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# INFINITE CHARACTERS ON $GL_n(\mathbf{Q})$ , ON $SL_n(\mathbf{Z})$ , AND ON GROUPS ACTING ON TREES

BACHIR BEKKA

ABSTRACT. Answering a question of J. Rosenberg from [Ros–89], we construct the first examples of infinite characters on  $GL_n(\mathbf{K})$  for a global field  $\mathbf{K}$  and  $n \geq 2$ . The case  $n = 2$  is deduced from the following more general result. Let  $G$  a non amenable countable subgroup acting on locally finite tree  $X$ . Assume either that the stabilizer in  $G$  of every vertex of  $X$  is finite or that the closure of the image of  $G$  in  $\text{Aut}(X)$  is not amenable. We show that  $G$  has uncountably many infinite dimensional irreducible unitary representations  $(\pi, \mathcal{H})$  of  $G$  which are traceable, that is, such that the  $C^*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$  generated by  $\pi(G)$  contains the algebra of the compact operators on  $\mathcal{H}$ . In the case  $n \geq 3$ , we prove the existence of infinitely many characters for  $G = GL_n(R)$ , where  $n \geq 3$  and  $R$  is an integral domain such that  $G$  is not amenable. In particular, the group  $SL_n(\mathbf{Z})$  has infinitely many such characters for  $n \geq 2$ .

## 1. INTRODUCTION

Let  $G$  be a countable discrete group and  $\widehat{G}$  the unitary dual of  $G$ , that is, the set of equivalence classes of irreducible unitary representations of  $G$ . The space  $\widehat{G}$ , equipped with a natural Borel structure, is a standard Borel space exactly when  $G$  is virtually abelian, by results of Glimm and Thoma (see [Gli–61] and [Tho–68]). So, unless  $G$  is virtually abelian (in which case the representation theory of  $G$  is well understood), a description of  $\widehat{G}$  is hopeless or useless. There are at least two other dual objects of  $G$ , which seem to be more accessible than  $\widehat{G}$ :

- **Thoma’s dual space**  $E(G)$ , that is, the set of indecomposable positive definite central functions on  $G$ ;
- the space  $\text{Char}(G)$  of **characters** of  $G$ , that is, the space of lower semi-continuous semi-finite (not necessarily finite) traces  $t$  on the **maximal  $C^*$ -algebra**  $C^*(G)$  of  $G$  (see Subsection 2.1)

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which satisfies the following extremality condition: every lower semi-continuous semi-finite trace on  $C^*(G)$  dominated by  $t$  on the ideal of definition of  $t$  is proportional to  $t$ .

The space  $\text{Char}(G)$  parametrizes the quasi-equivalence classes of factorial representations of  $C^*(G)$  which are **traceable**; recall that a unitary representation  $\pi$  is factorial if the von Neumann algebra  $\mathcal{M}$  generated by  $\pi(G)$  is a factor and that a factorial representation  $\pi$  is traceable if there exists a faithful normal (not necessarily finite) trace  $\tau$  on  $\mathcal{M}$  and a positive element  $x \in C^*(G)$  such that  $0 < \tau(\pi(x)) < +\infty$ . If this is the case, then  $t = \tau \circ \pi$  belongs to  $\text{Char}(G)$ . Conversely, every element of  $\text{Char}(G)$  is obtained in this way. Traceable representations are also called *normal* representations.

Two traceable factorial representations  $\pi_1$  and  $\pi_2$  are quasi-equivalent if there exists an isomorphism  $\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$  such that  $\Phi(\pi_1(g)) = \pi_2(g)$  for all  $g \in G$ , where  $\mathcal{M}_i$  is the factor generated by  $\pi_i(G)$ .

Observe that an *irreducible* unitary representation  $(\pi, \mathcal{H})$  of  $G$  is traceable if and only if  $\pi(C^*(G))$  contains the algebra of compact operators on  $\mathcal{H}$ . The character associated to such a representation is given by the usual trace on  $\mathcal{B}(\mathcal{H})$  and so does not belong to  $E(G)$  whenever  $\mathcal{H}$  is infinite dimensional; in this case, the character is said to be of type  $I_\infty$ , in accordance with the type classification of von Neumann algebras. Observe also that two irreducible traceable representations of a group  $G$  are quasi-equivalent if and only if they are unitarily equivalent.

Thoma's dual space  $E(G)$  is a subspace of  $\text{Char}(G)$  and classifies the quasi-equivalence classes of the factorial representations  $\pi$  of  $C^*(G)$  for which the factor  $\mathcal{M}$  generated by  $\pi(G)$  is *finite*, that is, such that the trace  $\tau$  on  $\mathcal{M}$  takes only finite values (for more detail on all of this, see Chapters 6 and 17 in [Dix-77]).

Thoma's dual space  $E(G)$  was determined for several examples of countable groups  $G$ , among them  $G = GL_n(\mathbf{K})$  or  $G = SL_n(\mathbf{K})$  for an infinite field  $\mathbf{K}$  and  $n \geq 2$  ([Kiri-65]; see also [PeT-16]), and  $G = SL_n(\mathbf{Z})$  for  $n \geq 3$  ([Bek-07]); a procedure is given in [How-77, Proposition 3] to compute  $E(G)$  when  $G$  is a nilpotent finitely generated group.

The space  $\text{Char}(G)$  has been described for some *amenable* groups  $G$ :

- when  $G$  is nilpotent, we have  $E(G) = \text{Char}(G)$  (see [CaM-84, Theorem 2.1]);
- the space  $\text{Char}(G)$  is determined in [Guic-63] for the Baumslag-Solitar group  $BS(1, 2)$  and in [VeK-91] for the infinite symmetric group;

- for  $G = GL_n(\mathbf{K})$  and  $n \geq 2$ , it is shown in [Ros–89] that  $E(G) = \text{Char}(G)$  in the case where  $\mathbf{K}$  an algebraic extension of a finite field. (Observe that  $GL_n(\mathbf{K})$  is amenable if and only if  $\mathbf{K}$  an algebraic extension of a finite field; see Proposition 9 in [HoR–89] or Proposition 11 below.)

J. Rosenberg asked in [Ros–89, Remark after Théorème 1] whether there exists an *infinite* character on  $G = GL_n(\mathbf{K})$ , that is, whether  $\text{Char}(G) \neq E(G)$ , for a field  $\mathbf{K}$  which is not an algebraic extension of a finite field. We will show below that the answer to this question is positive, by exhibiting as far we know the first examples of such characters. The case where  $n = 2$  and  $\mathbf{K}$  is a global field (see below) will be deduced from a general result concerning groups acting on trees, which we now state.

Recall that a graph  $X$  is locally finite if every vertex on  $X$  has only finitely many neighbours. In this case, the group  $\text{Aut}(X)$  of automorphisms of  $X$ , equipped with the topology of pointwise convergence, is a locally compact group for which the vertex stabilizers are compact. Concerning the notion of weakly equivalent representations, see Chapters 3 and 18 in [Dix–77] (see also Section 2.1).

**Theorem 1.** *Let  $X$  be a tree and  $G$  a countable subgroup acting on  $X$ . Assume that*

- (a) *either  $G$  is not amenable and the stabilizer in  $G$  of every vertex of  $X$  is finite, or*
- (b)  *$X$  is locally finite and the closure of the image of  $G$  in  $\text{Aut}(X)$  is not amenable.*

*There exists an uncountable family  $(\pi_t)_t$  of irreducible unitary representations of  $G$  with the following properties:  $\pi_t$  is infinite dimensional, is traceable and is not weakly equivalent to  $\pi_{t'}$  for  $t' \neq t$ .*

Recall that a global field is a finite extension of either the field  $\mathbf{Q}$  of rational numbers or of the field  $\mathbf{F}_p(T)$  of rational functions in  $T$  over the finite field  $\mathbf{F}_p$  (see Chapter III in [Wei–67]).

**Corollary 2.** *Let  $G$  be either*

- (i)  $GL_2(\mathbf{K})$  or  $SL_2(\mathbf{K})$  for a global field  $\mathbf{K}$ , or
- (ii)  $SL_2(\mathbf{Z})$ , or
- (iii)  $F_n$ , the free non abelian group over  $n \in \{2, \dots, +\infty\}$  generators.

*There exists an uncountable family  $(\pi_t)_t$  of unitary representations of  $G$  with the properties from Theorem 1; moreover, in case  $G = F_n$ , the representations  $\pi_t$  are all faithful.*

Turning to the case  $n \geq 3$ , we prove a result for  $G = \mathrm{GL}_n(R)$  or  $G = \mathrm{SL}_n(R)$ , valid for every integral domain  $R$  such that  $G$  is not amenable.

**Theorem 3.** *Let  $R$  be a countable unital commutative ring which is an integral domain; in case the characteristic of  $R$  is positive, assume that the field of fractions of  $R$  is not an algebraic extension of its prime field. For  $n \geq 3$ , let  $G = \mathrm{GL}_n(R)$  or  $G = \mathrm{SL}_n(R)$ . There exists an infinite dimensional irreducible unitary representation of  $G$  which is traceable.*

In the case where  $R$  is a field or the ring of integers, we can even produce infinitely many non equivalent representations as in Theorem 3.

**Corollary 4.** (i) *For  $n \geq 3$ , let  $G = \mathrm{GL}_n(\mathbf{K})$  for a countable field  $\mathbf{K}$  which is not an algebraic extension of a finite field. There exists an uncountable family  $(\pi_t)_t$  of pairwise non equivalent infinite dimensional irreducible unitary representations of  $G$  which are traceable. Moreover, the representations  $\pi_t$  all have a trivial central character, that is, the  $\pi_t$ 's are representations of  $\mathrm{PGL}_n(\mathbf{K})$ .*

(ii) *Let  $G = \mathrm{SL}_n(\mathbf{Z})$  for  $n \geq 3$ . There exists an infinite family of pairwise non equivalent infinite dimensional irreducible unitary representations of  $G$  which are traceable.*

The methods of proofs of Theorem 1 and Theorem 3 are quite different in nature:

- the proof of Theorem 1 is based on properties of a remarkable family of unitary representations of groups acting on trees constructed in [JuV-84] and used to show their  $K$ -theoretic amenability, a notion which originated from [Cun-83] in the case of free groups;
- the traceable representations we construct in Theorem 3 are induced representations from suitable subgroups. The case  $n \geq 4$  uses the existence of appropriate subgroups of  $\mathrm{GL}_n(R)$  with Kazhdan's Property (T).

**Remark 5.** For a group  $G$  as in Theorem 1 or Theorem 3, our results show that the set  $\mathrm{Char}(G)$  contains characters of type  $I_\infty$ .

For, say,  $G = \mathrm{GL}_n(\mathbf{Q})$ , we do not know whether  $\mathrm{Char}(G)$  contains characters of type  $II_\infty$ , that is, characters for which the corresponding factorial representation generates a factor of type  $II_\infty$ .

This paper is organized as follows. In Section 2, we establish some preliminary facts which are necessary to the proofs of our results. Section 3 is devoted to the proofs of Theorem 1 and Corollary 2; Theorem 3 and Corollary 4 are proved in Section 4.

## 2. SOME PRELIMINARY RESULTS

**2.1.  $C^*$ -algebras.** Let  $G$  be a countable group. Recall that a unitary representation of  $G$  is a homomorphism  $\pi : G \rightarrow U(\mathcal{H})$  from  $G$  to the unitary group of a complex separable Hilbert space  $\mathcal{H}$ . From now on, we will simply write **representation** of  $G$  instead of “unitary representation of  $G$ ”.

Every representation  $(\pi, \mathcal{H})$  of  $G$  extends naturally to a  $*$ -representation, denoted again by  $\pi$ , of the group algebra  $\mathbf{C}[G]$  by bounded operators on  $\mathcal{H}$ .

Recall that the maximal  $C^*$ -algebra  $C^*(G)$  of  $G$  is the completion of  $\mathbf{C}[G]$  of  $G$  with respect to the norm

$$f \mapsto \sup_{\pi \in \text{Rep}(G)} \|\pi(f)\|,$$

where  $\text{Rep}(G)$  denotes the set of representations  $(\pi, \mathcal{H})$  of  $G$  in a separable Hilbert space  $\mathcal{H}$ .

We can view  $G$  as subset of  $\mathbf{C}[G]$  and hence as a subset of  $C^*(G)$ . The  $C^*$ -algebra  $C^*(G)$  has the following universal property: every representation  $(\pi, \mathcal{H})$  of  $G$  extends to a unique representation (that is,  $*$ -homomorphism)  $\pi : C^*(G) \rightarrow \mathcal{B}(\mathcal{H})$ . The correspondence  $G \rightarrow C^*(G)$  is functorial: every homomorphism  $\varphi : G_1 \rightarrow G_2$  between two countable groups  $G_1$  and  $G_2$  extends to a unique morphism

$$\varphi_* : C^*(G_1) \rightarrow C^*(G_2)$$

of  $C^*$ -algebras. In particular, given a subgroup  $H$  of a group  $G$ , the injection map  $i : H \rightarrow G$  extends to a morphism  $i_* : C^*(H) \rightarrow C^*(G)$ ; the map  $i_*$  is injective and so  $C^*(H)$  can be viewed naturally as a subalgebra of  $C^*(G)$ : indeed, this follows from the fact that every representation  $\sigma$  of  $H$  occurs as subrepresentation of the restriction to  $H$  of some representation  $\pi$  of  $G$  (one may take as  $\pi$  the induced representation  $\text{Ind}_H^G \sigma$ , as shown below in Proposition 9).

The following simple lemma will be one of our tools in order to show that  $\pi(C^*(G))$  contains a non-zero compact operator for a representation  $\pi$  of  $G$ .

Let  $\mathcal{A}$  be a  $C^*$ -algebra. Recall that a representation  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  weakly contains another representation  $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$  if

$$\|\rho(a)\| \leq \|\pi(a)\| \quad \text{for all } a \in \mathcal{A},$$

or, equivalently,  $\ker \pi \subset \ker \rho$  (see Chapter 3 in [Dix-77]). Two representations  $\pi$  and  $\rho$  are weakly equivalent if  $\pi$  weakly contains  $\rho$  and  $\rho$  weakly contains  $\pi$ , that is, if  $\ker \pi = \ker \rho$ .

**Lemma 6.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  a representation of  $\mathcal{A}$ . Assume that  $\mathcal{H}$  contains a non-zero finite dimensional  $\pi(\mathcal{A})$ -invariant subspace  $\mathcal{K}$  and that the restriction  $\pi_1$  of  $\pi$  to  $\mathcal{K}$  is not weakly contained in the restriction  $\pi_0$  of  $\pi$  to the orthogonal complement  $\mathcal{K}^\perp$ . Then  $\pi(\mathcal{A})$  contains a non-zero compact operator.*

*Proof.* The ideal  $\ker \pi_0$  is not contained in  $\ker \pi_1$ , since  $\pi_1$  is not weakly contained in  $\pi_0$ . Hence, there exists  $a \in \mathcal{A}$  with  $\pi_0(a) = 0$  and  $\pi_1(a) \neq 0$ . Then  $\pi(a) = \pi_1(a)$  has a finite dimensional range and is non-zero.  $\square$

Knowing that a representation of  $\mathcal{A}$  contains in its image a non-zero compact operator, the following lemma enables us to construct an *irreducible* representation of  $\mathcal{A}$  with the same property.

**Lemma 7.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  a representation of  $\mathcal{A}$  in a separable Hilbert space  $\mathcal{H}$ . Let  $a \in \mathcal{A}$  be such that  $\pi(a)$  is a non-zero compact operator. Then there exists an irreducible subrepresentation  $\sigma$  of  $\pi$  such that  $\sigma(a)$  is a compact operator and such that  $\|\sigma(a)\| = \|\pi(a)\|$ .*

*Proof.* We can decompose  $\pi$  as a direct integral  $\int_\Omega^\oplus \pi_\omega d\mu(\omega)$  of irreducible representations  $\pi_\omega$ ; thus, we can find a probability measure  $\mu$  on a standard Borel space  $\Omega$ , a measurable field  $\omega \rightarrow \pi_\omega$  of irreducible representations of  $\mathcal{A}$  in a measurable field  $\omega \rightarrow \mathcal{H}_\omega$  of separable Hilbert spaces on  $\Omega$ , and a Hilbert space isomorphism  $U : \mathcal{H} \rightarrow \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$  such that

$$U\pi(x)U^{-1} = \int_\Omega^\oplus \pi_\omega(x) d\mu(\omega). \quad \text{for all } x \in \mathcal{A}$$

(see [Dix-77, §8.5]). Without loss of generality, we will identify  $\pi$  with  $\int_\Omega^\oplus \pi_\omega d\mu(\omega)$ .

Let  $a \in \mathcal{A}$  be such that  $\pi(a)$  is a non-zero compact operator. Since  $\|\sigma(a^*a)\| = \|\sigma(a)\|^2$  for every representation  $\sigma$  of  $\mathcal{A}$ , upon replacing  $a$  by  $a^*a$ , we can assume that  $a$  is a positive element of  $\mathcal{A}$ . So  $\pi(a)$  is a positive selfadjoint compact operator on  $\mathcal{H}$  with  $\pi(a) \neq 0$ .

There exists an orthonormal basis  $(F_n)_{n \geq 1}$  of  $\mathcal{H} = \int_{\Omega}^{\oplus} \mathcal{H}_{\omega} d\mu(\omega)$  consisting of eigenvectors of  $\pi(a)$ , with corresponding eigenvalues  $(\lambda_n)_{n \geq 1}$ , counted with multiplicities. For every  $\omega \in \Omega$  and every  $n \geq 1$ , we have

$$(*) \quad \pi_{\omega}(a)(F_n(\omega)) = \lambda_n F_n(\omega).$$

Let  $n_0 \geq 1$  be such that  $\lambda_{n_0} = \max\{\lambda_n \mid n \geq 1\}$ . Then  $\|\pi(a)\| = \lambda_{n_0}$ . Set

$$\Omega_0 = \{\omega \in \Omega \mid F_{n_0}(\omega) \neq 0\}.$$

Since  $F_{n_0} \neq 0$ , we have  $\mu(\Omega_0) > 0$ . We claim that  $\Omega_0$  is a finite subset of  $\Omega$  consisting of atoms of  $\mu$ . Indeed, assume by contradiction that is not the case. Then there exists an infinite sequence  $(A_k)_k$  of pairwise disjoint Borel subsets of  $\Omega_0$  with  $\mu(A_k) > 0$ . Observe that  $\mathbf{1}_{A_k} F_{n_0}$  is a non-zero vector in  $\mathcal{H}$  and that  $\langle \mathbf{1}_{A_k} F_{n_0}, \mathbf{1}_{A_l} F_{n_0} \rangle = 0$  for every  $k \neq l$ . Moreover, we have

$$\begin{aligned} \pi(a)(\mathbf{1}_{A_k} F_{n_0}) &= \int_{A_k}^{\oplus} \pi_{\omega}(a)(F_{n_0}(\omega)) d\mu(\omega) \\ &= \lambda_{n_0} \int_{A_k}^{\oplus} F_{n_0}(\omega) d\mu(\omega) \\ &= \lambda_{n_0} \mathbf{1}_{A_k} F_{n_0}. \end{aligned}$$

Since  $\pi(a)$  is a compact operator and  $\lambda_{n_0} \neq 0$ , this is a contradiction.

Let  $\omega_0 \in \Omega_0$  be such that  $\mu(\{\omega_0\}) > 0$ . We claim that the linear span of  $\{F_n(\omega_0) \mid n \geq 1\}$  is dense in  $\mathcal{H}_{\omega_0}$ . Indeed, let  $v \in \mathcal{H}_{\omega_0}$  be such that

$$\langle v \mid F_n(\omega_0) \rangle = 0 \quad \text{for all } n \geq 1.$$

Let  $F = \mathbf{1}_{\omega_0} \otimes v \in \mathcal{H}$  be defined by  $F(\omega_0) = v$  and  $F(\omega) = 0$  for  $\omega \neq \omega_0$ . Then  $\langle F \mid F_n \rangle = 0$  for all  $n \geq 1$ . Hence,  $F = 0$ , that is,  $v = 0$ , since  $(F_n)_{n \geq 1}$  is a basis of  $\mathcal{H}$ .

By (\*),  $F_n(\omega_0)$  is an eigenvector of  $\pi_{\omega_0}(a)$  with eigenvalue  $\lambda_n$  for every  $n \geq 1$  such that  $F_n(\omega_0) \neq 0$ . Since  $\{F_n(\omega_0) \mid n \geq 1\}$  is a total subset of  $\mathcal{H}_{\omega_0}$ , it follows that there exists a basis of  $\mathcal{H}_{\omega_0}$  consisting of eigenvectors of  $\pi_{\omega_0}(a)$ . As

$$\lim_{n \rightarrow \infty} \lambda_n = 0$$

(in case the sequence  $(\lambda_n)_{n \geq 1}$  is infinite), it follows that  $\pi_{\omega_0}(a)$  is a compact operator on  $\mathcal{H}_{\omega_0}$ . Moreover, we have

$$\|\pi_{\omega_0}(a)\| = \max\{\lambda_n \mid n \geq 1\} = \lambda_{n_0} = \|\pi(a)\|.$$

Finally, an equivalence between  $\pi_{\omega_0}$  and a subrepresentation of  $\pi$  is provided by the unitary linear map

$$\mathcal{H}_{\omega_0} \rightarrow \mathcal{H}, v \mapsto \mathbf{1}_{\omega_0} \otimes v.$$



□

**2.2. Induced representations of groups.** In the sequel, we will often consider group representations which are induced representations. Let  $G$  be a countable group,  $H$  a subgroup of  $G$  and  $(\sigma, \mathcal{K})$  a representation of  $H$ . Recall that the induced representation  $\text{Ind}_H^G \sigma$  of  $G$  may be realized as follows. Let  $\mathcal{H}$  be the Hilbert space of maps  $f : G \rightarrow \mathcal{K}$  with the following properties

- (i)  $f(hx) = \sigma(h)f(x)$  for all  $x \in G, h \in H$ ;
- (ii)  $\sum_{x \in H \backslash G} \|f(x)\|^2 < \infty$ . (Observe that  $\|f(x)\|$  only depends on the coset of  $x$  in  $H \backslash G$ .)

The induced representation  $\pi = \text{Ind}_H^G \sigma$  is given on  $\mathcal{H}$  by right translation:

$$(\pi(g)f)(x) = f(xg) \quad \text{for all } g \in G, f \in \mathcal{H} \text{ and } x \in G.$$

Recall that the **commensurator** of  $H$  in  $G$  is the subgroup, denoted by  $\text{Comm}_G(H)$ , of the elements  $g \in G$  such that  $gHg^{-1} \cap H$  is of finite index in both  $H$  and  $g^{-1}Hg$ .

The following result appeared in [Mac–51] in the case where  $\sigma$  is of dimension 1 and was extended to its present form in [Kle–61] and [Cor–75].

**Theorem 8.** *Let  $G$  be a countable group and  $H$  a subgroup of  $G$  such that  $\text{Comm}_G(H) = H$ .*

- (i) *For every finite dimensional irreducible representation  $\sigma$  of  $H$ , the induced representation  $\text{Ind}_H^G \sigma$  is irreducible.*
- (ii) *Let  $\sigma_1$  and  $\sigma_2$  be non equivalent finite dimensional irreducible representations of  $H$ . The representations  $\text{Ind}_H^G \sigma_1$  and  $\text{Ind}_H^G \sigma_2$  are non equivalent.*

We will need to decompose the restriction to a subgroup of an induced representation  $\text{Ind}_H^G \sigma$  as in Theorem 8. For  $g \in G$ , we denote by  $\sigma^g$  the representation of  $g^{-1}Hg$  defined by  $\sigma^g(x) = \sigma(gxg^{-1})$  for  $x \in g^{-1}Hg$ .

For the convenience of the reader, we give a short and elementary proof of the following special case of the far more general result [Mac–52, Theorem 12.1].

**Proposition 9.** *Let  $G$  be a countable group,  $H, L$  subgroups of  $G$  and  $(\sigma, \mathcal{K})$  a representation of  $H$ . Let  $S$  be a system of representatives for the double coset space  $H \backslash G / L$ . The restriction  $\pi|_L$  to  $L$  of the induced*

representation  $\pi = \text{Ind}_H^G \sigma$  is equivalent to the direct sum

$$\bigoplus_{s \in S} \text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L})$$

*Proof.* Let  $\mathcal{H}$  be the Hilbert space of  $\pi$ , as described above. For every  $s \in S$ , let  $\mathcal{H}_s$  be the space of maps  $f \in \mathcal{H}$  such that  $f = 0$  outside the double coset  $HsL$ . We have an orthogonal  $L$ -invariant decomposition

$$\mathcal{H} = \bigoplus_{s \in S} \mathcal{H}_s.$$

Fix  $s \in S$ . The Hilbert space  $\mathcal{H}'_s$  of  $\text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L})$  consists of the maps  $f : L \rightarrow \mathcal{K}$  such that

- $f(tx) = \sigma(sts^{-1})f(x)$  for all  $t \in s^{-1}Hs \cap L, x \in L$ ;
- $\sum_{x \in s^{-1}Hs \cap L \setminus L} \|f(x)\|^2 < \infty$ .

Define a linear map  $U : \mathcal{H}_s \rightarrow \mathcal{H}'_s$  by

$$Uf(x) = f(sx) \quad \text{for all } f \in \mathcal{H}_s, x \in L.$$

Observe that, for  $t \in s^{-1}Hs \cap L, x \in L$  and  $f \in \mathcal{H}_s$ , we have

$$Uf(tx) = f(stx) = f((sts^{-1})sx) = \sigma(sts^{-1})f(sx) = \sigma^s(t)Uf(x)$$

and that

$$\sum_{x \in s^{-1}Hs \cap L \setminus L} \|Uf(x)\|^2 = \sum_{x \in s^{-1}Hs \cap L \setminus L} \|f(sx)\|^2 = \sum_{y \in H \setminus G} \|f(y)\|^2 < \infty,$$

so that  $Uf \in \mathcal{H}'_s$  and  $U$  is an isometry. It is easy to check that the map  $U$  is invertible, with inverse given by

$$U^{-1}f(y) = \begin{cases} \sigma(h)f(x) & \text{if } y = hsx \in HsL \\ 0 & \text{otherwise,} \end{cases}$$

for  $f \in \mathcal{H}'_s$ . Moreover,  $U$  intertwines the restriction of  $\pi|_L$  to  $\mathcal{H}_s$  and  $\text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L})$ : for  $g, x \in L$  and  $f \in \mathcal{H}'_s$ , we have

$$\begin{aligned} (U\pi(g)U^{-1}f)(x) &= (\pi(g)U^{-1}f)(sx) \\ &= (U^{-1}f)(sxs) \\ &= f(xg) \\ &= (\text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L})(g)f)(x) \end{aligned}$$

□

We will need the following elementary lemma about induced representations containing a finite dimensional representation. Recall that a representation  $\pi$  of a group  $G$  contains another representation  $\sigma$  of  $G$  if  $\sigma$  is equivalent to a subrepresentation of  $\pi$ . Recall also that, if  $\pi$  is

finite dimensional representation of a group  $G$ , then  $\pi \otimes \bar{\pi}$  contains the trivial representation  $1_G$ , where  $\bar{\pi}$  is the conjugate representation of  $\pi$  and  $\pi \otimes \rho$  denotes the (inner) tensor product of the representations  $\pi$  and  $\rho$  (see [BHV–08, Proposition A. 1.12]).

**Proposition 10.** *Let  $G$  be a countable group,  $H$  a subgroup of  $G$ , and  $\sigma$  a representation of  $H$ . Assume that the induced representation  $\text{Ind}_H^G \sigma$  contains a finite dimensional representation of  $G$ . Then  $H$  has finite index in  $G$ .*

*Proof.* By assumption,  $\pi := \text{Ind}_H^G \sigma$  contains a finite dimensional representation  $\sigma$ . Hence,  $\pi \otimes \bar{\pi}$  contains  $1_G$ . On the other hand,

$$\pi \otimes \bar{\pi} = (\text{Ind}_H^G \sigma) \otimes \bar{\pi}$$

is equivalent to  $\text{Ind}_H^G(\rho)$ , where  $\rho = \sigma \otimes (\bar{\pi}|_H)$ ; see [BHV–08, Proposition E. 2.5]. So, there exists a non-zero map  $f : G \rightarrow \mathcal{K}$  in the Hilbert space of  $\text{Ind}_H^G(\rho)$  which is  $G$ -invariant, that is, such that  $f(xg) = f(x)$  for all  $g, x \in G$ . This implies that the  $L^2$ -function  $x \mapsto \|f(x)\|^2$  is constant on  $H \backslash G$ . This is only possible if  $H \backslash G$  is finite.  $\square$

**2.3. Amenability.** Let  $\mathbf{G}$  be a topological group and  $UCB(\mathbf{G})$  the Banach space of the left uniformly continuous bounded functions on  $\mathbf{G}$ , equipped with the uniform norm. Recall that  $\mathbf{G}$  is amenable if there exists a  $\mathbf{G}$ -invariant mean on  $UCB(\mathbf{G})$  (see Appendix G in [BHV–08]).

The following proposition characterizes the integral domains  $R$  for which  $\text{GL}_n(R)$  or  $\text{SL}_n(R)$  is amenable; the proof is an easy extension of the proof given in Proposition 9 in [HoR–89] for the case where  $R$  is a field.

**Proposition 11.** *Let  $R$  be a countable unital commutative ring which is an integral domain. Let  $\mathbf{K}$  be the field of fractions of  $R$  and  $G = \text{GL}_n(R)$  or  $G = \text{SL}_n(R)$  for an integer  $n \geq 2$ . The following properties are equivalent:*

- (i)  $G$  is not amenable.
- (ii)  $\mathbf{K}$  is not an algebraic extension of a finite field.
- (iii)  $R$  contains  $\mathbf{Z}$  if the characteristic of  $\mathbf{K}$  is 0 or the polynomial ring  $\mathbf{F}_p[T]$  if the characteristic of  $\mathbf{K}$  is  $p > 0$ .

*Proof.* Assume that  $\mathbf{K}$  is an algebraic extension of a finite field  $\mathbf{F}_q$ . Then  $\mathbf{K} = \bigcup_m \mathbf{K}_m$  for an increasing family of finite extensions  $\mathbf{K}_m$  of  $\mathbf{F}_q$ ; hence,  $\text{GL}_n(\mathbf{K}) = \bigcup_m \text{GL}_n(\mathbf{K}_m)$  is the inductive limit of the finite and hence amenable groups  $\text{GL}_n(\mathbf{K}_m)$ ; it follows that  $\text{GL}_n(\mathbf{K})$  is amenable and therefore  $\text{GL}_n(R)$  and  $\text{SL}_n(R)$  are amenable. This shows that (i) implies (ii).

Assume that (ii) holds. If the characteristic of  $\mathbf{K}$  is 0, then  $\mathbf{K}$  contains  $\mathbf{Q}$  and hence  $R$  contains  $\mathbf{Z}$ . So, we can assume that the characteristic of  $\mathbf{K}$  is  $p > 0$ . We claim that  $R$  contains an element which is not algebraic over the prime field  $\mathbf{F}_p$ . Indeed, otherwise, every element in  $R$  is algebraic over  $\mathbf{F}_p$ . As the set of elements in  $\mathbf{K}$  which are algebraic over  $\mathbf{F}_p$  is a field, it would follow that the field fraction field  $\mathbf{K}$  is algebraic over  $\mathbf{F}_p$ . This contradiction shows that (ii) implies (iii).

Assume that (iii) holds. Then  $\mathrm{SL}_n(R)$  contains a copy of  $\mathrm{SL}_2(\mathbf{Z})$  or a copy of  $\mathrm{SL}_2(\mathbf{F}_p[T])$ . It is well-known that both  $\mathrm{SL}_2(\mathbf{Z})$  and  $\mathrm{SL}_2(\mathbf{F}_p[T])$  contain a subgroup which is isomorphic to the free group on two generators. Therefore,  $G$  is not amenable and so (iii) implies (i).  $\square$

Let  $\mathbf{G}$  be a *locally compact* group, with Haar measure  $m$ . Recall that the amenability of  $\mathbf{G}$  is characterized by the Hulanicki-Reiter theorem (see [BHV–08, Theorem G.3.2]):  $\mathbf{G}$  is amenable if and only if the regular representation  $(\lambda_{\mathbf{G}}, L^2(\mathbf{G}, m))$  weakly contains the trivial representation  $1_{\mathbf{G}}$ , where  $m$  is Haar measure on  $\mathbf{G}$ ; when this is the case,  $\lambda_{\mathbf{G}}$  weakly contains every representation of  $\mathbf{G}$ .

The following result shows the amenability of  $\mathbf{G}$  can be detected by the restriction of  $\lambda_{\mathbf{G}}$  to a dense subgroup; for a more general result, see [Guiv–80, Proposition 1] or [Bek–16, Theorem 5.5].

**Proposition 12.** *Let  $\mathbf{G}$  be a locally compact group and  $G$  a countable dense subgroup of  $\mathbf{G}$ . Assume that the restriction to  $G$  of the regular representation  $\lambda_{\mathbf{G}}$  of  $\mathbf{G}$  weakly contains the trivial representation  $1_G$ . Then  $\mathbf{G}$  is amenable.*

*Proof.* By assumption, there exists a sequence  $(f_n)_n$  in  $L^2(\mathbf{G}, m)$  with  $\|f_n\| = 1$  such that

$$\lim_n \|\lambda_{\mathbf{G}}(g)f_n - f_n\| = 0 \quad \text{for all } g \in G.$$

Then, since  $\| |f_n(g^{-1}x)| - |f_n(x)| \| \leq |f_n(g^{-1}x) - f_n(x)|$  for  $g, x \in \mathbf{G}$ , we have

$$(*) \quad \lim_n \|\lambda_{\mathbf{G}}(g)|f_n| - |f_n|\| = 0 \quad \text{for all } g \in G.$$

Set  $\varphi_n := \sqrt{|f_n|}$ . Then  $\varphi_n \geq 0$  and  $\int_{\mathbf{G}} \varphi_n dm = 1$ . Every  $\varphi_n$  defines a mean  $M_n : f \mapsto \int_{\mathbf{G}} f \varphi_n dm$  on  $UCB(\mathbf{G})$ . Let  $M$  be a limit of  $(M_n)_n$  for the weak- $*$ -topology on the dual space of  $UCB(\mathbf{G})$ . It follows from (\*) that  $M$  is invariant under  $G$ . Since, for every  $f \in UCB(\mathbf{G})$ , the map

$$\mathbf{G} \rightarrow UCB(\mathbf{G}), \quad g \mapsto_g f$$

is continuous (where  $_g f$  denotes left translation by  $g \in \mathbf{G}$ ), it follows that  $M$  is invariant under  $\mathbf{G}$ . Hence,  $\mathbf{G}$  is amenable.

**2.4. Special linear groups over a subring of a field.** We will use the following elementary lemma about subgroups of  $\mathrm{SL}_n(\mathbf{K})$  which stabilize a line in  $\mathbf{K}^n$ .

**Lemma 13.** *For an infinite field  $\mathbf{K}$  and  $n \geq 2$ , let  $L$  be a subgroup of  $\mathrm{SL}_n(\mathbf{K})$  which stabilizes a line  $\ell$  in  $\mathbf{K}^n$ . Then  $L \cap \mathrm{SL}_n(R)$  has infinite index in  $\mathrm{SL}_n(R)$  for every infinite unital subring  $R$  of  $\mathbf{K}$ .*

*Proof.* Let  $\{v_1, \dots, v_n\}$  be a basis of  $\mathbf{K}^n$  with  $\ell = \mathbf{K}v_1$ . Fix  $i, j \in \{1, \dots, n\}$  with  $i \neq j$  and, for  $\lambda \in \mathbf{K}$ , let  $E_{ij}(\lambda)$  be the corresponding elementary matrix in  $\mathrm{SL}_n(\mathbf{K})$ , that is,

$$E_{ij}(\lambda) = I_n + \lambda \Delta_{ij},$$

where  $\Delta_{ij}$  denotes the matrix with 1 at the position  $(i, j)$  and 0 otherwise.

For every  $l = 1, \dots, n$ , let  $\varphi_l : \mathbf{K} \rightarrow \mathbf{K}$  be defined by

$$E_{ij}(\lambda)(v_1) = \sum_{i=1}^n \varphi_l(\lambda) v_i \quad \text{for } \lambda \in \mathbf{K}.$$

Every  $\varphi_l$  is a polynomial function (in fact, an affine function) on  $\mathbf{K}$  and, for  $l = 2, \dots, n$ , we have  $\varphi_l(\lambda) = 0$  for every  $\lambda \in \mathbf{K}$  such that  $E_{ij}(\lambda) \in L$ .

Assume, by contradiction, that  $L \cap \mathrm{SL}_n(R)$  has finite index in  $\mathrm{SL}_n(R)$  for an infinite subring  $R$  of  $\mathbf{K}$ . Then the subgroup

$$L_{i,j}(R) := L \cap \{E_{ij}(\lambda) \mid \lambda \in R\}$$

has finite index in the subgroup  $\{E_{ij}(\lambda) \mid \lambda \in R\}$  of  $\mathrm{SL}_n(R)$ . In particular,  $L_{i,j}(R)$  is infinite. It follows that  $\varphi_l$  has infinitely many roots in  $\mathbf{K}$  and hence that  $\varphi_l = 0$ , for every  $l = 2, \dots, n$ . Therefore, every elementary matrix  $E_{ij}(\lambda)$  fixes the line  $\ell$ , for  $i, j \in \{1, \dots, n\}$  and  $\lambda \in \mathbf{K}$ . Since  $\mathrm{SL}_n(\mathbf{K})$  is generated by elementary matrices, it follows that every matrix in  $\mathrm{SL}_n(\mathbf{K})$  fixes the line  $\ell$ ; this of course is impossible.  $\square$

### 3. PROOFS OF THEOREM 1 AND COROLLARY 2

**3.1. Proof of Theorem 1.** Let  $X$  be a tree, with  $X^0$  the set of vertices and  $X^1$  the set of edges of  $X$ . Let  $\mathbf{G}$  be a locally compact group acting on  $X$ .

Julg and Valette constructed in [JuV–84] (see also [Szw–91] and [Jul–15]) a remarkable family of representations  $(\pi_t)_{t \in [0,1]}$  of  $\mathbf{G}$ , all defined on  $\ell^2(X^0)$ , with the following properties:

- (i)  $\pi_0$  is the natural representation of  $\mathbf{G}$  on  $\ell^2(X^0)$  and  $\pi_1$  is equivalent to  $1_{\mathbf{G}} \oplus \rho_1$ , where  $\rho_1$  is the natural representation of  $\mathbf{G}$  on  $\ell^2(X^1)$ ;
- (ii) for every  $t \in [0, 1]$ , there exists a bounded operator  $T_t$  on  $\ell^2(X^0)$  with inverse  $T_t^{-1}$  defined on the subspace of functions of  $X^0$  with finite support such that  $\pi_t(g) := T_t^{-1}\pi_0(g)T_t$  extends to a unitary operator on  $\ell^2(X^0)$  for every  $g \in \mathbf{G}$ ; so, a unitary representation  $g \mapsto \pi_t(g)$  of  $\mathbf{G}$  is defined on  $\ell^2(X^0)$ ;
- (iii)  $\pi_t(g) - \pi_0(g)$  is a finite-rank operator on  $\ell^2(X^0)$ , for every  $t \in [0, 1]$  and  $g \in \mathbf{G}$ ;
- (iv) we have

$$\langle \pi_t(g)T_t^{-1}\delta_x \mid T_t^{-1}\delta_y \rangle = t^{d(gx,y)},$$

for every  $t \in (0, 1)$ ,  $g \in \mathbf{G}$  and  $x, y \in X^0$ , where  $d$  denotes the natural distance on  $X^0$ ;

- (v) the map

$$[0, 1] \rightarrow \mathbf{R}^+, t \mapsto \|\pi_t(g) - \pi_0(g)\|$$

is continuous for every  $g \in \mathbf{G}$ .

(Our representation  $\pi_t$  is  $g \mapsto U_\lambda \rho_\lambda(g) U_\lambda^{-1}$  with  $\lambda = -\log t$ , for the representation  $\rho_\lambda$  and the operator  $U_\lambda$  appearing in §2 of [JuV–84].)

Let  $G$  be a countable group acting on  $X$ . Assume that

- (a) either  $G$  is not amenable and the stabilizer in  $G$  of every vertex of  $X$  is finite or
- (b)  $X$  is locally finite and the closure of the image of  $G$  in  $\text{Aut}(X)$  is not amenable.

Set  $\mathbf{G} = G$  in case (a) and let  $\mathbf{G}$  be the closure of  $G$  in  $\text{Aut}(X)$  in case (b). Let  $(\pi_t)_{t \in [0,1]}$  be the family of representations of  $\mathbf{G}$  as above.

• *First step.* For every  $a \in C^*(G)$  and every  $t \in [0, 1]$ , the operator  $\pi_t(a) - \pi_0(a)$  is compact and the map

$$[0, 1] \rightarrow \mathbf{R}^+, t \mapsto \|\pi_t(a) - \pi_0(a)\|$$

is continuous.

Indeed, this follows from Properties (iii) and (v) of the family  $(\pi_t)_t$  and from the fact that  $\mathbf{C}[G]$  is dense in  $C^*(G)$ .

• *Second step.* The restriction  $\pi_0|_G$  of  $\pi_0$  to  $G$  does not weakly contain the trivial representation  $1_G$ .

Indeed, the representation  $\pi_0$  of  $\mathbf{G}$  is equivalent to the direct sum  $\bigoplus_{s \in T} \lambda_{\mathbf{G}/\mathbf{G}_s}$ , where  $S$  is a system of representatives for the  $\mathbf{G}$ -orbits in  $X^0$  and  $\mathbf{G}_s$  is the stabilizer in  $\mathbf{G}$  of  $s \in S$ . Since  $\mathbf{G}_s$  is compact (and even finite in case (a)) and hence amenable,  $\lambda_{\mathbf{G}/\mathbf{G}_s} = \text{Ind}_{\mathbf{G}_s}^{\mathbf{G}} 1_{\mathbf{G}_s}$

is weakly contained in the regular representation  $\lambda_{\mathbf{G}}$  of  $\mathbf{G}$  and so  $\pi_0$  is weakly contained in  $\lambda_{\mathbf{G}}$ . Hence,  $\pi_0$  does not weakly contain the trivial representation  $1_G$  in case (a). In case (b), the claim follows from Proposition 12, since  $\mathbf{G}$  is not amenable and  $G$  is dense in  $\mathbf{G}$ .

- *Third step.* There exists an element  $a \in C^*(G)$  and  $0 \leq t_0 < 1$  with the following properties:  $\pi_{t_0}(a) = 0$ ,  $\pi_t(a)$  is a non zero compact operator for every  $t \in (t_0, 1]$ , and the map

$$[t_0, 1] \rightarrow \mathbf{R}_+, t \mapsto \|\pi_t(a)\|$$

is continuous.

Indeed, by the second step, there exists  $a \in C^*(G)$  such that  $\pi_0(a) = 0$  and  $1_G(a) \neq 0$ . Therefore,  $\pi_1(a) \neq 0$  and  $\pi_t(a) = \pi_t(a) - \pi_0(a)$  for every  $t \in [0, 1]$  and so the claim follows from the first step.

- *Fourth step.* Let  $a \in C^*(G)$  and  $0 \leq t_0 < 1$  be as in the third step. There exists an irreducible infinite dimensional subrepresentation  $\sigma_t$  of  $\pi_t$  such that  $\sigma_t(a)$  is a compact operator and such that  $\|\sigma_t(a)\| = \|\pi_t(a)\|$  for every  $t \in (t_0, 1)$ .

Indeed, it follows from the third step and Lemma 7 that  $\pi_t|_G$  contains an irreducible subrepresentation  $\sigma_t$  such that  $\sigma_t(a)$  is a compact operator with  $\|\sigma_t(a)\| = \|\pi_t(a)\|$ . It remains to show that  $\sigma_t$  is infinite dimensional for every  $t \in (t_0, 1)$ .

Assume, by contradiction,  $\sigma_t$  is finite dimensional for some  $t \in (t_0, 1)$ . Since  $G$  is dense in  $\mathbf{G}$ , the closed subspace  $\mathcal{K}_t$  of  $\ell^2(X^0)$  defining  $\sigma_t$  is invariant under  $\mathbf{G}$  and so  $\sigma_t$  is the restriction to  $G$  of a finite dimensional subrepresentation of  $\pi_t$ , again denoted by  $\sigma_t$ . On the one hand,  $\mathbf{G}$  acts properly on  $X^0$ , since the stabilizers of vertices are compact (and even finite in case (a)). So, we have

$$\lim_{g \rightarrow +\infty: g \in \mathbf{G}} d(gx, x) = 0 \quad \text{for all } x \in X^0.$$

It follows from Property (iv) of the family  $(\pi_t)_t$  that  $\pi_t$  (and hence  $\sigma_t$ ) is a  $C_0$ -representation, that is,

$$\lim_{g \rightarrow +\infty: g \in \mathbf{G}} \langle \pi_t(g)v \mid w \rangle = 0$$

for every  $v, w \in \ell^2(X^0)$ . On the other hand, since  $\sigma_t$  is finite dimensional,  $\sigma_t \otimes \overline{\sigma_t}$  contains  $1_{\mathbf{G}}$ . As  $\mathbf{G}$  is not compact, this is a contradiction to the fact that  $\sigma_t$  is a  $C_0$ -representation.

- *Fifth step.* There exists uncountably many real numbers  $t \in (t_0, 1)$  such that the subrepresentations  $\sigma_t$  of  $\pi_t|_G$  as in the fourth step are pairwise non weakly equivalent.

Indeed, by the third step, the function  $f : t \mapsto \|\pi_t(a)\|$  is continuous on  $[t_0, 1]$ , with  $f(t_0) = 0$  and  $f(1) > 0$ . So, the range of  $f$  contains a whole interval. Let  $t, s \in (t_0, 1)$  be such that  $f(t) \neq f(s)$ . Then

$$\|\sigma_t(a)\| = \|\pi_t(a)\| = f(t) \neq f(s) = \|\pi_s(a)\| = \|\sigma_s(a)\|,$$

and so  $\sigma_t$  and  $\sigma_s$  are not weakly equivalent.

**3.2. Proof of Corollary 2.** The following remarks show how Corollary 2 follows from Theorem 1.

(i) Let  $\mathbf{K}$  be global field  $\mathbf{K}$ . Choose a non trivial discrete valuation  $v : \mathbf{K}^* \rightarrow \mathbf{Z}$ . The completion of  $\mathbf{K}$  at  $v$  is a non archimedean local field  $\mathbf{K}_v$ . The tree  $X_v$  associated to  $v$  (see Chapter II in [Ser–80]) is a locally finite regular graph. The group  $G = \mathrm{GL}_2(\mathbf{K})$  acts as a group of automorphisms of  $X_v$ , with vertex stabilizers conjugate to  $\mathrm{GL}_2(\mathcal{O}_v \cap \mathbf{K})$ , where  $\mathcal{O}_v$  is the compact subring of the integers in  $\mathbf{K}_v$ . The closure of the image of  $G$  in  $\mathrm{Aut}(X_v)$  coincides with  $\mathrm{PGL}_2(\mathbf{K}_v)$  and is therefore non amenable. A similar remark applies to  $G = \mathrm{SL}_2(\mathbf{K})$ .

(ii) As is well-known, the group  $G = \mathrm{SL}_2(\mathbf{Z})$  is an amalgamated product  $\mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$ . It follows that  $G$  acts on a tree with vertices of valence 2 or 3 with vertex stabilizers of order 4 or 6 (see Chapter I, Examples 4.2. in [Ser–80])

(iii) The free non abelian group  $F_2$  acts freely on its Cayley graph  $X$ , which is a 4-regular tree. It follows that  $F_n$  acts freely on  $X$  for every  $n \in \{2, \dots, +\infty\}$ . Observe that, in this case, the representations  $\pi_t$  and  $\sigma_t$  as in the proof of Theorem 1 are faithful for  $t \neq 1$  (since there are even  $C_0$ -representations).  $\square$

#### 4. PROOFS OF THEOREM 3 AND COROLLARY 4

**4.1. Proof of Theorem 3.** Let  $R$  be a countable unital commutative ring which is an integral domain and  $\mathbf{K}$  its field of fractions. In case the characteristic of  $\mathbf{K}$  is positive, assume that  $\mathbf{K}$  is not an algebraic extension of its prime field.

Let  $n \geq 3$  and  $G = \mathrm{GL}_n(R)$ . We consider the natural action of  $G$  on the projective space  $\mathbf{P}(\mathbf{K}^n)$ . Let  $\ell_0 = \mathbf{K}e_1 \in \mathbf{P}(\mathbf{K}^n)$  be the line defined by the first unit vector  $e_1$  in  $\mathbf{K}^n$ . The stabilizer of  $\ell_0$  in  $G$  is

$$H = \begin{pmatrix} R^\times & R^{n-1} \\ 0 & \mathrm{GL}_{n-1}(R) \end{pmatrix}.$$



Let  $\sigma$  be a finite dimensional representation of  $H$  and  $\pi := \text{Ind}_H^G \sigma$ . We claim that  $\pi$  is irreducible and that  $\pi(C^*(G))$  contains a non zero compact operator. For the proof of this claim, we have to treat separately the cases  $n = 3$  and  $n \geq 4$ .

4.1.1. **Case  $n = 3$ .** • *First step.* We claim that  $gHg^{-1} \cap H$  is amenable, for every  $g \in G \setminus H$ .

Indeed, let  $g \in G \setminus H$ . Then  $\ell_0$  and  $g\ell_0$  are distinct lines in  $\mathbf{K}^n$  and are both stabilized by  $gHg^{-1} \cap H$ . Hence,  $gHg^{-1} \cap H$  is isomorphic to a subgroup of the solvable group

$$\begin{pmatrix} \mathbf{K}^* & 0 & \mathbf{K} \\ 0 & \mathbf{K}^* & \mathbf{K} \\ 0 & 0 & \mathbf{K}^* \end{pmatrix}$$

and is therefore amenable.

• *Second step.* We claim that the representation  $\pi$  is irreducible.

Indeed, in view of Theorem 8, we have to show that  $\text{Comm}_G(H) = H$ . Let  $g \in G \setminus H$ . On the one hand,  $gHg^{-1} \cap H$  is amenable, by the first step. On the other hand,  $H$  is non amenable, by Proposition 11. This implies that  $gHg^{-1} \cap H$  is not of finite index in  $H$  and so  $g$  is not in the commensurator of  $H$  in  $G$ .

• *Third step.* We claim that the  $C^*$ -algebra  $\pi(C^*(G))$  contains a non-zero compact operator.

Indeed, let  $S$  be a system of representatives for the double cosets space  $H \backslash G / H$  with  $e \in S$ . By Proposition 9, the restriction  $\pi|_H$  of  $\pi$  to  $H$  is equivalent to the direct sum

$$\bigoplus_{s \in S} \text{Ind}_{s^{-1}Hs \cap H}^H(\sigma^s|_{s^{-1}Hs \cap H}) = \sigma \oplus \bigoplus_{s \in S \setminus \{e\}} \text{Ind}_{s^{-1}Hs \cap H}^H(\sigma^s|_{s^{-1}Hs \cap H})$$

Let  $s \in S \setminus \{e\}$ . By the first step,  $s^{-1}Hs \cap H$  is amenable and hence  $\sigma^s|_{s^{-1}Hs \cap H}$  is weakly contained in the regular representation  $\lambda_{s^{-1}Hs \cap H}$  of  $s^{-1}Hs \cap H$ , by the Hulanicki-Reiter theorem. By continuity of induction (see [BHV–08, Theorem F.3.5]), it follows that  $\text{Ind}_{s^{-1}Hs \cap H}^H(\sigma^s|_{s^{-1}Hs \cap H})$  is weakly contained in the regular representation  $\lambda_H$  of  $H$ . Therefore,

$$\pi_0 := \bigoplus_{s \in S \setminus \{e\}} \text{Ind}_{s^{-1}Hs \cap H}^H(\sigma^s|_{s^{-1}Hs \cap H})$$

is weakly contained in  $\lambda_H$ . It follows that  $\pi_0$  does not weakly contain  $\sigma$ ; indeed, assume by contradiction that  $\sigma$  is weakly contained in  $\pi_0$ . Then  $\lambda_H \otimes \overline{\lambda_H}$ , which is a multiple of  $\lambda_H$ , weakly contains  $\sigma \otimes \overline{\sigma}$ . However, since  $\sigma$  is finite dimensional,  $\sigma \otimes \overline{\sigma}$  contains  $1_H$ . Hence,  $1_H$  is weakly

contained in  $\lambda_H$  and this is a contradiction to the non amenability of  $H$ .

It follows from Lemma 6 that  $\pi(C^*(H))$  contains a non-zero compact operator. Since  $C^*(H)$  can be viewed a subalgebra of  $C^*(G)$ , the claim is proved for  $G = \mathrm{GL}_3(R)$ .

4.1.2. **Case**  $n \geq 4$ . For every unital subring  $R'$  of  $R$ , set

$$L(R') := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SL}_{n-1}(R') \end{pmatrix},$$

which is a subgroup of  $H$  isomorphic to  $SL_{n-1}(R')$ .

• *First step.* Let  $g_0 \in G \setminus H$  and  $R'$  an infinite unital subring of  $R$ . We claim that  $g_0 H g_0^{-1} \cap L(R')$  has infinite index in  $L(R')$ .

Indeed, the group  $L := g_0 H g_0^{-1} \cap L(R')$  stabilizes the two lines  $\ell_0$  and  $g_0 \ell_0$ . Let  $V$  be the linear span of the  $n - 1$  unit vectors  $e_2, \dots, e_n$ . Denote by  $\ell$  the projection on  $V$  of the line  $g_0 \ell_0$ , parallel to  $\ell_0$ . As  $g_0 \ell_0 \neq \ell_0$ , we have  $\ell \neq \{0\}$ . Moreover,  $L$  stabilizes  $\ell$ , since  $L$  stabilizes  $\ell_0$  and  $V$ . So, identifying  $L(R')$  with the group  $\mathrm{SL}_{n-1}(R')$ , we see can view  $L$  as a subgroup of  $\mathrm{SL}_{n-1}(\mathbf{K})$  which stabilizes a line in  $\mathbf{K}^{n-1}$ . Lemma 13 shows that  $L$  has infinite index in  $\mathrm{SL}_{n-1}(R')$ , as claimed.

• *Second step.* We claim that the representation  $\pi$  is irreducible. In view of Theorem 8, it suffices to show that  $\mathrm{Comm}_G(H) = H$ .

Let  $g_0 \in G \setminus H$ . By the first step,  $g_0 H g_0^{-1} \cap L(R)$  has infinite index in  $L(R)$ ; hence,  $g_0 H g_0^{-1} \cap H$  has infinite index in  $H$ , since  $L(R)$  is a subgroup of  $H$ .

• *Third step.* We claim that  $\pi(C^*(G))$  contains a non-zero compact operator.

Indeed, since  $\mathbf{K}$  is not an algebraic extension over its prime field,  $R$  contains a subring  $R'$  which is a copy  $\mathbf{Z}$  or a copy of the polynomial ring  $\mathbf{F}_p[T]$ , by Proposition 11. The corresponding subgroup

$$L := L(R')$$

of  $G$  is isomorphic to  $\mathrm{SL}_{n-1}(\mathbf{Z})$  or  $\mathrm{SL}_{n-1}(\mathbf{F}_p[T])$ . Observe that  $L$  is a lattice in the locally group  $\mathbf{G} = \mathrm{SL}_{n-1}(\mathbf{R})$  or  $\mathbf{G} = \mathrm{SL}_{n-1}(\mathbf{F}_p((T^{-1})))$ , where  $\mathbf{F}_p((T^{-1}))$  is the local field of Laurent series over  $\mathbf{F}_p$ . Since  $n - 1 \geq 3$ , the group  $\mathbf{G}$  and hence  $L$  has Kazhdan's Property (T); see [BHV-08, §. 1.4, 1.7].

Let  $S$  be a system of representatives for the double cosets space  $H \backslash G / H$  with  $e \in S$ . By Proposition 9, the restriction  $\pi|_L$  to  $L$  of  $\pi$  is

equivalent to the direct sum  $\sigma|_L \oplus \pi_0$ , where

$$\pi_0 := \bigoplus_{s \in S \setminus \{e\}} \text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L}).$$

We claim that  $\pi_0$  does not weakly contain  $\sigma|_L$ . Indeed, assume by contradiction that  $\pi_0$  weakly contains  $\sigma|_L$ . Since  $\sigma$  is finite dimensional and  $L$  has Property (T), it follows that  $\pi_0$  contains  $\sigma|_L$  (see [BHV–08, Theorem 1.2.5]). Therefore,  $\text{Ind}_{s^{-1}Hs \cap L}^L(\sigma^s|_{s^{-1}Hs \cap L})$  contains a subrepresentation of  $\sigma|_L$  for some  $s \in S \setminus \{e\}$ . Hence,  $s^{-1}Hs \cap L$  has finite index in  $L$ , by Proposition 10. Since  $L = L(R')$  for an infinite unital subring  $R'$  of  $R$ , this is a contradiction to the first step.

As in the proof for the case  $n = 3$ , we conclude that  $\pi(C^*(G))$  contains a non-zero compact operator.

This proves Theorem 3 for  $G = \text{GL}_n(R)$  when  $n \geq 3$ . The case  $G = \text{SL}_n(R)$  is proved in exactly the same way.

**4.2. Proof of Corollary 4.** For  $n \geq 3$ , let  $G = \text{GL}_n(R)$  for a ring  $R$  as above. The irreducible traceable representations of  $G$  constructed in the proof of Theorem 3 are of the form  $\pi = \text{Ind}_H^G \sigma$  for a finite dimensional representation of the subgroup  $H = \begin{pmatrix} R^\times & R^{n-1} \\ 0 & \text{GL}_{n-1}(R) \end{pmatrix}$ .

Observe that,  $\pi = \text{Ind}_H^G \sigma$  is trivial on the center  $Z$  of  $G$ , since  $H$  contains  $Z$ .

By Theorem 8, there are infinitely (respectively, uncountably) many non equivalent such representations  $\pi$ , provided there exists infinitely (respectively, uncountably) non equivalent finite dimensional irreducible representations of  $H$ . This will be the case if  $\text{GL}_{n-1}(R) \rtimes R^{n-1}$ , which is a quotient of  $H$ , has infinitely (respectively, uncountably) many non equivalent finite dimensional irreducible representations.

(i) Assume that  $R = \mathbf{K}$  is an infinite field. It is easy to show that the finite dimensional irreducible representations of  $\text{GL}_{n-1}(\mathbf{K}) \rtimes \mathbf{K}^{n-1}$  are all of the form

$$\begin{pmatrix} * & * \\ 0 & A \end{pmatrix} \mapsto \chi(\det A), \quad A \in \text{GL}_{n-1}(\mathbf{K})$$

for some  $\chi$  in the unitary dual  $\widehat{\mathbf{K}^*}$  of  $\mathbf{K}^*$ ; as  $\mathbf{K}^*$  is infinite,  $\widehat{\mathbf{K}^*}$  is a compact infinite group and is therefore uncountable.

(ii) Assume that  $R = \mathbf{Z}$ .

• *Case  $n = 3$ .* The free group  $F_2$  is a subgroup of finite index in  $\text{GL}_2(\mathbf{Z})$ . There exists uncountably many unitary characters (that is one-dimensional unitary representations) of  $F_2$ . For every such unitary

character  $\chi$ , the representation  $\text{Ind}_{F_2}^{\text{GL}_2(\mathbf{Z})} \chi$  is finite dimensional and so has a decomposition  $\oplus_i \sigma_i^{(\chi)}$  as a direct sum of finite dimensional irreducible representations  $\sigma_i^{(\chi)}$  of  $\text{GL}_2(\mathbf{Z})$ . One can choose uncountably many pairwise non equivalent representations among the  $\sigma_i^{(\chi)}$ 's and we obtain in this way uncountably many non equivalent finite dimensional irreducible representations of  $\text{GL}_2(\mathbf{Z})$  and hence of  $\text{GL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ .

• *Case  $n \geq 4$ .* The group  $\text{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$  has Kazhdan's property (T) and so has at most countably many non equivalent finite dimensional representations (see [Wan-75, Theorem 2.1]). There are indeed *infinitely* many such representations: for every integer  $N \geq 1$ , the finite group

$$G_N = \text{GL}_{n-1}(\mathbf{Z}/N\mathbf{Z}) \ltimes (\mathbf{Z}/N\mathbf{Z})^{n-1}$$

is a quotient of  $\text{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$ ; infinitely many representations among the irreducible representations of the  $G_N$ 's are pairwise non equivalent when viewed as representations of  $\text{GL}_{n-1}(\mathbf{Z}) \ltimes \mathbf{Z}^{n-1}$ .

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