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This manuscript was compiled on June 25, 2018

Path integrals are a central tool when it comes to describing quantum or thermal fluctuations of particles or fields. Their success dates back to Feynman who showed how to use them within the framework of quantum mechanics. Since then, path integrals have pervaded all areas of physics where fluctuation effects, quantum and/or thermal, are of paramount importance. Their appeal is based on the fact that one converts a problem formulated in terms of operators into one of sampling classical paths with a given weight. Path integrals are the mirror image of our conventional Riemann integrals, with functions replacing the real numbers one usually sums over. However, unlike conventional integrals, path integration suffers a serious drawback: in general, one cannot make non-linear changes of variables without committing an error of some sort. Thus, no path-integral based calculus is possible. Here we identify which are the deep mathematical reasons causing this important caveat, and we come up with cures for systems described by one degree of freedom. Our main result is a construction of path integration free of this longstanding problem.

Path integrals | Discretization | Functional calculus | Multiplicative Langevin processes

Though the notion of path integration can be traced back to Wiener (1, 2), it is fair to credit Feynman (3) for making path integrals one of the daily tools of theoretical physics. The idea is to express the transition amplitude of a particle between two states as an integral over all possible trajectories between these states with an appropriate weight for each of them. After such a formulation of quantum mechanics was proposed, path integrals turned out to provide a set of methods that are now ubiquitous in Physics (see (4, 5) for reviews) and they have become the language of choice for quantum field theory. But path integrals reach out well beyond quantum physics and they are also a versatile instrument to study stochastic processes. Beyond Wiener's original formulation of Brownian motion, Onsager and Machlup (6, 7), followed by Janssen (8, 9), and De Dominicis (10, 11) [based on the operator formulation of Martin, Siggia and Rose (12)], have contributed to establish path integrals as a useful tool, on equal footing with the Fokker–Planck and Langevin equations. Interestingly, mathematicians have mostly stayed a safe distance away from path integrals. Indeed, it has been known for many years that path integrals cannot be manipulated without extra caution in a vast category of problems. These problems, in the stochastic language, involve the notion of multiplicative noise (that we describe in detail below), and their counterpart in the quantum world has to do with quantization on curved spaces (13). The late seventies witnessed an important step in the understanding of the subtleties of path integrals: the authors of (14–19) found how to modify a posteriori and phenomenologically path integrals to make them visually consistent with differential calculus. Yet, a path integral acquires a definite meaning only as the continuum limit of a discretized expression (20) and this step was not achieved. The goal of this article is to come up with the missing link: we construct path integrals for stochastic and/or quantum trajectories, free of any mathematical hitch, by a direct time-discretization procedure which endows them with a well-defined mathematical meaning.

Quantum or Classical Fluctuations and Path Integrals

Physical context. Multiplicative noise is involved in a flurry of physical problems ranging from soft matter (*e.g.,* diffusion in microfluidic devices (21)), to condensed matter (*e.g.,* super paramagnets (22, 23)) or even inflational cosmology (24, 25). It also appears in other areas of science where Langevin equation are present (*e.g.*, Black–Scholes equation for option pricing (26)). Quantization on curved spaces (*e.g.,* a particle on a sphere (27, 28) or more generic manifolds (29–32)) pertains to the same mathematical class of problems, even though their physical motivation has a different origin. Connections between thermal and quantum noises were noted long ago by Nelson (33), and it is therefore no surprise that our discussion addresses both class of problems simultaneously. To illustrate how deep-set the problem that path integrals suffer is, we now turn to the simplest conceivable example of such.

A simple example of a failure of path integrals. Consider a Brownian particle with position $x(t)$ whose velocity $\dot{x}(t) = v + \eta(t)$ is subjected to both a thermal noise $\eta(t)$ and an external force imposing a constant velocity v. Here η is a Gaussian white noise with zero mean and correlations $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$ with $D > 0$. The path integral describing the trajectories is given in its Onsager–Machlup (6) form by

$$\int \mathcal{D}x \; \mathrm{e}^{-S_v[x]} \,, \qquad S_v[x] = \int \mathrm{d}t \, \frac{(\dot{x}-v)^2}{4D} \,. \qquad [1]$$

Significance Statement

Path integrals are ubiquitous in theoretical physics because they allow for a vivid and technically efficient description of fluctuating objects, whether they are quantum or stochastic. However, it has been known almost since the beginning that path integrals do not lend themselves to the same intuitive rules as those used in ordinary differential calculus, based on derivatives and changes of variables. We identify what the hitherto overlooked missing ingredients are in constructing such well-behaved path integrals. With our construction, the validity of which we mathematically establish, one can thoughtlessly work with functional integrals over paths as one likes working with regular integrals of functions over the real axis. This opens the road for a sound path-integral based calculus.

LFC, VL and FvW designed and performed the research and wrote the paper.

The authors declare no conflict of interest.

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Exploiting Galilean invariance, we change fields to $\tilde{x}(t) = x(t) - vt$ and we arrive at a problem with action $S_0[\tilde{x}] = \int \mathrm{d}t\, \dot{\tilde{x}}^2/4D$ that does not depend on v, as expected.

Suppose now that our interest goes towards the quantity $y(t) = \frac{1}{3}x(t)^3$. It is trickier but well-known how to handle y at the Langevin level. In Stratonovich discretization (34, 35) $\dot{y}(t) = v[3y(t)]^{2/3} + [3y(t)]^{2/3}\eta(t)$, and changing variables from y to $\tilde{y} = \frac{1}{3}[x - vt]^3 = \frac{1}{3}([3y]^{1/3} - vt)^3$, a Galilean-invariant equation is recovered: $\dot{\tilde{y}}(t) = [3\tilde{y}(t)]^{2/3}\eta(t)$.

Instead, the functional approach would be to apply existing recipes (9, 36) to convert the Langevin equation for y to a Stratonovich-discretized path integral $\int \mathcal{D}y\, \mathrm{e}^{-\mathcal{S}_v[y]}$ with

$$\mathcal{S}_v[y] = \int \mathrm{d}t \left\{ \frac{[\dot{y} - v(3y)^{\frac{2}{3}} + 2D(3y)^{\frac{1}{3}}]^2}{4D(3y)^{\frac{4}{3}}} + v(3y)^{-\frac{1}{3}} \right\}. \qquad [2]$$

One would expect that changing from $y(t)$ to $\tilde{y}(t)$ the action for $\tilde{y}(t)$ be independent of v. However, this is not so,

$$\mathcal{S}_v[\tilde{y}] = \int \mathrm{d}t \left\{ \frac{\left[\dot{\tilde{y}}[(3\tilde{y})^{\frac{1}{3}} + vt] + 2D(3\tilde{y})^{\frac{2}{3}}\right]^2}{4D(3\tilde{y})^{\frac{4}{3}}\left[(3\tilde{y})^{\frac{1}{3}} + vt\right]^2} + \frac{v}{[(3\tilde{y})^{\frac{1}{3}} + vt]} \right\} \quad [3]$$

and the v-dependence that Galilean invariance tells us should disappear, simply remains (even if one adds to [3] the Jacobian of the change of variables, see **SI** for computational details). This means that Eq. [3] is not the correct action for $\tilde{y}(t)$.

Of course, none of the details of this example matter. The lesson to draw is actually simple: either one sticks to stochastic calculus and forgets about path integrals that cannot accommodate nonlinear changes of integration fields, or one attempts to cure path integrals. It is not clear, historically, when such problems with path integrals were first realized, but there is a long list of works that point to their occurrence (31, 37–45). Varied strategies (14, 15) have led to a modified action that is manifestly covariant upon continuous-time changes of variables, but only within a "phenomenological" (16) description. Despite some attempts (17–19), an unambiguous discretization scheme of such a modified action has not been achieved.

In this paper we construct a non-ambiguous covariant path integral. This not only requires to focus on hitherto overlooked contributions in slicing up time-evolution, but also to resort to a new adaptive slicing of time. It is the combination of these two ingredients that allows us to immunize path integrals against the problems caused by nonlinear manipulations.

In what follows, we first recall the well-known discretization problems encountered when providing multiplicative Langevin equations with a non-ambiguous definition. With this settled, we present the main outcome of our paper, a path-integral weight (that includes a carefully defined normalization prefactor) that allows one to use the standard rule of calculus inside the action when changing variables. The rest of the paper explains how such an action corresponds to an actual time-discrete path weight, that we construct following a procedure that relies on the identification of crucial discretization issues that go well beyond the usual Itō-Stratonovich dilemma.

Langevin equations and their covariant action

What we know on discretization issues, in a nutshell. Discussions of discretization issues are not commonly found in the quantum literature (see however (20)). This is a question that has to do with the writing and the manipulation of Langevin equations. For arbitrary functions f and g, the process $x(t)$ evolving according to the Langevin equation

$$\dot{x}(t) = f(x(t)) + g(x(t))\eta(t) \qquad [4]$$

where $\langle \eta(t)\eta(t')\rangle = D\delta(t - t')$, is simply not defined unless a specific way to understand the product in the right-hand side is given. An ambiguity-free writing of Eq. [4] looks at a time evolution over an infinitesimal interval of duration Δt:

$$\Delta x = x(t + \Delta t) - x(t) = f(x(t))\Delta t + g(x(t))\Delta\eta \qquad [5]$$

where $\Delta\eta = \int_t^{t+\Delta t} \mathrm{d}\tau\, \eta(\tau)$ is Gaussian distributed with variance $2D\Delta t$. This is called the Itō discretization of the Langevin equation and it is the mathematicians' favorite (46). Another scheme adopted by physicists is the so-called Stratonovich rule, according to which Eq. [4] represents an implicit equation for the increment $\Delta x = x(t + \Delta t) - x(t)$

$$\Delta x = f\big(x(t) + \tfrac{1}{2}\Delta x\big)\Delta t + g\big(x(t) + \tfrac{1}{2}\Delta x\big)\Delta\eta\,. \qquad [6]$$

Although the naive continuum limits of Eq. [5] and Eq. [6] may well be visually identical to Eq. [4], they actually describe different physical processes, and their corresponding evolution equations for the probability distribution of x differ. We stress, however, that the ambiguity of Eq. [4] is only superficial: a Langevin equation describes a limit process in which some time scales, related to memory and elimination of degrees of freedom, have been sent to zero. Hence, a physicist writing a Langevin equation with multiplicative noise as in Eq. [4] knows how to understand the equation. This being said, once a Langevin equation is derived for a physical process, with a given discretization rule, we are free to transform the equation into some equivalent one endowed with an alternative discretization rule. For instance, Eq. [5] is equivalent to a Stratonovich-discretized equation like Eq. [6] in which $f - Dg'g$ is substituted for f (35, 45). From a numerical standpoint, Eq. [5] has an obvious advantage: solving for $x(t + \Delta t)$ can be done by simple recursion, while in the Stratonovich scheme, an implicit equation for Δx must be solved at each time step. However, an important advantage of the Stratonovich discretization is that for an arbitrary function $U(t) = u(x(t))$, if x evolves according to Eq. [6], one can write in the same Stratonovich discretization that $\dot{U}(t) = u'(x(t))\dot{x}(t) = u'(x(t))f(x(t)) + u'(x(t))g(x(t))\eta(t)$. In short, within the Stratonovich discretization, the standard chain rule of differential calculus can be used without caution most of the time, even though none of the manipulated objects is actually differentiable! The celebrated Itō lemma (47) teaches us how to modify the chain rule when working with the Itō-discretized Eq. [5] instead. Other schemes exist and a plethoric literature has been devoted to this subject (48–50). Yet, the discretization of the integral appearing in the action of the path integral has little been discussed, although it is known that the expression of the action actually depends on the scheme chosen to write it (45, 48, 51–53). We now give the form of the action arising from our new adaptive covariant discretization scheme, before explaining its construction.

Our result. The weight $\mathrm{e}^{-S[x]}$ of a trajectory $[x(t)]_{0\le t\le t_\mathrm{f}}$ that evolves according to the Langevin equation [4] understood in the Stratonovich sense [6], if endowed with an action (14, 16)

$$S[x] = \int_0^{t_\mathrm{f}} \mathrm{d}t \left\{ \frac{1}{4D}\left[\frac{\dot{x} - f(x)}{g(x)}\right]^2 + \frac{1}{2}f'(x) - \frac{1}{2}\frac{g'(x)}{g(x)}f(x) \right\}, \quad [7]$$

benefits from an essential feature: it is *covariant* under arbitrary changes of variables, in the sense that the trajectory weight of a process $U(t) = u(x(t))$ defined from $x(t)$ is equal to $\mathrm{e}^{-S[U]}$ with $S[U]$ inferred from the action Eq. [7] for $x(t)$ by merely passing from the variable x to U through the use of the standard chain rule of calculus. This comes in obvious contrast to the historical Stratonovich-discretised expression

$$\int_0^{t_\mathrm{f}} \mathrm{d}t \left\{ \frac{1}{4D} \left[\frac{\dot{x} - f(x) + D\, g(x) g'(x)}{g(x)} \right]^2 + \frac{1}{2} f'(x) \right\} \qquad [8]$$

for the action (8, 36, 45, 51–53). As illustrated on the example above, the latter does not enjoy the required covariance property, while the covariant action [7] for $y = \frac{1}{3}x^3$ does. It now becomes $\mathcal{S}_v[y] = \int \mathrm{d}t\, \frac{[\dot{y} - v(3y)^{2/3}]^2}{4D(3y)^{4/3}}$ instead of [2] and one checks that the action for $\tilde{y}$ is $\mathcal{S}_v[\tilde{y}] = \int \mathrm{d}t\, \frac{\dot{\tilde{y}}^2}{4D(3\tilde{y})^{4/3}}$ which, in contrast to Eq. [3], verifies the Galilean invariance as well as being covariant. Generalizing to any Langevin equation of the form [4] one checks as in (14, 16) that the action [7] is indeed covariant under an arbitrary invertible change of variables.

Yet, in the same way as a Langevin equation must be endowed with a discretization rule, the covariant action [7] acquires a definite meaning only if endowed with a (yet unknown) discretization. Our main result is to fill this gap by building a complete description of the path weight with action [7].

Covariant discretization

What is a covariant discretization? Constructing a path integral invariably involves some discretization procedure in which time is divided into tiny slices. Our goal is to resort to a scheme that is fully consistent with the rules of differential calculus (like the chain rule or integration by parts), also at the level of paths. The naive answer, based on the Stratonovich scheme, simply fails to possess the required property, as illustrated by the example in the introduction. The reason is rather subtle (19, 43) and was identified only recently (45). It has been known for decades that a path integral is more sensitive to discretization issues than, say, a Fokker–Planck equation (as discussed by Janssen (8, 48)). But it has been realized only very recently that the Stratonovich discretization, which allows for the blind use of differential calculus at the level of a Langevin equation, actually fails to extend its properties to path integration (44, 45). Moreover, establishing the compatibility of the chain rule with the Stratonovich discretization at the level of a Langevin equation only works to leading order in the discretization timescale Δt. Since the path-integral formulation requires higher orders in Δt to be included, it appears natural that this discretization will poorly fare regarding changes of variables and differentiation inside path integrals. What we need, thus, is a discretization scheme that is consistent with the chain rule to a high-enough order (up to the order needed in constructing a path integral). Fortunately, such a scheme can be found, and this is our first important result. The inspiration comes from the field of calculus with Poisson point processes (54–57), though our solution departs from anything that has already been proposed. Let us postulate that Eq. [4] is to be understood in the form

$$\Delta x = \mathbb{T}_{f,g} f(x(t))\Delta t + \mathbb{T}_{f,g} g(x(t))\Delta\eta \qquad [9]$$

where the operator $\mathbb{T}_{f,g}$ acts on an arbitrary function h as

$$\mathbb{T}_{f,g} h(x) = \frac{\mathrm{e}^{\mathcal{D}(x)\frac{\mathrm{d}}{\mathrm{d}x}} - \mathbf{1}}{\mathcal{D}(x)\frac{\mathrm{d}}{\mathrm{d}x}} h(x) = \sum_{n\geq 0} \frac{\left(\mathcal{D}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)^n}{(n+1)!} h(x)\,. \qquad [10]$$

Here* $\mathcal{D}(x) = f(x)\Delta t + g(x)\Delta\eta$ acts as an operator, and it does not commute with $\frac{\mathrm{d}}{\mathrm{d}x}$. When acting on f the operator $\mathbb{T}_{f,g}$ leaves us with a complicated function of both $x(t)$ and $\Delta\eta$, which, in an implicit fashion through Eq. [9], is then a function of $x(t)$ and $\Delta x = x(t+\Delta t) - x(t)$. As is perhaps less obvious than in previous discretization schemes, the $\Delta t \to 0$ limit also gets us back to Eq. [4]. This is because $\Delta\eta$, which is of order $\Delta t^{1/2}$, also typically goes to 0. The complex appearance of this discretization rule should not conceal its central property: it is consistent with the chain rule *for any finite* Δt. In other words, when the evolution of x is understood with Eq. [9], one can manipulate a function $U(t) = u(x(t))$ as if it were differentiable, and $\dot{U} = \frac{\mathrm{d}U}{\mathrm{d}t} = u'(x)\dot{x}$ holds in the sense that

$$\frac{U(t+\Delta t) - U(t)}{\Delta t} = \mathbb{T}_{F,G} F(U(t)) + \mathbb{T}_{F,G} G(U(t))\Delta\eta \qquad [11]$$

where $F(U)$ and $G(U)$ are the force and the noise amplitude of the Langevin equation verified by $U(t)$, defined as $F(u(x)) = u'(x)f(x)$ and $G(u(x)) = u'(x)g(x)$.

The unpleasant feature of the discretization rule in Eq. [10] is that it is expressed in terms of $\Delta\eta$ rather than in terms of Δx, as we did in Eq. [6]. This means that such a discretization cannot be used as such in the definition of the path integral in which the noise $\eta(t)$ is eliminated in favor of $x(t)$. We would rather express Eq. [9] in terms of a function $\delta(\Delta x)$ such that

$$\mathbb{T}_{f,g} h(x) = h(x + \delta(\Delta x))\,. \qquad [12]$$

Though this cannot be done explicitly to arbitrary order, an expansion of δ in powers of Δx can be found:

$$\delta(\Delta x) = \alpha\Delta x + \beta(x)\Delta x^2 + \gamma(x)\Delta x^3 + \dots \qquad [13]$$

where $\alpha = \frac{1}{2}$, $\beta = \beta_g = \frac{1}{24}\frac{g''}{g'} - \frac{1}{12}\frac{g'}{g}$, and $\gamma = \gamma_g = -\frac{g''}{24g} + \frac{g'^2}{24g^2} + \frac{g^{(3)}}{48g'} - \frac{g''^2}{48g'^2}$, *etc.* We shall henceforth keep the functional dependence of these functions on g explicit. Keeping in mind that $\Delta x = O(\Delta t^{1/2})$ as $\Delta t \to 0$, at minimal order $\delta(\Delta x) = \frac{1}{2}\Delta x$ and we recover the Stratonovich discretization [6]. For the Stratonovich discretization, the chain rule in Eq. [11] is valid with up to an error of order $\Delta t^{1/2}$, while including the β term in Eq. [13] with $\beta = \beta_g$ renders the error of order Δt (and so on when increasing the order of the expansion). Terms of order higher than β in [13] will prove useless for our purpose.

Covariant discretization for the path integral. We thus have shown that a discretization scheme of the form

$$\mathcal{T}_g h(x) = h\big(x + \tfrac{1}{2}\Delta x + \beta_g(x)\Delta x^2\big) \qquad [14]$$

$$\beta_g(x) = \frac{1}{24}\frac{g''(x)}{g'(x)} - \frac{1}{12}\frac{g'(x)}{g(x)} \qquad [15]$$

yields a Langevin equation for which the chain rule [11] is valid up to order Δt, namely one more order in $\Delta t^{1/2}$ than the Stratonovich one. Such a scheme, that we call *covariant discretization*, will serve as a starting point for our construction

*In the study of Poisson point-processes with multiplicative noise, the appropriate discretization restricts to $\mathcal{D}(x) = g(x)\Delta\eta$, but in our context the supplemental term $f(x)\Delta t$ is needed.

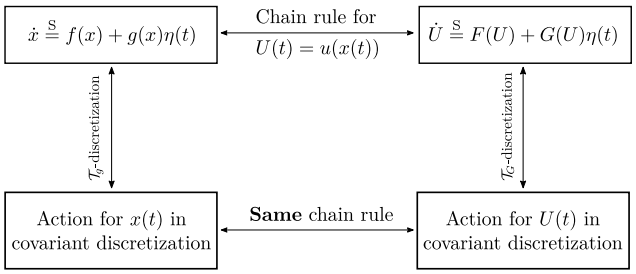

Fig. 1. Schematic representation, for a change of variables $x \mapsto u(x)$, of how the covariant discretization scheme allows one to use the same rules of calculus for a Stratonovich-discretized Langevin equation and for their corresponding covariant Onsager–Machlup and MSRJD actions [7] and [24]. Such a use of the chain rule would be incorrect in the traditional Stratonovich-discretized actions [8] and [25].

of the path integral, where every function in the action is understood as discretized according to Eq. [14]. It is truncated after order Δx^2 compared with the full expansions of Eqs. [10] or [13], but we will prove that the additional contribution $\beta_g(x)\Delta x^2$ in Eq. [14] is sufficient to cure the path integral from its problems upon changing paths. As we now explain, the covariant discretization scheme yields, in the $\Delta t \to 0$ limit, a path integral which possesses the *modified action* in Eq. [7] instead with the Stratonovich-discretised one in Eq. [8].

From the infinitesimal propagator to the path integral

Using the notation x_t for the variable defined in discrete time $t = 0$, Δt, $2\Delta t$, ... the path-integral weight of a trajectory is inferred from the infinitesimal propagator $\mathbb{P}(x_{\Delta t}|x_0) \equiv \mathbb{P}(x_{\Delta t}, \Delta t|x_0, 0)$ for the first time step, defined as the conditional probability of $x_{\Delta t}$ at time Δt, given x_0 at time 0.

On the importance of the prefactor. Following a well-known route (36), one finds that

$$\mathbb{P}(x_{\Delta t}|x_0) \overset{\mathcal{T}_g}{=} \frac{\mathcal{N}}{|g(\bar{x}_0)|}\, \mathrm{e}^{-\frac{1}{2}\frac{\Delta t}{2D}\left[\frac{\Delta x}{\Delta t} - f(\bar{x}_0) + D\, g(\bar{x}_0) g'(\bar{x}_0)}{g(\bar{x}_0)}\right]^2 - \frac{1}{2}\Delta t f'(\bar{x}_0)} \times \mathrm{e}^{-4D\Delta t\, \beta_g(\bar{x}_0) g(\bar{x}_0) g'(\bar{x}_0)} \qquad [16]$$

with $\mathcal{N} = 1/(4\pi D\Delta t)^{1/2}$, where $\overset{\mathcal{T}_g}{=}$ indicates that a function $h(\bar{x}_t)$ is evaluated in the covariant discretization [14], *i.e.*

$$\bar{x}_t = x_t + \frac{1}{2}\Delta x + \beta_g(x_t)\Delta x^2 \quad \text{with} \quad \Delta x = x_{t+\Delta t} - x_t\,. \qquad [17]$$

Compared with the standard Stratonovich scheme ($\beta_g \equiv 0$) one observes that an additional contribution arises from the β_g term. . The second line in [16] is thus a signature of the higher sensitivity of the path integral to the details of the discretization. Had we kept a $\gamma(x)\Delta x^3$ in the expansion of Eq. [14], this would not have changed the form of the propagator [16] to the order relevant for the path integral (namely, up to $O(\Delta t)$ included). The covariant discretization [14] thus goes up to the optimal order in powers of Δx.

The prefactor of the exponential in Eq. [16] was arbitrarily discretized at $\bar{x}_0$ following a standard convention, but this choice actually presents a bothering practical drawback. Indeed, when changing paths from x_t to $U_t = u(x_t)$ according to

$$\mathbb{P}(x_{\Delta t}|x_0) = |u'(x_{\Delta t})|\, \mathbb{P}_U(U_{\Delta t}|U_0) \qquad [18]$$

(where $\mathbb{P}_U$ is the propagator for the process U_t), the Jacobian $|u'(x_{\Delta t})|$ brings a contribution into the exponential weight of Eq. [16] whereas we require the continuous-time limit we wish to establish to exhibit none. We found that our results are better formulated when adopting an endpoint discretization for the prefactor. We will then use $\frac{\mathcal{N}}{|g(x_{\Delta t})|}$ rather than $\frac{\mathcal{N}}{|g(\bar{x}_0)|}$ in Eq. [16] (this induces extra terms in the exponential of the infinitesimal propagator, see **SI**). With the β_g in Eq. [15],

$$\mathbb{P}(x_{\Delta t}|x_0) \overset{\mathcal{T}_g}{=} \frac{\mathcal{N}}{|g(x_{\Delta t})|}\, \mathrm{e}^{-\frac{1}{2}\frac{\Delta t}{2D}\left[\frac{\Delta x}{\Delta t} - f(\bar{x}_0)}{g(\bar{x}_0)}\right]^2} \times \mathrm{e}^{-\frac{1}{2}\Delta t\left[f'(\bar{x}_0) - \frac{g'(\bar{x}_0)}{g(\bar{x}_0)} f(\bar{x}_0)\right]}\,. \qquad [19]$$

The integration measure: path integral in time slices. Putting these bits together, one writes

$$\prod_{0\le t<t_\mathrm{f}/\Delta t} \mathrm{d}x_t\, \mathbb{P}(x_{t+\Delta t}, t+\Delta t|x_t, t) \overset{\Delta t\to 0}{\longrightarrow} \mathcal{D}x\, \mathcal{N}[x]\, \mathrm{e}^{-S[x]} \qquad [20]$$

which defines the Onsager–Machlup probability of a path $[x(t)]_{0\le t\le t_\mathrm{f}}$:

$$\mathrm{Prob}[x] = \mathcal{N}[x]\, \mathrm{e}^{-S[x]}\,. \qquad [21]$$

Here $S[x]$ is the action announced in Eq. [7] and its argument is evaluated in $\mathcal{T}_g$-discretization according to Eq. [19], and $\mathcal{N}[x]$ is a normalization prefactor that is endpoint-evaluated:

$$\mathcal{N}[x] = \prod_{0\le t<t_\mathrm{f}/\Delta t} \frac{1}{\sqrt{4\pi D\Delta t}}\, \frac{1}{|g(x_{t+\Delta t})|}\,. \qquad [22]$$

The average of a functional $\mathcal{F}[x]$ is given by $\big\langle \mathcal{F}[x] \big\rangle = \int \mathcal{D}x\, \mathcal{F}[x]\, \mathrm{Prob}[x]\, P_\mathrm{i}\big(x(0)\big)$ and is interpreted in the Feynman sense (3): a sum over all possible trajectories in discrete time. The initial condition is sampled by P_i.

The time-discrete framework provided in this paragraph fully describes what we believe to be the most natural *covariant* path-integral representation of the trajectory weight of a multiplicative Langevin process. However the core part of the path we started to follow awaits one further step on our side: We must now prove that changing variables in the infinitesimal propagator [19] is equivalent to changing variables by a blind application of the chain rule in the continuous-time action [7].

Covariance of the time-discrete weight. Let us then prove the covariance of the action under a change of path $x(t) \mapsto U(t) = u(x(t))$, proceeding, for convenience, backwards from U to x (see Fig. 1). We have to show that the time-discrete propagator $\mathbb{P}_U(U_{\Delta t}|U_0)$ for the variable $U(t)$ yields back the corresponding propagator [19] for the variable $x(t)$. This has to hold irrespective of whether one follows the correct time-discrete procedure to change variables or the naive continuous-time chain rule. Starting from the now $\mathcal{T}_G$-discretized expression[†]

$$\mathbb{P}_U(U_{\Delta t}|U_0) \overset{\mathcal{T}_G}{=} \frac{\mathcal{N}}{|G(U_{\Delta t})|}\, \mathrm{e}^{-\frac{1}{2}\frac{\Delta t}{2D}\left[\frac{\Delta U}{\Delta t} - F(\bar{U}_0)\right]^2}{G(\bar{U}_0)} \times \mathrm{e}^{-\frac{1}{2}\Delta t\left[F'(\bar{U}_0) - \frac{G'(\bar{U}_0)}{G(\bar{U}_0)} F(\bar{U}_0)\right]}\,, \qquad [23]$$

one first notices (15) using [18] that the prefactor of the propagator becomes the expected one of Eq. [19] for the variable $x(t)$,

[†] with $G(U) = u'(u^{-1}(U))\, G(u^{-1}(U))$ where u^{-1} is the inverse function of u.

Situation	Required discretization
$x(t)$ is differentiable	Any can work
$x(t)$ is a Langevin process, Eq. [4]	Stratonovich, Eq. [6]
$x(t)$ is a path in the covariant action, Eq. [7] or [24]	Covariant, Eq. [14]
$x(t)$ is a path in the standard action, Eq. [8] or [25]	None works

Table 1. Minimal required discretization for the chain rule of standard calculus to hold upon a change of variables $U(t) = u(x(t))$.

thanks to the end-point discretized prefactor we have chosen. Then, the difficulty is to shift from the $\mathcal{T}_G$-discretized variable $U(t)$ to the $\mathcal{T}_g$-discretized variable $x(t)$, but this only requires a correct expansion at $O(\Delta t)$. With the recipe presented in the *Methods*, one compares the two following routes:

(A) in Eq. [23], express $\bar{U}_0$ as a function of $\bar{x}_0$ and Δx; expand in powers of $\Delta x = O(\Delta t^{1/2})$ up to order $O(\Delta t)$; use substitution rules (derived in Ref. (45) and recalled in **SI**) in order to handle powers of Δx of degree higher than 1;

(B) naively replace $\frac{\Delta U}{\Delta t}$ in Eq. [23] by $u'(\bar{x}_0)\frac{\Delta x}{\Delta t}$; $F(\bar{U}_0)$ by $u'(\bar{x}_0)f(\bar{x}_0)$; and $G(\bar{U}_0)$ by $u'(\bar{x}_0)g(\bar{x}_0)$.

Route (B) is in principle completely faulty because it misses many terms of orders $O(\Delta t^{1/2})$ and $O(\Delta t)$, as discussed in Ref. (45). However, for the chosen covariant discretization of Eq. [14] it correctly matches the outcome of route (A) – which happens to be the expected infinitesimal propagator $\mathbb{P}(x_{\Delta t}|x_0)$ of Eq. [19]. This completes the proof that the covariant action [7], formally introduced in Refs. (14, 15), actually corresponds to a non-ambiguous time-discrete weight. For other choices of time discretization, including the Stratonovich one, route (B) does not yield the correct result, which is illustrated by the failure of the manipulations of our toy example.

Since taking route (B) amounts to using the standard rules of calculus in the action, we have thus shown that, for the covariant discretization scheme of Eq. [14], the correct rules of calculus in the infinitesimal propagator at small but finite Δt become identical to the standard rules of calculus in the action [7] when taking the continuous-time limit $\Delta t \to 0$. Such a limiting procedure, which is simple for differentiable functions, and significantly more intricate in a Langevin equation (where discretization issues matter), has demanded an even higher degree of caution in order to manipulate fields inside the action, through the use of the covariant discretization [14] (see Table 1). We now describe how this procedure extends to a formally different but physically equivalent path integral formulation of the same original problem.

Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) path-integral formulation

Since the early formulation of quantum mechanics in terms of path integrals, there have been two equivalent expressions for the transition amplitudes. One, that we have just discussed extensively, involves a single position field. An alternative one also involves a conjugate momentum field. The latter can be removed or included at will by Gaussian integration. A mirror image of the auxiliary momentum field exists for stochastic dynamics: the alternative to the original Onsager–Machlup formulation is the MSRJD approach (8, 10–12, 58) and involves an additional so-called response field. The purpose of this section is to extend our findings in the two-field path integral formalism. Again, we adopt the language of stochastic dynamics, but our results equally apply to quantum mechanics.

Response fields as an answer to non-linearities. In the MSRJD approach one introduces a response field $\hat{x}(t)$ to represent the trajectory weight in a manner that allows one, for instance, to get rid of some non-linearities of the action [7]. Physics-wise, this setting facilitates the study of correlations and response functions on an equal footing, and to linearize (to some extent) possible symmetries of the process under scrutiny (time-reversal, rapidity reversal, *etc*). We now present our result for the covariant MSRJD action before describing its construction and its full time-discrete implementation.

In the covariant discretization scheme of Eq. [14], the action

$$S[\hat{x},x] \stackrel{\mathcal{T}_g}{=} \int_0^{t_f} \mathrm{d}t \left\{ \hat{x}\big(\dot{x} - f(x) + D\, g(x)g'(x)\big) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) + \frac{1}{4}g'(x)^2 + \frac{1}{2}\frac{g'(x)}{g(x)}\,\dot{x} \right\} \qquad [24]$$

describes the path measure as $\propto \mathcal{D}x\,\mathcal{D}\hat{x}\,\mathrm{e}^{-S[\hat{x},x]}$.

In this path integral one can directly change variables covariantly using the standard chain rule and avoiding any Jacobian contribution. In continuous time, this property is tediously checked by direct computation using the chain rule of calculus together with the correspondence $\hat{x}(t) = u'(x(t))\,\hat{U}(t)$ between response fields. In contrast, the historically derived MSRJD action in Stratonovich discretization reads

$$\int_0^{t_f} \mathrm{d}t \left\{ \hat{x}\big(\dot{x} - f(x) + D\, g(x)g'(x)\big) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) \right\} \qquad [25]$$

and applying the chain rule to it leads to inconsistencies (52).

Sketch of the derivation of the covariance. The actual derivation of the covariance property involves a careful handling of the time-discrete infinitesimal propagator, by analyzing the contributions that arise order by order in powers of Δt upon the change of variables $U(t) = u(x(t))$. To construct the MSRJD representation, one rewrites the infinitesimal propagator [19] by using at every time step a Hubbard–Stratonovich transformation of the form $\sqrt{2\pi/a}\,\mathrm{e}^{-\frac{1}{2}\frac{b^2}{a}} = \int_{i\mathbb{R}} d\hat{x}\,\mathrm{e}^{\frac{1}{2}a\hat{x}^2 - b\hat{x}}$ for the following choice of parameters a and b

$$a = 2Dg(\bar{x}_t)^2\,\Delta t\,, \quad b = \left[\frac{\Delta x}{\Delta t} - f(\bar{x}_t)\right]\Delta t\,. \qquad [26]$$

The infinitesimal propagator for the first time step, as inferred from Eq. [19], is now represented as

$$\mathbb{P}(x_{\Delta t}|x_0) \stackrel{\mathcal{T}_g}{=} \left|\tfrac{g(\bar{x}_0)}{g(x_{\Delta t})}\right| \int_{i\mathbb{R}} d\hat{x}_0\,\mathrm{e}^{-\delta S[\hat{x}_0,\bar{x}_0]}\,, \qquad [27]$$

$$\delta S[\hat{x}_0,\bar{x}_0] \stackrel{\mathcal{T}_g}{=} \Delta t \left\{ \hat{x}_0\left[\tfrac{\Delta x}{\Delta t} - f(\bar{x}_0)\right] - Dg(\bar{x}_0)^2\hat{x}_0^2 + \frac{1}{2}f'(\bar{x}_0) - \frac{1}{2}\frac{g'(\bar{x}_0)}{g(\bar{x}_0)}f(\bar{x}_0) \right\}, \qquad [28]$$

which completely encodes the continuous-time expression

$$\tilde{S}[\hat{x},x] \stackrel{\mathcal{T}_g}{=} \int_0^{t_f} \mathrm{d}t \left\{ \hat{x}\left(\dot{x} - f(x)\right) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) - \frac{1}{2}\frac{g'(x)}{g(x)}f(x) \right\}. \qquad [29]$$

Note from Eq. [27] the appearance, in the discretized expression for the probability of a path, of a normalization prefactor $\mathcal{N}_{\rm MSR}[x(t)] = \prod_{0\leq t<t_f/\Delta t} \left|\frac{g(\bar{x}_t)}{g(x_{t+\Delta t})}\right|$ in front of the exponential weight. This $\mathcal{N}_{\rm MSR}$ warrants that a change of path in the action [24] induces no spurious contribution coming from the Jacobian of [18]. Up to a translation of the field $\hat{x}(t)$ by $g'/(2g)$, one recovers Eq. [24]. The symbol $\mathcal{T}_g$ over the equality sign means that functions of the variable x are $\mathcal{T}_g$-discretized, *i.e.* evaluated at $\bar{x}_t$. The field $\hat{x}(t)$ is not discretized: a variable $\hat{x}_t$ is introduced at each t and merely associated to $\bar{x}_t$.

One proves that only for the covariant discretization it is valid to naively change variables: namely, going from the fields $(\hat{U}, U)$ to $(\hat{x}, x)$, one can replace $\frac{\Delta U}{\Delta t}$ by $u'(\bar{x}_0)\frac{\Delta x}{\Delta t}$, $F(\bar{U}_0)$ by $u'(\bar{x}_0)f(\bar{x}_0)$, and $G(\bar{U}_0)$ by $u'(\bar{x}_0)g(\bar{x}_0)$. Such operations, combined with $\hat{U}_0 = \hat{x}_0/u'(\bar{x}_0)$, would normally yield an incorrect result by missing essential contributions of order $O(\Delta t^{1/2})$ and $O(\Delta t)$. Satisfactorily, these manipulations are correct for our chosen covariant discretization. The proof follows a procedure similar to the one we presented for the Onsager–Machlup case by comparing a correct route (A) with a naive route (B), with three important caveats: (*i*) One has to pay attention to the fact that $\hat{x}_t \sim \Delta t^{-1/2}$ at every time step, as inferred from the scaling of a in the Hubbard–Stratonovich transform [26], implying that the expansions in powers of $O(\Delta t)$ bring in terms that one can be tempted to throw away at first sight; (*ii*) One has to design additional substitution rules in order to handle powers of $\hat{x}_0$ larger than 1. This is done following a procedure similar to the one of Ref. (45) (see **SI**); (*iii*) Unexpectedly, in contrast to the Onsager–Machlup case exposed previously, the prefactor $\left|\frac{g(\bar{x}_0)}{g(x_{\Delta t})}\right|$ in [27] *brings a Jacobian contribution into the action* upon the time-discrete change of variables of route (A), which compensates precisely a term that is missing when naively substituting $\frac{\Delta U}{\Delta t}$ by $u'(\bar{x}_0)\frac{\Delta x}{\Delta t}$ along route (B).

To summarize, we have shown that changing variables in the MSRJD action [24] can be done following the standard rules of differential calculus, provided that the discrete-time construction of the path-integral weight is performed according to the covariant discretization [14] – leading to a modified action as compared to the historical Stratonovich-discretized one.

Summary and outlook

When dealing with fluctuating signals as encountered in quantum mechanics or stochastic processes, physicists rely on a triptych of methods: solving a linear problem involving an operator (Schrödinger or Fokker–Planck equations), resorting to stochastic calculus (Langevin equations), or using path integrals (field theory). As we have discussed, there is a vast number of operations for which path integrals have been known to be badly flawed. This surely explains why path integrals never became a tool of choice for mathematicians working on similar problems. What we have shown in the present work is how to construct a path-integral calculus devoid of what we view as its biggest flaw. It is now possible to manipulate well-defined path integrals with nonlinear changes of fields making no errors. It is our belief that our proposed construction should not only trigger a revival of interest on the mathematics side, but also on the physics one. Mathematics-wise, though we would not blush with embarrassment about our physicist's derivation, it is almost certain that many more steps are needed to bring path integrals on a rigorous par with other aspects of stochastic calculus. Physics-wise, we see immediate consequences, and open questions. Among the former, given the pedagogical importance of path integrals in higher education, we would advocate strongly in favor of our presentation (which time and efforts will surely smoothen and hopefully simplify) rather than in existing ones which suffer from well-known obvious problems (we refer to the telling example of our introduction). Second, given the lack of control, so far, in nonlinear manipulations of fields, which have been put to work in some many areas, it seems like a necessity to return to these and sort out whether and how path-integral based results are altered by taking our corrected formalism into account. Transformations of the action based on the chain rule, as simple as integrations by parts for instance, are in principle forbidden unless one uses the covariant discretization. This is especially important in areas of physics where no alternative to path integrals exist (like in path-integral based quantization issues). This brings us to future research directions, which we briefly list: What about higher space dimensions?, What about supersymmetries?, What about field theories expressed in second quantized form with coherent-states fields?

Methods

We explain here the methodology used to manipulate the infinitesimal propagator in the small Δt limit, following Ref. (45).

Changing variables while respecting the discretization. When passing from one infinitesimal propagator to another, one needs to reconstitute the $\mathcal{T}_g$-discretization of the variable $x(t)$ in $\mathbb{P}(x_{\Delta t}|x_0)$ (Eq. [19]) from the $\mathcal{T}_G$-discretization of the variable $U(t)$ in $\mathbb{P}_U(U_{\Delta t}|U_0)$ (Eq. [23]). The idea is to express the time-discrete values $U_0 = u(x_0)$, $U_{dt} = u(x_{\Delta t})$ and $\bar{U}_0$ appearing in the r.h.s. of Eq. [23] as a function of $\bar{x}_0$ and Δx, using

$$\bar{U}_0 = u(x_0) + \tfrac{1}{2}[u(x_{\Delta t}) - u(x_0)] + \beta_G(u(x_0))\,[u(x_{\Delta t}) - u(x_0)]^2\,,$$
$$x_0 = \bar{x}_0 - \tfrac{1}{2}\Delta x - \beta_g(\bar{x}_0)\,\Delta x^2\,, \qquad [30]$$

$$x_{\Delta t} = \bar{x}_0 + \frac{1}{2}\Delta x - \beta_g(\bar{x}_0)\,\Delta x^2\,. \qquad [31]$$

The strategy is the following: first, one uses these expressions in Eq. [23]; second, one expands this equation in powers of Δt and Δx, keeping in mind that the latter is $O(\Delta t^{1/2})$. The result takes the form

$$\frac{\mathcal{N}}{|g(\bar{x}_0)|}\,\mathrm{e}^{-\frac{1}{2}\frac{\Delta x^2}{2Dg(\bar{x}_0)^2\Delta t}} \times \Big[1 + \text{polynomial in } \Delta x \text{ and } \Delta t\Big] \qquad [32]$$

The fraction in the exponential is $O(\Delta t^0)$ and cannot be expanded; in fact, it defines the dominant order $O(\Delta t^{1/2})$ of Δx. The polynomial contains terms of the form $\Delta t^n \Delta x^m$ which are of order $O(\Delta t^{1/2})$ and $O(\Delta t)$. Higher-order terms ($O(\Delta t^{3/2})$ and higher) can be discarded because they do not contribute to the action. Many of the terms in the polynomial do not present an obvious $\Delta t \to 0$ limit (*e.g.* $\Delta x^4\,\Delta t^{-1}$) but the substitution rules derived in (45) allow one to take the continuous-time limit. For completeness, these are recalled (and slightly reformulated) in the **SI**. The last stage of the procedure consists in reexponentiating the resulting factor $[1 + \ldots]$ obtained from Eq. [32]. One then recovers the expected propagator $\mathbb{P}(x_{\Delta t}|x_0)$ of Eq. [19] as announced.

The same procedure allows one to change variables in the historical action [8] but this involves rules of calculus sharing little kindred with the standard ones [see Ref. (45)]. The covariant discretization, instead, yields back the usual chain rule as $\Delta t \to 0$.

ACKNOWLEDGMENTS. VL acknowledges financial support from the ERC Starting Grant No. 680275 MALIG, the UGA IRS PHEMIN project and the ANR-15-CE40-0020-03 Grant LSD. LFC is a member of Institut Universitaire de France. The authors thank H. J. Hilhorst and H. K. Janssen for very useful discussions.

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Supplementary information

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Contents

1 Some details on the toy model calculations

1.1 Model, action, and change of variables

The evolution equation of the position $x(t)$ is $\dot{x}(t) = v + \eta(t)$ with η a centered Gaussian white noise of correlation function $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. The Onsager–Machlup path-integral measure is

$$\mathcal{D}x\, \mathrm{e}^{-S_v[x]}, \qquad S_v[x] = \int_0^{t_\mathrm{f}} \mathrm{d}t\, \frac{(\dot{x}-v)^2}{4D}\,. \tag{1}$$

One considers the non-linear change of variables $y(t) = \frac{1}{3}x(t)^3$. In Stratonovich discretization, the variable $y(t)$ is governed by the Langevin equation

$$\dot{y} = v(3y)^{2/3} + (3y)^{2/3}\eta(t) \tag{2}$$

It takes the generic form $\dot{y} = F(y) + G(y)\eta(t)$ with $F(y) = v(3y)^{2/3}$ and $G(y) = (3y)^{2/3}$. The Galilean invariance of the underlying variable $x(t)$ tells us that $\tilde{y} = \frac{1}{3}(x-vt)^3 = \frac{1}{3}[(3y)^{1/3} - vt]^3$ should be independent of v; and indeed the variable $\tilde{y}$ is governed by the Langevin equation

$$\dot{\tilde{y}} = (3\tilde{y})^{2/3}\eta(t)\,. \tag{3}$$

1.2 (Historical) action for the variable $y(t)$

The path-integral weight for the multiplicative Langevin process $y(t)$ is now

$$\mathcal{D}y\, \mathcal{J}[y]\, \mathrm{e}^{-S_v[y]}, \qquad S_v[y] = \int_0^{t_\mathrm{f}} \mathrm{d}t \left\{ \frac{\left[\dot{y} - v(3y)^{2/3} + 2D(3y)^{1/3}\right]^2}{4D(3y)^{4/3}} + v\,(3y)^{-\frac{1}{3}} \right\} \tag{4}$$

with the associated normalization prefactor

$$\mathcal{J}[y] \overset{\mathrm{S}}{=} \prod_{0\le t<t_\mathrm{f}/\Delta t} \left\{ \frac{1}{\sqrt{4\pi D\Delta t}}\, \frac{1}{|G(\bar{y}_t)|} \right\} \tag{5}$$

which is Stratonovich-discretized (as the symbol $\overset{\mathrm{S}}{=}$ denotes).

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1.3 Naive change of variable from y to $\tilde{y}$

Changing variable from y to $\tilde{y}$, one can naively compute *via* the usual change of variables

$$S_v[y]\big|_{y=\frac{1}{3}[(3\tilde{y})^{1/3}+vt]^3} = \int_0^{t_{\rm f}} \mathrm{d}t \left\{ \frac{\dot{\tilde{y}}^2}{4D(3\tilde{y})^{4/3}} + \frac{\dot{\tilde{y}}}{vt\,(3\tilde{y})^{2/3}+3\tilde{y}} + \frac{v(3\tilde{y})^{1/3}+D+v^2t}{\big((3\tilde{y})^{1/3}+vt\big)^2} \right\} \tag{6}$$

which still depends on v and does not respect the Galilean invariance. Note that this action also factorizes as

$$S_v[y]\big|_{y=\frac{1}{3}[(3\tilde{y})^{1/3}+vt]^3} = \int_0^{t_{\rm f}} \mathrm{d}t \left\{ \frac{1}{4D} \Big[\frac{\dot{\tilde{y}}[(3\tilde{y})^{\frac{1}{3}}+vt]+2D(3\tilde{y})^{\frac{2}{3}}}{(3\tilde{y})^{\frac{2}{3}}[(3\tilde{y})^{\frac{1}{3}}+vt]} \Big]^2 + \frac{v}{(3\tilde{y})^{\frac{1}{3}}+vt} \right\} . \tag{7}$$

This is the result of Eq. [3] in the main text. One may of course complain that the normalization prefactor (5) and the Jacobian of the change of path measure induce supplementary terms in the action, two contributions that we analyze next.

1.4 Determination of the contributions arising from the Jacobian and from the normalization prefactor

Let us denote the change of variables between y and $\tilde{y}$ by $y = Y(\tilde{y}) = \frac{1}{3}[(3\tilde{y})^{1/3}+vt]^3$. Going to the infinitesimal propagator, one has, for the first time step (of duration Δt),

$$\tilde{\mathbb{P}}(\tilde{y}_{\Delta t}|\tilde{y}_0) = |Y'(\tilde{y}_{\Delta t})|\,\mathbb{P}(y_{\Delta t}|y_0) \quad \text{where generically} \quad y_t = Y(\tilde{y}_t)\,, \tag{8}$$

$\mathbb{P}$ is the propagator of the process $y(t)$ and $\tilde{\mathbb{P}}$ is the propagator of the process $\tilde{y}(t)$. Denoting δS and $\delta\tilde{S}$ the infinitesimal actions corresponding to these propagators, one obtains

$$\frac{1}{|\tilde{G}(\bar{\tilde{y}}_0)|}\,\mathrm{e}^{-\delta\tilde{S}} \overset{\rm s}{=} |Y'(\tilde{y}_{\Delta t})|\,\frac{1}{|G(\bar{y}_0)|}\,\mathrm{e}^{-\delta S}\,. \tag{9}$$

We thus see that when passing from the infinitesimal action for y to the one for $\tilde{y}$,

$$\delta\tilde{S} = \delta S\big|_{y\mapsto\tilde{y}} + \Delta\delta S\,, \tag{10}$$

$$\Delta\delta S = -\ln\left\{ |Y'(\tilde{y}_{\Delta t})|\,\frac{|\tilde{G}(\bar{\tilde{y}}_0)|}{|G(\bar{y}_0)|} \right\} . \tag{11}$$

The function $\tilde{G}(\tilde{y})$ is obtained as follows: differentiating $y(t) = Y(\tilde{y}(t))$ with respect to time t, one derives $\dot{y} = \dot{\tilde{y}}\,Y'(\tilde{y}(t))$, from which one infers that the noise amplitude of $\tilde{y}(t)$ is $\tilde{G}(\tilde{y}) = G(Y(\tilde{y}))/Y'(\tilde{y})$. Hence

$$\Delta\delta S = -\ln\left| \frac{Y'(\tilde{y}_{\Delta t})}{Y'(\bar{\tilde{y}}_0)}\,\frac{G(Y(\bar{\tilde{y}}_0))}{G(\bar{y}_0)} \right| . \tag{12}$$

To compute this "Jacobian shift" $\Delta\delta S$ as a function of the Stratonovich-discretized variable $\bar{\tilde{y}}(t)$ one then uses

$$\bar{y}_0 = y_0 + \tfrac{1}{2}[y_{\Delta t} - y_0] = Y(\tilde{y}_0) + \tfrac{1}{2}[Y(\tilde{y}_{\Delta t}) - Y(\tilde{y}_0)]\,, \tag{13}$$

$$\tilde{y}_0 = \bar{\tilde{y}}_0 - \tfrac{1}{2}\Delta\tilde{y}\,, \tag{14}$$

$$\tilde{y}_{\Delta t} = \bar{\tilde{y}}_0 + \tfrac{1}{2}\Delta\tilde{y}\,. \tag{15}$$

With these expressions, one then writes (12) as a function of the Stratonovich-discretized variable $\bar{\tilde{y}}_0$ and the increment $\Delta\tilde{y}$ only. Noting that $\Delta\tilde{y} = \tilde{y}_{\Delta t} - \tilde{y}_0 = O(\Delta t^{1/2})$, we can expand (12) up to order Δt included, to obtain

$$\Delta\delta S = -\frac{Y''(\bar{\tilde{y}}_0)}{2Y'(\bar{\tilde{y}}_0)}\Delta\tilde{y} + \frac{1}{8}\Big[\frac{G'(Y(\bar{\tilde{y}}_0))Y''(\bar{\tilde{y}}_0)}{G(Y(\bar{\tilde{y}}_0))} + \frac{Y''(\bar{\tilde{y}}_0)^2 - Y'(\bar{\tilde{y}}_0)Y^{(3)}(\bar{\tilde{y}}_0)}{Y'(\bar{\tilde{y}}_0)^2} \Big]\Delta\tilde{y}^2\,. \tag{16}$$

Here we can use the substitution rule (see Sec. 2 below)

$$\Delta\tilde{y}^2 \mapsto 2D\,\tilde{G}(\bar{\tilde{y}}_0)^2\Delta t\,. \tag{17}$$

Finally, using the explicit expressions of the functions $G(y)$ and $Y(\tilde{y})$ one obtains

$$\Delta\delta S = \frac{vt}{3vt\tilde{\bar{y}}_0 + (3\tilde{\bar{y}}_0)^{4/3}}\Delta\tilde{y} + \frac{D}{2}\Big[\frac{3}{(vt + (3\tilde{\bar{y}}_0)^{1/3})^2} - \frac{3}{(3\tilde{\bar{y}}_0)^{2/3}}\Big]\Delta t \tag{18}$$

to which corresponds a contribution in the action of the form

$$\Delta S \overset{\text{S}}{=} \int_0^{t_\text{f}} \mathrm{d}t \Big\{ \frac{vt}{3vt\tilde{y} + (3\tilde{y})^{4/3}}\dot{\tilde{y}} + \frac{D}{2}\Big[\frac{3}{(vt + (3\tilde{y})^{1/3})^2} - \frac{3}{(3\tilde{y})^{2/3}}\Big]\Big\} . \tag{19}$$

1.5 Incorporating the ΔS contribution to the action

Reading from the infinitesimal decomposition (10), we thus deduce a candidate for the action on the variable $\tilde{y}(t)$

$$\begin{aligned} S_v^{\text{naive}}[\tilde{y}] &= S_v[y]\big|_{y\mapsto\tilde{y}} + \Delta S \\ &= \int_0^{t_\text{f}} \mathrm{d}t \left\{ \frac{\dot{\tilde{y}}^2}{4D(3\tilde{y})^{4/3}} + \frac{5D}{2(tv + (3\tilde{y}(t))^{1/3})^2} + \frac{v}{tv + (3\tilde{y}(t))^{1/3}} - \frac{3D}{2(3\tilde{y}(t))^{2/3}} + \frac{\dot{\tilde{y}}(t)}{3\tilde{y}(t)} \right\} \\ &= \int_0^{t_\text{f}} \mathrm{d}t \left\{ \frac{1}{4D}\Big[\frac{\dot{\tilde{y}}(t) + 2D(3\tilde{y}(t))^{\frac{1}{3}}}{(3\tilde{y}(t))^{2/3}}\Big]^2 + \frac{5D}{2\big(vt + (3\tilde{y}(t))^{\frac{1}{3}}\big)^2} + \frac{v}{vt + (3\tilde{y}(t))^{\frac{1}{3}}} - \frac{5D}{2(3\tilde{y}(t))^{\frac{2}{3}}} \right\} \end{aligned} \tag{20}$$

where we have added the expressions (6) and (19) to write the second line. This expression still depends on v, meaning that even taking into account correctly how the normalization prefactor and the Jacobian of the change of variables add a shift contribution to the action, it is still impossible to change variable naively by applying the chain rule, when working with the action (4).

2 Substitution rules

Denoting $\lceil \Delta x^2 \rfloor = 2Dg(\bar{x}_0)^2\Delta t$, the substitution rules deduced in [1] can be reformulated as follows

$$\Delta x^2 \mapsto \lceil \Delta x^2 \rfloor , \tag{21}$$

$$\Delta x^3\, \Delta t^{-1} \mapsto 3\, \Delta x\, \lceil \Delta x^2 \rfloor\, \Delta t^{-1} , \tag{22}$$

$$\Delta x^4\, \Delta t^{-1} \mapsto 3\, \lceil \Delta x^2 \rfloor^2\, \Delta t^{-1} , \tag{23}$$

$$\Delta x^6\, \Delta t^{-2} \mapsto 15\, \lceil \Delta x^2 \rfloor^3\, \Delta t^{-2} . \tag{24}$$

Note that, as discussed in Ref. [1], the substitution rule (22) *cannot* be used inside the exponential of the infinitesimal propagator; indeed, since $\Delta x^3\, \Delta t^{-1} = O(\Delta t^{1/2})$ one has $\mathrm{e}^{h(x)\Delta x^3\, \Delta t^{-1}} = 1 + h(x)\Delta x^3\, \Delta t^{-1} + \frac{1}{2}[h(x)\Delta x^3\, \Delta t^{-1}]^2 + O(\Delta t^{3/2})$ and the second term of this expansion would be wrong if one had first applied the rule (22) and then expanded. This is the trivial but shrouded reason why the procedure exposed in the *Methods* of the main text has to be performed by expanding the terms of order $\Delta t^{>0}$ outside of the exponential of the infinitesimal propagator of Eq. [23] in the main text. This reflects the fact, known to mathematicians, that the validity of the continuous-time chain rule is relatively weak, even in the Stratonovich discretization: it cannot be manipulated without care by, for instance, taking its square and exponentiating it – as one would do by naively using it in the Onsager–Machlup action. For further discussion on this subject, see Ref. [1].

3 An example: changing the discretization of the normalization prefactor

The change of discretization of the prefactor from $\frac{\mathcal{N}}{|g(\bar{x}_0)|}$ to $\frac{\mathcal{N}}{|g(x_{\Delta t})|}$ is obtained by following the following one uses the representation in the *Methods* of the main text to change the discretization point and to

expand the resulting prefactor in powers of Δx up to order $O(\Delta t)$

$$\begin{aligned}\frac{1}{g(\bar{x}_0)} &= \frac{1}{g(x_{\Delta t})}\frac{g(x_{\Delta t})}{g(\bar{x}_0)} \\ &= \frac{1}{g(x_{\Delta t})}\left[1+\frac{g'(\bar{x}_0)}{2g(\bar{x}_0)}\Delta x + \frac{\frac{1}{4}g''(\bar{x}_0)-2\beta_g(\bar{x}_0)g'(\bar{x}_0)}{2g(\bar{x}_0)}\Delta x^2\right] \\ &= \frac{1}{g(x_{\Delta t})}\mathrm{e}^{\frac{g'(\bar{x}_0)\Delta x}{2g(\bar{x}_0)}+D\left[g(\bar{x}_0)\left(\frac{1}{4}g''(\bar{x}_0)-2\beta_g(\bar{x}_0)g'(\bar{x}_0)\right)-\frac{1}{4}g'(\bar{x}_0)^2\right]\Delta t}.\end{aligned} \tag{25}$$

To obtain the last equality, we used the substitution rule (21) for Δx^2 before exponentiating. The argument of the exponential is then added to the argument of the exponential in the original infinitesimal propagator to obtain Eq. [19] of the main text.

References

[1] L. F. Cugliandolo and V. Lecomte. Rules of calculus in the path integral representation of white noise Langevin equations: the Onsager–Machlup approach. *Journal of Physics A: Mathematical and Theoretical* **50**, 345001 (2017).