

Given three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$, we define $\mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$ as the set of configurations (X^N, λ^N) such that

$$(34) \quad \frac{M[X^N, \lambda^N]}{N|\lambda^N|^3} \leq \bar{M},$$

$$(35) \quad \frac{|\lambda^N|}{N^{2/3}d[X^N]^2} \leq \mathcal{E}_1,$$

$$(36) \quad \frac{|\lambda^N|^2}{d[X^N]} \leq \mathcal{E}_2,$$

Remark 3.1. Note that, according to the definition of $M[X^N, \lambda^N]$, assumption (34) ensures that:

$$(37) \quad \frac{1}{N|\lambda^N|^3} \leq \bar{M},$$

which yields thanks to assumption (35)

$$(38) \quad d[X^N] \geq \frac{1}{\sqrt{\mathcal{E}_1} \bar{M}^{1/6}} \frac{1}{\sqrt{N}}.$$

In what follows, we keep the same notations as before by setting:

$$d_{\min}(t) := d[X^N(t)] \quad , \quad M^N(t) := M[X^N(t), \lambda^N],$$

where $X^N(t) = (x_1(t), \dots, x_N(t))$ for $t \geq 0$.

We will see that, at each fixed time $t \geq 0$, the convergence of the method of reflections toward the unique solution u^N hold true in the case where $(X^N(t), \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$.

Remark 3.2. Unlike the initial assumption (6), the constant \mathcal{E}_2 in Definition 3.1 does not have to be small. Precisely, we do not need to propagate this property in time. Assumption $\frac{|\lambda^N|^2}{d_{\min}(0)} \ll 1$ ensures the convergence of the Wasserstein distance $W_1(\rho^N, \rho)$.

3.1. The method of reflections. In this part, we present and prove the convergence of a modified method of reflections for the velocity field u^N for arbitrary $N \in \mathbb{N}^*$, we remind that u^N is the unique solution to the stationary Stokes problem (1), (2), where (V_i, Ω_i) are the unique velocities satisfying

$$F_i + mg = 0 \quad , \quad T_i = 0 \quad , \quad \forall 1 \leq i \leq N.$$

The main idea is to express u^N as the superposition of N fields produced by the isolated N particle. Thanks to the superposition principle, we know that the velocity field

$$\sum_{i=1}^N (U_{x_j,R}[V_j](x) + A_{x_j,R}[\Omega_j](x)),$$

satisfies a Stokes equation on $\mathbb{R}^3 \setminus \bigcup_i \overline{B_i}$. But this velocity field does not match the boundary conditions of u^N . Indeed, for all $1 \leq i \leq N$ and $x \in B_i$ we have:

$$\begin{aligned} u_*^{(1)}(x) &:= u^N(x) - \sum_{j=1}^N (U_{x_j,R}[V_j](x) + A_{x_j,R}[\Omega_j](x)) \\ &= - \sum_{i \neq j}^N (U_{x_j,R}[V_j](x) + A_{x_j,R}[\Omega_j](x)), \end{aligned}$$

which represents the error committed on the boundary conditions when approaching u^N by the sum of the particular Stokes solutions. In this paper, for all $u_* \in \mathcal{C}^\infty(\bigcup_i \overline{B_i})$ we use the notation $U[u_*]$ to define the unique solution of the Stokes problem

$$(39) \quad \begin{cases} -\Delta u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed by the boundary conditions:

$$(40) \quad \begin{cases} u &= u_*(x), \text{ on } B_i, \\ \lim_{|x| \rightarrow \infty} |u(x)| &= 0, \end{cases}$$

hence, we can write

$$u^N = \sum_{i=1}^N U_{x_i,R}[V_i] + A_{x_i,R}[\Omega_i](x) + U[u_*^{(1)}].$$

Note that the boundary condition $u_*^{(1)}$ is not constant on each particle B_i , thus, the idea is to approach $u_*^{(1)}$ by

$$(41) \quad u_*^{(1)}(x) \sim u_*^{(1)}(x_i) + \nabla u_*^{(1)}(x_i) \cdot (x - x_i),$$

on each particle B_i and write $U[u_*^{(1)}]$ as follows:

$$U[u_*^{(1)}] = \sum_{j=1}^N \left(U_{x_j,R}[V_j^{(1)}] + A_{x_j,R}[\nabla_j^{(1)}] \right) + U[u_*^{(2)}],$$

where

$$V_i^{(1)} := u_*^{(1)}(x_i) = - \sum_{j \neq i} (U_{x_j,R}[V_j](x_i) + A_{x_j,R}[\Omega_j](x_i)),$$

$$\nabla_i^{(1)} := \nabla u_*^{(1)}(x_i) = - \sum_{j \neq i} \left(\nabla U_{x_j, R}[V_j](x_i) + \nabla A_{x_j, R}[\Omega_j](x_i) \right),$$

remark that $\nabla_i^{(1)}$ has null trace due to the fact that

$$\operatorname{div} u_*^{(1)}(x_i) = 0.$$

We have then for all $1 \leq i \leq N$, and $x \in B_i$

$$\begin{aligned} u_*^{(2)}(x) &= u_*^{(1)}(x) - \sum_{j=1}^N \left(U_{x_j, R}[V_j^{(1)}](x) + A_{x_j, R}[\nabla_j^{(1)}](x) \right) \\ &= u_*^{(1)} - V_i^{(1)} - \nabla_i^{(1)}(x - x_i) - \sum_{j \neq i}^N \left(U_{x_j, R}[V_j^{(1)}](x) + A_{x_j, R}[\nabla_j^{(1)}](x) \right). \end{aligned}$$

Which yields the following development:

$$u^N = \sum_{j=1}^N \left(U_{x_j, R}[V_j] + A_{x_j, R}[\Omega_j] \right) + \sum_{j=1}^N \left(U_{x_j, R}[V_j^{(1)}] + A_{x_j, R}[\nabla_j^{(1)}] \right) + U[u_*^{(2)}].$$

We iterate then the process by setting for all $1 \leq i \leq N$

$$(42) \quad V_i^{(0)} := V_i, \quad \nabla_i^{(0)} := \Omega_i,$$

and for $p \geq 1$,

$$(43) \quad V_i^{(p)} := u_*^{(p)}(x_i), \quad \nabla_i^{(p)} := \nabla u_*^{(p)}(x_i),$$

for the error term we set

$$(44) \quad u_*^{(0)}(x) := \sum_i^N (V_i + \Omega_i \times (x - x_i)) \mathbf{1}_{B_i},$$

and define for all $p \geq 0$, $1 \leq i \leq N$, $x \in B_i$

$$\begin{aligned} (45) \quad u_*^{(p+1)}(x) &= u_*^{(p)}(x) - \sum_{j=1}^N \left(U_{x_j, R}[V_j^{(p)}](x) + A_{x_j, R}[\nabla_j^{(p)}](x) \right) \\ &= u_*^{(p)}(x) - u_*^{(p)}(x_i) - \nabla u_*^{(p)}(x_i)(x - x_i) - \sum_{j \neq i}^N \left(U_{x_j, R}[V_j^{(p)}](x) + A_{x_j, R}[\nabla_j^{(p)}](x) \right). \end{aligned}$$

With this construction the following equality holds true for all $k \geq 1$

$$(46) \quad u^N = \sum_{p=0}^k \sum_{j=1}^N \left(U_{x_j, R}[V_j^{(p)}] + A_{x_j, R}[\nabla_j^{(p)}] \right) + U[u_*^{(k+1)}].$$

Remark 3.3. *This method of reflection is obtained by expanding the error term u_* up to the first-order:*

$$u_*(x) = u_*(x_i) + \nabla u_*(x_i)(x - x_i) + o(|x - x_i|^2),$$

which leads us to formula (46). If one consider an expansion of u_ up to the zeroth-order then one obtain only the Stokeslet development:*

$$u^N = \sum_{p=0}^k \sum_{j=1}^N U_{x_j, R}[V_j^{(p)}] + U[u_*^{(k+1)}].$$

The main difference between these two expansions is that the first one allows us to tackle the particle rotation. It also helps us to obtain a converging method of reflections for a more general assumption on the minimal distance. We emphasize that we only need to show that the series $(\sum_{p=0}^k V_i^{(p)})_{k \in \mathbb{N}}$ and $(\sum_{p=0}^k \nabla_i^{(p)})_{k \in \mathbb{N}}$ for all $1 \leq i \leq N$ converge to obtain the convergence of the expansion (46). The second step is to show that the expansion converges to the unique solution u^N . Precisely, the only assumptions needed to obtain the convergence of the series are assumption (7), (34) and the fact that:

$$\frac{|\lambda^N|^2}{N^{1/3}d_{\min}} \ll 1 \quad , \quad \frac{R|\lambda^N|}{N^{2/3}d_{\min}^2} \ll 1,$$

which is less restrictive than (35) and (36), see Lemma 3.2 and Remark 3.4. In addition, the following assumption

$$\frac{(\log M^N)^{1/3}}{N^2 d_{\min}^3} < +\infty,$$

ensures the convergence of the expansion to the velocity field u^N , see Proposition 3.4. One can show that this assumption is less restrictive than (35) and allows us to consider smaller minimal distance. To reach lower bound for the minimal distance, one may develop u_ at higher orders.*

3.1.1. Preliminary estimates. Recall that the dependence in time is implicit in this section. All the following estimates hold true under the assumption that there exists three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N, \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$ (see Definition 3.1).

Lemma 3.2. *Under assumption (7) there exists a constant $K < 1$ such that:*

$$\max_i |V_i^{(p+1)}| + R \max_i |\nabla_i^{(p+1)}| \leq K \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right),$$

for N large enough.

Proof. Using formulas (43) and (45) we get:

$$(47) \quad V_i^{(p+1)} = u_*^{(p+1)}(x_i)$$

$$(48) \quad = - \sum_{j \neq i}^N \left(U_{x_j, R}[V_j^{(p)}](x_i) + A_{x_j, R}[\nabla_j^{(p)}](x_i) \right),$$

and

$$\begin{aligned}\nabla_i^{(p+1)} &= \nabla u_*^{(p+1)}(x_i) \\ &= -\sum_{j \neq i}^N \left(\nabla U_{x_j, R}[V_j^{(p)}](x_i) + \nabla A_{x_j, R}[\nabla_j^{(p)}](x_i) \right).\end{aligned}$$

This yields using the decay-rate of the special solutions (29), (15) and Lemma A.1 with $k = 1$ and $k = 2$

$$\begin{aligned}\max_i |V_i^{(p+1)}| &\leq C \sum_{j \neq i} R \frac{|V_j^{(p)}|}{d_{ij}} + R^3 \frac{|\nabla_j^{(p)}|}{d_{ij}^2} \\ &\leq C \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right) \left(\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + NR + \frac{R|\lambda^N|}{N^{2/3} d_{\min}^2} + R^2 N \right) \bar{M},\end{aligned}$$

similarly we have:

$$\begin{aligned}\max_i |\nabla_i^{(p+1)}| &\leq C \sum_{j \neq i} R \frac{|V_j^{(p)}|}{d_{ij}^2} + R^3 \frac{|\nabla_j^{(p)}|}{d_{ij}^3} \\ &\leq C \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right) \left(\sum_{j \neq i} \frac{R}{d_{ij}^2} + \frac{1}{d_{\min}} \sum_{j \neq i} \frac{R^2}{d_{ij}^2} \right) \\ &= C \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right) \left(\sum_{j \neq i} \frac{R}{d_{ij}^2} \right) \left(1 + \frac{R}{d_{\min}} \right) \\ &\leq C \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right) \left(\frac{|\lambda^N|}{N^{2/3} d_{\min}^2} + NR \right) \bar{M}.\end{aligned}$$

Finally

$$\begin{aligned}\max_i |V_i^{(p+1)}| + R \max_i |\nabla_i^{(p+1)}| &\leq C \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right) \\ &\quad \times \left(\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + NR + \frac{R|\lambda^N|}{N^{2/3} d_{\min}^2} + R^2 N \right) \bar{M},\end{aligned}$$

we have, according to assumptions (35) and (36):

$$\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + \frac{R|\lambda^N|}{N^{2/3} d_{\min}^2} \leq \frac{1}{N^{1/3}} \mathcal{E}_2 + \frac{r_0}{N} \mathcal{E}_1,$$

which vanishes for N large enough. On the other hand, according to assumption (7), $\bar{M}RN = \bar{M}r_0 \ll 1$. This ensures the existence of a positive constant $K < 1$ such that

$$\max_i |V_i^{(p+1)}| + R \max_i |\nabla_i^{(p+1)}| \leq K \left(\max_i |V_i^{(p)}| + R \max_i |\nabla_i^{(p)}| \right).$$

for N large enough and depending on $r_0, \bar{M}, \mathcal{E}_1$ and \mathcal{E}_2 . \square

Remark 3.4. *Note that the following assumptions*

$$\frac{|\lambda^N|^2}{N^{1/3}d_{\min}} \ll 1, \quad \frac{R|\lambda^N|}{N^{2/3}d_{\min}^2} \ll 1,$$

ensures the convergence of the sequences $(V_i^{(p)}, \nabla_i^{(p)})_{p \in \mathbb{N}}$ for all $1 \leq i \leq N$. These assumptions are less restrictive than assumptions (35) and (36).

We have also the following estimates.

Proposition 3.3. *For all $k \geq 1$ we set*

$$\eta^{(k)} := \max_j |V_j^{(k)}| + R \max_j |\nabla_j^{(k)}|.$$

We have for all $1 \leq i \leq N$

$$\begin{aligned} \|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)} &\lesssim \left(\frac{\log M^N}{Nd_{\min}^3} + M^N \frac{|\log \lambda^N|}{N|\lambda^N|^3} \right) \max_i (|V_i| + R|\Omega_i|), \\ \|\nabla u_*^{(k+1)}\|_{L^\infty(B_i)} &\lesssim R \|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)} + \left(\frac{\lambda^N}{N^{2/3}d_{\min}^2} + r_0 \right) \eta^{(k)}, \\ \|u_*^{(k)}\|_{L^\infty(B_i)} &\lesssim R^2 \|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)} + \eta^{(k)}. \end{aligned}$$

Proof.

1. Estimate of $\|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)}$

Let $x \in B_i$, as $R \ll d_{\min}$, we recall that for $i \neq j$

$$|x - x_j| \geq |x_i - x_j| - |x - x_i| \geq \frac{1}{2}d_{ij}.$$

Applying this and the iteration formula (45) together with Lemma A.1 for $k = 3$ yields

$$\begin{aligned} |\nabla^2 u_*^{(k+1)}(x)| &\leq |\nabla^2 u_*^{(k)}(x)| + \sum_{j \neq i} |\nabla^2 U_{x_j, R}[V_j^{(k)}](x)| + |\nabla^2 A_{x_j, R}[\nabla_j^{(k)}](x)| \\ &\lesssim \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \sum_{j \neq i} \frac{|V_j^{(k)}|}{d_{ij}^3} R + \frac{|\nabla_j^{(k)}|}{d_{ij}^4} R^3 \\ &\lesssim \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \left(\sum_{j \neq i} \frac{R}{d_{ij}^3} + \frac{R}{d_{\min}} \sum_{j \neq i} \frac{R}{d_{ij}^3} \right) \left(\max_j |V_j^{(k)}| + R \max_j |\nabla_j^{(k)}| \right) \\ &= \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \left(\sum_{j \neq i} \frac{R}{d_{ij}^3} \right) \left(1 + \frac{R}{d_{\min}} \right) \eta^{(k)} \\ &\lesssim \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \left(\frac{\log M^N}{Nd_{\min}^3} + M^N \frac{|\log \lambda^N|}{N|\lambda^N|^3} \right) \eta^{(k)}, \end{aligned}$$

hence, we iterate the formula and use the fact that $\nabla^2 u_*^{(0)} = 0$ according to formula (44), to get

$$\|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)} \lesssim \left(\frac{\log M^N}{Nd_{\min}^3} + M^N \frac{|\log \lambda^N|}{N|\lambda^N|^3} \right) \sum_{p=0}^{k+1} \eta^{(p)},$$

which yields the expected result by applying Lemma 3.2.

2. Estimate of $\|\nabla u_*^{(k+1)}\|_{L^\infty(B_i)}$

Let $x \in B_i$, again, formula (45) and Lemma A.1 yields

$$\begin{aligned} |\nabla u_*^{(k+1)}(x)| &\leq |\nabla u_*^{(k)}(x) - \nabla u_*^{(k)}(x_i)| + \sum_{j \neq i} |\nabla U_{x_j, R}[V_j^{(p)}](x_i)| + |\nabla A_{x_j, R}[\nabla_j^{(p)}](x_i)| \\ &\lesssim R \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \sum_{j \neq i} \frac{|V_j^{(k)}|}{d_{ij}^2} R + \frac{|\nabla_j^{(k)}|}{d_{ij}^3} R^3 \\ &\lesssim R \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \left(\sum_{j \neq i} \frac{R}{d_{ij}^2} + \frac{R}{d_{\min}} \sum_{j \neq i} \frac{R}{d_{ij}^2} \right) \left(\max_j |V_j^{(k)}| + R \max_j |\nabla_j^{(k)}| \right) \\ &\lesssim R \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \left(1 + \frac{R}{d_{\min}} \right) \left(\frac{\lambda^N}{N^{2/3} d_{\min}^2} + RN \right) \bar{M} \eta^{(k)}, \end{aligned}$$

again note that for N large enough, $1 + \frac{R}{d_{\min}} \leq 2$.

3. Estimate of $\|u_*^{(k+1)}\|_{L^\infty(B_i)}$

Let $x \in B_i$, again, formula (45) yields

$$\begin{aligned} |u_*^{(k+1)}(x)| &\leq R^2 \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \sum_{j \neq i} |U_{x_j, R}[V_j^{(p)}](x)| + |A_{x_j, R}[\nabla_j^{(p)}](x)| \\ &\lesssim R^2 \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \sum_{j \neq i} \frac{|V_j^{(k)}|}{d_{ij}} R + \frac{|\nabla_j^{(k)}|}{d_{ij}^2} R^3 \\ &\lesssim R^2 \|\nabla^2 u_*^{(k)}\|_{L^\infty(B_i)} + \bar{M} \left(\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + RN + \frac{R\lambda^N}{N^{2/3} d_{\min}^2} + R^2 N \right) \eta^{(k)}. \end{aligned}$$

Since $\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + \frac{R\lambda^N}{N^{2/3} d_{\min}^2}$ is bounded, the right hand side can be bounded by $\eta^{(k)}$. Note that this is ensured by the same assumptions as before (see Remark 3.4). \square

3.1.2. *Convergence result.* We can now state the main result of this section.

Proposition 3.4. *Given $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N, \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$, there exists a positive constant $C = C(\bar{r}_0, \bar{M}, \mathcal{E}_1, \mathcal{E}_2)$ satisfying*

$$\lim_{k \rightarrow \infty} \|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)} \leq CR \max_i (|V_i| + R|\Omega_i|),$$

for N large enough.

Proof. The aim is to estimate $\|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)}$. To this end, we construct a suitable extension $E[u_*^{(k+1)}]$ of the boundary conditions $u_*^{(k+1)}$ and apply the variational principle (10). By construction, $u_*^{(k+1)}$ is regular and well defined on each particle $B(x_i, R)$. Hence, we construct the extension piecewise in each $B(x_i, 2R)$. Let $1 \leq i \leq N$, for all $x \in B(x_i, 2R)$ we set

$$v^i(x) := u_1^{(i)}(x) + u_2^{(i)}(x),$$

where the first term $u_1^{(i)}$ matches the boundary condition on $B(x_i, R)$ and vanishes outside $B(x_i, 2R)$. The second term is the correction needed to get $\operatorname{div} v_i = 0$. In order to obtain an extension of $u_*^{(k)}$ on $B(x_i, 2R)$ we set

$$\begin{aligned} u_1^{(i)}(x) &= u_*^{(k)}\left(x_i + R \frac{x - x_i}{|x - x_i|}\right) \chi\left(\left|\frac{x - x_i}{R}\right|\right), \text{ if } |x - x_i| \geq R, \\ u_1^{(i)}(x) &= u_*^{(k)}(x), \text{ if } x \in B(x_i, R), \end{aligned}$$

with χ a truncation function such that $\chi = 1$ on $[0, 1]$ and $\chi = 0$ outside $[0, 2]$. We have then

$$(49) \quad \|\nabla u_1^{(i)}\|_{L^\infty(B(x_i, 2R))} \leq K_\chi \left(\|\nabla u_*^{(k)}\|_{L^\infty(B(x_i, R))} + \frac{1}{R} (\|u_*^{(k)}\|_{L^\infty(B(x_i, R))}) \right).$$

In what follows we introduce the notation $A(x, r, R) := B(x, R) \setminus \overline{B(x, r)}$ for $r < R$. For the second term we set:

$$u_i^{(2)} = \mathcal{B}_{x_i, R, 2R}(-\operatorname{div} u_i^{(1)}),$$

where \mathcal{B} is the Bogovskii operator (see [9, Appendix A Lemma 15 and 16] for more details).

The construction satisfies:

- $\operatorname{supp} u_i^{(2)} \subset A(x_i, R, 2R)$
- $\operatorname{div} v_i = 0$
- $v_i = u_i^{(1)} = u_*^{(k)}$ on $B(x_i, R)$

We set then

$$E[u_*^{(k+1)}] = \sum_i^N v^i(x) 1_{B(x_i, 2R)},$$

and thanks to the variational formulation we have

$$\begin{aligned} \|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)}^2 &\leq \|\nabla E[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)}^2 \\ &= \sum_i^N \|\nabla v_i\|_{L^2(A(x_i, R, 2R))}^2, \end{aligned}$$

where we used the fact that the v_i have disjoint support.

Thanks to the properties of the Bogovskii operator $\mathcal{B}_{x_i, R, 2R}$ we get:

$$\begin{aligned} \|\nabla v_i\|_{L^2(B(x_i, R))}^2 &\lesssim \int_{A(x_i, R, 2R)} |\nabla u_1^{(i)}|^2 \\ &\lesssim R^3 \|\nabla u_1^{(i)}\|_{L^\infty(B(A(x_i, R, 2R)))}^2 \\ &\lesssim R^3 \left(\|\nabla u_*^{(k)}\|_{L^\infty(B_i)} + \frac{1}{R} \|u_*^{(k)}\|_{L^\infty(B_i)} \right)^2. \end{aligned}$$

Finally

$$\|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)}^2 \lesssim \sum_i^N R^3 \left(\|\nabla u_*^{(k)}\|_{L^\infty(B_i)} + \frac{1}{R} \|u_*^{(k)}\|_{L^\infty(B_i)} \right)^2.$$

Thanks to Proposition 3.3 we have

$$\begin{aligned} \|\nabla u_*^{(k)}\|_{L^\infty(B_i)} + \frac{1}{R} \|u_*^{(k)}\|_{L^\infty(B_i)} &\lesssim \max_i (|V_i| + R|\Omega_i|) \left(\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \right) \\ &\quad + \left(\frac{1}{R} + \frac{\lambda^N}{N^{2/3} d_{\min}^2} \right) \eta^{(k)}. \end{aligned}$$

Since

$$\eta^{(k)} \leq K^k \max_i (|V_i| + R|\Omega_i|),$$

with $K < 1$ according to Lemma (3.2), we get

$$\begin{aligned} \|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)}^2 &\lesssim \max_i (|V_i| + R|\Omega_i|)^2 \left\{ R \left(\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \right) \right. \\ &\quad \left. + \left(1 + \frac{R\lambda^N}{N^{2/3} d_{\min}} \right) K^k \right\}^2. \end{aligned}$$

Since $K < 1$, for N large enough, the second term vanishes when $k \rightarrow \infty$. This yields

$$\lim_{k \rightarrow \infty} \|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)} \lesssim R \max_i (|V_i| + R|\Omega_i|) \left(\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \right).$$

The second term on the right hand side can be bounded using assumptions (34) and (35)

$$\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \leq \frac{\log N}{N^2 d_{\min}^3} + \frac{\bar{M} \log(CN)}{N} \lesssim 1,$$

where we used the fact that $d_{\min} \geq \frac{1}{M^{1/6} \sqrt{\mathcal{E}_1}} \frac{1}{N^{1/2}}$. Finally we obtain the convergence result

$$\lim_{k \rightarrow \infty} \|\nabla U[u_*^{(k+1)}]\|_{L^2(\mathbb{R}^3 \setminus \cup B_i)} \lesssim R \max_i (|V_i| + R|\Omega_i|).$$

□

Remark 3.5. *In addition of assumption (34), the only assumptions needed to obtain the convergence of the method of reflections to the unique solution u^N are*

$$\frac{\log M^N}{N^2 d_{\min}^3} < +\infty \quad , \quad \frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + \frac{R \lambda^N}{N^{2/3} d_{\min}^2} < +\infty$$

which are less restrictive than assumption (35) and (36).

Remark 3.6. *According to Proposition 3.3 we have for all $1 \leq i \leq N$*

$$\begin{aligned} \|u_*^{(k+1)}\|_{L^\infty(B_i)} &\lesssim R^2 \|\nabla^2 u_*^{(k+1)}\|_{L^\infty(B_i)} + \eta^{(k)} \\ &\lesssim \max_i (|V_i| + R|\Omega_i|) \left\{ R \left(\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \right) + K^k \right\}. \end{aligned}$$

as for the proof of Proposition 3.4 the second term vanishes when $k \rightarrow \infty$ and we obtain

$$\lim_{k \rightarrow \infty} \|u_*^{(k+1)}\|_{L^\infty(B_i)} \lesssim \max_i (|V_i| + R|\Omega_i|) R.$$

3.1.3. Some associated estimates. We recall that we aim to compute the velocities (V_i, Ω_i) associated to the unique solution u^N of the Stokes equation:

$$\begin{cases} -\Delta u^N + \nabla p^N &= 0, \\ \operatorname{div} u^N &= 0, \end{cases} \quad \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed with the no-slip boundary conditions:

$$\begin{cases} u^N &= V_i + \Omega_i \times (x - x_i), \text{ on } \partial B_i, \\ \lim_{|x| \rightarrow \infty} |u^N(x)| &= 0. \end{cases}$$

With

$$F_i + mg = 0 \quad , \quad T_i = 0 \quad , \quad \forall 1 \leq i \leq N.$$

The method of reflections obtained in this section helps us to describe the velocity field u^N in terms of explicit flows

$$u^N = \lim_{k \rightarrow \infty} \sum_{j=1}^N \left(U_{x_j, R} \left[\sum_{p=0}^k V_j^{(p)} \right] + A_{x_j, R} \left[\sum_{p=0}^k \nabla_j^{(p)} \right] \right).$$

In order to extract a formula for the unknown velocities (V_i, Ω_i) , $1 \leq i \leq N$ we need to compute the series

$$V_i^\infty := \sum_{p=0}^{\infty} V_i^{(p)} \quad , \quad \nabla_i^\infty := \sum_{p=0}^{\infty} \nabla_i^{(p)}.$$

Applying the method of reflections and writing the force, torque and stress associated to the unique solution u^N in two different ways we get the following result.

Lemma 3.5. *For all $1 \leq i \leq N$:*

$$\begin{aligned} V_i^\infty &= \kappa g + O\left(\max_i(|V_i| + R[\Omega_i]) \frac{R}{\sqrt{d_{\min}}}\right). \\ R(|\operatorname{asym}(\nabla_1^\infty)| + |\operatorname{sym}(\nabla_1^\infty)|) &= O\left(\max_i(|V_i| + R[\Omega_i]) \frac{R}{\sqrt{d_{\min}}}\right). \end{aligned}$$

Proof. For the sake of clarity we fix $i = 1$ and the same result holds for all $1 \leq i \leq N$. Let $V \in \mathbb{R}^3$, D a trace-free 3×3 matrix.

The main idea is to apply an integration by parts with a suitable test function $v \in D_\sigma(\mathbb{R}^3 \setminus \bigcup_i \overline{B_i})$ such that $v = V + D(x - x_1)$ on ∂B_1 and $v = 0$ on the other ∂B_j , $j \neq 1$.

We choose v the unique solution to the Stokes equation:

$$(50) \quad \begin{cases} -\Delta v + \nabla p &= 0, \\ \operatorname{div} v &= 0, \end{cases} \text{ on } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

completed by the boundary conditions:

$$(51) \quad \begin{cases} v &= V + D(x - x_1), \text{ on } \partial B_1, \\ v &= 0 \text{ on } \partial B_i, i \neq 1, \\ \lim_{|x| \rightarrow \infty} |v(x)| &= 0. \end{cases}$$

We extend u^N and v by their boundary values on all B_i , $1 \leq i \leq N$. We set $E = \operatorname{sym}(D)$, $\Omega = \operatorname{asym}(D)$. An integration by parts yields:

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(u^N) : D(v) &= - \sum_i \int_{\partial B_i} [\sigma(u^N, p^N)n] \cdot v \\ &= - \int_{\partial B_1} [\sigma(u^N, p^N)n] \cdot (V + \Omega \times (x - x_1) + E(x - x_1)) \\ &= -V \cdot \int_{\partial B_1} \sigma(u^N, p^N)n - \Omega \cdot \int_{\partial B_1} (x - x_1) \times [\sigma(u^N, p^N)n] \\ &\quad - E : \int_{\partial B_1} (x - x_1) \otimes [\sigma(u^N, p^N)n] \\ (52) \quad &= -V \cdot F_1 - \Omega \cdot T_1 - E : S_1, \end{aligned}$$

see (11) and (12) for the definition of the force F_1 , torque T_1 and strain S_1 . On the other hand, we apply the method of reflections to get

$$\begin{aligned} (53) \quad \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(u^N) : D(v) &= \\ \sum_{j=1}^N \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} (D(U_{x_j, R}[V_j^\infty]) + D(\nabla A_{x_j, R}[\nabla_j^\infty])) : D(v) &+ \lim_{k \rightarrow \infty} \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : D(v). \end{aligned}$$

For the first term we integrate by parts to get for all $1 \leq j \leq N$

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U_{x_j,R}[V_j^\infty]) : D(v) &= - \sum_{i=1}^N \int_{\partial B_i} [\sigma(U_{x_j,R}[V_j^\infty], P_{x_j,R}[V_j^\infty])n] \cdot v. \\ \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(A_{x_j,R}[\nabla_j^\infty]) : D(v) &= - \sum_{i=1}^N \int_{\partial B_i} [\sigma(A_{x_j,R}[\nabla_j^\infty], P_{x_j,R}[\nabla_j^\infty])n] \cdot v. \end{aligned}$$

Recall that v vanishes on ∂B_i , $i \neq 1$ and hence, the sums above are reduced to the first term. Applying (25) (22) and (16) there holds for all $1 \leq j \leq N$

$$\begin{aligned} \int_{\partial B_1} [\sigma(U_{x_j,R}[V_j^\infty], P_{x_j,R}[V_j^\infty])n] \cdot v &= -6\pi R V_1^\infty \cdot V \delta_{1j}, \\ \int_{\partial B_1} [\sigma(A_{x_j,R}[\nabla_j^\infty], P_{x_j,R}[\nabla_j^\infty])n] \cdot v &= -\pi R^3 \left(8 \operatorname{asym}(\nabla_1^\infty) \cdot \Omega + \frac{20}{3} \operatorname{sym}(\nabla_1^\infty) : E \right) \delta_{1j}, \end{aligned}$$

where δ_{1j} is the Kronecker symbol.

For the second term we consider $v_1 := U_{x_1,R}[V] + A_{x_1,R}[D]$ and write

$$\int_{\mathbb{R}^3 \setminus \bigcup_i B_i} DU[u_*^k] : D(v) = \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : D(v_1) + \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : D(v - v_1).$$

To bound the last term we apply Lemma 2.2 and the convergence result for the method of reflections 3.4

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : D(v - v_1) \right| &\lesssim \max_i (|V_i| + R[\Omega_i]) R \left(\frac{R}{\sqrt{d_{\min}}} |V| + \frac{R^3}{d_{\min}^{3/2}} |D| \right) \\ &\lesssim \frac{R^2}{\sqrt{d_{\min}}} (|V| + R|D|) \max_i (|V_i| + R[\Omega_i]). \end{aligned}$$

We focus now on the first term, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} DU[u_*^{(k)}] : D(v_1) \right| &= \left| \sum_i \int_{\partial B_i} [\sigma(v_1, p_1) \cdot n] \cdot u_*^{(k)} \right| \\ &\leq \sum_i 4\pi R^2 \|\sigma(v_1, p_1)\|_{L^\infty(B_i)} \|u_*^{(k)}\|_{L^\infty(B_i)}, \end{aligned}$$

recall that:

$$\begin{aligned} \|\sigma(v_1, p_1)\|_{L^\infty(B_i)} &\lesssim \frac{R|V|}{d_{i1}^2} + \frac{R^3}{d_{i1}^3} |D|, \text{ for } i \neq 1, \\ \|\sigma(v_1, p_1)\|_{L^\infty(B_1)} &\lesssim \frac{|V|}{R} + |D|, \end{aligned}$$

hence

$$\begin{aligned}
\left| \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^{(k)}]) : D(v_1) \right| &\lesssim R(|V| + R|\Omega|) \|u_*^{(k)}\|_{L^\infty(B_1)} \\
&+ R \sum_{i \neq 1} \left(\frac{R^2|V|}{d_{i1}^2} + \frac{R^4|D|}{d_{i1}^3} \right) \max_i \|u_*^{(k)}\|_{L^\infty(B_i)} \\
&\lesssim R(|V| + R|D|) \max_i \|u_*^{(k)}\|_{L^\infty(B_i)}.
\end{aligned}$$

According to Remark 3.6, we have for all $1 \leq i \leq N$

$$\lim_{k \rightarrow \infty} \|u_*^{(k)}\|_{L^\infty(B_i)} \lesssim R \max_i (|V_i| + R|\Omega_i|).$$

Finally we get

$$\begin{aligned}
(54) \quad \lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : \nabla v_1 \right| &+ \left| \int_{\mathbb{R}^3 \setminus \bigcup_i B_i} D(U[u_*^k]) : \nabla(v - v_1) \right| \lesssim \\
&\max_i (|V_i| + R|\Omega_i|) \frac{R^2}{\sqrt{d_{\min}}} (|V| + R|D|).
\end{aligned}$$

Identifying formula (52) and (53) and gathering all the inequalities above we have for all $V, \Omega \in \mathbb{R}^3$:

$$\begin{aligned}
&-V \cdot F_1 - \Omega \cdot T_1 - E : E_1 = \\
&6\pi R V_1^\infty \cdot V + 8\pi R^3 \text{asym}(\nabla_1^\infty) \cdot \Omega + O \left(\max_i (|V_i| + R|\Omega_i|) \frac{R^2}{\sqrt{d_{\min}}} (|V| + R|D|) \right),
\end{aligned}$$

with $F_1 + mg = 0$, $T_1 = E_1 = 0$. We conclude by identifying the terms involving $V \in \mathbb{R}^3$ to obtain

$$V_i^\infty := \sum_{p=0}^{\infty} V_i^{(p)} = \frac{m}{6\pi R} g + O \left(\max_i (|V_i| + R|\Omega_i|) \frac{R}{\sqrt{d_{\min}}} \right),$$

and analogously we obtain

$$R(|\text{asym}(\nabla_1^\infty)| + |\text{sym}(\nabla_1^\infty)|) = O \left(\max_i (|V_i| + R|\Omega_i|) \frac{R}{\sqrt{d_{\min}}} \right),$$

which concludes the proof. \square

Corollary 3.6. *There exists a positive constant independent of N such that for all $1 \leq i \leq N$ we have*

$$\max_{1 \leq i \leq N} (|V_i| + R|\Omega_i|) < C,$$

Proof. recall that $V_i^{(0)} = V_i$, $\nabla_i^{(0)} = \Omega_i$ for all $1 \leq i \leq N$, according to Lemma 3.5 and Lemma 3.2 we obtain for all $1 \leq i \leq N$.

$$\begin{aligned} |V_i| + R|\Omega_i| &\leq |V_i^\infty| + R|\nabla_i^\infty| + \sum_{p=1}^{\infty} \left(|V_i^{(p)}| + R|\nabla_i^{(p)}| \right) \\ &\leq |V_i^\infty| + R|\nabla_i^\infty| + K \left(\sum_{p=0}^{\infty} K^p \right) \max_i (|V_i| + R|\Omega_i|) \\ &\lesssim \kappa|g| + \left(\frac{R}{\sqrt{d_{\min}}} + \frac{K}{1-K} \right) \max_i (|V_i| + R|\Omega_i|). \end{aligned}$$

Hence, according to assumption (7), if $\bar{M}r_0$ is small enough we can choose $K < \frac{1}{2}$. Moreover, assumptions (35) and (36) ensures that

$$\frac{R}{\sqrt{d_{\min}}} \lesssim \min \left(\frac{\mathcal{E}_2^{1/2}}{N^{1/3}}, \frac{\mathcal{E}_1^{1/4}}{N^{3/4}} \right) \ll 1.$$

This completes the proof. \square

3.2. Extraction of the first order terms for the velocities (V_i, Ω_i) . In order to control the motion of the particles, we want to provide a good approximation of the unknown velocities (V_i, Ω_i) . Thanks to the method of reflections, the velocity field u^N can be approached by a superposition of analytical solutions to a Stokes flow generated by a translating, a rotating and a straining sphere (See Proposition 3.4) with the associated velocities $(V_i^\infty, \nabla_i^\infty)$. This allows us to compute the first order terms for (V_i, Ω_i) applying 3.5 and Corollary 3.6. Keeping in mind that all the computations are done for a fixed time $t \geq 0$, the main result of this section is the following Proposition:

Proposition 3.7. *Assume that for a fixed time $t \geq 0$ we have the existence of three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N, \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$. Assume moreover that assumption (7) is satisfied. Then, for all $1 \leq i \leq N$ we have*

$$V_i = \kappa g + 6\pi R \sum_{j \neq i}^N \Phi(x_i - x_j) \kappa g + O(d_{\min}) \quad , \quad R\Omega_i = O(d_{\min}),$$

for N large enough.

We begin by the following lemma:

Lemma 3.8. *For all trace-free 3×3 matrices $(D_i)_{1 \leq i \leq N}$, for all $W \in \mathbb{R}^3$ and $1 \leq i \leq N$ we have:*

$$\begin{aligned} \sum_{j \neq i}^N |6\pi R \Phi(x_i - x_j) W - U_{x_j, R}[W](x_i)| &\lesssim R|W|. \\ \sum_{j \neq i}^N |A_{x_j, R}[D_j](x_i)| &\lesssim R \left(1 + \frac{\lambda^N}{N^{2/3} d_{\min}^2} \right) \max_j R|D_j|. \end{aligned}$$

Proof. Thanks to formula (17) We have for $i \neq j$

$$U_{x_j,R}[W](x_i) = 6\pi R\Phi(x_j - x_i)W + \frac{1}{4} \frac{R^3}{|x_j - x_i|^3} W - \frac{3}{4} R^3 \frac{(x_j - x_i) \cdot W}{|x_j - x_i|^5} (x_j - x_i),$$

this yields

$$|U_{x_j,R}[W](x_i) - 6\pi R\Phi(x_j - x_i)W| \lesssim \frac{R^3}{d_{ij}^3} |W|.$$

Applying Lemma A.1 with $k = 3$ yields

$$\sum_{j \neq i}^N |U_{x_j,R}[W](x_i) - 6\pi R\Phi(x_j - x_i)W| \lesssim \sum_{j \neq i}^N \frac{R^3}{d_{ij}^3} |W| \lesssim R \left(\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \right) |W|.$$

We have thanks to assumptions (34), (35)

$$\frac{\log M^N}{N^2 d_{\min}^3} + M^N \frac{|\log \lambda^N|}{N^2 |\lambda^N|^3} \leq \frac{\log N}{N^2 d_{\min}^3} + \frac{\bar{M} \log CN}{N} \lesssim 1,$$

where we used the fact that $d_{\min} \geq \frac{1}{\bar{M}^{1/6} \varepsilon_1^{1/2}} \frac{1}{N^{1/2}}$ according to assumption (35). Analogously, we obtain the second bound by applying A.1 with $k = 2$ this time. \square

We can now prove the main result.

Proof. Let fix $1 \leq i \leq N$. According to Lemma 3.5 and Corollary 3.6 we have

$$V_i^\infty := \sum_{p=0}^{\infty} V_i^{(p)} = \frac{m}{6\pi R} g + O\left(\frac{R}{\sqrt{d_{\min}}}\right).$$

As $V_i^{(0)} = V_i$ we get:

$$V_i = -\sum_{p=1}^{\infty} V_i^{(p)} + \frac{m}{6\pi R} g + O\left(\frac{R}{\sqrt{d_{\min}}}\right).$$

Formula (47) for the velocities $V_j^{(p)}$ yields

$$\begin{aligned} V_i &= \sum_{p=1}^{\infty} \sum_{j \neq i} \left(U_{x_j,R}[V_j^{(p-1)}](x_i) + A_{x_j,R}[\nabla_j^{(p-1)}](x_i) \right) + \frac{m}{6\pi R} g + O\left(\frac{R}{\sqrt{d_{\min}}}\right) \\ &= \frac{m}{6\pi R} g + \sum_{j \neq i} \left(U_{x_j,R}[V_j^\infty](x_i) + A_{x_j,R}[\nabla_j^\infty](x_i) \right) + O\left(\frac{R}{\sqrt{d_{\min}}}\right), \end{aligned}$$

we apply Lemma 3.8 and Lemma 3.5 to get:

$$\begin{aligned} \sum_{j \neq i} |A_{x_j, R}[\nabla_j^\infty](x_i)| &\lesssim \left(R + \frac{R\lambda^N}{N^{2/3}d_{\min}^2} \right) \max_j R |\nabla_j^\infty| \\ &\lesssim \left(R + \frac{R\lambda^N}{N^{2/3}d_{\min}^2} \right) \frac{R}{\sqrt{d_{\min}}} \\ &\lesssim d_{\min} \left(\frac{R}{d_{\min}^{3/2}} \right). \end{aligned}$$

As for the proof of the convergence of the method of reflections, we used the fact that $\frac{R\lambda^N}{N^{2/3}d_{\min}^2}$ is uniformly bounded. Moreover, according to assumption (35) and (37)

$$\frac{R}{d_{\min}^{3/2}} \leq r_0 \mathcal{E}_2^{3/2} \frac{1}{N|\lambda^N|^3} \leq \bar{M} r_0 \mathcal{E}_2^{3/2}.$$

Now, we rewrite the sum on the Stokeslets as follows:

$$\sum_{j \neq i} U_{x_j, R}[V_j^\infty](x_i) = \sum_{j \neq i} U_{x_j, R}[\kappa g](x_i) + \sum_{j \neq i} U_{x_j, R}[V_j^\infty - \kappa g](x_i),$$

and we bound the error term using Lemma 3.5 and Lemma A.1 with $k = 1$

$$\begin{aligned} \left| \sum_{j \neq i} U_{x_j, R}[V_j^\infty - \kappa g](x_i) \right| &\lesssim \left(\sum_{j \neq i} \frac{R}{d_{ij}} \right) \max_j |V_j^\infty - \kappa g| \\ &\lesssim \frac{R}{\sqrt{d_{\min}}} \\ &\lesssim d_{\min}. \end{aligned}$$

We conclude by replacing the Stokeslets by the Oseen tensor thanks to Lemma 3.8. Finally we have for all $1 \leq i \leq N$

$$V_i = \kappa g + 6\pi R \sum_{j \neq i}^N \Phi(x_i - x_j) \kappa g + O(d_{\min}).$$

For the angular velocities we obtain thanks to Lemma 3.5 and formula (43) for $\nabla_1^{(p)}$, $p \geq 1$

$$\begin{aligned} R\Omega_1 &= -\sum_{p=1}^{\infty} R \text{asym} \nabla_1^{(p)} + O\left(\frac{R}{\sqrt{d_{\min}}}\right) \\ &= R \text{asym} \left(\sum_{j \neq 1} \nabla U_{x_j, R}[V_j^\infty](x_1) + \nabla A_{x_j, R}[\nabla_j^\infty](x_1) \right) + O\left(\frac{R}{\sqrt{d_{\min}}}\right). \end{aligned}$$

As before, we bound the first term by

$$\begin{aligned}
R \left| \sum_{j \neq 1} \nabla U_{x_j, R}[V_j^\infty](x_1) + \nabla A_{x_j, R}[\nabla_j^\infty](x_1) \right| &\lesssim R \left(\sum_{j \neq 1} \frac{R}{d_{1j}^2} + \frac{R^2}{d_{1j}^3} \right) \max_j (|V_j^\infty|, R|\nabla_j^\infty|) \\
&\lesssim R \left(\sum_{j \neq 1} \frac{R}{d_{1j}^2} \right) \left(1 + \frac{R}{d_{\min}} \right) \max_j (|V_j^\infty|, R|\nabla_j^\infty|) \\
&\lesssim R \left(\frac{\lambda^N}{N^{2/3} d_{\min}^2} + RN \right) \\
&\lesssim d_{\min} \left(\frac{R\lambda^N}{N^{2/3} d_{\min}^3} + r_0 \right),
\end{aligned}$$

where we used the fact that $\frac{R\lambda^N}{N^{2/3} d_{\min}^3}$ is uniformly bounded according to (35). \square

4. CONTROL OF THE PARTICLE DISTANCE AND CONCENTRATION

In this section, we make precise the particle behaviour in time. Precisely we want to prove that if initially there exists three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N(0), \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$ (see Definition 3.1), then the same holds true for a finite time. Recall that the initial distribution of particles satisfies:

- The minimal distance is at least of order $\max \left(\frac{|\lambda^N|^{1/2}}{N^{1/3}}, |\lambda^N|^2 \right)$.
- The maximal number of particles concentrated in a cube of width λ^N satisfies assumption (34).

We aim to show that the particle distance and concentration stay at the same order in finite time. The idea is to use a Gronwall argument and the computation of the velocities $(V_i)_{1 \leq i \leq N}$ at each fixed time $t \geq 0$.

4.1. Proof of Theorem 1.1. We assume that initially there exists three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N(0), \lambda^N) \in \mathcal{X}(\bar{M}, \mathcal{E}_1, \mathcal{E}_2)$. Let $T > 0$ be such that

$$(55) \quad d_{ij}(t) \geq \frac{1}{2} d_{ij}(0), \forall 1 \leq i \neq j \leq N, \forall t \in [0, T[.$$

This maximal time $T > 0$ exists and we aim to prove that it is independent of N . As long as $t < T$ we have a control on the particle concentration.

Lemma 4.1 (Control of particle concentration M^N). *As long as $t \in [0, T[$ we have:*

$$M^N(t) \leq 8^4 M^N(0).$$

Proof. We recall the definition of $M^N(t)$:

$$M^N(t) := \sup_{x \in \mathbb{R}^3} \left\{ \# \{ i \in \{1, \dots, N\} \text{ such that } x_i(t) \in \overline{B_\infty(x, \lambda^N)} \} \right\}.$$

We introduce the following quantity:

$$(56) \quad L^N(t) := \max_i \# \{ j \in \{1, \dots, N\} \text{ such that } |x_i(t) - x_j(t)|_\infty \leq \lambda^N \}.$$

One can show that the two definitions of concentration $L^N(t)$ and $M^N(t)$ are equivalent in the sense that:

$$L^N(t) \leq M^N(t) \leq 8L^N(t)$$

see Lemma A.2 for the proof. We also need to introduce the following notation for all $\beta > 0$:

$$L_\beta^N(t) := \max_i \# \{j \in \{1, \dots, N\} \text{ such that } |x_i(t) - x_j(t)|_\infty \leq \beta \lambda^N\},$$

and

$$M_\beta^N(t) := \sup_x \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i(t) \in \overline{B_\infty(x, \beta \lambda^N)}\} \right\},$$

with the notation

$$M_1^N(t) := M^N(t) \text{ , } L_1^N(t) := L^N(t).$$

We have for all $\beta > 0$ and all $\alpha > 1$:

$$L_{\alpha\beta}^N(t) \leq 8\lceil\alpha\rceil^3 L_\beta^N(t),$$

where $\lceil \cdot \rceil$ denotes the ceiling function. See Corollary A.3 for the proof.

The idea is to show that the concentration L^N is controlled in time and hence, the same applies to M^N according to Lemma A.2. Recall that we have for all $t \in [0, T[$:

$$d_{ij}(t) \geq \frac{1}{2} d_{ij}(0).$$

Now, fix $1 \leq i \leq N$ and consider $j \neq i$ satisfying $|x_i(0) - x_j(0)|_\infty > \lambda^N$, then

$$\begin{aligned} |x_i(t) - x_j(t)|_\infty &\geq \frac{1}{\sqrt{3}} |x_i(t) - x_j(t)| \\ &\geq \frac{1}{2\sqrt{3}} |x_i(t) - x_j(0)| \\ &> \frac{\lambda^N}{2\sqrt{3}}. \end{aligned}$$

Which means that

$$j \notin \left\{ 1 \leq k \leq N, \text{ such that } |x_i(t) - x_k(t)| \leq \frac{\lambda^N}{2\sqrt{3}} \right\}.$$

We obtain

$$\begin{aligned} (57) \quad &\left\{ 1 \leq j \leq N, \text{ such that } |x_i(t) - x_j(t)| \leq \frac{\lambda^N}{2\sqrt{3}} \right\} \\ &\subset \left\{ 1 \leq j \leq N, \text{ such that } |x_i(0) - x_j(0)| \leq \lambda^N \right\}. \end{aligned}$$

Hence taking the maximum over $1 \leq i \leq N$ we obtain

$$L_{\frac{1}{2\sqrt{3}}}^N(t) \leq L^N(0),$$

thus, we apply Lemma A.3 with $\beta = \frac{1}{2\sqrt{3}}$ and $\alpha = \beta^{-1} = 2\sqrt{3}$ to get

$$L^N(t) \leq 8^3 L^N(0).$$

According to Lemma A.2, the equivalence between M^N and L^N yields finally for all $t \in [0, T[$

$$M^N(t) \leq 8^4 M^N(0).$$

□

This shows that as long as $t < T$ we have $(X^N(t), \lambda^N) \in \mathcal{X}(8^4 \bar{M}, 4\mathcal{E}_1, 2\mathcal{E}_2)$. This implies the following control.

Proposition 4.2. *Given a fixed time $t \geq 0$, assume that there exists three positive constants $\bar{M}, \mathcal{E}_1, \mathcal{E}_2$ such that $(X^N(t), \lambda^N) \in \mathcal{X}(8^4 \bar{M}, 4\mathcal{E}_1, 2\mathcal{E}_2)$. Then, there exists a positive constant $C > 0$ independent of N such that for all $i \neq j$ we have for N large enough:*

$$|V_i - V_j| \leq C d_{ij},$$

with $C = C(r_0, \bar{M}, \mathcal{E}_1, \mathcal{E}_2)$.

Proof. For the sake of clarity we fix $i = 1$ and $j = 2$. The computations below are independent of this choice. Thanks to Proposition 3.7 we obtain :

$$V_1 - V_2 = 6\pi R \sum_{i \neq 1, 2}^N (\Phi(x_1 - x_i) - \Phi(x_2 - x_i)) \kappa g + O(d_{\min}).$$

Hence, according to assumption (35) we obtain:

$$\begin{aligned} |V_1 - V_2| &\lesssim R \sum_{i \neq 1, 2}^N \left(\frac{1}{d_{1i}^2} + \frac{1}{d_{2i}^2} \right) |x_1 - x_2| + O(d_{\min}) \\ &\lesssim \bar{M} \left(\frac{\lambda^N}{N^{2/3} d_{\min}^2} + RN \right) |x_1 - x_2| + O(d_{\min}) \\ &\lesssim d_{12}. \end{aligned}$$

We set then $C > 0$ the universal constant implicit in the above estimate. □

We have the following control.

Lemma 4.3 (Control of particle distance). *For all $1 \leq i \neq j \leq N$, for all $t \in [0, T[$ we have:*

$$d_{ij}(t) \geq d_{ij}(0) e^{-Ct}.$$

Proof. Thanks to (55) and Lemma 4.1 we have for all $t < T$ that

$$(X^N(t), \lambda^N) \in \mathcal{X}(8^4 \bar{M}, 4\mathcal{E}_1, 2\mathcal{E}_2).$$

Hence, all computations from proposition 4.2 hold true up to time T . In other words, there exists a positive constant $C = C(r_0, \bar{M}, \mathcal{E}_1, \mathcal{E}_2)$ such that for all indices $1 \leq i \neq j \leq N$ we have

$$|V_i(t) - V_j(t)| \leq C d_{ij}(t) \forall t \in [0, T[,$$

thus,

$$\begin{aligned} \frac{d}{dt}d_{ij}(t) &\geq -|V_i(t) - V_j(t)|, \\ &\geq -C d_{ij}(t). \end{aligned}$$

This entails

$$d_{ij}(t) \geq d_{ij}(0)e^{-Ct}.$$

□

Conclusion. Thanks to Lemma 4.3 and Lemma 4.1 we have for all $1 \leq i \neq j \leq N$, $t \in [0, T[$

$$\begin{aligned} d_{ij}(t) &\geq d_{ij}(0)e^{-Ct}, \\ M^N(t) &\leq 8^4 M^N(0), \end{aligned}$$

this shows that T is independent of N and is at least of order

$$T \geq \frac{\log(2)}{C}.$$

5. REMINDER ON THE VLASOV-STOKES EQUATION AND WASSERSTEIN DISTANCE

In this part we recall some important results of existence, uniqueness, regularity and stability concerning the Vlasov-Stokes equations. We recall also the definition of the Monge-Kantorovich-Wasserstein distance of order one and infinite. We refer to [17, Part I, chapter 6] for definition and properties of the order one distance W_1 . To define the infinite Wasserstein distance we start with some associated notions. We refer to [3] for more details.

Definition 5.1 (Transference plane). *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^3)$ be two probability measures. The set of transference planes from μ to ν denoted $\Pi(\mu, \nu)$ is the set of all probability measures $\pi \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ having μ for first marginal and ν the second one i.e:*

$$\pi \in \Pi(\mu, \nu) \Leftrightarrow \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\phi(x) + \psi(y)) \pi(dxdy) = \int_{\mathbb{R}^3} \phi(x) \mu(dx) + \int_{\mathbb{R}^3} \psi(y) \nu(dy),$$

for all $\phi, \psi \in \mathcal{C}_b(\mathbb{R}^3)$.

Recall that for all probability measure $\lambda \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ we have:

Definition 5.2 (Essential supremum).

$$\lambda - \text{esssup } |x - y| := \inf\{t \geq 0 : \lambda(\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - y| > t\}) = 0\}.$$

We recall also the definition of the support for a (non-negative) measure.

Definition 5.3 (Measure support). *Given $\mu \in \mathcal{P}(\mathbb{R}^3)$ a non-negative measure, then the support of μ is defined as the set of all points x for which every open neighbourhood of x has positive measure:*

$$\text{supp } \mu = \{x \in \mathbb{R}^3 : \forall V \in \mathcal{V}(x), \mu(V) > 0\},$$

where $\mathcal{V}(x)$ denotes the set of open neighbourhoods of x .

With this definition for the support one can show that there holds

$$\lambda - \text{esssup } |x - y| := \sup\{|x - y| : (x, y) \in \text{supp } \lambda\}.$$

We can now define the infinite Wasserstein distance W_∞ :

Definition 5.4 (Infinite Wasserstein distance). *The infinite Wasserstein distance between two probability measures μ and ν is defined as follows:*

$$M_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \{\pi - \text{esssup } |x - y|\}.$$

A transference plan $\pi^* \in \Pi(\mu, \nu)$ satisfying

$$M_\infty(\mu, \nu) = \pi^* - \text{esssup } |x - y|,$$

is called an optimal transference plan.

We recall also the definition of a transport map.

Definition 5.5 (Transport map). *Given two probability measures μ and ν , a transport map T is a measurable mapping $T : \text{supp } \mu \rightarrow \mathbb{R}^3$ such that*

$$\nu = T_\# \mu.$$

We emphasize that $T(\mathbb{R}^3) \subset \text{supp } \nu$ μ -almost everywhere. Indeed

$$\begin{aligned} \mu\{x \in \mathbb{R}^3 : T(x) \notin \text{supp } \nu\} &= \mu\{T^{-1}({}^c \text{supp } \nu)\} \\ &= \nu\{{}^c \text{supp } \nu\} \\ &= 0, \end{aligned}$$

in the case where $\text{supp } \nu$ is measurable which is satisfied if ν is absolutely continuous with respect to the Lebesgue measure or if ν is a finite sum of diracs for instance.

Remark 5.1. *Note that, for all transport map T from μ to ν one may associate a transference plane $(Id, T)_\# \mu \in \Pi(\mu, \nu)$ i.e the pushforward of μ by the map $x \mapsto (x, T(x))$ and we have*

$$\begin{aligned} &(Id, T)_\# \mu - \text{esssup } |x - y| \\ &:= \inf\{t \geq 0 : (Id, T)_\# \mu(\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - y| \geq t\}) = 0\} \\ &= \inf\{t \geq 0 : \mu((Id, T)^{-1}\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - y| \geq t\}) = 0\} \\ &= \inf\{t \geq 0 : \mu(\{x \in \mathbb{R}^3 : |x - T(x)| \geq t\}) = 0\} \\ &= \mu - \text{esssup } |x - T(x)|. \end{aligned}$$

Note that this yields

$$\inf_{\pi \in \Pi(\mu, \nu)} \{\pi - \text{esssup } |x - y|\} \leq \inf\{\mu - \text{esssup } |T(x) - x|, T : \text{supp } \mu \rightarrow \mathbb{R}^3, \nu = T_\# \mu\}.$$

It is then natural to investigate in which conditions one has the existence of a transport map T associated to an optimal transference plan. As in [8] we refer to [3] for the following existence result.

Theorem 5.6 (Champion, De Pascale, Juutinen). *Assume that μ is absolutely continuous with respect to the Lebesgue measure. Then there exists optimal transference plans, and at least one of them is given by a transport map T . If moreover ν is a finite sum of Dirac masses, this optimal transport map is unique.*

5.1. Existence, uniqueness and stability of Vlasov-Stokes solution. Consider the Vlasov-Stokes problem

$$(58) \quad \begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}((\kappa g + \mathcal{K}\rho)\rho) &= 0, \\ \rho(0, \cdot) &= \rho_0, \end{cases}$$

recall the definition of the kernel \mathcal{K} :

$$\mathcal{K}\eta(x) = 6\pi r_0 \kappa \int \Phi(x - y) g \eta(y) dy,$$

for all $\eta \in L^\infty(\mathbb{R}^3) \times L^1(\mathbb{R}^3)$. We refer to the existence and uniqueness result due to Höfer [10, Theorem 9.2] in the case where the initial data ρ_0 and its gradient $\nabla \rho_0$ are in the space X_β for some $\beta > 2$ where

$$X_\beta := \{h \in L^\infty(\mathbb{R}^3), \|h\|_{X_\beta} < \infty\},$$

with

$$\|h\|_{X_\beta} := \sup_x (1 + |x|^\beta) |h(x)|.$$

Theorem 5.7 (Höfer). *Assume that $\rho_0, \nabla \rho_0 \in X_\beta$ for $\beta > 2$. There exists a unique solution $\rho \in W^{1,\infty}((0, T), X_\beta)$ to equation (58) for all $T > 0$ and a unique well defined flow X satisfying*

$$(59) \quad \begin{cases} \partial_s X(s, t, x) &= \kappa g + \mathcal{K}\rho(s, X(s, t, x)), & \forall s, t \in [0, +\infty[, \\ X(t, t, x) &= x, & \forall t \in [0, +\infty[, \end{cases}$$

such that

$$(60) \quad \rho(t, x) = \rho_0(X(0, t, x)), \quad \forall (t, x) \in [0, +\infty[\times \mathbb{R}^3.$$

Moreover, in the case where ρ_0 is compactly supported, $\operatorname{supp} \rho_0 \subset B(0, \bar{R})$, we have for all $T \geq 0$ and $0 \leq t \leq T$

$$\operatorname{supp} \rho(t, \cdot) \subset B(0, \bar{R} + T(\kappa|g| + C\bar{R}\|\rho_0\|_{L^\infty})).$$

Remark 5.2. *The flow X is measure-preserving i.e for a test function $\phi \in \mathcal{C}_b(\mathbb{R}^3)$ we have*

$$\int \phi(y) \rho(s, y) dy = \int \phi(X(s, t, y)) \rho(t, y) dy,$$

for all $s, t \in [0, T]$. This allows us to separate the dependence of time s in the integral with respect to the measure $\rho(t, \cdot)$.

Remark 5.3. *Note that for all $\eta \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, the velocity field $\mathcal{K}\eta$ is Lipschitz:*

$$|\mathcal{K}(\eta)(x) - \mathcal{K}(\eta)(y)| \lesssim (\|\eta\|_{L^1} + \|\eta\|_{L^\infty}) |x - y|, \quad \forall x \neq y \in \mathbb{R}^3.$$

Moreover, if one assume that ρ_0 is only Lipschitz and compactly supported, then one can show the existence and uniqueness of the solution ρ to equation (58) in the space $L^\infty((0, T); L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$. The method of proof is related to the stability result due to G. Loeper in [14] which gives a stability estimate in terms of Wasserstein distance for the Vlasov-Poisson equation. This result is adapted by M. Hauray in [7, Theorem 4] for a more general class of kernels K satisfying a (C^α) condition with $\alpha < d - 1$ where d is the dimension space.

$$(C^\alpha) \quad \operatorname{div} K = 0, \forall x \neq 0, |K(x)|, |x| |\nabla K(x)| < \frac{C}{|x|^\alpha},$$

see [7]. This condition being satisfied by the Oseen tensor Φ we have the following result.

Theorem 5.8 (Hauray-Loeper). *Given $T > 0$, consider two solutions $\rho_i \in L^\infty((0, T), L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$, $i = 1, 2$, to the Vlasov-Stokes equation (58) associated to two initial data $\rho_0^i \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $i = 1, 2$. There holds:*

$$(61) \quad W_1(\rho_1(t, \cdot), \rho_2(t, \cdot)) \leq W_1(\rho_0^1, \rho_0^2) e^{C \max(\|\rho_1^0\|_{L^\infty \cap L^1}, \|\rho_2^0\|_{L^\infty \cap L^1}) t}.$$

We refer to [7, Theorem 4] for a complete proof which introduces the main ideas used also in [8] for the mean field approximation result.

5.2. ρ^N as a weak solution to a Vlasov-Stokes equation. According to Theorem 1.1, there exists a time $T > 0$ independent of N for which the particles do not overlap. This shows that the empirical measure

$$\rho^N(t, x) := 6\pi R \sum_{i=1}^N \delta_{x_i(t)}(x),$$

is well defined on $[0, T]$. Recall that we are interested in the limiting behaviour of $\rho^N \in \mathcal{P}([0, T] \times \mathbb{R}^3)$ when $N \rightarrow \infty$. According to Proposition 3.7, particles $(x_i)_{1 \leq i \leq N}$ satisfy the following system:

$$\begin{cases} \dot{x}_i &= V_i, \\ V_i &\sim \kappa g + 6\pi R \sum_{i \neq j} \Phi(x_i - x_j). \end{cases}$$

In order to prove Theorem 1.2 we want to compare the particle system to the continuous density ρ which is solution to the Vlasov-Stokes equation (58). Hence, we need to express ρ^N as a weak solution to a Vlasov-Stokes equation. The end of this section is devoted to prove it.

Analogously to the continuous case, we are interested in giving a sense to the quantity

$$\mathcal{K} \rho^N = 6\pi r_0 \kappa \int \Phi(x - y) g \rho^N(t, dy),$$

which is not well defined because Φ is singular. On the other hand, as the only values of Φ that matters are the terms $\Phi(x_i - x_j)$, $i \neq j$ we define the following regularization:

$$\psi^N \Phi(x) := \Phi(x) \psi^N(x),$$

where $\psi^N(x) := \psi\left(\frac{x}{d_{\min}(0)}\right)$ and ψ is a truncation function such that $\psi = 0$ on $B(0, 1/4)$ and $\psi = 1$ outside $B(0, 1/2)$. We can now define the operator \mathcal{K}^N :

$$\begin{aligned}\mathcal{K}^N \rho^N(t, x) &:= 6\pi R_0 \kappa \int_{\mathbb{R}^3} \psi^N \Phi(x - y) g \rho^N(t, dy) \\ &= \frac{6\pi r_0 \kappa}{N} \sum_i \psi^N \Phi(x - x_i(t)) g.\end{aligned}$$

With this construction we have for $x = x_i(t)$

$$\mathcal{K}^N \rho^N(t, x_i(t)) = \frac{6\pi r_0 \kappa}{N} \sum_{j \neq i} \Phi(x_j(t) - x_i(t)) g.$$

Indeed, theorem 1.1 ensures that the particles satisfy

$$|x_i(t) - x_j(t)| \geq \frac{1}{2} d_{\min}(0), \quad \forall i \neq j,$$

for all $t \in [0, T]$ and $T > 0$ independent of N as stated in 1.1. Hence $\mathcal{K}^N \rho^N$ is well defined on $[0, T]$.

Now, it remains to check that ρ^N is a weak solution of a Vlasov-Stokes equation. We recall that ρ^N is a weak solution of a transport equation $\frac{\partial}{\partial t} + \text{div}(V \rho^N)$ with $V \in \mathcal{C}([0, T], \mathcal{C}^1(\mathbb{R}^3))$ if for all test function $\phi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R}^3)$ we have

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t \phi(t, x) + \nabla \phi(t, x) \cdot V(t, x)) \rho^N(dx, t) dt = 0.$$

Note that this integral yields

$$\begin{aligned}& \int_0^T \int_{\mathbb{R}^3} (\partial_t \phi(t, x) + \nabla \phi(t, x) \cdot V(t, x)) \rho^N(dx, t) dt \\ &= \int_0^T \frac{1}{N} \sum_i (\partial_t \phi(t, x_i(t)) + \nabla \phi(t, x_i(t)) \cdot V(t, x_i(t))) dt.\end{aligned}$$

In particular if we choose V such that $V(t, x_i(t)) = V_i(t)$ one has:

$$\begin{aligned}&= \int_0^T \frac{1}{N} \sum_i \partial_t \phi(t, x_i(t)) + \nabla \phi(t, x_i(t)) \cdot V_i \\ &= \frac{1}{N} \sum_i \int_0^T \frac{d}{dt} (\phi(t, x_i(t))) dt \\ &= 0.\end{aligned}$$

On the other hand, we recall that from 3.7 we can write for all $1 \leq i \leq N$

$$\begin{aligned} V_i &= \kappa g + 6\pi R \sum_{j \neq i}^N \Phi(x_i - x_j) \kappa g + E_i(t) \\ &= \kappa g + \mathcal{K}^N \rho^N(t, x_i(t)) + E_i(t), \end{aligned}$$

with $E_i(t) = O(d_{\min})$. Hence if we construct a divergence-free vector field E^N such that

$$E^N(t, x_i(t)) = E_i(t),$$

we can define V as

$$V(t, x) = \kappa g + \mathcal{K}^N \rho^N(t, x) + E^N(t, x).$$

Construction of E^N . We fix χ a truncation function such that $\chi = 1$ on $B(0, 1)$ and $\chi = 0$ on ${}^c B(0, 2)$. For all i we set

$$\mathcal{E}_i(t, x) := \operatorname{curl} \left(\frac{x - x_i(t)}{2} \times E_i(t) \chi \left(\frac{x - x_i(t)}{R} \right) \right).$$

By construction, \mathcal{E}_i is a divergence-free compactly supported vector field satisfying

$$\mathcal{E}_i(t, x_i(t)) = E_i(t).$$

Furthermore, \mathcal{E}_i is supported in $B(x_i(t), 2R)$. Thanks to Theorem 1.1, this entails that $\operatorname{supp}(\mathcal{E}_i) \cap \operatorname{supp}(\mathcal{E}_j) = \emptyset$ for $i \neq j$. We set then

$$E^N(t, x) := \sum_i \mathcal{E}_i(t, x),$$

which satisfies for all $N \geq 1$

- $E^N \in \mathcal{C}([0, T] \times \mathbb{R}^3)$, $E^N(t, \cdot) \in \mathcal{C}^1(\mathbb{R}^3)$ for all $0 \leq t \leq T$,
- $\operatorname{div} E^N = 0$,
- $E^N(t, x_i(t)) = E_i(t)$ for all $1 \leq i \leq N$
- $\|E^N(t, \cdot)\|_\infty \leq C_\chi \max_i |E_i(t)|$ for all $0 \leq t \leq T$.

The only statement that needs further explanation is the last one. For all $x \in B(x_i(t), R_i)$ we have

$$\mathcal{E}_i(t, x) = E_i(t),$$

and for all $x \in B(x_i, 2R) \setminus B(x_i, R)$, direct computations yields

$$\begin{aligned} \mathcal{E}_i(t, x) &= \frac{1}{2} \left[2\chi \left(\frac{x - x_i(t)}{R} \right) I_3 - \frac{1}{R} \nabla \chi \left(\frac{x - x_i(t)}{R} \right) \otimes (x - x_i(t)) \right. \\ &\quad \left. + \frac{1}{R} (x - x_i(t)) \cdot \nabla \chi \left(\frac{x - x_i(t)}{R} \right) I_3 \right] E_i(t). \end{aligned}$$

Therefore

$$|\mathcal{E}_i(t, x)| \leq C [\|\chi\|_\infty + \|\nabla \chi\|_\infty] |E_i(t)|.$$

We can now state the following proposition.

Proposition 5.9. *For arbitrary N we have that $\kappa g + \mathcal{K}^N \rho^N + E^N \in \mathcal{C}([0, T] \times \mathbb{R}^3)$ and $\nabla \mathcal{K}^N \rho^N + \nabla E^N \in \mathcal{C}([0, T] \times \mathbb{R}^3)$. Moreover, the velocity field satisfies*

$$(62) \quad |\kappa g + \mathcal{K}^N \rho^N(t, x) + E^N(t, x)| \leq C(1 + |x|), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^3,$$

for some constant C independent of N .

Proof. As the kernel is regularized, the two first properties are satisfied by construction. For all $(t, x) \in [0, T] \times \mathbb{R}^3$ we have

$$\begin{aligned} \mathcal{K}^N \rho^N(x) &= \frac{6\pi r_0 \kappa}{N} \sum_i \psi^N(x) \Phi(x - x_i(t)) \\ &= \frac{6\pi r_0 \kappa}{N} \sum_i \psi^N(x) 1_{\{|x_i(t) - x| > \frac{d_{\min}(0)}{2}\}} \Phi(x - x_i(t)). \end{aligned}$$

We set $\mathcal{I}(t, x) = \{1 \leq i \leq N, |x_i(t) - x| > \frac{d_{\min}(0)}{2}\}$, which yields

$$\begin{aligned} |\mathcal{K}^N \rho^N(x)| &\lesssim \frac{1}{N} \sum_{\mathcal{I}(t, x)} \frac{1}{|x - x_i(t)|} \\ &\lesssim \bar{M} \left(\frac{|\lambda^N|^2}{N^{1/3} d_{\min}} + 1 \right), \end{aligned}$$

thanks to Lemma A.1 for $k = 1$. For the velocity field E^N we have also by construction that $\sup_{t \in [0, T]} \|E^N(t, \cdot)\|_\infty$ is uniformly bounded. \square

This allows us to state the following result.

Theorem 5.10. ρ^N is a weak solution of

$$(63) \quad \begin{cases} \frac{\partial \rho^N}{\partial t} + \operatorname{div}((\kappa g + \mathcal{K}^N \rho^N + E^N) \rho^N) &= 0, \\ \rho^N(0, \cdot) &= \rho_0^N, \end{cases}$$

on $[0, T] \times \mathbb{R}^3$. Moreover, the characteristic flow associated to the velocity $\kappa g + \mathcal{K}^N \rho^N + E^N$ is of class \mathcal{C}^1 for all $N \geq 1$

$$(64) \quad \begin{cases} \partial_s X^N(s, t, x) &= \kappa g + \mathcal{K}^N \rho^N(s, X^N(s, t, x)) + E^N(s, X^N(s, t, x)) & \forall s, t \in [0, T], \\ X^N(t, t, x) &= x, & \forall t \in [0, T], \end{cases}$$

and the following classical formula holds true:

$$(65) \quad \rho^N(t, \cdot) = X^N(t, 0, \cdot) \# \rho_0^N \quad \forall t \in [0, T].$$

Proof. As $V(t, x) := \kappa g + \mathcal{K}^N \rho^N(t, x) + E^N(t, x) \in \mathcal{C}^1([0, T] \times \mathbb{R}^3)$ is defined such that $V(t, x_i(t)) = V_i$, $\forall 1 \leq i \leq N$ this ensures that for all test function $\phi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R}^3)$:

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t \phi(t, x) + \nabla \phi(t, x) \cdot [\kappa g + \mathcal{K}^N \rho^N(t, x) + E^N(t, x)]) \rho^N(dx, t) dt = 0,$$

thus, ρ^N is a weak solution for (63).

According to Proposition 5.9, the ode governing the characteristic flow satisfies the assumptions of the Cauchy-Lipschitz theorem. Therefore, the ode admits a unique maximal solution $X^N \in \mathcal{C}^1([0, T] \times [0, T] \times \mathbb{R}^3)$ thanks to formula (62). Equality (65) holds true thanks to the classical theory for transport equations. \square

6. CONVERGENCE TO THE VLASOV-STOKES EQUATION

At this point, we proved that the particles interact two by two with an interaction force given by the Oseen-tensor with an additional error term.

$$(66) \quad \begin{cases} \dot{x}_i(t) &= V_i(t), \\ V_i(t) &= \kappa g + 6\pi R \sum_{i \neq j} \Phi(x_i(t) - x_j(t)) + E^N(t, x_i(t)). \end{cases}$$

We want to estimate the Wasserstein distance $W_1(\rho^N(t, \cdot), \rho(t, \cdot))$ for all time $0 \leq t \leq T$. To this end, we follow the ideas of [7] and [8] and show that the additional error term E^N can be controled. As in [8], we introduce an intermediate density $\bar{\rho}^N$.

6.1. Step 1. Estimate of the distance between ρ and $\bar{\rho}^N$. We define $\bar{\rho}_0^N$ as the regularized density of ρ_0^N :

$$\bar{\rho}_0^N := \rho_0^N * \chi_{\lambda^N}$$

where $\chi_{\lambda^N}(x) := \frac{1}{|\lambda^N|^3} \chi\left(\frac{x}{\lambda^N}\right)$ a mollifier compactly supported in $B(0, \lambda^N)$ (for instance). We emphasize that the regularized density is uniformly bounded:

$$\begin{aligned} \bar{\rho}_0^N(x) &= \int \frac{1}{|\lambda^N|^3} \chi\left(\frac{x-y}{\lambda^N}\right) \rho_0^N(dy) \\ &= \frac{1}{N|\lambda^N|^3} \sum_{i=1}^N \chi\left(\frac{x-x_i(0)}{\lambda^N}\right) \\ &\leq \frac{1}{N|\lambda^N|^3} \|\chi\|_\infty \sup_x \#\{i \in \{1, \dots, N\}, x_i(0) \in B(x, \lambda^N)\} \\ &\leq \|\chi\|_\infty \bar{M}, \end{aligned}$$

according to assumption (34). Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \bar{\rho}_0^N(x) dx &= \frac{1}{N|\lambda^N|^3} \sum_{i=1}^N \int_{B(x_i(0), \lambda^N)} \chi\left(\frac{x-x_i(0)}{\lambda^N}\right) dx \\ &= 1. \end{aligned}$$

Remark also that $\bar{\rho}_0^N$ is uniformly compactly supported thanks to assumption (4):

$$\text{supp } \bar{\rho}_0^N \subset \bigcup_i B(x_i(0), \lambda^N) \subset B(0, \bar{R} + 1), \quad \forall N \in \mathbb{N}.$$

Now, we define $\bar{\rho}^N$ as the unique solution to the Vlasov stokes equation (58) associated to the initial data $\bar{\rho}_0^N$. The stability Theorem 5.8 allows us to compare ρ and $\bar{\rho}^N$:

$$W_1(\rho(t, \cdot), \bar{\rho}^N(t, \cdot)) \leq W_1(\rho_0, \bar{\rho}_0^N) e^{Ct},$$

where $C = C(\bar{R}, \|\chi\|_\infty, \bar{M}, \|\rho_0\|_{X_\beta})$. We split the distance $W_1(\rho_0, \bar{\rho}_0^N)$ as follows

$$W_1(\rho_0, \bar{\rho}_0^N) \leq W_1(\rho_0, \rho_0^N) + W_1(\rho_0^N, \bar{\rho}_0^N),$$

and use the fact that

$$W_1(\rho_0^N, \bar{\rho}_0^N) \leq W_\infty(\rho_0^N, \bar{\rho}_0^N),$$

together with [8, Proposition 1]:

$$W_\infty(\rho_0^N, \bar{\rho}_0^N) \leq C\lambda^N,$$

to get

$$(67) \quad W_1(\rho(t, \cdot), \bar{\rho}^N(t, \cdot)) \lesssim (\lambda^N + W_1(\rho_0, \rho_0^N)) e^{Ct}$$

6.2. Step 2. Estimate of the distance between $\bar{\rho}^N$ to ρ^N . It remains to estimate $W_1(\rho^N(t, \cdot), \bar{\rho}^N(t, \cdot))$. We have the following result:

Lemma 6.1. *For arbitrary N there holds,*

$$W_1(\rho^N(t, \cdot), \bar{\rho}^N(t, \cdot)) \lesssim W_\infty(\rho_0^N, \bar{\rho}_0^N) e^{Ct} + t e^{Ct} d_{\min}.$$

Proof. According to Theorems 5.7 and 5.10 we have the explicit formulas for all $s, t \in [0, T]$:

$$\begin{aligned} \bar{\rho}^N(t, \cdot) &= X(t, s, \cdot) \# \bar{\rho}_s^N, \\ \rho^N(t, \cdot) &= X^N(t, s, \cdot) \# \rho_s^N. \end{aligned}$$

At $t = 0$ we have the existence of an optimal transport map T_0 from $\bar{\rho}_0^N$ to ρ_0^N thanks to Theorem 5.6

$$\rho_0^N = T_0 \# \bar{\rho}_0^N,$$

satisfying

$$W_\infty(\bar{\rho}_0^N, \rho_0^N) = \bar{\rho}_0^N - \text{esssup} |T_0(x) - x|.$$

We construct then a transport map T_t from $\bar{\rho}^N$ to ρ^N at all time $t \in [0, T]$ by following T_0 along the two flows X and X^N :

$$T_t = X^N(t, 0, \cdot) \circ T_0 \circ X(0, t, \cdot).$$

One can remark that for all $0 \leq s \leq t$:

$$\begin{aligned} T_t &= X^N(t, s, \cdot) \circ T_s \circ X(s, t, \cdot) \\ \rho^N(t, \cdot) &= T_t \# \bar{\rho}^N(t, \cdot). \end{aligned}$$

As in [8] we set then

$$f(t) := \sup_{s \leq t} \bar{\rho}^N(t, \cdot) - \text{esssup} |T_s(x) - x|,$$

so that

$$W_\infty(\rho^N(t, \cdot), \bar{\rho}^N(t, \cdot)) \leq f(t),$$

and

$$f(0) = W_\infty(\bar{\rho}_0^N, \rho_0^N).$$

We reproduce the same steps as in [8] and introduce the following notation for a generic "particle" of the continuous system with position x_t at time t such that

$$x_s = X(s, t, x_t),$$

we fix in what follows $0 \leq t_2 \leq t_1$ and recall the following formula

$$T_{t_1} \circ X(t_1, t_2, \cdot) = X^N(t_1, t_2, \cdot) \circ T_{t_2}.$$

We aim now to estimate $|T_{t_1}(x_{t_1}) - x_{t_1}|$ for all test particle x_{t_1} :

$$\begin{aligned} T_{t_1}(x_{t_1}) - x_{t_1} &= X^N(t_1, t_2, T_{t_2}(x_{t_2})) - X(t_1, t_2, x_{t_2}), \\ &= T_{t_2}(x_{t_2}) - x_{t_2} + \int_{t_2}^{t_1} \dot{X}^N(s, t_2, T_{t_2}(x_{t_2})) - \dot{X}(s, t_2, x_{t_2}) ds, \\ &= T_{t_2}(x_{t_2}) - x_{t_2} + \int_{t_2}^{t_1} ([\mathcal{K}^N \rho^N + E^N](s, X^N(s, t_2, T_{t_2}(x_{t_2}))), \\ &\quad - \mathcal{K} \bar{\rho}^N(s, x_s)) ds, \\ &= T_{t_2}(x_{t_2}) - x_{t_2} + \int_{t_2}^{t_1} ([\mathcal{K}^N \rho^N + E^N](s, T_s(x_s)) - \mathcal{K} \bar{\rho}^N(s, x_s)) ds, \\ &= T_{t_2}(x_{t_2}) - x_{t_2} + \int_{t_2}^{t_1} E^N(s, T_s(x_s)) ds, \\ &\quad + \int_{t_2}^{t_1} \int_{\mathbb{R}^3} 6\pi r_0 \kappa (\psi^N \Phi(T_s(x_s) - T_s(y)) - \Phi(x_s - y)) g \bar{\rho}^N(s, dy) ds, \end{aligned}$$

where we used the fact that $\rho_s^N = T_s \# \bar{\rho}_s^N$ to get:

$$\begin{aligned} \mathcal{K}^N \rho^N(s, T_s(x_s)) &= 6\pi r_0 \kappa \int_{\mathbb{R}^3} \psi^N \Phi(T_s(x_s) - y) g \rho^N(s, dy) \\ &= 6\pi r_0 \kappa \int_{\mathbb{R}^3} \psi^N \Phi(T_s(x_s) - T_s(y)) g \bar{\rho}^N(s, dy). \end{aligned}$$

We set then $t_1 = t$ and $t_2 = t_1 - \tau = t - \tau$, $\tau > 0$. We obtain for almost every x_t and $x_{t-\tau}$

$$\begin{aligned} |T_t(x_t) - x_t| &\leq |T_{t-\tau}(x_{t-\tau}) - x_{t-\tau}| + \tau \|E^N(t)\|_\infty, \\ &\quad + 6\pi r_0 \kappa |g| \int_{t-\tau}^t \int_{\mathbb{R}^3} |\psi^N \Phi(T_s(x_s) - T_s(y)) - \Phi(x_s - y)| \bar{\rho}^N(s, dy) ds, \\ &\leq f(t - \tau) + \tau \|E^N(t)\|_\infty \\ &\quad + C \int_{t-\tau}^t \int_{\mathbb{R}^3} |\psi^N \Phi(T_s(x_s) - T_s(y_s)) - \Phi(x_s - y_s)| \bar{\rho}^N(t, dy_t) ds, \end{aligned}$$

here we used remark 5.2 with $y_s = X(s, t, y_t)$. In addition we defined

$$\|E^N(t)\|_\infty := \sup_{0 \leq s \leq t} \|E^N(s, \cdot)\|_\infty.$$

This being true for almost every x_t we obtain:

$$(68) \quad f(t) \leq f(t - \tau) + \tau \|E^N(t)\|_\infty + C \operatorname{esssup}_{x_t} \int_{t-\tau}^t \int_{\mathbb{R}^3} |\psi^N \Phi(T_s(x_s) - T_s(y_s)) - \Phi(x_s - y_s)| \bar{\rho}^N(t, dy_t) ds.$$

Hence, it remains to control the last quantity. We split the integral on \mathbb{R}^3 into two terms: the first one denoted J_1 is the integral over the subset I and the second one denoted J_2 the integral over $\mathbb{R}^3 \setminus I$ where

$$I = \{y_t : |x_t - y_t| \geq 4f(t)e^{\tau L}\},$$

where L will be defined later.

Step 1: Estimate of J_1 .

For all $t - \tau \leq s \leq t$, we have:

$$\begin{aligned} |x_s - y_s| &\geq |x_t - y_t| - \int_s^t |\dot{X}(t', t, x_t) - \dot{X}(t', t, y_t)| dt', \\ &\geq |x_t - y_t| - \int_s^t |\mathcal{K} \bar{\rho}^N(t', X(t', t, x_t)) - \mathcal{K} \bar{\rho}^N(t', X(t', t, y_t))| dt', \\ &\geq |x_t - y_t| - \operatorname{Lip}(\mathcal{K} \bar{\rho}^N) \int_s^t |X(t', t, x_t) - X(t', t, y_t)| dt'. \end{aligned}$$

According to Remark 5.3, the Lipschitz constant of $\mathcal{K} \bar{\rho}^N$ is bounded by

$$\operatorname{Lip}(\mathcal{K} \bar{\rho}^N) \leq C \|\bar{\rho}^N\|_{L^\infty(L^\infty \cap L^1)}.$$

Thanks to (60) and the fact that $\bar{\rho}_0^N$ is uniformly bounded and compactly supported we obtain

$$\operatorname{Lip}(\mathcal{K} \bar{\rho}^N) \leq C(\|\bar{\rho}_0^N\|_\infty, \operatorname{supp} \bar{\rho}_0^N) \leq C.$$

Applying Gronwall's inequality yields for all $0 \leq t - \tau \leq s \leq t$

$$|x_s - y_s| \geq |x_t - y_t| e^{-L(t-s)}.$$

We can make precise now the constant $L := \operatorname{Lip}(\mathcal{K} \bar{\rho}^N)$ which is uniformly bounded for all $N \geq 0$ and $t \in [0, T]$.

According to the definition of $I = \{y_t : |x_t - y_t| \geq 4f(t)e^{\tau L}\}$ we have for all $0 \leq t - \tau \leq s \leq t$ and τ small enough:

$$(69) \quad |x_s - y_s| \geq |x_t - y_t| e^{-L(t-s)} \geq |x_t - y_t| e^{-L\tau} \geq \frac{1}{2} |x_t - y_t|.$$

Analogously, for almost all x_s and y_s

$$|T_s(x_s) - T_s(y_s)| \geq |x_s - y_s| - |T_s(x_s) - x_s| - |T_s(y_s) - y_s| \geq |x_s - y_s| - 2f(s) \geq |x_s - y_s| - 2f(t),$$

where we used the fact that $f(t) \geq f(s)$. This yields for τ small enough

$$(70) \quad |T_s(x_s) - T_s(y_s)| \geq \frac{1}{4}|x_t - y_t|.$$

Moreover, recall that $T_s(x_s)$ and $T_s(y_s)$ are in the support of $\rho^N(s, \cdot)$ i.e there exists i, j such that $T_s(x_s) = x_i(s)$ and $T_s(y_s) = x_j(s)$. In addition, estimate (70) and the definition of I ensures that $i \neq j$. We have then

$$(71) \quad \psi^N \Phi(T_s(x_s) - T_s(y_s)) = \Phi(T_s(x_s) - T_s(y_s)).$$

Finally, using estimates (69), (70), formula (71) and the Lipschitz-like estimate (19) for Φ we obtain:

$$\begin{aligned} J_1 &= \int_I \int_{t-\tau}^t |\Phi(T_s(x_s) - T_s(y_s)) - \Phi(x_s - y_s)| ds \bar{\rho}^N(t, dy_t), \\ &\leq C \int_I \int_{t-\tau}^t \frac{|x_s - T_s(x_s)| + |y_s - T_s(y_s)|}{\min(|x_s - y_s|, |T_s(x_s) - T_s(y_s)|)^2} ds \bar{\rho}^N(t, dy_t), \\ &\leq C f(t) \tau \int_I \frac{1}{|x_t - y_t|^2} \bar{\rho}^N(t, dy_t), \\ &\leq C t \tau f(t) \|\bar{\rho}^N(t)\|_{L^\infty}. \end{aligned}$$

Recall that $\|\bar{\rho}^N(t)\|_{L^\infty} \leq \|\bar{\rho}_0^N\|_{L^\infty} \leq \|\chi\|_\infty \bar{M}$ and

$$\text{supp } \bar{\rho}^N(t, \cdot) \subset B(0, \bar{R} + t(\kappa|g| + C\bar{R}\|\bar{\rho}_0^N\|_{L^\infty})).$$

Step 2: Estimate of J_2 .

We focus now on

$$J_2 := \text{esssup}_{x_t} \int_{t-\tau}^t \int_{cI} |\psi^N \Phi(T_s(x_s) - T_s(y_s)) - \Phi(x_s - y_s)| \bar{\rho}^N(t, dy_t) ds$$

Again $T_s(x_s)$ and $T_s(y_s)$ are in the support of $\rho^N(s, \cdot)$ i.e there exists i, j such that $T_s(x_s) = x_i(s)$ and $T_s(y_s) = x_j(s)$. Moreover if $i = j$ then $\psi^N \Phi(T_s(x_s) - T_s(y_s)) = 0$. Hence in all cases we have

$$\begin{aligned} |\Phi(x_s - y_s) - \psi^N \Phi(T_s(x_s) - T_s(y_s))| &\leq |\Phi(x_s - y_s)| + |\psi^N \Phi(T_s(x_s) - T_s(y_s))| \\ &\leq C \left(\frac{1}{|x_s - y_s|} + \frac{1}{d_{\min}(s)} \right), \end{aligned}$$

applying the change of variable $y_t = X(t, s, y_s)$ we get

$$\begin{aligned} \int_{cI} \int_{t-\tau}^t \frac{1}{|x_s - y_s|} ds \bar{\rho}^N(t, dy_t) &\leq \|\bar{\rho}^N\|_\infty \int_{t-\tau}^t \int_{cI} \frac{1}{|x_s - y_s|} dy_t ds, \\ &= C \int_{t-\tau}^t \int_{X(t, s, cI)} \frac{1}{|x_s - y_s|} dy_s ds. \end{aligned}$$

Denote $K = X(t, s, {}^cI)$, as the flow X preserves the Lebesgue measure we have $|K| = |{}^cI|$. For all $s \in [t - \tau, t]$ and $a > 0$ a direct computation yields

$$\begin{aligned} \int_K \frac{1}{|x_s - y_s|} dy_s &= \left(\int_{K \cap B(x_s, a)} + \int_{K \cap {}^cB(x, a)} \right) \frac{1}{|x_s - y_s|} dy_s, \\ &\leq Ca^2 + \frac{1}{a}|K|, \end{aligned}$$

we choose then $a^3 = |K| = |{}^cI| \leq Cf(t)^3 e^{3L\tau}$ to get

$$(72) \quad \int_{{}^cI} \int_{t-\tau}^t \frac{1}{|x_s - y_s|} ds \bar{\rho}^N(t, dy_t) \leq C\tau f(t)^2 e^{2L\tau}.$$

For the remaining term we apply Theorem 1.1 and get for all $t - \tau \leq s \leq t$

$$\begin{aligned} \int_{{}^cI} \int_{t-\tau}^t \frac{1}{d_{\min}(s)} ds \bar{\rho}^N(t, dy_t) &\leq \frac{2}{d_{\min}(0)} \int_{{}^cI} \int_{t-\tau}^t ds \bar{\rho}^N(t, dy_t), \\ &\leq C\tau \frac{2e^{3\tau L}}{d_{\min}(0)} f(t)^3. \end{aligned}$$

Conclusion.

Gathering these bounds, there exists a universal constant $K > 0$ independent of N such that for τ small enough and $0 < t \leq T$

$$f(t) \leq f(t - \tau) + \tau \|E^N(t)\|_\infty + K\tau f(t) \left[1 + Tf(t) + \frac{f(t)^2}{d_{\min}(0)} \right].$$

We can now apply a discrete Gronwall argument: Note that at time $t = 0$, assumption (6) ensures the existence of a positive constant $C_1 > 1$ such that

$$1 + Tf(0) + \frac{f(0)^2}{d_{\min}(0)} \leq \frac{C_1}{K},$$

hence, we define $T^* \leq T$ as the maximal time for which

$$1 + Tf(t) + \frac{f(t)^2}{d_{\min}(0)} \leq \frac{C_1}{K} \quad \forall t \in [0, T^*].$$

We obtain

$$f(t) \leq f(t - \tau) + C_1\tau f(t) + \tau \|E^N\|_\infty.$$

If τ is small enough we can write

$$f(t) \leq (1 - C_1\tau)^{-1} f(t - \tau) + \frac{\tau}{1 - C_1\tau} \|E^N\|_\infty,$$

iterating the formula we obtain for $M \in \mathbb{N}^*$

$$\begin{aligned} f(t) &\leq (1 - C_1\tau)^{-M} f(t - M\tau) + \tau \sum_{k=1}^M \frac{1}{(1 - C_1\tau)^k} \|E^N\|_\infty, \\ &\leq (1 - C_1\tau)^{-M} f(t - M\tau) + \tau \sum_{k=1}^M e^{2C_1\tau k} \|E^N\|_\infty. \end{aligned}$$

Thanks to the bound $\frac{1}{1-C_1\tau} \leq e^{2C_1\tau}$ for τ small enough. We set then $t - M\tau = 0$ to get

$$f(t) \leq (1 - C_1 \frac{t}{M})^{-M} f(0) + \frac{t}{M} \sum_{k=1}^M e^{2C_1 \frac{t}{M} k} \|E^N\|_\infty.$$

As $e^{2C_1 \frac{t}{M} k} \leq e^{2C_1 t}$ for all $1 \leq k \leq M$ the second term yields

$$\frac{t}{M} \sum_{k=1}^M e^{2C_1 \frac{t}{M} k} \|E^N\| \leq t e^{2C_1 t} \|E^N\|_\infty,$$

and for M arbitrary large:

$$(1 - C_1 \frac{t}{M})^{-M} \leq e^{2C_1 t}.$$

Finally for all $t \in [0, T^*[$

$$f(t) \leq f(0) e^{2C_1 t} + C t e^{C_1 t} \|E^N\|_\infty.$$

In particular we have for all $t \in [0, T^*[$

$$\begin{aligned} T f(t) + \frac{f(t)^2}{d_{\min}(0)} &\leq T f(0) e^{2C_1 t} + \|E^N\|_\infty T^2 e^{C_1 t} + \frac{2}{d_{\min}(0)} (f(0)^2 e^{4C_1 t} + \|E^N\|_\infty^2 t^2 e^{2C_1 t}) \\ &\leq e^{4C_1 T} (2 + T + 2T^2) \left(f(0) + \|E^N\|_\infty + \frac{f(0)^2 + \|E^N\|_\infty^2}{d_{\min}(0)} \right). \end{aligned}$$

Since we have $f(0) = O(\lambda^N)$ and

$$\begin{aligned} \frac{f(0)^2 + \|E^N\|_\infty^2}{d_{\min}(0)} &\lesssim \frac{|\lambda^N|^2}{d_{\min}(0)} + d_{\min}, \\ &<< 1, \end{aligned}$$

according to assumption (6), this shows that for N large enough, we have for all $t \in [0, T^*[$

$$1 + T f(t) + \frac{f(t)^2}{d_{\min}(0)} \leq \frac{C_1}{K},$$

where N depends on T^* . Hence, we can choose N large enough so that $T^* \rightarrow T$. \square

APPENDIX A. TECHNICAL LEMMAS

We state here an important lemma which is the extension of [12, lemma 2.1] to the new assumptions on the dilution regime introduced in [9].

Lemma A.1 (Adapted from P. E. Jabin and F. Otto). *Under assumptions 1.1, for all $k \in [0, 2]$ there exists a positive constant $C > 0$ such that for all fixed $1 \leq i \leq N$:*

$$(73) \quad \sum_{j \neq i} \frac{1}{d_{ij}^k} \leq C\bar{M} \left(\frac{|\lambda^N|^{3-k} N^{1-\frac{k}{3}}}{d_{\min}^k} + N \right).$$

Moreover, if $k = 3$ we have

$$\sum_{j \neq i} \frac{1}{d_{ij}^3} \lesssim \frac{\log M^N}{d_{\min}^3} + M^N \frac{\log |\lambda^N|^{-3}}{|\lambda^N|^3}.$$

Proof. We fix $i = 1$ and the same holds true for all $1 \leq i \leq N$. We introduce the shortcut:

$$I_1 := \{1 \leq j \leq N, |x_1 - x_j| \leq \lambda^N\},$$

we have :

$$\sum_{j \neq 1} \frac{1}{d_{1j}^k} = \left(\sum_{\substack{j \in I_1 \\ j \neq 1}} + \sum_{j \notin I_1} \right) \frac{1}{d_{1j}^k}.$$

As $\#I_1 \leq M^N$ we apply [12, Lemma 2.1] to the set of particles I_1 with minimal distance d_{\min} to get:

$$\begin{aligned} \sum_{\substack{j \in I_1 \\ j \neq 1}} \frac{1}{d_{1j}^k} &\leq C \frac{|M^N|^{1-k/3}}{d_{\min}^k} \\ &\leq C\bar{M} \frac{|\lambda^N|^{3-k} N^{1-\frac{k}{3}}}{d_{\min}^k}. \end{aligned}$$

In the special case where $k = 3$, reproducing the proof of [12, Lemma 2.1] with $k = 3$ yields:

$$\sum_{\substack{j \in I_1 \\ j \neq 1}} \frac{1}{d_{1j}^3} \leq \frac{4^3}{d_{\min}^3} \int_1^{M^N} x^{-1} dx \lesssim \frac{\log M^N}{d_{\min}^3}.$$

For the second sum we choose a covering $(C^q)_{1 \leq q \leq |\lambda^N|^{-3}}$ of $\bigcup_i \{x_i\}$ with disjoint cubes C^q of width $2\lambda^N$ such that C^1 is the cube centered in x_1 with width $2\lambda^N$. We have then

$${}^c I_1 \subset \bigcup_{q \neq 1} C^q,$$

we use then the shortcut

$$I_q := \{1 \leq j \leq N \text{ such that } x_j \in C^q\},$$

and set \bar{x}_q the center of the cube C^q . We have then for all $j \in I_q$:

$$\begin{aligned} |x_1 - x_j| &\geq |x_1 - \bar{x}_q| - |\bar{x}_q - x_j|, \\ &\geq |x_1 - \bar{x}_q| - \lambda^N, \\ &\geq \frac{1}{2}|x_1 - \bar{x}_q|, \end{aligned}$$

where we used the fact that each center \bar{x}_q is at least $2\lambda^N$ far away from the center x_1 . This yields

$$\begin{aligned} \sum_{j \notin I_1} \frac{1}{d_{1j}^k} &= \sum_{q \neq 1}^{|\lambda^N|^{-3}} \sum_{j \in C^q} \frac{1}{d_{1j}^k}, \\ &\leq M^N \sum_{q \neq 1}^{|\lambda^N|^{-3}} \frac{1}{|x_1 - \bar{x}_q|^k}. \end{aligned}$$

Now, we apply [12, lemma 2.1] for the set of centers $(\bar{x}_q)_{1 \leq q \leq |\lambda^N|^{-3}}$ having minimal distance $2\lambda^N$ to get the existence of a positive constant $C > 0$ satisfying

$$\sum_{q \neq 1}^{|\lambda^N|^{-3}} \frac{1}{|x_1 - \bar{x}_q|^k} \leq C \frac{(|\lambda^N|^{-3})^{1-k/3}}{|\lambda^N|^k}.$$

Again, if $k = 3$ we apply the same method of proof as [12, Lemma 2.1] to get

$$\sum_{q \neq 1}^{|\lambda^N|^{-3}} \frac{1}{|x_1 - \bar{x}_q|^3} \leq C \frac{|\log \lambda^N|}{|\lambda^N|^3},$$

which completes the proof. \square

The following results are used for the control of the particle concentration M^N :

$$M^N(t) := \sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i(t) \in \overline{B_\infty(x, \lambda^N)}\} \right\}.$$

We recall the definition of L^N introduced in (56):

$$L^N(t) := \max_i \# \{j \in \{1, \dots, N\} \text{ such that } |x_i(t) - x_j(t)|_\infty \leq \lambda^N\}.$$

The following lemma shwos that the two definitions are equivalent.

Lemma A.2. *We have*

$$L^N(t) \leq M^N(t) \leq 8L^N(t).$$

Proof. The first inequality is trivial. To prove the second one note that we have:

$$\sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N)}\} \right\} \leq, \\ 8 \sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N/2)}\} \right\}.$$

Indeed, for all $x \in \mathbb{R}^3$ there exists \bar{x}_k , $k = 1, \dots, 8$ such that

$$\overline{B_\infty(x, \lambda^N)} \subset \bigcup_k^8 \overline{B_\infty\left(\bar{x}_k, \frac{\lambda^N}{2}\right)},$$

this yields

$$\{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N)}\} \\ \subset \bigcup_k^8 \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(\bar{x}_k, \lambda^N/2)}\}.$$

Taking the supremum in the right hand side and then in the left one we obtain:

$$(74) \quad \sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N)}\} \right\} \leq \\ 8 \sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N/2)}\} \right\}.$$

Moreover, we remark that the supremum over all $x \in \mathbb{R}^3$ can be reduced to the supremum over $\bigcup_i \overline{B_\infty(x_i, \frac{\lambda^N}{2})}$. Now consider $x \in \bigcup_i \overline{B_\infty(x_i, \frac{\lambda^N}{2})}$, there exists $1 \leq i_0 \leq N$ such that $|x - x_{i_0}|_\infty \leq \frac{\lambda^N}{2}$, we have then for all $j \neq i_0$ such that $|x - x_j|_\infty \leq \frac{\lambda^N}{2}$:

$$|x_j - x_{i_0}|_\infty \leq |x_j - x|_\infty + |x - x_{i_0}|_\infty \leq \lambda^N,$$

which means that for all $x \in \bigcup_i \overline{B_\infty(x_i, \frac{\lambda^N}{2})}$ there exists $1 \leq i_0 \leq N$ such that

$$\{1 \leq j \leq N, \text{ such that } x_j \in \overline{B_\infty(x, \lambda^N/2)}\} \subset \{1 \leq j \leq N, \text{ such that } |x_j - x_{i_0}|_\infty \leq \lambda^N\}.$$

Taking the maximum over all i_0 in the right hand side, and then the supremum over all $x \in \bigcup_i \overline{B_\infty(x_i, \frac{\lambda^N}{2})}$ we obtain:

$$(75) \quad \sup_x \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i \in \overline{B_\infty(x, \lambda^N/2)}\} \right\} \leq \\ \max_i \# \{j \in \{1, \dots, N\} \setminus \{i\} \text{ such that } |x_i - x_j|_\infty \leq \lambda^N\}.$$

Gathering inequality (74) and (75) concludes the proof. \square

More generally we define for all $\beta > 0$:

$$L_\beta^N(t) := \max_i \# \{j \in \{1, \dots, N\} \text{ such that } |x_i(t) - x_j(t)|_\infty \leq \beta \lambda^N\},$$

and

$$M_\beta^N(t) := \sup_{x \in \mathbb{R}^3} \left\{ \# \{i \in \{1, \dots, N\} \text{ such that } x_i(t) \in \overline{B_\infty(x, \beta \lambda^N)}\} \right\},$$

with the notation

$$M_1^N(t) := M^N(t), \quad L_1^N(t) := L^N(t).$$

The previous results yields

Corollary A.3. *For all $\beta > 0$ and all $\alpha > 1$ we have:*

$$L_{\alpha\beta}^N(t) \leq 8[\alpha]^3 L_\beta^N(t),$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

Proof. For sake of clarity we set $\beta = 1$ and the proof remains the same for all $\beta > 0$. The idea is to show an equivalent formula for M^N and use Lemma A.2. Analogously to the proof of Lemma A.2, for all $x \in \mathbb{R}^3$ there exists \bar{x}_k , $k = 1, \dots, \lfloor \lambda \rfloor^3$ such that

$$\overline{B_\infty(x, \alpha \lambda^N)} \subset \bigcup_{k=1}^{\lceil \alpha \rceil^3} \overline{B_\infty(\bar{x}_k, \lambda^N)}.$$

Which yields, with the definition of M_λ^N :

$$M_\alpha^N \leq \lceil \alpha \rceil^3 M^N(t).$$

Finally, we apply Lemma A.2 to get

$$L_\alpha^N(t) \leq M_\alpha^N(t) \leq \lceil \alpha \rceil^3 M^N(t) \leq 8[\alpha]^3 L^N(t).$$

□

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REFERENCES

- [1] G. K. BATCHELOR, *Sedimentation in a dilute dispersion of spheres*, J. Fluid Mech. vol. 52, part 2, (1972), pp. [245,268].
- [2] L. BOUDIN, L. DESVILLETES, C. GRANDMONT, AND A. MOUSSA, *Global existence of solutions for the coupled Vlasov and Navier-Stokes equations*, Differential Integral Equations, 22(11-12), (2009), pp. [1247,1271].
- [3] T. CHAMPION, L. D. PASCALE, AND P. JUUTINEN, *The ∞ -Wasserstein distance: local solutions and existence of optimal transport maps*, SIAM J. Math. Anal. 40, (2008), pp. [1,20].
- [4] G. P. GALDI, *An introduction to the mathematical theory of the Navier-Stokes equations*, Springer Monographs in Mathematics. Springer, New York, second edition ed., 2011. Steady-state problems.
- [5] E. GUAZZELLI AND J. F. MORRIS, *A Physical Introduction To Suspension Dynamics*, Cambridge Texts In Applied Mathematics, 2012.

- [6] K. HAMDACHE, *Global existence and large time behaviour of solutions for the Vlasov-Stokes equations*, Japan J. Indust. Appl. Math., 15(1), (1998), pp. [51,74].
- [7] M. HAURAY, *Wasserstein distances for vortices approximation of Euler-type equations*, Math. Models Methods Appl. Sci. 19, (2009), pp. [1357,1384].
- [8] M. HAURAY AND P. E. JABIN, *Particle approximation of Vlasov equations with singular forces : propagation of chaos*, Ann. Sci. Éc. Norm. Supér. (4), (2015).
- [9] M. HILLAIRET, *On the homogenization of the Stokes problem in a perforated domain*, <https://hal.archives-ouvertes.fr/hal-01302560>, (2016).
- [10] R. M. HÖFER, *Sedimentation of inertialess particles in Stokes flows*, arXiv:1610.03748, (2016).
- [11] R. M. HÖFER AND J. J. L. VELÀZQUEZ, *The method of reflections, homogenization and screening for Poisson and Stokes equations in perforated domains*, arXiv:1603.06750, (2016).
- [12] P. E. JABIN AND F. OTTO, *Identification of the dilute regime in particle sedimentation*, Communications in Mathematical Physics, (2004).
- [13] S. KIM AND S. J. KARRILA, *Microhydrodynamics : Principles and Selected Applications*, Courier Corporation, 2005.
- [14] G. LOEPER, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl. 86, (2006), pp. [68,79].
- [15] J. H. C. LUKE, *Convergence of a multiple reflection method for calculating Stokes flow in a suspension*, Society for Industrial and Applied Mathematics, (1989).
- [16] M. SMOLUCHOWSKI, *ber die wechselwirkung von kugeln, die sich in einer zhen flssigkeit bewegen*, Bull. Acad. Sci. Cracovie A 1, (1911), pp. [28,39].
- [17] C. VILLANI, *Optimal transport, old and new*, Springer-Verlag, Berlin, 2009.

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