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Estimation of position and resistance of a sensorless PMSM : a nonlinear Luenberger approach for a non-observable system

Pauline Bernard and Laurent Praly*

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Abstract

We show that the pair (*rotor position, resistance*) of a non-salient Permanent Magnet Synchronous Motor (PMSM) is not observable in the sensorless setting, namely when only the electrical signals (voltages and intensities) are measured. However, if the rotation speed ω and the current in rotating frame i_d are simultaneously non zero, we prove that there can only exist six indistinguishable solutions, the resistance being the root of a polynomial of degree six. Besides, in the particular case where ω , i_d and i_q are constant, we show that the number of possible solutions is reduced to two, with two well-identified possible values for the resistance. Those two solutions can be dissociated if the sign of i_q (namely the mode of use of the motor) is known. Inspired by the nonlinear Luenberger methodology, this property enables us to propose an estimation algorithm which relies on a one-dimensional minimization of a criterion involving the voltages and intensities filtered at three distinct sufficiently large frequencies. Its performances are tested and illustrated in simulations on real data.

1 Introduction

For an observer to give an asymptotically converging estimate of the system state, this system must be detectable if not observable. So for any application, before attempting any design of an observer, observability should be studied. The application we are concerned with in this paper is the estimation of the rotor position of a Permanent Magnet Synchronous Motors (PMSM) with nearly no saliency, with known inductance L , known magnet flux Φ , but unknown stator resistance R , and in the so-called sensorless setting, namely when only electric variables – currents and voltages – are measured.

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1.1 Context

In order to reduce their cost and increase their sobriety and reliability, Permanent Magnet Synchronous Motors (PMSM) are not always equipped with mechanical sensors. As a consequence, their control and supervision require observers to estimate the mechanical variables – rotor position and speed – from the measurement of the electric variables only, and using a model of the dynamics which typically involves R , L and Φ .

A review of the first steps towards sensorless control was given in [1] and a Luenberger observer was proposed in [2] in order to estimate the position of a PMSM from electrical measurements only. Then, in [3], pursuing the same goal, was proposed a very simple gradient observer which turned out to be extremely effective in practice. However, from a theoretical view point, it was proved in [4] to be only conditionally convergent : it may admit several equilibrium points depending on the rotation speed ω . Fortunately, later in [5], the authors showed how a minor modification enables to achieve global asymptotic stability thanks to convexity arguments.

All these observers typically require the knowledge of the parameters R , Φ and L . Unfortunately, while L may be considered known and constant (as long as there is no magnetic saturation), R and Φ do vary significantly with the temperature and these variations should be taken into account in the observer. For example, for a given injected current, when the magnet's temperature increases, its magnetic flux decreases, and the produced torque becomes smaller. Similarly, an online estimation of the resistance could enable to :

- adapt the control law in real time and thus ensure a torque control which is robust to the machine's temperature ;
- if a robust control already exists, have an estimation of the rotor's temperature and use this knowledge for supervision and fault diagnosis.

That is why efforts have been made to look for position observers which do not rely on the knowledge of those parameters, or even better, which also estimates them. For instance, [6, 7] have proposed sensorless position observers which are independent from the magnet flux. This line of research was completed in [8] by directly extending the gradient observer proposed in [3] with the estimation of Φ . But as far as the resistance is concerned, we are not aware of any sensorless observation algorithm with proved global convergence. A first attempt appeared in [9] using an adaptive sliding mode scheme. Then, in [10] or later in [11], observers for the pair (*rotor position, resistance*) were proposed, but guaranteeing only local stability. In [12], the authors proposed a global adaptive observer to make the gradient observer previously mentioned independent from the resistance, but only a qualitative convergence to a neighborhood of the solution is ensured; quantitatively, there is no control on the size of this neighborhood.

But along those attempts at observer design, we have not found any report on the study of observability of the pair (*rotor position, resistance*) for a PMSM. We show in this paper that,

actually, this pair is not observable unless other information are added. Nevertheless, it is a peculiar kind of non observability because we prove that when the rotation speed and the current in rotating frame i_d are simultaneously nonzero, there are at most six possible values for the pair (*rotor position, resistance*). This phenomenon could not appear in a linear context where non observability necessarily implies an infinite number of indistinguishable trajectories. Besides, when the rotation speed and the currents evaluated in the rotating frame are constant, the number of indistinguishable solutions is reduced to two, and can be dissociated thanks to the extra knowledge of the motor mode of use (motor or generator).

Note that pairs of indistinguishable trajectories have also been observed in the context of sensorless observation of induction motors in [13]. A multi-valued sliding mode observer was proposed and later generalized in [14]. Unfortunately, this approach does not apply here and in this paper, we rather propose an estimation algorithm based on a nonlinear Luenberger design.

The Luenberger design consists in transforming the system dynamics into asymptotically stable linear dynamics with the measurements as input and for which a trivial observer exists. Then, an estimate in the system coordinates is obtained by inverting the transformation. This approach was first developed by Luenberger himself for linear systems in [15], then extended to stationary nonlinear systems ([16, 17] and references therein), and finally to time-varying systems in [18, 19]. It was also successfully applied to the sensorless position estimation of a PMSM (with known resistance) in [5]. In all those papers, this design is used for observable systems where the invertibility of the transformation is guaranteed. But we show in this paper how it can still be useful in the absence of observability and invertibility. In our particular example, it enables to transform the problem of estimating the pair (*rotor position, resistance*) from a single output equation valid at all times into the one of solving at each time a set of three equations which are valid only asymptotically as time goes to infinity, but exponentially fast. They involve the voltages and currents filtered at three distinct sufficiently large frequencies and their solvability can be deduced from the observability analysis. Of course, due to non observability, the solutions are not unique but we recover the at most six possible values of the pair (*rotor position, resistance*), which can then be a posteriori discriminated according to extra knowledge of those quantities, such as bounds for the resistance, mode of use of the motor, etc.

1.2 Problem statement

Using Joule's and Faraday's laws, a PMSM model expressed in a fixed stator frame reads

$$\dot{\Psi} = u - R i \quad (1)$$

where Ψ is the total flux generated by the stator windings and the permanent magnet, (u, i) are voltage and current, evaluated in the fixed stator frame, and R the stator winding resistance. The quantities u , i and Ψ are two dimensional vectors. The way the total flux Ψ is related to the rotor angle θ_r differs depending on the geometry of the rotor and stator. When the repartition of the

windings and the profile of the magnet are perfectly symmetric, the motor is said to be *non-salient* and the total flux may be expressed simply as

$$\Psi = L i + \Phi \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2)$$

where L is the stator inductance, Φ the magnet's flux, and $\theta = n_p \theta_r$ the electrical phase, with n_p the number of poles (winding pairs) of the stator. Plugging (2) into (1) leads to the usual PMSM model

$$\begin{aligned} L \dot{i} &= u - R i - \Phi n_p \omega_r \begin{pmatrix} -\sin(n_p \theta_r) \\ \cos(n_p \theta_r) \end{pmatrix} \\ \dot{\theta}_r &= \omega_r \\ \mathcal{I} \dot{\omega}_r &= \Phi n_p i^\top \begin{pmatrix} -\sin(n_p \theta_r) \\ \cos(n_p \theta_r) \end{pmatrix} - \tau_L \end{aligned} \quad (3)$$

where \mathcal{I} is the inertia of the rotor and τ_L the load torque. But this operation makes $\omega_r = \dot{\theta}_r$ appear in the model along with two new parameters \mathcal{I} and τ_L which must be either known or estimated. Fortunately, as noticed in [2], (2) implies

$$|\Psi - L i|^2 - \Phi^2 = 0 \quad (4)$$

$$\theta = \arg(\Psi - L i) . \quad (5)$$

So, according to (5), in the case where L and i are known, an estimate of θ can be simply recovered from an estimate of the total flux Ψ . Therefore, as suggested in [2], it is preferable to consider the model made of (1)-(4) instead of (3), namely the system given by

$$\dot{\Psi} = u - R i \quad , \quad \Phi^2 - L|i|^2 = \Psi^\top \Psi - 2L i^\top \Psi \quad (6)$$

with known inputs (u, i) and where the information given by (4) is used as a measurement. This system is independent from the parameters \mathcal{I} and τ_L .

It is proved in [8] that it is possible to estimate both Ψ and Φ at the same time. In this paper, we rather suppose the magnet flux Φ is known, and the resistance R unknown. We want to know if it is possible to estimate both Ψ (and thus θ) and R . So we consider the extended system

$$\begin{cases} \dot{\Psi} &= u - R i \\ \dot{R} &= 0 \\ y &= \Phi^2 - L|i|^2 = \Psi^\top \Psi - 2L i^\top \Psi \end{cases} \quad (7)$$

with inputs (u, i) , measured output y , known parameters (Φ, L) , and state (Ψ, R) .

In Section 2, we show that the knowledge of (u, i) , y and L is generally not enough to determine uniquely the state (Ψ, R) . However, when the rotation speed and the currents evaluated in the

rotating frame are non zero, there are at most six possible solutions, which may be dissociated by the extra knowledge of the mode of use of the machine (motor or generator). Then, we exploit this knowledge to propose an estimation algorithm for (Ψ, R) (and thus (θ, R)) based on a nonlinear Luenberger observer design in Section 3. In Section 4, we propose a possible implementation in practice and show how it can be adapted in the presence of saliency. We finally illustrate its performance and robustness on a simulated scenario, as well as on real data in Section 5.

Notations

1. the rotation matrix of angle θ

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (8)$$

The particular one for $\theta = -\frac{\pi}{2}$ is

$$J = \mathcal{R}(-\frac{\pi}{2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2. the current in a rotated frame of angle θ

$$I_{dq}(\theta) = \begin{pmatrix} I_d(\theta) \\ I_q(\theta) \end{pmatrix} = \mathcal{R}(-\theta) i \quad (9)$$

Its evaluation along a solution is

$$i_{dq}(t) = I_{dq}(\theta(t)) = \begin{pmatrix} i_d(t) \\ i_q(t) \end{pmatrix} = \begin{pmatrix} I_d(\theta(t)) \\ I_q(\theta(t)) \end{pmatrix}$$

3. the angle corresponding to a flux Ψ

$$\Theta(\Psi) = \arg(\Psi - Li) \quad (10)$$

2 Observability analysis

Inspired by (7), we consider the time-varying system

$$\begin{cases} \dot{x} &= u - x_3 i \\ \dot{x}_3 &= 0 \\ y &= \Phi^2 - L|i|^2 = x^\top x - 2Li^\top x \end{cases} \quad (11)$$

with L and Φ given. The time signals u and i are assumed to be consistent with this model, i.e. such that there exists a solution $(x = \Psi, x_3 = R)$ of (11). As a consequence, there exists a unique time-signal θ such that the component x of this solution can be written as

$$\Psi(t) = L i(t) + \Phi \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}. \quad (12)$$

We want to know whether, given consistent time signals (u, i) and parameters (L, Φ) , the particular solution $(x, x_3) = (\Psi, R)$ is the unique solution to System (11).

In the following, with $t \mapsto \theta(t)$ being the function associated with this particular solution according to (12), we denote

$$z(t) = \begin{pmatrix} \cos \theta(t) \\ \sin \theta(t) \end{pmatrix}, \quad i_{dq}(t) = I_{dq}(\theta(t)), \quad \omega(t) = \dot{\theta}(t).$$

Also, in some cases, it is appropriate to rewrite the last equation of (11) equivalently as :

$$0 = |x - Li|^2 - \Phi^2 \quad (13)$$

In this case the output function h and its corresponding output y are respectively

$$h(x, t) = |x - Li(t)|^2 - \Phi^2, \quad y = 0. \quad (14)$$

2.1 A first observability result

We start with the following result :

Theorem 1. *If*

a) *for all t , $\omega(t) = 0$*

or

b) *there exists t such that $\omega(t) \neq 0$, but for all t , $i_d(t) = 0$, and $\frac{\omega}{i_q}(t)$ is defined and constant,*

then there exists an infinite number of solutions to System (11).

Otherwise, there exist at most 6 solutions.

Proof. Consider a solution (x, x_3) to System (11). x is necessarily of the form

$$x(t) = x_0 + \int_0^t u(\tau) d\tau - x_3 \int_0^t i(\tau) d\tau$$

with

$$\dot{x}_0 = 0, \quad \dot{x}_3 = 0,$$

and finding (x, x_3) is equivalent to finding (x_0, x_3) . From (13), it follows for all t

$$\begin{aligned} 0 &= |x(t) - Li(t)|^2 - |x_0 - Li(0)|^2 \\ &= [x(t) - x_0 - L(i(t) - i(0))]^\top [x(t) + x_0 - L(i(t) + i(0))] \\ &= \tilde{\eta}(x_3, t)^\top [2(x_0 - Li(0)) + \tilde{\eta}(x_3, t)] \end{aligned}$$

where we have defined the following affine function of x_3

$$\tilde{\eta}(x_3, t) = \int_0^t u(\tau) d\tau - x_3 \int_0^t i(\tau) d\tau - L(i(t) - i(0)). \quad (15)$$

We deduce that for any time t ,

$$2\tilde{\eta}(x_3, t)^\top (x_0 - Li(0)) = -\tilde{\eta}(x_3, t)^\top \tilde{\eta}(x_3, t) = -|\tilde{\eta}(x_3, t)|^2 .$$

Therefore, unless x_3 makes $\tilde{\eta}(x_3, t_1)$ and $\tilde{\eta}(x_3, t_2)$ colinear for any (t_1, t_2) , there exists at most one possible value of x_0 for each x_3 .

The rest of the proof then consists in showing that:

1. for x_3 such that the function $t \mapsto \tilde{\eta}(x_3, t)$ is not constant, there exist couples (t_1, t_2) such that $\tilde{\eta}(x_3, t_1)$ and $\tilde{\eta}(x_3, t_2)$ are not colinear. x_0 is then uniquely determined, as a rational fraction of x_3 with a numerator of degree 3 and a denominator of degree 2, and is solution of $|x_0 - Li(0)|^2 = \Phi^2$. This implies that x_3 must be the root of a polynomial of degree 6. Therefore, there are at most 6 solutions (x, x_3) such that the function $t \mapsto \tilde{\eta}(x_3, t)$ is not constant.
2. to the values of x_3 such that the function $t \mapsto \tilde{\eta}(x_3, t)$ is constant, is associated an infinite number of solutions (x, x_3) .
3. x_3 makes the function $t \mapsto \tilde{\eta}(x_3, t)$ constant if and only if it satisfies for all t

$$\begin{aligned} (R - x_3)i_d(t) &= 0 \\ (R - x_3)i_q(t) &= -\omega(t)\Phi . \end{aligned} \tag{16}$$

This is in particular the case for $x_3 = R$ when $\omega(t) = 0$ for all t .

Let us start by Point 1. Take x_3 such that $\tilde{\eta}(x_3, \cdot)$ is not constant. There exists t_1 such that $\tilde{\eta}(x_3, t_1) \neq 0$. Thus, for some $t_2 \neq t_1$, $\tilde{\eta}(x_3, t_2)$ colinear to $\tilde{\eta}(x_3, t_1)$ implies that there exists λ such that $\tilde{\eta}(x_3, t_2) = \lambda \tilde{\eta}(x_3, t_1)$. But then, we have

$$2\lambda \tilde{\eta}(x_3, t_1)^\top (x_0 - Li(0)) = -\lambda^2 |\tilde{\eta}(x_3, t_1)|^2 = -\lambda |\tilde{\eta}(x_3, t_1)|^2$$

and necessarily $\lambda = 1$ or $\lambda = 0$, i.e $\tilde{\eta}(x_3, t_2) = \tilde{\eta}(x_3, t_1)$ or $\tilde{\eta}(x_3, t_2) = 0$. But, since $\tilde{\eta}(x_3, \cdot)$ is continuous and not constant, there exists t_2 such that $\tilde{\eta}(x_3, t_2) \neq \tilde{\eta}(x_3, t_1)$ and $\tilde{\eta}(x_3, t_2) \neq 0$. Actually, still by continuity, we can even say that there exist two intervals I_1 and I_2 such that for all (t_1, t_2) in $I_1 \times I_2$, we have $\tilde{\eta}(x_3, t_1) \neq 0$, $\tilde{\eta}(x_3, t_2) \neq 0$ and $\tilde{\eta}(x_3, t_2) \neq \tilde{\eta}(x_3, t_1)$, i.e such that $\tilde{\eta}(x_3, t_1)$ and $\tilde{\eta}(x_3, t_2)$ are not colinear. For each such couple (t_1, t_2) , x_0 is uniquely determined by the value of x_3 . Indeed, denoting

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

we have $\tilde{\eta}(x_3, t_1)^\top J \tilde{\eta}(x_3, t_2) \neq 0$ and necessarily

$$x_0 = Li(0) + \frac{1}{2} \frac{J(\tilde{\eta}(x_3, t_1), \tilde{\eta}(x_3, t_2))}{\tilde{\eta}(x_3, t_1)^\top J \tilde{\eta}(x_3, t_2)} \begin{pmatrix} |\tilde{\eta}(x_3, t_2)|^2 \\ -|\tilde{\eta}(x_3, t_1)|^2 \end{pmatrix} .$$

Replacing this expression in the constraint

$$|x_0 - Li(0)|^2 - \Phi^2 = 0 ,$$

we obtain a polynomial of degree 6 in x_3 for each couple (t_1, t_2) in $I_1 \times I_2$. In order to deduce that there are at most 6 solutions x_3 making $\tilde{\eta}(x_3, \cdot)$ not constant, we need to prove that at least one of these polynomials is not a constant. It is possible to show that the coefficient of highest degree is given by $I(t_1)^2 I(t_2)^2 (I(t_1) - I(t_2))$ with $I(t) = \int_0^t i(\tau) d\tau$. If $I(t_1) = 0$ for all t_1 in I_1 , then i is zero on I_1 which is excluded by assumption, thus there exists t_1 in I_1 such that $I(t_1) \neq 0$. Now assume $I(t_2) = I(t_1)$ or $I(t_2) = 0$ for all t_2 in I_2 . Again this means that i is zero on I_2 , which is impossible. We conclude that there exists (t_1, t_2) in $I_1 \times I_2$ such that the corresponding polynomial is "truly" of order 6 (i-e with a nonzero coefficient of order 6) and therefore, there are at most 6 solutions x_3 making $\tilde{\eta}(x_3, \cdot)$ not constant, and for each of these values, there is a unique corresponding x_0 . This characterizes at most 6 solutions (x, x_3) .

Now, for Point 2., take x_3 such that $\tilde{\eta}(x_3, \cdot)$ is constant. Since $\tilde{\eta}(x_3, 0) = 0$, $\tilde{\eta}(x_3, t) = 0$ for all t . It follows that any x_0 verifying $|x_0 - Li(0)| = \Phi$ is solution, and there exists an infinity of solutions associated to this value of x_3 .

Finally, for Point 3., x_3 makes $t \mapsto \tilde{\eta}(x_3, t)$ constant if and only if for all t , differentiating with respect to time,

$$u(t) - x_3 i(t) - L \dot{\hat{i}}(t) = 0 .$$

But differentiating (12) with respect to time, we also know that

$$u(t) - Ri(t) = L \dot{\hat{i}}(t) + \omega \Phi \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}$$

so that, by combining the two equations and multiplying by $\mathcal{R}(-\theta(t))$, we get system (16).

We can thus distinguish the following cases :

- if $\omega(t) = 0$ for all t , the function $t \mapsto \tilde{\eta}(R, t)$ is constant and, from 2), there is an infinite number of solutions (x, x_3) .
- if there exists t such that $\omega(t) \neq 0$, and for all t , $i_d(t) = 0$ and $\frac{\omega(t)}{i_q(t)}$ is defined and constant, then $x_3 = R + \frac{\omega \Phi}{i_q}$ is the only solution to System (16) for all t and from 2) there is an infinity of solutions (x, x_3) .
- otherwise, there exist no solution to System (16). Therefore, $\tilde{\eta}(x_3, \cdot)$ cannot be constant and from 1) there are at most 6 solutions (x, x_3) to our observability problem.

□

We recover the well-known fact that the system is not observable when the rotating speed is zero (this is true even when R is known). In the usual case where there exists at least a time t_1 for

which $\omega(t_1) \neq 0$ and at least a time t_2 such that $i_d(t_2) \neq 0$, this result says that there exist at most six possible solutions (x, x_3) , with x_3 given by the roots of a polynomial of degree six. Because this polynomial has an even degree, it has at least two real roots. But in order to conclude that the system is not observable, we need to prove that they are distinct, and constant with time.

To get more information about those roots, one could study in detail this polynomial of degree six obtained in the proof. But its expression is too complex. In the next section, we show that a more useful expression of this polynomial can be obtained when a stronger differential observability holds.

2.2 Differential observability of order 3

We consider now the following more stringent observability question expressed using the notation (14) for the output:

How many solutions in (x, x_3) may the following equation have

$$\overline{\mathbf{H}}_3(x, x_3, t) = 0 = \begin{pmatrix} h(x, t) \\ \dot{\overline{h(x, t)}} \\ \ddot{\overline{h(x, t)}} \end{pmatrix} ? \quad (17)$$

If there is only one solution the system is said differentially observable of order 3. Of course, in the cases of non observability identified in Theorem 1, the answer is *more than* 1. But we want to study in more details what happens in the other cases, in particular when there exists a time t such that

$$\omega(t) \neq 0 \quad \text{and} \quad i_d(t) \neq 0 .$$

By differentiations in (13), we obtain

$$\overline{\mathbf{H}}_3(x, x_3, t) = \begin{pmatrix} |x - L i(t)|^2 - \Phi^2 \\ 2\eta(x_3, t)^\top (x - L i(t)) \\ 2\dot{\eta}(x_3, t)^\top (x - L i(t)) + 2|\eta(x_3, t)|^2 \end{pmatrix} ,$$

where we have denoted

$$\eta(x_3, t) = u(t) - x_3 i(t) - L \hat{i}(t) . \quad (18)$$

We have the following result :

Theorem 2. *Consider a time t such that $\omega(t) \neq 0$ and $i_d(t) \neq 0$. There are as many solutions (x, x_3) to the equation*

$$\overline{\mathbf{H}}_3(x, x_3, t) = 0 ,$$

as the number of distinct real roots of the following polynomial of degree six :

$$P(x_3, t) = \omega(t)^6 \Phi^6 \left[\left(1 - \frac{(R - x_3)}{\omega(t)\Phi} \left(\overline{\left(\frac{i_d}{\omega} \right)}(t) - 2i_q(t) \right) + \frac{(R - x_3)^2}{\omega(t)^2 \Phi^2} \mu(t) |i(t)|^2 \right)^2 - \left(1 + \frac{(R - x_3)}{\omega(t)\Phi} 2i_q + \frac{(R - x_3)^2}{\omega(t)^2 \Phi^2} |i(t)|^2 \right)^3 \right] \quad (19)$$

where¹

$$\mu(t) = \frac{1}{\omega(t)} \frac{\left[i(t)^\top J \overline{\dot{i}(t)} \right]}{|i(t)|^2}.$$

If besides $i_q(t) \neq -\overline{\left(\frac{i_d}{\omega} \right)}(t)$, then $P(x_3, t)$ admits at least two distinct real roots.

Proof. It is interesting to note that $\eta(x_3, t) = \overline{\tilde{\eta}(x_3, t)}$, where $\tilde{\eta}$ is defined in (15), so that what is done in this proof is somehow the differential version of the proof of Theorem 1. To study how many solutions in (x, x_3) the equation (17) has, we remark that the second and third component of $\overline{\mathbf{H}}_3$ give a linear system in $x - Li$. So our approach is to solve this system and replace it in the first component. This gives a function of x_3 only. Hence the first question is invertibility of the linear system, i.e. colinearity of $\eta(x_3, t)$ and $\dot{\eta}(x_3, t)$.

Assume that, for t given in the statement and some x_3 , $\eta(x_3, t)$ is non zero and colinear with $\dot{\eta}(x_3, t)$, namely $\dot{\eta}(x_3, t) = \lambda \eta(x_3, t)$. Then, (17) gives

$$\eta(x_3, t)^\top (x - Li(t)) = 0, \quad \lambda \eta(x_3, t)^\top (x - Li(t)) = -|\eta(x_3, t)|^2$$

and necessarily $\eta(x_3, t) = 0$ which is a contradiction. Therefore, colinearity can only happen if $\eta(x_3, t) = 0$. But differentiating (12) with respect to time and combining this expression with (18), we get

$$\eta(x_3, t) = -\omega(t)\Phi Jz(t) + (R - x_3)i(t). \quad (20)$$

By imposing $\eta(x_3, t)$ to be zero and multiplying by $\mathcal{R}(-\theta)$, we recover System (16). But this system has no solution since, at the time t considered in the statement, $\omega(t)$ and $i_d(t)$ are assumed to be nonzero. Therefore, for all x_3 , $\eta(x_3, t)$ and $\dot{\eta}(x_3, t)$ are not colinear.

It follows that we can get x from the second and third components of $\overline{\mathbf{H}}_3$, namely

$$\begin{aligned} (x - Li(t))^\top \eta(x_3, t) &= 0 \\ (x - Li(t))^\top \dot{\eta}(x_3, t) &= -|\eta(x_3, t)|^2 \end{aligned} \quad (21)$$

¹ μ is the ratio between ω and the rotation speed of i .

i.e.

$$x - Li(t) = \frac{|\eta(x_3, t)|^2}{\eta(x_3, t)^\top J \dot{\eta}(x_3, t)} J \eta(x_3, t) . \quad (22)$$

Inserting this expression in the first component of $\bar{\mathbf{H}}_3$ gives

$$\begin{aligned} \Phi^2 &= \left| \frac{|\eta(x_3, t)|^2}{\eta(x_3, t)^\top J \dot{\eta}(x_3, t)} J \eta(x_3, t) \right|^2 \\ &= \frac{|\eta(x_3, t)|^6}{[\eta(x_3, t)^\top J \dot{\eta}(x_3, t)]^2} \end{aligned}$$

so that x_3 is a root of the following polynomial

$$P(x_3, t) = \Phi^2 [\eta(x_3, t)^\top J \dot{\eta}(x_3, t)]^2 - |\eta(x_3, t)|^6 .$$

Differentiating (20), we get

$$\dot{\eta}(x_3, t) = -\dot{\omega}(t) \Phi J z(t) - \omega(t)^2 \Phi z(t) + (R - x_3) \dot{\hat{i}}(t) , \quad (23)$$

which yields

$$\begin{aligned} \eta(x_3, t)^\top J \dot{\eta}(x_3, t) &= \det(\eta(x_3, t), \dot{\eta}(x_3, t)) \\ &= \omega^3 \Phi^2 - (R - x_3) \Phi \left[\omega^2 i^\top J z - \dot{\omega} i^\top z + \omega \dot{\hat{i}}^\top z \right] + (R - x_3)^2 i^\top J \dot{\hat{i}} \\ &= \omega^3 \Phi^2 - (R - x_3) \Phi \omega^2 \left[-2i_q + \overline{\left(\frac{i_d}{\omega} \right)} \right] + (R - x_3)^2 i^\top J \dot{\hat{i}} \end{aligned} \quad (24)$$

where we have used the fact that $i^\top z = i_d$ and $i^\top J z = -i_q$. Inserting those expressions in the expression of P , we get the polynomial (19). The coefficient of degree 6 is $|i|^6$ which is non zero by assumption. Since the value of x is imposed by that of x_3 , we get the first statement.

R is one real root of $P(x_3, t)$. Because P has an even degree, there is at least another real root.

But we have $\frac{\partial P}{\partial x_3}(R, t) = 0$ iff $i_q(t) = -\overline{\left(\frac{i_d}{\omega} \right)}(t)$. Therefore, if this condition is not satisfied, R is a simple root and there exist at least two distinct real roots. \square

With this result, we recover the fact that there exist at most 6 possible values for (x, x_3) . Also, because $i_q(t) = -\overline{\left(\frac{i_d}{\omega} \right)}(t)$ is likely to happen for specific inputs only, we can expect the system to be generally not differentially observable, with at least two candidates for (x, x_3) at each time t . We may wonder if P admits other real roots, or if its other 4 roots are complex. To answer this question, we restrict our attention to the particular case where ω , i_d and i_q are constant.

2.3 Particular case where ω , i_d and i_q are constant

In this case, we already see from Theorem 2 that differential observability is possible only if $i_q = 0$. Let us look more closely at the roots of P .

Corollary 1. *If ω , i_d and i_q are constant with $\omega \neq 0$ and $i_d \neq 0$, then the polynomial P in (19) has only two roots in x_3 given by*

$$x_3 = R \quad , \quad x_3 = R + \frac{2\Phi\omega i_q}{|i|^2} .$$

Therefore, the equation $\bar{\mathbf{H}}_3(x, x_3, t) = 0$ admits one solution if $i_q = 0$ and two distinct solutions if $i_q \neq 0$.

Proof. We have :

$$i^\top J \dot{\hat{i}} = i_{dq}^\top \mathcal{R}(-\theta) \mathcal{R}(-\frac{\pi}{2}) \overline{\mathcal{R}(-\theta) i_{dq}}$$

So when i_{dq} is constant, we obtain :

$$\begin{aligned} i^\top J \dot{\hat{i}} &= i_{dq}^\top \mathcal{R}(-\theta) \mathcal{R}(-\frac{\pi}{2}) \overline{\mathcal{R}(-\theta) i_{dq}} \\ &= \omega i_{dq}^\top \mathcal{R}(-\theta) \mathcal{R}(-\frac{\pi}{2}) \mathcal{R}(\frac{\pi}{2}) \mathcal{R}(\theta) i_{dq} \\ &= \omega |i_{dq}|^2 = \omega |i|^2 . \end{aligned}$$

This implies $\mu(t) = 1$ and

$$P(x_3) = -\omega^6 \Phi^6 \left(1 + \frac{(R - x_3)}{\omega \Phi} 2i_q + \frac{(R - x_3)^2}{\omega^2 \Phi^2} |i|^2 \right)^2 \frac{(R - x_3)}{\omega \Phi} \left(2i_q + \frac{(R - x_3)}{\omega \Phi} |i|^2 \right) .$$

The polynomial $1 + 2i_q X + X^2 |i|^2$ has a discriminant equal to $-4i_d^2 < 0$ and does not admit any real root. The conclusion follows. Note that in this case, according to (24), P also writes

$$P(x_3) = -\Phi^2 \det \left(\eta(x_3), \dot{\eta}(x_3) \right)^2 \times \frac{(R - x_3)}{\omega \Phi} \left(2i_q + \frac{(R - x_3)}{\omega \Phi} |i|^2 \right) . \quad (25)$$

□

This theorem says that the system is not differentially observable of order 3 unless $i_q = 0$. This does not mean that the system is not observable because the solution corresponding to $x_3 = R + \frac{2\Phi\omega i_q}{|i|^2}$ may not be admissible for System (11). Actually, it turns out that both solutions are truly indistinguishable :

Theorem 3. *Assume ω , i_d and i_q are constants such that $\omega \neq 0$ and $i_d \neq 0$. There exist exactly two indistinguishable solutions (x, x_3) to System (11). They are of the form (Ψ, R) and (Ψ_δ, R_δ)*

with

$$\begin{aligned}
R_\delta &= R + \frac{2\Phi\omega i_q}{|i|^2}, \quad i_{d,\delta} = i_d, \quad i_{q,\delta} = -i_q, \\
\omega_\delta &= \dot{\theta}_\delta = \dot{\theta} = \omega, \\
\Psi_\delta(t) &= L i(t) + \frac{|\eta(R_\delta, t)|^2}{\eta(R_\delta, t)^\top J \dot{\eta}(R_\delta, t)} J \eta(R_\delta, t),
\end{aligned} \tag{26}$$

with η defined in (18), $\theta_\delta = \Theta(\Psi_\delta)$ and $i_{dq,\delta} = I_{dq}(\theta_\delta)$. Besides, if $(\hat{R}, \hat{\theta})$ is one of the solutions (R, θ) or $(R_\delta, \theta_\delta)$, then the other solution is

$$\left(\hat{R} + \frac{2\Phi\hat{\omega}\hat{i}_q}{|\hat{i}|^2}, \hat{\theta} + \arctan_2 \left(2\hat{i}_q\hat{i}_d, 1 - 2\hat{i}_q^2 \right) \right),$$

where $\hat{i}_{dq} = I_{dq}(\hat{\theta})$ and $\hat{\omega} = \dot{\hat{\theta}}$.

Proof. See Appendix .1. □

We conclude that the system is not observable if $i_q \neq 0$. However, the problem is well-identified with only two possible solutions and they can be dissociated with the additional information of the sign of i_q . In fact, the sign of i_q determines the mode of use of the machine : if $i_q > 0$, the torque is positive and the machine acts as a motor, whereas if $i_q < 0$, the torque is negative and the machine acts as a generator. In other words, both solutions can be distinguished if we know the mode of use of the motor.

This result also says that, in practice, if an estimation \hat{R} among $\{R, R_\delta\}$ is available (for instance thanks to an observer), it is possible to find the other candidate. However for this, we need as estimation of the rotation speed ω . This will be exploited in the estimation algorithm presented below.

Remark 1. In fact, from a physical point of view, those two values of R correspond to two systems with same total energy but with different energy repartition. Indeed, the total energy is

$$E = \frac{L}{2}|i|^2 + \frac{\mathcal{I}}{2}\omega_r^2$$

From (3) and the fact that $\omega = n_p\omega_r$, it satisfies :

$$\dot{E} = -R|i|^2 + u^\top i - \frac{\omega}{n_p}\tau_L$$

When i_d, i_q and ω are constant, E is constant and we have:

$$-R|i|^2 + u^\top i - \frac{1}{n_p}\omega\tau_L = 0, \quad \Phi n_p i_q = -\Phi n_p i^\top J z = \tau_L$$

So when u , i and ω are known but the sign of the mechanical power $\frac{\omega}{n_p}\tau_L = \Phi i_q \omega$ is unknown, there are two possible values for R

$$R = \frac{u^\top i - \Phi |i_q| |\omega|}{|i|^2} \quad \text{or} \quad \frac{u^\top i + \Phi |i_q| |\omega|}{|i|^2}$$

The difference $2\frac{\Phi |i_q| |\omega|}{|i|^2}$ between the two values is exactly what we have found above. And the fact that $\frac{u^\top i}{|i|^2}$ is larger or smaller than R depends only the sign the mechanical power.

We conclude from this observability analysis that System (11) is not observable when ω or i_d remains at 0. However, when ω and i_d are nonzero, the number of indistinguishable trajectories is reduced to maximum six : the possible values of R are the roots of a polynomial P of degree six given by (19). This polynomial is likely to have at least two real roots in the general case, and only two if ω , i_d and i_q are constant. In that latter case, there are exactly two indistinguishable trajectories and they can be distinguished with the additional information of the sign of i_q . Note that by continuity, if ω , i_d and i_q vary slowly, we may expect P to have only two real roots at each time t .

3 From solving a single equation at different times to several equations at the same time

With the observability study in hand, we would like now to design an estimation algorithm. For this, we follow the Luenberger methodology. It consists in finding a transformation $(x, t) \mapsto T(x, t)$ that maps the system dynamics into a Hurwitz form

$$\dot{z} = Az + B(u, y)$$

of a certain dimension with A a Hurwitz matrix. Indeed, implementing

$$\dot{\hat{z}} = A\hat{z} + B(u, y)$$

for any initial conditions then gives an estimate \hat{z} which asymptotically converges to $z = T(x, t)$. If $x \mapsto T(x, t)$ is besides injective (uniformly in time t), it is possible to deduce an estimate \hat{x} of x by inverting the transformation. Although the injectivity of T typically requires the observability of the system, we show here that this design may still be fruitful in the context of non observable systems like the PMSM under study here.

In this spirit, for any complex number λ with strictly positive real part, we define the function

$$T_\lambda(x, x_3, t) = \lambda^2 x^\top x + \lambda c_\lambda(t)^\top x + \lambda x_3 b_\lambda(t)^\top x + a_\lambda(t)x_3 + d_\lambda(t)x_3^2 - e_\lambda \quad (27)$$

on $\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}$, with a_λ , b_λ , c_λ , d_λ , and e_λ the outputs of the following filters :

$$\dot{a}_\lambda = -\lambda (a_\lambda - c_\lambda^\top i + b_\lambda^\top u) \quad (28)$$

$$\dot{b}_\lambda = -\lambda (b_\lambda - 2i) \quad (29)$$

$$\dot{c}_\lambda = -\lambda (c_\lambda + 2u + 2\lambda Li) \quad (30)$$

$$\dot{d}_\lambda = -\lambda (d_\lambda - b_\lambda^\top i) \quad (31)$$

$$\dot{e}_\lambda = -\lambda (e_\lambda - c_\lambda^\top u + \lambda^2 L^2 |i|^2 - \lambda^2 \Phi^2) . \quad (32)$$

We have the following result :

Lemma 1. *For any complex number λ with strictly positive real part, for any initial conditions in the filters (28)-(32), any solution (Ψ, R) to System (7) verifies*

$$T_\lambda(\Psi(t), R, t) = \exp(-\lambda t) T_\lambda(\Psi(0), R, 0) .$$

Proof. Straightforward computations show that the function $t \rightarrow T_\lambda(\Psi(t), R, t)$ is solution to $\dot{z} = -\lambda z$, hence the result. \square

This means that, up to neglecting terms decaying with time t as $\exp(-\lambda t)$, we can replace the single measurement equation

$$\Phi^2 - L|i(t)|^2 = x^\top x - 2Li(t)^\top x ,$$

exact at each time t , by as many following equations as the number of λ we choose to pick

$$T_\lambda(x, x_3, t) = 0$$

which (exponentially fast) become exact as t goes to infinity.

This leads to the following question : can we choose some number m of λ_i , such that, given t , $(x, x_3) \mapsto T(x, x_3, t) = (T_{\lambda_1}(x, x_3, t), \dots, T_{\lambda_m}(x, x_3, t))$ is injective ? Indeed, in this case, along any solution we would have for this t , the implication

$$T(x, x_3, t) = T(\Psi, R, t) \implies x = \Psi , \quad x_3 = R .$$

Unfortunately, as we have established, when ω , i_d and i_q are constant, two solutions (Ψ, R) and (Ψ_δ, R_δ) are not distinguishable by the dynamics, and thus necessarily, whatever m is, we have

$$T(\Psi(t), R, t) = T(\Psi_\delta(t), R_\delta, t) \quad \forall t .$$

This means that it is hopeless to prove the injectivity of T , but it may still be possible to recover the (at least two and at most six) possible values of (Ψ, R) .

3.1 Inversion of T

To investigate in this direction, we restrict our attention to the case with $m = 3$ and select $\lambda_1, \lambda_2, \lambda_3$ as strictly positive real numbers to simplify the notations. We could as well pick one real number and one complex number.

We denote :

$$\begin{aligned} T(x, x_3, t) &= \begin{pmatrix} T_{\lambda_1}(x, x_3, t) \\ T_{\lambda_2}(x, x_3, t) \\ T_{\lambda_3}(x, x_3, t) \end{pmatrix} \\ &= m_\lambda x^\top x + \Lambda (c(t) + x_3 b(t)) x + a(t) x_3 + d(t) x_3^2 - e(t) \end{aligned} \quad (33)$$

where

$$\begin{aligned} \Lambda &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \\ m_\lambda &= \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \end{pmatrix}, \quad a = \begin{pmatrix} a_{\lambda_1} \\ a_{\lambda_2} \\ a_{\lambda_3} \end{pmatrix}, \quad b = \begin{pmatrix} b_{\lambda_1}^\top \\ b_{\lambda_2}^\top \\ b_{\lambda_3}^\top \end{pmatrix} \\ c &= \begin{pmatrix} c_{\lambda_1}^\top \\ c_{\lambda_2}^\top \\ c_{\lambda_3}^\top \end{pmatrix}, \quad d = \begin{pmatrix} d_{\lambda_1} \\ d_{\lambda_2} \\ d_{\lambda_3} \end{pmatrix}, \quad e = \begin{pmatrix} e_{\lambda_1} \\ e_{\lambda_2} \\ e_{\lambda_3} \end{pmatrix} \end{aligned}$$

Because of Lemma 1, we are interested in finding the solutions (x, x_3) to :

$$T(x, x_3, t) = 0 \quad (34)$$

at each time t .

To compact the notations, we let

$$M_\lambda = \begin{pmatrix} \lambda_2^2 & -\lambda_1^2 & 0 \\ 0 & \lambda_3^2 & -\lambda_2^2 \end{pmatrix}$$

which is such that

$$M_\lambda m_\lambda = 0, \quad (35)$$

and introduce the following functions of (x_3, t)

$$\mathcal{M}(x_3, t) = M_\lambda \Lambda (c(t) + x_3 b(t)) \quad (36)$$

$$\chi(x_3, t) = \mathcal{M}(x_3, t)^{-1} M_\lambda (e(t) - a(t) x_3 - d(t) x_3^2) \quad (37)$$

$$J(x_3, t) = m_\lambda^\top (e(t) - T(\chi(x_3, t), x_3, t)). \quad (38)$$

For (x, x_3, t) verifying (34) and such that $\mathcal{M}(x_3, t)$ is invertible, we have

$$J(x_3, t) = 0 \quad , \quad x = \chi(x_3, t)$$

i.e., the roots of $J(\cdot, t)$ give the possible values of x_3 and $\chi(x_3, t)$ the corresponding value of x .

In fact, for all times t , except maybe for those unlikely ones where both $\det(M_\lambda \Lambda c(t))$ and $\det(M_\lambda \Lambda b(t))$ are zero, the determinant of the 2×2 -matrix $\mathcal{M}(x_3, t)$, affine in x_3 , is a polynomial of degree 2 in x_3 , so that $\mathcal{M}(x_3, t)$ is invertible for all x_3 except maybe for at most two bad values denoted $\{x_{3a}(t), x_{3b}(t)\}$. Then, $\chi(x_3, t)$ is a two-dimensional matrix made of rational fractions in x_3 with numerator of degree 3 and denominator of degree 2, defined everywhere except at $\{x_{3a}(t), x_{3b}(t)\}$. Similarly $J(x_3, t)$ is a rational fraction in x_3 , with numerator of degree 6 and denominator of degree 4 defined everywhere except maybe at $\{x_{3a}(t), x_{3b}(t)\}$. We have the following result :

Theorem 4. *Consider any $(\lambda_1, \lambda_2, \lambda_3)$ in $(\mathbb{R}_+^*)^3$, any initial conditions of the filters (28)-(32). Assume the input (u, i) is bounded. Then, any solution (Ψ, R) to System (7) such that there exist positive real numbers \bar{t} and $\underline{\delta}$ such that*

$$\left| \det \left(\mathcal{M}(R, t) \right) \right| \geq \underline{\delta} > 0 \quad \forall t \geq \bar{t}$$

verifies

$$\lim_{t \rightarrow +\infty} \Psi(t) - \chi(R, t) = 0 \quad , \quad \lim_{t \rightarrow +\infty} J(R, t) = 0 \quad .$$

Proof. See Appendix .2. □

This result says that, if $\mathcal{M}(R, t)$ is invertible for all t , the model (7) seen as equation in Ψ and R is equivalent, at least asymptotically (and exponentially fast) to

$$\begin{cases} 0 &= J(R, t) \\ \Psi &= \chi(R, t) \end{cases} \quad (39)$$

We have thus replaced the scalar equation $y(t) = h(\Psi(t), t)$, written with the compact notation (14), valid at all times t , by the three equations in (39) to be solved at each time t . Because Ψ is explicitly given by the last two equations, the problem of estimating (Ψ, R) reduces to finding the roots of $x_3 \mapsto J(x_3, t)$ at each time t .

In order to deduce from this a possible estimation algorithm, we must therefore answer the following two questions :

- Is the matrix $\mathcal{M}(R, t)$, involved in the expression of $\chi(R, t)$, invertible, or, more precisely, is $\left| \det \left(\mathcal{M}(R, t) \right) \right|$ lower-bounded at least after a certain time ?
- If yes, is R the only root of $J(\cdot, t)$ (on its domain of definition) at least after a certain time ?
If not, which are the other solutions ?

It turns out that the observability analysis carried out above gives some insight about those questions.

3.2 Link with observability

The following technical lemma shows that there is a tight link between $\mathcal{M}(x_3, t)$ and J we have here and $(\eta(x_3, t), \dot{\eta}(x_3, t))$ and P encountered during the observability study above.

Lemma 2. *Assume the inputs (u, i) and their time derivatives are bounded. We have the following relations :*

$$\det \left(\mathcal{M}(x_3, t) \right) = O(\lambda^5) \det \left(\eta(x_3, t), \dot{\eta}(x_3, t) \right) + O(\lambda^4) , \quad (40)$$

and if (x_3, t) is such that the matrices $\mathcal{M}(x_3, t)$ and $(\eta(x_3, t), \dot{\eta}(x_3, t))$ are invertible

$$\begin{pmatrix} \eta(x_3, t)^\top \\ \dot{\eta}(x_3, t)^\top \end{pmatrix} (\chi(x_3, t) - Li) = \begin{pmatrix} 0 \\ -|\eta(x_3, t)|^2 \end{pmatrix} + O\left(\frac{1}{\lambda}\right) \quad (41)$$

$$J(x_3, t) = O(\lambda^4) \frac{P(x_3, t)}{\det \left(\eta(x_3, t), \dot{\eta}(x_3, t) \right)^2} + O(\lambda^3) \quad (42)$$

with η defined in (18), P in (19), and the notation $O(\lambda^k)$ indicates a term $f(\lambda_1, \lambda_2, \lambda_3, x_3, t)$ such that, for any fixed $(\lambda_1, \lambda_2, \lambda_3, x_3)$, $t \mapsto f(\lambda_1, \lambda_2, \lambda_3, x_3, t)$ is bounded and for any $t \geq 0$,

$$\limsup_{\alpha \rightarrow +\infty} \left| \frac{f(\alpha\lambda_1, \alpha\lambda_2, \alpha\lambda_3, x_3, t)}{\alpha^k} \right| < +\infty .$$

Proof. This is obtained by expanding the filter solutions as (positive or negative) powers of λ . See Appendix .3. \square

It follows that, when the λ_i are sufficiently large, \mathcal{M} is tightly related to $(\eta(x_3, t), \dot{\eta}(x_3, t))$ and J is tightly related to P . We can thus hope to transfer the known properties of those functions to \mathcal{M} and J .

3.2.1 About Equation (40)

From (40), we get the impression that the invertibility of $\mathcal{M}(x_3, t)$ is related to that of $(\eta(x_3, t), \dot{\eta}(x_3, t))$, at least for λ_i sufficiently large. Actually, we have a more precise result :

Theorem 5. *Assume the inputs (u, i) and their derivatives are bounded. Pick three distinct strictly positive real numbers $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$ and select the λ_i in the observer as*

$$(\lambda_1, \lambda_2, \lambda_3) = (\alpha\tilde{\lambda}_1, \alpha\tilde{\lambda}_2, \alpha\tilde{\lambda}_3) .$$

with α a strictly positive real number to be tuned. Then, for any initial conditions in the filters, there exists $\bar{t} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying

$$\lim_{\alpha \rightarrow +\infty} \bar{t}(\alpha) = 0 ,$$

and,

1.) for each x_3 and each \underline{d} , satisfying for all t ,

$$\left| \det \left(\eta(x_3, t), \dot{\eta}(x_3, t) \right) \right| \geq \underline{d} > 0 ,$$

there exists $\underline{\alpha} > 0$ and $\underline{\delta} > 0$, such that for any $\alpha \geq \underline{\alpha}$,

$$\left| \det \left(\mathcal{M}(x_3, t) \right) \right| \geq \underline{\delta} \quad \forall t \geq \bar{t}(\alpha) .$$

2.) If there exists $\underline{\omega} > 0$ such that $|\omega(t)| \geq \underline{\omega}$ for all t , there exists $\underline{\alpha} > 0$ and $\underline{\delta} > 0$ such that, for any $\alpha \geq \underline{\alpha}$,

$$\left| \det \left(\mathcal{M}(R, t) \right) \right| \geq \underline{\delta} \quad \forall t \geq \bar{t}(\alpha) .$$

In particular, if ω , i_d and i_q are constant with $\omega \neq 0$ and $i_d \neq 0$, then, for any $0 < \varepsilon < 1$, there exists $\underline{\alpha} > 0$ such that for all $\alpha \geq \underline{\alpha}$ and for all $t \geq \bar{t}(\alpha)$, the only two roots of $\det \left(\mathcal{M}(x_3, t) \right)$ are complex and located in the annulus² $C(R, \underline{r}_\varepsilon, \overline{r}_\varepsilon)$ with

$$\underline{r}_\varepsilon = \frac{\omega \Phi}{|i|} (1 - \varepsilon) \quad , \quad \overline{r}_\varepsilon = \frac{\omega \Phi}{|i|} (1 + \varepsilon)$$

Proof. This is done by bounding the terms $O(\lambda^k)$ in Lemma 2 uniformly in time, and applying Rouché's theorem. See Appendix .4. \square

We conclude that, if ω is lower-bounded away from zero, it is possible to guarantee the invertibility of $\mathcal{M}(R, t)$ at least after a certain time \bar{t} by taking the λ_i sufficiently large. Also, the larger the λ_i , the shorter the time needed before $\mathcal{M}(R, t)$ becomes invertible. Therefore, at least after some time, any value of x_3 making $\mathcal{M}(x_3, t)$ non invertible (and thus $J(x_3, t)$ not defined) cannot be R and can be put aside in the algorithm. Note that according to Theorem 5, if ω , i_d and i_q are constant with $\omega \neq 0$ and $i_d \neq 0$, $\mathcal{M}(x_3, t)$ becomes invertible for all x_3 in \mathbb{R} for all $t \geq \bar{t}$ and $J(x_3, t)$ is then defined also for all x_3 in \mathbb{R} .

3.2.2 About Equation (41)

(41) implies that $\chi(x_3, t)$ is solution to the same system (21) (at the first order of $\frac{1}{\lambda}$) as x in the observability analysis. Therefore, whenever $(\eta(x_3, t), \dot{\eta}(x_3, t))$ is invertible, $\chi(x_3, t)$ corresponds to x in the observability analysis, and further $||\chi(x_3, t) - Li|^2 - \Phi^2|$ corresponds to $P(x_3, t)$, still at the first order in $\frac{1}{\lambda}$. Thus, in order to find x_3 , one could minimize $||\chi(x_3, t) - Li|^2 - \Phi^2|$ instead of finding the roots of J . But the injection of the input i in the criteria increases its sensitivity to noise.

²The annulus $C(a, r_0, r_1)$ is the set of points such that $r_0 < |x - a| < r_1$.

3.2.3 About Equation (42)

(42) implies that, for large values of λ_i , the $J(x_3, t)$ can be approximated by $\frac{P(x_3, t)}{\det(\eta(x_3, t), \dot{\eta}(x_3, t))^2}$

which is also a rational fraction with numerator of degree 6 and denominator of degree 4. Therefore, we can hope that, by choosing λ_i sufficiently large, one can ensure that J does not have more roots than P , and the roots of J are close to those of P . Since P is perfectly known with Corollary 1 when ω , i_d and i_q are constant, it is possible to state the following result :

Theorem 6. *Assume the inputs (u, i) and their derivatives are bounded, and that ω , i_d and i_q are constant with $\omega \neq 0$ and $i_d \neq 0$. Pick three distinct strictly positive real numbers $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$ and select the λ_i in the observer as*

$$(\lambda_1, \lambda_2, \lambda_3) = (\alpha \tilde{\lambda}_1, \alpha \tilde{\lambda}_2, \alpha \tilde{\lambda}_3) .$$

with α a strictly positive real number to be tuned. Then, for any initial conditions in the filters, there exists $\bar{t}_0 > 0$, and for any $0 < \varepsilon < 1$, there exist $\underline{\alpha} > 0$, such that for all $\alpha \geq \underline{\alpha}$, for all $t \geq \bar{t}_0 \frac{\ln \alpha}{\alpha}$, by defining r_ε as in Theorem 5, the function $x_3 \mapsto J(x_3, t)$ admits in $[R - r_\varepsilon, R + r_\varepsilon]$

- only one zero $\hat{R}_1(t)$ if $i_q > \frac{1-\varepsilon}{2}|i|$;
- two distinct zeros $(\hat{R}_1(t), \hat{R}_2(t))$ if $0 < i_q < \frac{1-\varepsilon}{2}|i|$.
- only one zero $\hat{R}_1(t)$ if $i_q = 0$

Proof. The proof of this result relies on Rouché's theorem. See Appendix .5. □

Remark 2. Unfortunately, we cannot say anything about the number of zeros of $x_3 \mapsto J(x_3, t)$ outside of $[R - r_\varepsilon, R + r_\varepsilon]$. Indeed, $x_3 \mapsto J(x_3, t)$ admits (complex) poles outside of $B_{r_\varepsilon}(R)$ (the roots of $x_3 \mapsto \det(\mathcal{M}(x_3, t))$), and Rouché's theorem would only tell us that it admits at most 6 zeros, which we already know.

We conclude from this study that when $|\omega|$ is lower-bounded away from zero, the invertibility of $\mathcal{M}(R, t)$ (and lower-boundedness of $|\det(\mathcal{M}(R, t))|$) is ensured arbitrarily fast by taking the λ_i sufficiently large. According to Theorem 4, this means that $\lim_{t \rightarrow +\infty} J(R, t) = 0$.

In particular, when ω , i_d and i_q are constant with $\omega \neq 0$ and $i_d \neq 0$, J has only one or two zeros in the vicinity of R . Note that (Ψ, R) and (Ψ_δ, R_δ) identified in Corollary 1 are both solution to the dynamics. Therefore, Theorem 4 apply to both and we have in fact :

$$\lim_{t \rightarrow +\infty} J(R, t) = \lim_{t \rightarrow +\infty} J(R_\delta, t) = 0 .$$

This means that the two zeros of J expected with Theorem 6 are likely to be R and $R_\delta = R + \frac{2\omega\Phi i_q}{|i|^2}$ asymptotically. In fact, although we are not able to prove it theoretically at this point, simulations seem to indicate that $P(\cdot, t)$ has always only two roots, as soon as $i_d(t) \neq 0$ and $\omega(t) \neq 0$. Therefore, $J(\cdot, t)$ has, at least after a certain time, also two roots, with one converging to R .

4 Estimation algorithm

Following the results of the previous section, we want to solve (39), namely find the roots of $J(\cdot, t)$. Since x_3 is one-dimensional and we often have a fairly good idea of the interval in which R lies, this can be managed with a one-dimensional grid, which can either be fixed around the initial guess $\hat{R}(0)$ or placed at each iteration around the previously found value $\hat{R}(t)$. This latter option enables to follow the slow variations of R with the temperature. Also, unlike $\hat{\Psi}$ (and thus $\hat{\theta}$) which must be computed at every time, it may not be necessary to update \hat{R} at each iteration since R is fairly constant : the minimization of $|J(\cdot, t)|$ can be carried out only every $dt_R > 0$.

The problem of course is that a numerical minimization of $|J(\cdot, t)|$ might return the "wrong" root. So, how to detect this situation, and how to deduce the "right" root ? Here are some elements of solution exploited in our algorithm :

- most of the time, an interval for the value of R is known and the minimization can be carried out on this interval. When both roots are far apart, there might be only one in the interval of interest.
- in the case where ω and i_{dq} are constant, Theorem 3 shows how to detect whether the solution is the "right" one, if the sign of i_q is known. It also provides the exact expression of the other candidate, which can be computed by estimating the rotation speed, i.e. $\dot{\hat{\theta}}$.
- even in the general case, when ω and i_{dq} are not moving too fast, the two solutions may still be associated to two values of i_q of opposite sign. Therefore, the detection may still be possible if this sign is known. As for computing the other candidate, although the value given by Theorem 3 is not exact, it can enable to switch the basin of attraction and obtain the right estimate at the following iteration.

As seen in Theorem 5, another problem is that, depending on the initial conditions of the filters, a certain time has to elapse before $\mathcal{M}(R, t)$ becomes invertible and thus $J(R, t)$ defined. A solution is to launch the filters with any initial condition, and wait for them to reach their steady state and "forget" their initial condition before starting the estimation process. This happens exponentially fast, depending on the chosen eigenvalues in the filters. This is done in the following algorithm.

4.1 Observer algorithm.

The various results we have obtained motivate the following observer algorithm. Our goal is only to make precise the various steps and components that are used in the simulations in Section 5. A real implementation would need more work to reduce the computational burden.

Choose strictly positive real numbers G and dt_R , a one-dimensional grid \mathcal{G} of the interval $[-G, G]$, three distinct sufficiently large strictly positive real numbers $\lambda_1, \lambda_2, \lambda_3$. We assume the machine is used as a motor, i.e. that i_q is positive. The observer consists of the following modules :

- From $t = 0$, implementation of filters (28)-(32) for any arbitrary initial condition. Denote \bar{t} the duration of the transient.
- At each time $t \geq \bar{t}$, computation of

$$\hat{\Psi}(t) = \chi(\hat{R}, t) \quad , \quad \hat{\theta}(t) = \Theta(\Psi(t)) \quad ,$$

given the current value of \hat{R} , with Θ defined in (10).

- At each time t , estimation of $\hat{\omega}(t) = \dot{\hat{\theta}}(t)$ (see next section).
- Every $dt_R > 0$, update of the value of \hat{R} with the following algorithm :

```

 $\hat{R}_1 = \text{Argmin}_{x_3 \in \hat{R} + \mathcal{G}} |J(x_3, t)|$ 
 $\hat{\Psi}_1 = \chi(\hat{R}_1, t)$ 
 $\hat{\theta}_1 = \Theta(\hat{\Psi}_1)$ 
 $i_{q,1} = I_q(\hat{\theta}_1)$ 
if  $i_{q,1} \geq 0$  then
     $\hat{R} = \hat{R}_1$ 
else
     $\hat{R}_2 = \hat{R}_1 + \frac{2\Phi\hat{\omega}(t)i_{q,1}}{|\hat{i}(t)|^2}$ 
     $\delta = \hat{R}_2 - \hat{R}_1$ 
    if  $|\delta| > G$  then
         $\hat{R} = \hat{R}_2$ 
    else if  $\delta > 0$  then
         $\hat{R} = \text{Argmin}_{x_3 \in \hat{R} + (\mathcal{G} \cap [\frac{\delta}{2}, G])} |J(x_3, t)|$ 
    else
         $\hat{R} = \text{Argmin}_{x_3 \in \hat{R} + (\mathcal{G} \cap [-G, -\frac{\delta}{2}])} |J(x_3, t)|$ 
    end if
end if

```

In other words, the minimum of J is computed on the grid $\hat{R} + \mathcal{G}$ centered at the current value \hat{R} and, if the corresponding i_q is positive, this value is kept. Otherwise, we take the other candidate given by Theorem 3, or rather, if this other value is in the grid where J has already been computed, the true minimum of J around this value is computed. This latter step can be removed and postponed to the next iteration, but it offers the possibility of correcting the estimate given by Theorem 3 when ω , i_d and i_q are not constant and/or when \hat{w} is not exact.

4.2 Estimation of $\hat{\omega} = \dot{\hat{\theta}}$.

In order to implement the previous algorithm, ω and thus $\hat{\omega} = \dot{\hat{\theta}}$ needs to be estimated. This can be done in numerous ways including dirty derivatives, exact differentiators etc. For example in [20,

(18)], the following speed estimator is derived from a position one :

$$\begin{cases} \dot{\hat{\theta}}_e &= \hat{\omega} - k_p(\hat{\theta}_e - y) \\ \dot{\hat{\omega}} &= -k_i(\hat{\theta}_e - y) \\ \dot{\hat{\omega}}_e &= \hat{\omega} - k_p(\hat{\theta}_e - y) \end{cases}, \quad y = \hat{\theta}$$

with k_p and k_i sufficiently large to compensate for the neglected $\dot{\hat{\omega}}$. Because of the problem of having the angles modulo $2k\pi$ only, the speed estimator we have chosen to use for our experiments is³ :

$$\begin{cases} \dot{\hat{\chi}} = -(\hat{\gamma} - k + \ell^2) \cos \hat{\theta} - \ell \hat{\chi} \\ \dot{\hat{\chi}} = -(\hat{\gamma} - k + \ell^2) \tilde{\cos} \hat{\theta} - \ell \hat{\chi} \\ \dot{\hat{\gamma}} = 2k(\hat{\chi} + \ell \cos \hat{\theta}) \cos \hat{\theta} + 2k(\hat{\chi} + \ell \sin \hat{\theta}) \sin \hat{\theta} \end{cases} \quad (43)$$

and

$$\hat{\omega}_e = \hat{\chi}^2 + \hat{\chi}^2 - \ell^2 \quad \text{or} \quad \hat{\omega}_e = \tilde{\chi} \cos \hat{\theta} - \chi \sin \hat{\theta}.$$

Notice that $\hat{\theta}$ is used to estimate $\hat{\omega}$ which in turn is used in the estimation algorithm of θ . This may appear circular. However, the value $\hat{\omega}$ is only used to move the grid and "switch" the basin of attraction when the sign of \hat{i}_q is not right. This is in particular useful when R is not in the grid and thus enables to take a smaller grid width and have more precision with fewer computations. Alternatively, if we are confident the true value of R lies in the initial grid, this "switching" no longer is necessary, and one can rather look for the possible two roots of the criteria in the grid and select the one which gives the right sign for \hat{i}_q . With this latter solution, it is no longer necessary to estimate $\hat{\omega}$.

4.3 In presence of saliency ?

In practice, the repartition of the windings or the geometry of the magnet are never perfectly symmetric : the motor is said to be *salient*. According to [21], this means that (12) is not verified and is replaced by

$$\Psi = L_0 i + L_1 \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} i + \Phi \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (44)$$

with L_0 and L_1 given by

$$L_0 = \frac{L_d + L_q}{2}, \quad L_1 = \frac{L_d - L_q}{2},$$

³When $\hat{\omega}$ is constant, convergence of $\hat{\omega}_e$ to $\hat{\omega}$, can be established by letting

$$\begin{aligned} x_2 &= \dot{x}_1 = -\hat{\omega} \sin \hat{\theta}, \quad \tilde{x}_2 = \dot{\tilde{x}}_1 = \hat{\omega} \cos \hat{\theta}, \quad \chi = x_2 - \ell x_1, \quad \tilde{\chi} = \tilde{x}_2 - \ell \tilde{x}_1 \\ \gamma &= \hat{\omega}^2 + kx_1^2 + k\tilde{x}_1^2 = \hat{\omega}^2 + k, \quad e_\chi = \hat{\chi} - \chi, \quad \tilde{e}_\chi = \hat{\chi} - \tilde{\chi}, \quad e_\gamma = \hat{\gamma} - \gamma \end{aligned}$$

and obtaining $\overbrace{ke_\chi^2 + k\tilde{e}_\chi^2 + \frac{1}{2}e_\gamma^2} = -2k\ell(e_\chi^2 + \tilde{e}_\chi^2)$. This implies the convergence of e_χ and \tilde{e}_χ to zero and therefore the one of $\hat{\omega}_e$ to $\hat{\omega}$.

where L_d and L_q are the inductances in the rotating frame. When there is no saliency, $L_d = L_q$, $L_1 = 0$ and we recover our model (12). But as noticed in [8], (44) is equivalent to

$$\Psi = (L_0 - L_1)i + (\Phi + 2L_1i_d(\theta)) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (45)$$

and therefore, when i_d is constant, (Ψ, R) is still solution to our model (7) provided we replace L and Φ by

$$L_s = L_0 - L_1 \quad , \quad \Phi_s = |\Phi + 2L_1i_d| \quad ,$$

namely

$$|\Psi - L_si|^2 - \Phi_s^2 = 0 \quad . \quad (46)$$

Therefore, we could adapt our algorithm replacing L by L_s and Φ by Φ_s . The problem is that Φ_s is unknown because i_d depends on the unknown θ . However, it was observed in [8] that for any Ψ , there exists a unique $\theta = \Theta_s(\Psi)$ verifying (45) which can be computed without the knowledge of Φ_s by⁴:

$$\Theta_s(\Psi) = \begin{cases} \arg(\Psi - L_si) \quad , & \text{if } \Phi - 2L_1I_d(\arg(\Psi - L_si)) > 0 \\ \arg(\Psi - L_si) + \pi \quad , & \text{otherwise} \end{cases} \quad (47)$$

We can thus consider the following modified function

$$\tilde{T}_\lambda(x, x_3, t) = T_\lambda(x, x_3, t) + m_\lambda(\Phi^2 - \Phi_s(x)^2)$$

where

$$\Phi_s(x) = |\Phi + 2L_1I_d(\Theta_s(x))| \quad .$$

We can check that, if Ψ verifies (46) instead of (12) with Φ_s (i.e. i_d) constant, $t \rightarrow \tilde{T}_\lambda(\Psi(t), R, t)$ defined above verifies the dynamics (32) when taking L_s instead of L . It follows that

$$\lim_{t \rightarrow +\infty} Z_\lambda(t) - \tilde{T}_\lambda(\Psi(t), R, t) = 0 \quad .$$

Now, because $M_\lambda m_\lambda = 0$, $M_\lambda \tilde{T}_\lambda = M_\lambda T_\lambda$ and Theorem 4 still holds with the same function χ defined in (37) and criteria

$$\tilde{J}(x_3, t) = m_\lambda^\top \left(Z(t) - \tilde{T}_\lambda(\chi(x_3, t), x_3, t) \right) \quad (48)$$

instead of (38). Therefore, we can implement the algorithm presented in Section 4.1 but replacing

- L by L_s everywhere
- criteria $|J|$ by $|\tilde{J}|$
- the function Θ by Θ_s .

Of course, the convergence is ensured only for solutions (Ψ, R) giving a constant (or slowly varying) i_d . This is illustrated in Section 5.2 on real data.

⁴The sign of $\Phi + 2L_1i_d$ is unknown, so $\theta = \arg(\Psi - L_si)$ is no longer true.

5 Simulations

5.1 Ideal data with non-salient model

We start by testing our observer on ideal data produced by a PMSM model of the type (3), under a plausible torque scenario and where the input u is chosen to follow the rotation speed scenario ω_R shown on Figure 1. The corresponding signals (u, i) in the rotation frame are given in Figure 2. Note that at $t = 3$, although the speed setpoint is constant, an external torque is added, resulting in a transient behavior in the signals. This torque then remains constant throughout the simulation.

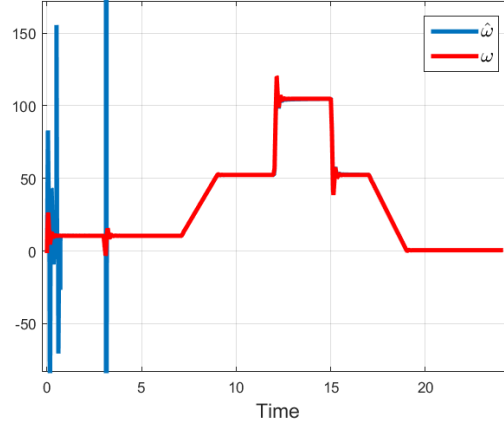
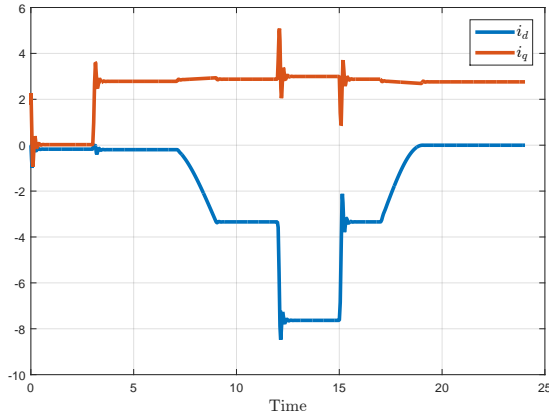
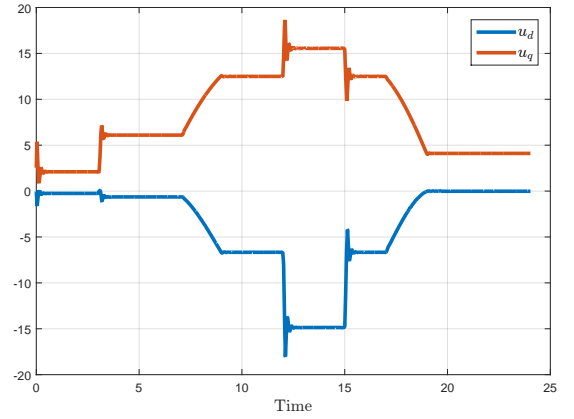


Figure 1: Ideal data : Rotation speed $\omega = \dot{\theta}$ and estimated rotation speed $\hat{\omega} = \dot{\hat{\theta}}$. The estimation algorithm starts at $\bar{t} = 0.5$.

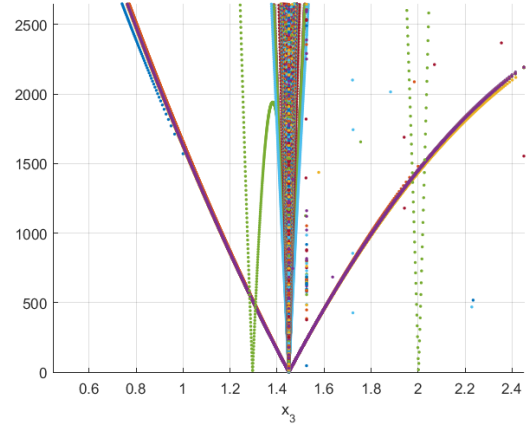
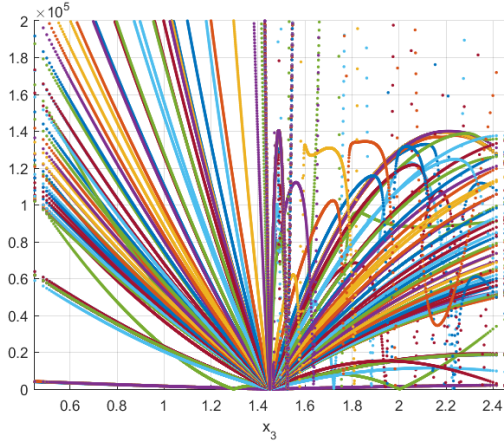


(a) $i_{dq} = \mathcal{R}(-\theta)i$



(b) $u_{dq} = \mathcal{R}(-\theta)u$

Figure 2: Ideal data : currents and voltages in the rotating frame.



(b) Zoom around $R = 1.45$

Figure 3: Ideal data : Plot of the criteria $J(\cdot, t)$ on the grid with $G = 1$ at each iteration where \hat{R} is updated, i.e. every $dt_R = 0.1$.

The results of the simulations are presented in Figure 4, for two resistance grids with amplitude $G = 1$ and $G = 0.1$ respectively. At the beginning, we wait for 0.5s before starting the estimation, to let the filters reach their steady state. The estimation of $\hat{\omega}$ (and thus ω) uses observer (43) with $\ell = 1000$ and $k = 500$ and is shown in Figure 1 .

Observe that with $G = 1$, the algorithm finds the right value of R in two iterations only, whereas with $G = 0.1$, it takes a longer time before R can appear in the grid. In fact, for a same precision, the broader the grid, the higher the chances of R appearing in it, but the larger the number of points and computation time, and also the higher the chances of having several minima in the grid. In practice, one know roughly well the initial value of the resistance, so that a grid with small amplitude around the initial guess can be chosen, which then follows R throughout the experiment (if it evolves due to temperature for instance).

The evolution of the criteria J during the simulation with $G = 1$ is shown in Figure 3. One can see that the minimum is well marked around $R = 1.45\Omega$.

As for the estimation of θ , it naturally converges once \hat{R} has converged. It is interesting to observe the peak in the error around $t = 3$ (which in turn appears on $\hat{\omega}$) shown on Figure 1). This is due to the sudden addition of a torque which destabilizes i_d and ω and makes them go through 0. We have seen that in this case, observability is lost and $\mathcal{M}(R, t)$ is likely to be non invertible (the assumption of Theorem 5 is no longer verified). This event is not visible on \hat{R} because it is not updated at those precise moments. A very interesting feature here is that this anomaly is not propagated in time. This is because it does not appear in the dynamics of the observer but only in its (static) output function transforming the observer state $(a_{\lambda_i}, b_{\lambda_i}, c_{\lambda_i}, d_{\lambda_i}, e_{\lambda_i})$ in the estimate \hat{R}_1 and $\hat{\theta}_1$.

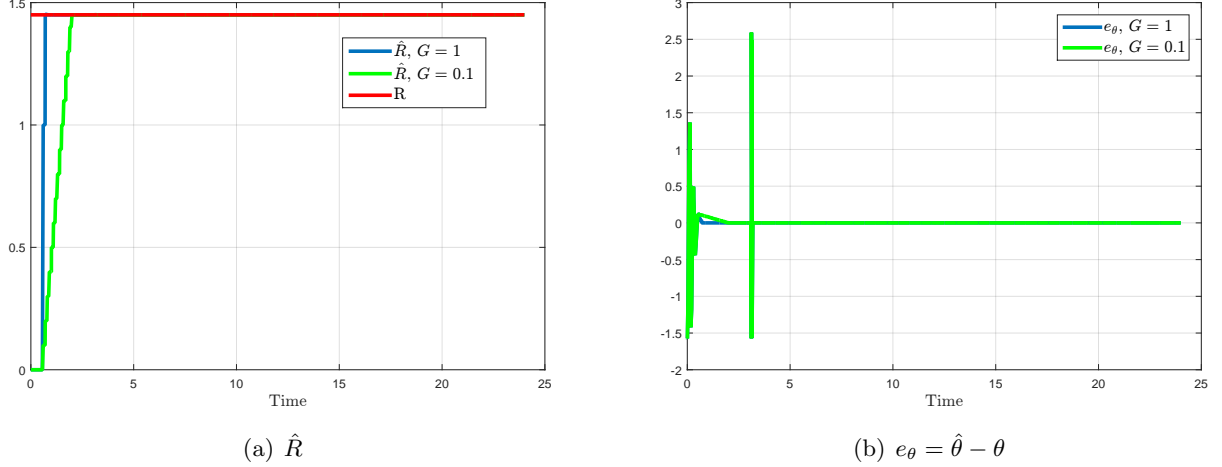


Figure 4: Ideal data : Results of the observer algorithm discussed in Section 4.1 with $\lambda_1 = 20$, $\lambda_2 = 30$, $\lambda_3 = 40$, $dt_R = 0.1$, and two grids with amplitude $G = 1$ and $G = 0.1$ respectively. The estimation starts at $\bar{t} = 0.5$.

5.2 Real data

In order to test the robustness of our algorithm with respect to noise and saliency, we apply it in open-loop (and offline) to real data obtained from a PMSM used in a test bed at IFPEN. For this motor an identification of the parameters involved in the model (1) with (44) has led to $L_d = 0.72\text{mH}$, $L_q = 0.78\text{mH}$, $\Phi = 8.94\text{mWb}$, and $R = 0.151\Omega$. Note that this motor presents saliency because $L_d \neq L_q$. The experiment from which the data comes from aimed at keeping the rotation speed constant at a regime of $w_r = 150$ rpm (with $n_p = 10$, this gives $\omega = 157$ rad/s). We have at our disposal the measurement of the rotor position θ_m for verification and transformations between rotating and fixed frame : it is not used in the algorithm. The recorded measurements of voltages u_m and currents i_m in the rotating frame are given in Figure 5.

The results of a simulation of the observer using the modifications discussed in Section 4.3 to take saliency into account, are presented in Figure 7. The observer manages to reconstruct \hat{R} rapidly and then $\hat{\theta} - \theta$ converges to 0, with some oscillations due to the noise. The amplitude of these oscillations depends on the chosen eigenvalues λ_i , and a more thorough analysis of this phenomenon should be carried out in order to select those parameters optimally.

Finally, the evolution of the criteria $|\tilde{J}|$ during the simulation is shown in Figure 6. Because ω , i_d and i_q are constant, the criteria $|\tilde{J}(\cdot, t)|$ tends to a constant criteria which has two roots given by R and $R_\delta = R + \frac{2\Phi_s\omega i_q}{|i|^2}$ as predicted by Theorems 3 and 6.

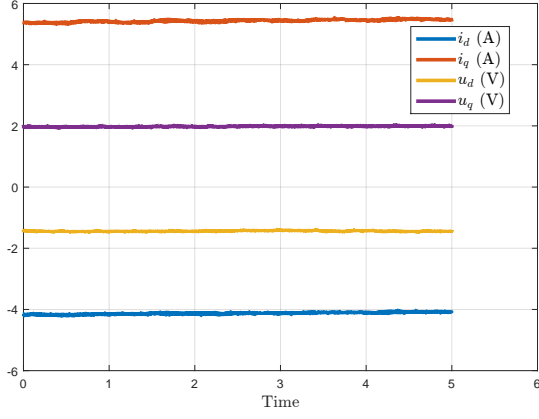


Figure 5: Real data : currents and voltages in the rotating frame.

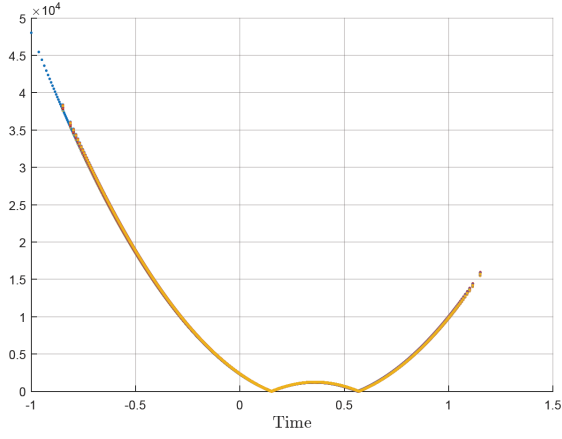


Figure 6: Real data : Plot of the criteria $|J(\cdot, t)|$ on the grid with $G = 1$ at each iteration where \hat{R} is updated, i.e. every $dt_R = 0.1$.

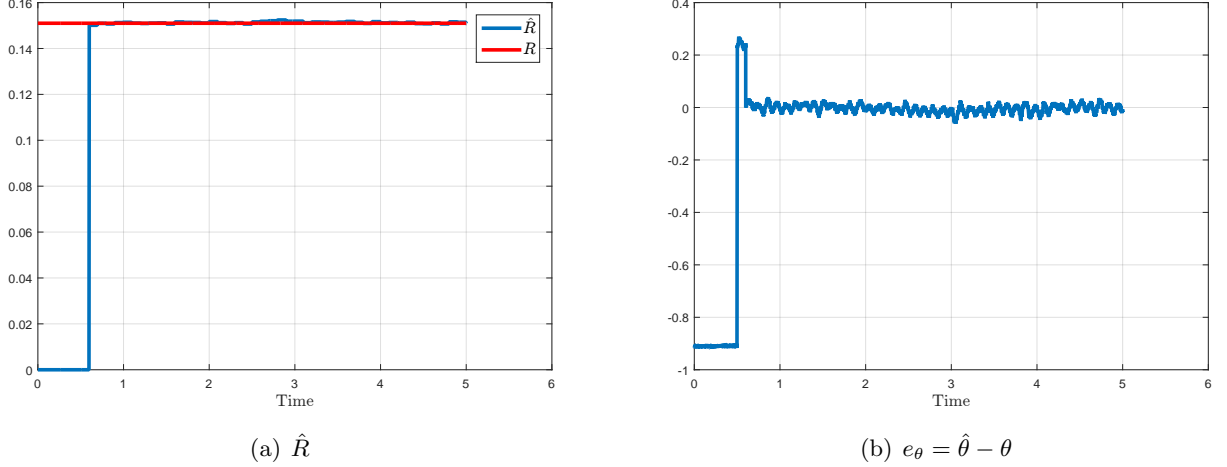


Figure 7: Real data : Results of the observer algorithm discussed in Section 4.3, with $\lambda_1 = 40$, $\lambda_2 = 50$, $\lambda_3 = 60$, $dt_R = 0.1$, and a grid with amplitude $G = 1$. The estimation starts at $t = 0.5$.

6 Conclusion

Unlike the pair (Ψ, Φ) , the pair (Ψ, R) is not observable for the electric equations (6) of the PMSM model. However, when ω and i_d are non zero, there are at most six indistinguishable solutions, the resistance being one of the roots of a polynomial of degree 6. Besides, in the particular case where ω , i_d and i_q are constant, the number of possible solutions is reduced to two, with two well-identified values for the resistance. Fortunately, those solutions turn out to be distinguishable if the sign of i_q (i.e. the mode of use of the machine) is known. This information has enabled us to propose an observation algorithm based on a Luenberger design. Its implementation involves a dynamic part made of filters of voltages and currents and a static part including on a one-dimensional minimization of a criterion. Its efficiency and robustness to saliency and noise were tested in simulations on ideal and real data.

The robustness to saliency requires either the saliency to be small or that i_d be constant/slowly-varying (see Section 4.3). A way to avoid those assumptions and obtain a more general design would be to carry out the same observability analysis but directly on the salient model (44). Note that simulations of our observer with a salient model in the case where i_d is not constant indicate that the criteria obtained in this paper still has a finite number of roots, with R among them. But we do not know how to distinguish them yet.

As for the impact of the noise, the choice of the eigenvalues λ_i seems to play an important role and a detailed analysis could help to determine how to choose those parameters optimally.

.1 Proof of Theorem 3

Consider a solution to System (11). According to Corollary 1, since $y(t) = \dot{y}(t) = \ddot{y}(t) = 0$, it is necessarily (Ψ, R) or (Ψ_δ, R_δ) where, according to (22), Ψ_δ is given by (26). It remains to show that (Ψ_δ, R_δ) is a solution to System (11).

Using (20), we get

$$|\eta(x_3, t)|^2 = \omega^2 \Phi^2 + 2(R - x_3)\Phi\omega i_q + (R - x_3)^2 |i|^2$$

and thus,

$$|\eta(R_\delta, t)| = \omega\Phi .$$

It follows that there exists θ_δ such that

$$\eta(R_\delta, t) = \omega\Phi \begin{pmatrix} -\sin \theta_\delta \\ \cos \theta_\delta \end{pmatrix} = -\omega\Phi J z_\delta$$

where we denote

$$z_\delta = \begin{pmatrix} \cos \theta_\delta \\ \sin \theta_\delta \end{pmatrix} .$$

We deduce according to (20) that

$$\begin{aligned} -\omega\Phi J z_\delta &= -\omega\Phi J z + (R - R_\delta)i \\ &= -\omega\Phi J z - \frac{2\Phi\omega i_q}{|i|^2} i \end{aligned}$$

and after a rotation of angle $-\theta$, we have

$$J \begin{pmatrix} \cos(\theta_\delta - \theta) \\ \sin(\theta_\delta - \theta) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{2 i_q}{|i|^2} i_{dq} . \quad (49)$$

Therefore, $\theta_\delta - \theta$ is a constant and

$$w_\delta = \dot{\theta}_\delta = w .$$

It follows that x_δ defined by

$$x_\delta = L i + \Phi z_\delta$$

verifies the dynamics :

$$\dot{x}_\delta = L \dot{\widehat{i}} - \Phi\omega_\delta J z_\delta = L \dot{\widehat{i}} - \Phi\omega J z_\delta = L \dot{\widehat{i}} + \eta(R_\delta, t) = u - R_\delta i$$

and

$$0 = y = |x_\delta - L i|^2 - \Phi .$$

Thus, (x_δ, R_δ) is a solution to (11), which must appear among $\{(\Psi, R), (\Psi_\delta, R_\delta)\}$, i.e. it is necessarily (Ψ_δ, R_δ) . Therefore, (Ψ, R) and (Ψ_δ, R_δ) are the only two indistinguishable solutions.

Now, according to (49),

$$\begin{aligned}\cos(\theta_\delta - \theta) &= 1 - 2i_q^2 \\ \sin(\theta_\delta - \theta) &= 2i_q i_d .\end{aligned}$$

But after a rotation of $-\theta_\delta$ instead of θ , we would have obtained :

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = J \begin{pmatrix} \cos(\theta - \theta_\delta) \\ \sin(\theta - \theta_\delta) \end{pmatrix} + \frac{2i_q}{|i|^2} i_{dq,\delta}$$

i.e.

$$\begin{aligned}\cos(\theta_\delta - \theta) &= 1 + 2i_q i_{q,\delta} \\ \sin(\theta_\delta - \theta) &= 2i_q i_{d,\delta} ,\end{aligned}$$

which gives the result.

Now, if $\hat{R} = R_\delta$, one can find R by computing

$$R = \hat{R} - \frac{2\Phi\omega i_q}{|i|^2} = \hat{R} + \frac{2\Phi\hat{\omega} \hat{i}_q}{|i|^2}$$

and if $\hat{R} = R$, one can find R_δ by computing

$$R_\delta = \hat{R} + \frac{2\Phi\omega i_q}{|i|^2} = \hat{R} + \frac{2\Phi\hat{\omega} \hat{i}_q}{|i|^2} .$$

This means that whatever the value of \hat{R} is, the value of the other candidate is $\hat{R} + \frac{2\Phi\hat{\omega} \hat{i}_q}{|i|^2}$. Similarly, if $\hat{\theta} = \theta_\delta$, then $\theta = \hat{\theta} + \Delta$ with Δ defined by

$$\begin{aligned}\cos(\Delta) &= 1 - 2i_q^2 = 1 - 2\hat{i}_q^2 \\ \sin(\Delta) &= -2i_q i_d = 2\hat{i}_q \hat{i}_d\end{aligned}$$

and if $\hat{\theta} = \theta$, then $\theta_\delta = \hat{\theta} - \Delta$ with $-\Delta$ defined by

$$\begin{aligned}\cos(-\Delta) &= 1 - 2i_q^2 = 1 - 2\hat{i}_q^2 \\ \sin(-\Delta) &= 2i_q i_d = 2\hat{i}_q \hat{i}_d .\end{aligned}$$

Therefore, whatever the value of $\hat{\theta}$, the value of the other solution is $\hat{\theta} + \arctan_2(2\hat{i}_q \hat{i}_d, 1 - 2\hat{i}_q^2)$.

.2 Proof of Theorem 4

Observe that

$$M_\lambda T(x, x_3, t) = M_\lambda \Lambda \left(c(t) + x_3 b(t) \right) x + M_\lambda a(t) x_3 + M_\lambda d(t) x_3^2 - M_\lambda e(t)$$

is linear in x . This means that for any x_3 and any t such that the matrix $\mathcal{M}(x_3, t)$ is invertible, x is solution of :

$$x = \mathcal{M}(x_3, t)^{-1} M_\lambda \left(T(x, x_3, t) - M_\lambda a(t) x_3 - M_\lambda d(t) x_3^2 + e(t) \right) .$$

Thus, $(x, x_3) = (\Psi(t), R)$ satisfies this equation for all $t \geq \bar{t}$ and we have

$$|\Psi(t) - \chi(R, t)| \leq |\mathcal{M}(R, t)^{-1}| |M_\lambda| |T(\Psi(t), R, t)| .$$

Lemma 1 gives the result if

$$|\mathcal{M}(R, t)^{-1}| = \frac{1}{\left| \det \left(\mathcal{M}(R, t) \right) \right|} |\mathcal{M}^*(R, t)|$$

is upper-bounded in time, where $\mathcal{M}^*(R, t)$ is the comatrix of $\mathcal{M}(R, t)$. $t \mapsto \mathcal{M}^*(R, t)$ is a continuous function of the coefficients of c and b which are filtered versions of the bounded input (u, i) and which are thus bounded. Since $\left| \det \left(\mathcal{M}(R, t) \right) \right|$ is lower-bounded away from 0, the conclusion follows.

.3 Proof of Lemma 2

It is straightforward to check that the quantity

$$\mu_\lambda(x_3, t) = -(c_\lambda + x_3 b_\lambda + 2\lambda L i) \quad (50)$$

verifies

$$\overline{\mu_\lambda(x_3, t)}^\cdot = -\lambda(\mu_\lambda(x_3, t) - 2\eta(x_3, t)) \quad (51)$$

with η defined in (18). This means that μ_λ is a filtered version of 2η . Besides, denoting $\mu = (\mu_{\lambda_1}, \mu_{\lambda_2}, \mu_{\lambda_3})^\top$ we have

$$\Lambda \mu(x_3, t) = -(\Lambda c + x_3 \Lambda b + 2L m_\lambda i^\top)$$

and thus since $M_\lambda m_\lambda = 0$,

$$\mathcal{M}(x_3, t) = -M_\lambda \Lambda \mu(x_3, t) . \quad (52)$$

Let us further investigate the link between μ and η . According to the dynamics (51), μ_λ can be developed in the following way :

$$\mu_\lambda(x_3, t) = 2\eta(x_3, t) - \frac{2\dot{\eta}(x_3, t)}{\lambda} + \frac{2r_{\mu, \lambda}(x_3, t)}{\lambda^2} \quad (53)$$

where the remainder $r_{\mu, \lambda}$ follows the dynamics :

$$\overline{r_{\mu, \lambda}(x_3, t)}^\cdot = -\lambda(r_{\mu, \lambda}(x_3, t) - \ddot{\eta}(x_3, t)) . \quad (54)$$

This yields

$$|r_{\mu,\lambda}(x_3, t)| \leq \exp(-\lambda t) |r_{\mu,\lambda}(x_3, 0)| + \sup_{s \in [0, t]} |\ddot{\eta}(x_3, s)| \quad (55)$$

Since $\ddot{\eta}$ is a polynomial in x_3 with coefficients depending on the bounded signals $(\ddot{u}, \ddot{\widehat{i}}, i^{(3)})$, we obtain

$$\mu_\lambda(x_3, t) = 2\eta(x_3, t) - \frac{2\dot{\eta}(x_3, t)}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

It follows that μ_λ can be approximated by $2\eta - \frac{2\dot{\eta}}{\lambda}$ for large values of λ . Therefore,

$$\begin{aligned} \mathcal{M} &= -2M_\lambda \Lambda \begin{pmatrix} \eta - \frac{\dot{\eta}}{\lambda_1} + O\left(\frac{1}{\lambda_1^2}\right) \\ \eta - \frac{\dot{\eta}}{\lambda_2} + O\left(\frac{1}{\lambda_2^2}\right) \\ \eta - \frac{\dot{\eta}}{\lambda_3} + O\left(\frac{1}{\lambda_3^2}\right) \end{pmatrix} \\ &= -2M_\lambda \begin{pmatrix} \lambda_1 \eta - \dot{\eta} + O\left(\frac{1}{\lambda_1}\right) \\ \lambda_2 \eta - \dot{\eta} + O\left(\frac{1}{\lambda_2}\right) \\ \lambda_3 \eta - \dot{\eta} + O\left(\frac{1}{\lambda_3}\right) \end{pmatrix} \\ &= -2 \begin{pmatrix} \lambda_1 \lambda_2 (\lambda_2 - \lambda_1) & \lambda_1^2 - \lambda_2^2 \\ \lambda_2 \lambda_3 (\lambda_3 - \lambda_2) & \lambda_2^2 - \lambda_3^2 \end{pmatrix} \begin{pmatrix} \eta^\top \\ \dot{\eta}^\top \end{pmatrix} + O(\lambda), \end{aligned} \quad (56)$$

and straightforward computations give

$$\det(\mathcal{M}(x_3, t)) = 4\lambda_2^2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \det(\eta(x_3, t), \dot{\eta}(x_3, t)) + O(\lambda^4)$$

i.e. (40).

Let us now develop $\chi(x_3, t)$ with respect to λ . To do that, we define ρ_λ with

$$\rho_\lambda(x_3, t) = -e_\lambda + a_\lambda x_3 + d_\lambda x_3^2 - \lambda L \mu_\lambda^\top i - \lambda^2 L^2 |i|^2 + \lambda^2 \Phi^2, \quad (57)$$

which follows the dynamics

$$\dot{\overline{\rho_\lambda(x_3, t)}} = -\lambda(\rho_\lambda(x_3, t) - \mu_\lambda(x_3, t)^\top \eta(x_3, t)). \quad (58)$$

In other words, by denoting $(\rho_{\lambda_1}, \rho_{\lambda_2}, \rho_{\lambda_3})^\top$, we obtain

$$\rho(x_3, t) = -e(t) + a(t)x_3 + d(t)x_3^2 - L\Lambda\mu i - m_\lambda L^2 |i|^2 + m_\lambda \Phi^2. \quad (59)$$

When \mathcal{M} is invertible, by definition of χ in (37), we have

$$\mathcal{M}(x_3, t)\chi(x_3, t) = M_\lambda(e(t) - a(t)x_3 - d(t)x_3^2)$$

and thus since $M_\lambda m_\lambda = 0$ and (52),

$$\mathcal{M}(x_3, t)(\chi(x_3, t) - Li) = -M_\lambda \rho .$$

As we did above for μ , it is possible to develop ρ thanks to (58). Indeed, it is straightforward to check that

$$\rho_\lambda(x_3, t) = 2|\eta(x_3, t)|^2 + \frac{r_{\rho, \lambda}(x_3, t)}{\lambda}$$

with

$$\overline{r_\rho(x_3, t)} = -\lambda(r_\rho(x_3, t) + 6\eta(x_3, t)^\top \dot{\eta}(x_3, t)) .$$

In other words

$$\rho_\lambda(x_3, t) = 2|\eta(x_3, t)|^2 + O\left(\frac{1}{\lambda}\right) \quad (60)$$

and we have with (56)

$$\begin{aligned} & \begin{pmatrix} \eta^\top \\ \dot{\eta}^\top \end{pmatrix} (\chi(x_3, t) - Li) \\ &= \frac{1}{2} \begin{pmatrix} \lambda_1 \lambda_2 (\lambda_2 - \lambda_1) & \lambda_1^2 - \lambda_2^2 \\ \lambda_2 \lambda_3 (\lambda_3 - \lambda_2) & \lambda_2^2 - \lambda_3^2 \end{pmatrix}^{-1} M_\lambda \rho + O\left(\frac{1}{\lambda}\right) \\ &= \begin{pmatrix} \lambda_1 \lambda_2 (\lambda_2 - \lambda_1) & \lambda_1^2 - \lambda_2^2 \\ \lambda_2 \lambda_3 (\lambda_3 - \lambda_2) & \lambda_2^2 - \lambda_3^2 \end{pmatrix}^{-1} M_\lambda \begin{pmatrix} |\eta|^2 \\ |\eta|^2 \end{pmatrix} + O\left(\frac{1}{\lambda}\right) \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} |\eta|^2 + O\left(\frac{1}{\lambda}\right) . \end{aligned}$$

Finally, according to the definitions (33), (50) and (59), it is straightforward to check that

$$T(\chi(x_3, t), x_3, t) = m_\lambda (|\chi(x_3, t) - Li|^2 - \Phi^2) - \Lambda \mu(x_3, t) (\chi(x_3, t) - Li) + \rho(x_3, t) .$$

It follows that for x_3 and t making $(\eta(x_3, t), \dot{\eta}(x_3, t))$ invertible

$$\begin{aligned} J(x_3, t) &= -(\lambda_1^4 + \lambda_2^4 + \lambda_3^4) (|\chi(x_3, t) - Li|^2 - \Phi^2) + O(\lambda^3) \\ &= (\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \frac{P(x_3, t)}{\det(\eta(x_3, t), \dot{\eta}(x_3, t))^2} + O(\lambda^3) . \end{aligned}$$

.4 Proof of Theorem 5

Assume $\left| \det(\eta(x_3, t), \dot{\eta}(x_3, t)) \right| \geq \underline{d}$. In order to deduce from (40) that $\left| \det(\mathcal{M}(x_3, t)) \right| \geq \underline{\delta}$ for all t , if α is sufficiently large, we need to bound the term $O(\lambda^4)$ uniformly in time. In the present case (55) is

$$|r_{\mu, \tilde{\lambda}}(x_3, t)| = |r_{\mu, \tilde{\lambda}}(x_3, 0)| e^{-\alpha \tilde{\lambda} t} + \sup_{t \in [0, t]} |\ddot{\eta}(x_3, t)| .$$

And, since $\ddot{\eta}$ is a polynomial in x_3 with coefficients depending on the bounded signals $(\ddot{u}, \widehat{i}, i^{(3)})$, there exists a polynomial \mathfrak{R}_r (time and α independent) such that

$$|r_{\mu, \tilde{\lambda}}(x_3, t)| \leq |r_{\mu, \tilde{\lambda}}(x_3, 0)|e^{-\alpha \tilde{\lambda} t} + \mathfrak{R}_r(|x_3|) .$$

But (53) gives

$$r_{\mu, \tilde{\lambda}}(x_3, 0) = \alpha^2 \left[\frac{\tilde{\lambda}^2 [\mu(x_3, 0) - 2\eta(x_3, 0)]}{2} \right] + \alpha \tilde{\lambda} \dot{\eta}(x_3, 0)$$

So with (50), if $\alpha \geq 1$, there exists a polynomial $\mathfrak{R}_{r,0}$ depending only on $\tilde{\lambda}$, the initialization of the filters and the input, such that

$$|r_{\mu, \tilde{\lambda}}(x_3, t)| \leq \alpha^2 \mathfrak{R}_{r,0}(|x_3|)e^{-\alpha \tilde{\lambda} t} + \mathfrak{R}_r(|x_3|) .$$

Therefore, there exist \bar{t}_0 and a polynomial $\bar{\mathfrak{R}}_r$, both depending only on $\tilde{\lambda}_i$ and the initial conditions in the filters, such that, denoting

$$\bar{t}(\alpha) = \bar{t}_0 \frac{\ln \alpha}{\alpha} ,$$

$$|r_{\mu, \tilde{\lambda}}(x_3, t)| \leq \bar{\mathfrak{R}}_r(|x_3|) \quad \forall t \geq \bar{t}(\alpha) .$$

Following this term in (56), and then in its determinant (40), the reader can check that $O(\lambda^4)$ in (40) is also a polynomial in x_3 with bounded (in time) coefficients, at least after \bar{t} . Thus, there exists a polynomial \mathfrak{R} (time and α independent) such that for all $\alpha \geq 1$ and all $t \geq \bar{t}(\alpha)$

$$\frac{1}{\alpha^5} \left| \det \left(\mathcal{M}(x_3, t) \right) \right| \geq 4\tilde{\lambda}_2^2(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_2 - \tilde{\lambda}_3)(\tilde{\lambda}_3 - \tilde{\lambda}_1) \underline{d} - \frac{\mathfrak{R}(|x_3|)}{\alpha} \quad (61)$$

which is strictly positive when α is sufficiently large.

If besides $|\omega(t)| \geq \underline{\omega} > 0$, according to (24),

$$\left| \det \left(\eta(R, t), \dot{\eta}(R, t) \right) \right| = |\omega^3 \Phi^2| \geq \underline{\omega}^3 \Phi^2 > 0$$

for all t , hence the second point of the result.

Assume now that ω , i_d and i_q are constant, with $\omega \neq 0$ and $i_d \neq 0$. As seen in the proof of Corollary 1, $\det(\eta, \dot{\eta})$ and P are time independent polynomials with

$$\det(\eta, \dot{\eta})(x_3) = \omega^3 \Phi^2 \left(1 + \frac{(R - x_3)}{\omega \Phi} 2i_q + \frac{(R - x_3)^2}{\omega^2 \Phi^2} |i|^2 \right)$$

$$P(x_3) = -\Phi^2 \det \left(\eta(x_3), \dot{\eta}(x_3) \right)^2 \frac{(R - x_3)}{\omega \Phi} \left(2i_q + \frac{(R - x_3)}{\omega \Phi} |i|^2 \right) .$$

Therefore, the roots of $\det \left(\eta(x_3), \dot{\eta}(x_3) \right)$ are the complex numbers $\frac{\Phi \omega}{|i|^2} (-i_q \pm j i_d)$, both situated on the circle with center R and radius $\frac{\Phi \omega}{|i|}$, and

$$\begin{aligned}
Q(x_3) &= \frac{P(x_3)}{\det \left(\eta(x_3), \dot{\eta}(x_3) \right)^2} \\
&= -\Phi^2 \frac{(R-x_3)}{\omega\Phi} \left(2i_q + \frac{(R-x_3)}{\omega\Phi} |i|^2 \right)
\end{aligned}$$

is a polynomial of degree 2, with the two roots $(R, R_\delta) = \left(R, R + \frac{2\omega\Phi i_q}{|i|^2} \right)$ identified in Corollary 1.

Now, take any $\varepsilon > 0$ and consider $\Gamma_{\underline{r}_\varepsilon}(R)$ and $\Gamma_{\overline{r}_\varepsilon}(R)$, the circles with center R and radius $\underline{r}_\varepsilon$ and \overline{r}_ε respectively. The polynomial $\det \left(\eta(x_3), \dot{\eta}(x_3) \right)$ has no root on those circles so that it can be lower-bounded by some $\underline{d} > 0$. Also, \Re introduced in (61) is continuous on those compact sets and is bounded by some $\overline{\Re}$. Choosing $(\lambda_1, \lambda_2, \lambda_3)$ as suggested in the theorem and denoting $\gamma = 4\tilde{\lambda}_2^2(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_2 - \tilde{\lambda}_3)(\tilde{\lambda}_3 - \tilde{\lambda}_1)$, we have for all $t \geq \bar{t}(\alpha)$ and all x_3 in $\Gamma_{\overline{r}_\varepsilon}(R) \cup \Gamma_{\underline{r}_\varepsilon}(R)$,

$$\left| \frac{1}{\alpha^5} \det \left(\mathcal{M}(x_3, t) \right) - \gamma \det \left(\eta(x_3), \dot{\eta}(x_3) \right) \right| \leq \frac{\Re(|x_3|)}{\alpha} \leq \frac{\overline{\Re}}{\alpha} \leq \gamma \underline{d} \leq \left| \gamma \det \left(\eta(x_3), \dot{\eta}(x_3) \right) \right|$$

for α sufficiently large. Both functions being holomorphic (polynomials), according to Rouché's theorem, $\det \left(\mathcal{M}(x_3, t) \right)$ and $\det \left(\eta(x_3), \dot{\eta}(x_3) \right)$ have the same number of roots in⁵ $B_{\underline{r}_\varepsilon}(R)$ (resp $B_{\overline{r}_\varepsilon}(R)$), namely no roots in $B_{\underline{r}_\varepsilon}(R)$ (resp 2 roots in $B_{\overline{r}_\varepsilon}(R)$). Since we know $\det \left(\mathcal{M}(x_3, t) \right)$ is a polynomial of degree 2, we deduce that its only two roots are situated in the annulus $C(R, \underline{r}_\varepsilon, \overline{r}_\varepsilon)$. Besides, since $\det \left(\eta(x_3), \dot{\eta}(x_3) \right)$ does not admit any real roots, its modulus is lower-bounded on the real axis and according to (61), one can make $\det \left(\mathcal{M}(x_3, t) \right)$ strictly positive for x_3 in the compact set $[R - \overline{r}_\varepsilon, R - \underline{r}_\varepsilon] \cup [R + \underline{r}_\varepsilon, R + \overline{r}_\varepsilon]$ by choosing α sufficiently large. In that case, its roots are in $C(R, \underline{r}_\varepsilon, \overline{r}_\varepsilon)$, but not in $C(R, \underline{r}_\varepsilon, \overline{r}_\varepsilon) \cap \mathbb{R}$: they are necessarily complex.

5.5 Proof of Theorem 6

The result is proved by applying Rouché's theorem on (42) with path $\Gamma_{\underline{r}_\varepsilon}(R)$. To do this, we need to lower-bound $|Q(x_3)|$ and upper-bound the term $O(\lambda^3)$ in (42). When $|i_q| \neq \frac{1-\varepsilon}{2}|i|$, Q does not have any root on $\Gamma_{\underline{r}_\varepsilon}(R)$ and $|Q|$ can thus be lower-bounded in this set by some $\underline{d}_2 > 0$. As for the term $O(\lambda^3)$, by following the proof of Lemma 2, the reader can check that it is actually a rational function in x_3 whose coefficients are bounded in time, at least for $t \geq \bar{t}_1 \frac{\ln \alpha}{\alpha}$ (with \bar{t}_1 some scalar independent from α) and whose denominator is necessarily of the form $\det \left(\mathcal{M}(x_3, t) \right)^{k_1} \det \left(\eta(x_3), \dot{\eta}(x_3) \right)^{k_2}$ (coming from the inversion of the corresponding matrices). On $\Gamma_{\underline{r}_\varepsilon}(R)$, $\det \left(\eta(x_3), \dot{\eta}(x_3) \right)$ does not have any root and can be lower-bounded. The same thing holds for $\det \left(\mathcal{M}(x_3, t) \right)$ uniformly in time for $t \geq \bar{t}_0 \frac{\ln \alpha}{\alpha}$ by taking α sufficiently large according to (61). Therefore, the term $O(\lambda^3)$ in (42) can be upper-bounded by a polynomial of $|x_3|$ for $t \geq \bar{t}(\alpha) = \max\{\bar{t}_0, \bar{t}_1\} \frac{\ln \alpha}{\alpha}$. More precisely, denoting $\gamma_2 = \tilde{\lambda}_1^4 + \tilde{\lambda}_2^4 + \tilde{\lambda}_3^4$, there exists a polynomial \Re_2 (time and α independent) such that for all $t \geq \bar{t}$ and all x_3 in $\Gamma_{\underline{r}_\varepsilon}(R)$,

⁵ $B_r(a)$ denotes the ball with center a and radius r .

$$\begin{aligned}
\left| \frac{1}{\alpha^4} J(x_3, t) - \gamma_2 Q(x_3) \right| &\leq \frac{\Re_2(|x_3|)}{\alpha} \\
&\leq \frac{\overline{\Re}_2}{\alpha} \leq \gamma_2 \underline{d}_2 \leq |\gamma_2 Q(x_3)| .
\end{aligned}$$

Since $\mathcal{M}(\cdot, t)^{-1}$ is defined on $B_{r_\varepsilon}(R)$ (where its determinant does not have any root), $J(\cdot, t)$ is holomorphic on that set and according to Rouché's theorem, it admits as many zeros as Q on $B_{r_\varepsilon}(R)$, i.e. either one or two depending on i_q .

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