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MAXIMAL DIFFERENTIAL UNIFORMITY POLYNOMIALS

YVES AUBRY, FABIEN HERBAUT, AND JOSÉ FELIPE VOLOCH

Abstract. We provide explicit infinite families of integers m such that all the polynomials of $\mathbb{F}_{2^n}[x]$ of degree m have maximal differential uniformity for n large enough. We also prove a conjecture of the third author for these families.

1. Introduction

Throughout this paper n is a positive integer and $q = 2^n$. For a polynomial $f \in \mathbb{F}_q[x]$ we define the differential uniformity $\delta(f)$ following Nyberg ([6]):

$$\delta(f) := \max_{(\alpha,\beta)\in\mathbb{F}_q^*\times\mathbb{F}_q} \sharp\{x \in \mathbb{F}_q \mid f(x+\alpha)+f(x) = \beta\}.$$

When $\delta(f) = 2$ the associated functions $f : \mathbb{F}_q \to \mathbb{F}_q$ are called APN (Almost Perfectly Nonlinear). These functions have been extensively studied as they offer good resistance against differential attacks (see [2]). Among them, those which are APN over infinitely many extensions of $\mathbb{F}_q$ have attracted special attention.

In the opposite direction the third author proved in [10] that most polynomials $f \in \mathbb{F}_q[x]$ of degree $m \equiv 0$ or $3 \pmod 4$ have differential uniformity equal to $m-1$ or $m-2$, the largest possible for polynomials of degree m. Precisely, he proved that for a given integer $m > 4$ such that $m \equiv 0 \pmod 4$ (respectively $m \equiv 3 \pmod 4$), if $\delta_0 = m-2$ (respectively $\delta_0 = m-1$) then

$$\lim_{n\to\infty} \frac{\sharp\{f \in \mathbb{F}_{2^n}[x] \mid \deg(f) = m,\ \delta(f) = \delta_0\}}{\sharp\{f \in \mathbb{F}_{2^n}[x] \mid \deg(f) = m\}} = 1.$$

The first two authors extended this result to the second order differential uniformity in [1].

The following conjecture is also stated in [10]:

Conjecture 1.1. *For a given integer $m > 4$, there exists $\varepsilon_m > 0$ such that for all sufficiently large n, if f is a polynomial of degree m over $\mathbb{F}_{2^n}$, for at least $\varepsilon_m 2^{2n}$ values of $(\alpha,\beta) \in \mathbb{F}_{2^n}^* \times \mathbb{F}_{2^n}$ we have $\sharp\{x \in \mathbb{F}_q \mid f(x+\alpha)+f(x) = \beta\} = \delta(f)$.*

Moreover, it was proved in [10] that all polynomials f of degree 7 have maximal differential uniformity (that is here $\delta(f) = 6$) if n is large enough.

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The aim of this paper is to exhibit an infinite set $\mathcal{M}$ (defined below) of integers m such that every polynomial $f \in \mathbb{F}_{2^n}[x]$ of degree m has maximal differential uniformity if n is large enough, that is $\delta(f)$ is equal to the degree of $D_\alpha f(x) = f(x+\alpha) + f(x)$, the derivative of f with respect to α. We stress that, for $m \in \mathcal{M}$, our results are much stronger than those of [10] as we prove maximality of differential uniformity for all polynomials of degree m, as opposed to most of them.

Definition. *(Definition 3.10 and Proposition 3.11) We denote by $\mathcal{M}$ the set of the odd integers m such that the unique polynomial g satisfying $g(x(x+1)) = D_1(x^m)$ has distinct critical values.*

We have that m belongs to $\mathcal{M}$ if and only if for any ζ_1 and ζ_2 in $\overline{\mathbb{F}}_2 \setminus \{1\}$, the equalities $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$ imply $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$.

Now we can state our main results.

Theorem. *(Theorem 5.3 and Theorem 5.7) Let $m \in \mathcal{M}$ such that $m \equiv 7 \pmod 8$. Then for n sufficiently large, for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m-1$. Furthermore, Conjecture 1.1. is true for such integers m.*

For example, we will prove that the previous theorem applies for the integers $m \in \{7, 23, 39, 47, 55, 79, 87, 95, 111, 119, 135, 143, 159, 167, 175, 191, 199\}$ (see Example 3.16). We also provide explicit infinite families of such integers m, namely the integers $m = 2\ell^{2k+1}+1$ for $k \geqslant 0$ and $\ell \in \{3, 11, 19, 23, 43, 47, 59, 67, 71, 79, 83, 103, 107, 131, 139, 151, 163, 167, 179, 191, 199\}$ (see Corollary 5.4).

When m is congruent to 3 modulo 8, we also obtain some results but we have conditions on the parity of n or we have to remove some polynomials.

Theorem. *(Theorem 5.5)*

Let $m \in \mathcal{M}$ such that $m \geqslant 7$ and $m \equiv 3 \pmod 8$.

(i) For n even and sufficiently large and for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m-1$.

(ii) For n sufficiently large and for all polynomials $f = \sum_{i=0}^{m} a_{m-i}x^i$ in $\mathbb{F}_{2^n}[x]$ of degree m such that $a_1^2 + a_0 a_2 \neq 0$, we have $\delta(f) = m-1$.

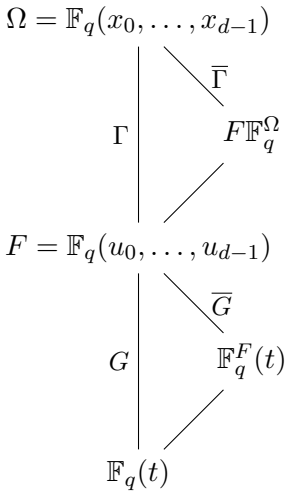
We also provide infinite families of integers $m \equiv 3 \pmod 8$ for which the previous theorem applies, namely the integers $m = 2\ell^k + 1$ for $k \geqslant 1$ and $\ell \in \{17, 41, 97, 113, 137, 193\}$ and the integers $m = 2\ell^{2k} + 1$ for $k \geqslant 1$ and $\ell \in \{23, 47, 71, 79, 103, 151, 167, 191, 199\}$ (see Corollary 5.6).

Let us explain the strategy of the proofs of the above theorems which has important similarities to that of [10] and [1]. For simplicity we consider in this sketch the case where m is congruent to 7 modulo 8.

If $f \in \mathbb{F}_q[x]$ is a polynomial of degree m and if $\alpha \in \mathbb{F}_q^*$, we introduce the unique polynomial $L_\alpha f$ of degree $d = (m-1)/2$ such that $L_\alpha f\,(x(x+\alpha)) =$

$D_\alpha f(x)$ (see Proposition 2.3). We consider the splitting field F of the polynomial $L_\alpha f(x) - t$ over the field $\mathbb{F}_q(t)$ with t transcendental over $\mathbb{F}_q$ and set $\mathbb{F}_q^F$ be the algebraic closure of $\mathbb{F}_q$ in F. The Galois groups $G = \mathrm{Gal}(F/\mathbb{F}_q(t))$ and $\overline{G} = \mathrm{Gal}(F/\mathbb{F}_q^F(t))$ are respectively the arithmetic and geometric monodromy groups of $L_\alpha f$.

If $u_0, \ldots, u_{d-1}$ are the roots of $L_\alpha f(x) = t$, then we will denote by x_i a root of $x^2 + \alpha x = u_i$. So the $2d$ elements $x_0, x_0 + \alpha, \ldots, x_{d-1}, x_{d-1} + \alpha$ are the solutions of $D_\alpha f(x) = t$. Thus we consider $\Omega = \mathbb{F}_q(x_0, \ldots, x_{d-1})$ the compositum of the fields $F(x_i)$ and $\mathbb{F}_q^\Omega$ the algebraic closure of $\mathbb{F}_q$ in Ω. We set also $\Gamma = \mathrm{Gal}(\Omega/F)$ and $\overline{\Gamma} = \mathrm{Gal}(\Omega/F\mathbb{F}_q^\Omega)$. Then we have the following diagram:



When the integer m belongs to $\mathcal{M}$ and is congruent to 7 modulo 8 we prove that for n sufficiently large and for any polynomial $f \in \mathbb{F}_{2^n}[x]$ of degree m, there exists α in $\mathbb{F}_{2^n}^*$ such that:

(1) $L_\alpha f$ is Morse
(2) the equation $x^2 + \alpha x = \frac{b_1}{b_0}$ has a solution in $\mathbb{F}_{2^n}$.

Now, condition (1) implies by Proposition 4.1 that the extension $F/\mathbb{F}_q(t)$ is regular. Condition (1) and (2) imply by Proposition 4.6 that the extension Ω/F is regular. It enables us to apply Chebotarev density theorem (see Proposition 5.1) to obtain, for n sufficiently large depending only on m, the existence of $\beta \in \mathbb{F}_{2^n}$ such that the polynomial $D_\alpha f(x) + \beta$ splits in $\mathbb{F}_{2^n}[x]$ with no repeated factors. The differential uniformity of f is thus equal to the degree of $D_\alpha f$.

The paper is organized as follows. Section 2 is devoted to the study of the operator L_α. Section 3 provides a detailed exposition of Morse polynomials in even characteristic. According to the appendix by Geyer in [5], Morse polynomials in this context are polynomials of odd degree satisfying two

conditions: their critical points are non degenerate and their critical values are distinct. The first condition leads to the study of the number of α such that the resultant of the derivative $(L_\alpha f)'$ with the second Hasse-Schmidt derivative $(L_\alpha f)^{[2]}$ does not vanish (Proposition 3.2). We give upper bounds for the number of exceptions in terms of m.

By contrast, we need additional requirements on m to guarantee that for enough α the polynomial $L_\alpha f$ has distinct critical values (see Proposition 3.6). Precisely, we will make the assumption that $L_1(x^m)$ has distinct critical values, this is that m belongs to $\mathcal{M}$ (Definition 3.10). We complete Section 3 by exhibiting some families of infinitely many integers belonging to $\mathcal{M}$.

Section 4 is devoted to the study of the Galois groups G, $\overline{G}$, Γ and $\overline{\Gamma}$. We prove in Proposition 4.6 that if the equation $x^2+\alpha x=\frac{b_1}{b_0}$ has a solution in $\mathbb{F}_{2^n}$ i.e. if $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\left(\frac{b_1}{b_0\alpha^2}\right)=0$ then the extension Ω/F is regular. The different expressions of b_1/b_0 we have obtained in Lemma 2.5, depending on the congruence of m modulo 8, induce differences in the treatment.

Section 5 deals with the Chebotarev density theorem and contains the statements and the proofs of the main results.

Let us stress the main difference between the common approach of [10] and [1] and the approach of the present paper. For simplicity, we consider again that $m\equiv 7 \pmod 8$. In [10] and [1], one of the key steps is to fix $\alpha_1,\dots,\alpha_k$ in $\mathbb{F}_{2^n}$ and to obtain a lower bound depending on n for the number of polynomials f in $\mathbb{F}_{2^n}[x]$ such that at least one of the $L_{\alpha_i}f$ is Morse. By contrast, we prove here that for n sufficiently large and for any polynomial f of degree m in $\mathbb{F}_{2^n}[x]$ there exists α such that $L_\alpha f$ is Morse.

2. The associated polynomial $L_\alpha f$

Let $f\in\mathbb{F}_q[x]$ be a polynomial of degree $m\geqslant 7$ (the cases where $m<7$ are handled in [10]) and $\alpha\in\mathbb{F}_q^*$. The derivative of a polynomial $f\in\mathbb{F}_q[x]$ along α is defined by:

$$D_\alpha f(x)=f(x)+f(x+\alpha).$$

If we set $f=\sum_{k=0}^m a_{m-k}x^k$, a straightforward computation gives that $D_\alpha f=\sum_{k=0}^m c_{m-k}x^k$ where $c_k=a_k+\sum_{i=m-k}^m a_{m-i}\binom{i}{m-k}\alpha^{i-m+k}$. As we work over an even characteristic field, we have $c_0=a_0+a_0=0$, $c_1=m\alpha a_0$ and $c_2=(m-1)\alpha a_1+\binom{m}{2}\alpha^2 a_0$. We deduce the following proposition.

Proposition 2.1. *Let $f\in\mathbb{F}_q[x]$ be a polynomial of degree m. If m is odd then the degree of $D_\alpha f$ is $m-1$. If m is even then the degree of $D_\alpha f$ is less than or equal to $m-2$, and equal to $m-2$ if and only if $a_1+a_0\alpha\binom{m}{2}\neq 0$.*

In the whole paper, we will associate to any integer m the following integer d.

Definition 2.2. Let m be an integer. Suppressing in our notation the dependence on m, we set $d = (m-1)/2$ if m is odd and $d = (m-2)/2$ is m is even.

2.1. Existence of $L_\alpha f$.

Proposition 2.3. *Let $\alpha \in \mathbb{F}_q^*$ and let $f \in \mathbb{F}_q[x]$ be a polynomial of degree m. Then there exists a unique polynomial $g \in \mathbb{F}_q[x]$ of degree less than or equal to d such that*

$$D_\alpha f(x) = g(x(x+\alpha)).$$

Furthermore, the map $L_\alpha : f \longmapsto g$ is linear and its restriction to the subspace of polynomials of degree at most m is surjective onto the subspace of polynomials of degree at most d.

Proof. The proof is similar to that of Proposition 2.2. of [1] dealing with the set Λ_k of roots of multiplicity k of $D_\alpha f$ and noticing that $x \mapsto x + \alpha$ is an involution of each set Λ_k. The surjectivity of L_α follows from the fact that the kernel of the restriction of L_α to the space of polynomials of degree at most m is the subspace of polynomials $g\left(x(x+\alpha)\right)$ where $g \in \mathbb{F}_q[x]$ has degree at most $[m/2]$ (see Lemma 2.3. of [1]). □

2.2. The coefficients b_i of $L_\alpha f$.

Let $f = \sum_{i=0}^{m} a_{m-i}x^i \in \mathbb{F}_q[x]$ be a polynomial of degree m and $L_\alpha f = \sum_{i=0}^{d} b_{d-i}x^i$ be the associated polynomial of degree d when m is odd and of degree less than or equal to d otherwise (see Proposition 2.1). To obtain information on the coefficients b_i, one can consider the triangular linear system with coefficients 1 on the diagonal arising when identifying the coefficients of $x^{2d}, x^{2d-2}, \ldots, x^2, x^0$ in $g\left(x(x+\alpha)\right)$ and in $D_\alpha f$. Note that this approach proves again the unicity of g claimed in Proposition 2.3.

More precisely, a necessary condition for the term $b_s x^t$ to appear in $g\left(x(x+\alpha)\right)$ is that $d - t \leqslant s \leqslant d - t/2$. In this case, it appears with the coefficient $\binom{d-s}{t-d+s}\alpha^{2(d-s)-t}$. So for each integer k between 0 and d, identifying the coefficient of $x^{2(d-k)}$ in $g\left(x(x+\alpha)\right)$ and in $D_\alpha f(x)$ gives

$$\sum_{s=\mathrm{Max}\{0,2k-d\}}^{k} \binom{d-s}{2k-2s}\alpha^{2k-2s}b_s = \sum_{i=2d-2k+1}^{m} \binom{i}{2d-2k}\alpha^{i-2d+2k}a_{m-i}. \tag{1}$$

We consider the polynomial ring $\mathbb{F}_2[\alpha, a_0, \ldots, a_m]$ where $\alpha, a_0, \ldots, a_m$ are indeterminates with the degree w such that $w(\alpha) = 1$ and $w(a_j) = j$. It means that the monomial $\alpha^{d_\alpha}a_0^{d_0}a_1^{d_1}a_2^{d_2}\ldots a_m^{d_m}$ has degree $d_\alpha + d_1 + 2d_2 + \cdots + md_m$. Then using the triangular system obtained from (1) and an induction on k prove the following homogeneity result.

Lemma 2.4. *For all integers i such that $0 \leqslant i \leqslant d$ we have $b_i \in \mathbb{F}_2[\alpha, a_0, \ldots, a_m]$ which is an homogeneous polynomial of degree $2i+1$ if m is odd and of degree $2i+2$ if m is even, when considering the degree w such that $w(\alpha) = 1$ and $w(a_j) = j$.*

The relations (1) also provide expressions of the first coefficients $b_0, b_1, \ldots$ of $L_\alpha f$ depending on the congruence class of m modulo 8, as made explicit in the next lemma which will be needed in the proof of Theorem 5.3. Note that formulas for b_1/b_0 appeared in [10] as well, but the last two had misprints.

Lemma 2.5. *Let m be an integer. If $m \equiv 0 \pmod 4$ then $b_0 = a_1\alpha$ and if $m \equiv 3 \pmod 4$ then $b_0 = a_0\alpha$. Moreover, we have the following expressions of b_1/b_0 depending on the congruence of m:*

$m \pmod 8$	b_1/b_0
3	$\alpha^2 + \frac{a_1\alpha+a_2}{a_0}$
7	$\frac{a_1\alpha+a_2}{a_0}$
0	$\frac{a_2\alpha+a_3}{a_1}$
4	$\alpha^2 + \frac{a_0\alpha^3+a_2\alpha+a_3}{a_1}$

3. For almost every α the polynomial $L_\alpha f$ is Morse

We will focus now on polynomials f of degree $m \equiv 3 \pmod 4$ and thus, for nonzero α, on polynomials $L_\alpha f$ of odd degree $d = (m-1)/2$.

3.1. Morse polynomials in even characteristic.

We consider the following notion of Morse polynomial given in all characteristic by Geyer in an appendix to the paper [5].

Definition 3.1. Let K be a field of characteristic $p \geqslant 0$. We say that a polynomial g over K is Morse if the three following conditions hold:

(a) the critical points of g, i.e the zeroes of g', are non degenerate,
(b) the critical values of g are distinct, i.e. $g'(\tau) = g'(\eta) = 0$ and $g(\tau) = g(\eta)$ imply $\tau = \eta$,
(c) if $p > 0$, then the degree of g is not divisible by p.

These conditions are chosen such that g corresponds to a covering with maximum Galois group, that is $\mathrm{Gal}\,(g(t) - x, K(x))$ is the symmetric group $\mathfrak{S}_d$ where d is the degree of g (see Proposition 4.2 in [5]). In the case where $p > 0$, the loci of non-Morse polynomials is described in the same appendix.

Let us sum up the situation in the case where $p = 2$. In this case one has to introduce the Hasse-Schmidt derivative $g^{[2]}$ which is defined by the equality $g(t+u) \equiv g(t) + g'(t)u + g^{[2]}(t)u^2 \pmod{u^3}$ where u and t are independent variables. If $g = \sum_{i=0}^{d} b_{d-i}x^i$ is a degree d polynomial of $\mathbb{F}_q[x]$

with q a power of 2, then the condition (a) above is fulfilled if and only if g' and $g^{[2]}$ have no common roots, that is if and only if the resultant

$$R := \mathrm{Res}(g', g^{[2]}) \in \mathbb{F}_2[b_0, \dots, b_d]$$

does not vanish. And the condition (b) above is fulfilled if and only if

$$\Pi(g) := \prod_{i \neq j} (g(\tau_i) - g(\tau_j))$$

does not vanish, where $\tau_1, \dots, \tau_{\left[\frac{d-1}{2}\right]}$ are the (double) roots of g'. Using the theorem on symmetric functions, one can obtain an expression of $\Pi(g)$ depending on the coefficients $b_0, \dots, b_d$ of g.

In order to calculate the second order Hasse-Schmidt derivative, we will make use of the following Lucas theorem about binomial coefficients (see for instance the introduction of [4]). For p a prime number, write $m = m_0 + m_1 p + m_2 p^2 + \cdots + m_r p^r$ and $k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r$ in base p. Then we have $\binom{m}{k} \equiv \binom{m_0}{k_0}\binom{m_1}{k_1} \cdots \binom{m_r}{k_r} \pmod{p}$.

3.2. **The condition (a).** In order to bound the number of α such that the critical values of $L_\alpha f$ are non degenerate, we study in this subsection $\mathrm{Res}((L_\alpha f)', (L_\alpha f)^{[2]}) \in \mathbb{F}_2[a_0, \dots, a_m]$.

We will need three lemmas to succeed in doing so. Lemma 3.3 enables us to study $\tilde{R} := \mathrm{Res}\left((D_\alpha f)', (D_\alpha f)^{[2]}\right)$ rather than $\mathrm{Res}\left((L_\alpha f)', (L_\alpha f)^{[2]}\right)$. Then Lemma 3.4 gives a result about the homogeneity and the degree of this polynomial if it is nonzero. To prove its non nullity we evaluate it in $a_0 = 1, a_1 = \cdots = a_m = 0$ which amounts to determining in Lemma 3.5 if the polynomial x^m has non degenerate critical points.

Proposition 3.2. *Let $m \geqslant 7$ such that $m \equiv 3 \pmod 4$ and let $f(x) = \sum_{k=0}^{m} a_{m-k} x^k$ be a polynomial of $\mathbb{F}_q[x]$ of degree m. Then the critical points of $L_\alpha f$ are non degenerate except for at most $m(m-3)$ values of $\alpha \in \overline{\mathbb{F}}_2$.*

Lemma 3.3. *Let $f \in \mathbb{F}_q[x]$ be a polynomial. For all $\alpha \in \mathbb{F}_q^*$ the polynomials $(L_\alpha f)'$ and $(L_\alpha f)^{[2]}$ have a common root in $\overline{\mathbb{F}}_2$ if and only if the polynomials $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ have a common root in $\overline{\mathbb{F}}_2$.*

Proof. Since $D_\alpha f = L_\alpha f \circ T_\alpha$ where $T_\alpha(x) := x(x + \alpha)$, we can prove the two following equalities:

$$(D_\alpha f)' = \alpha (L_\alpha f)' \circ T_\alpha$$

and

$$(D_\alpha f)^{[2]} = (L_\alpha f \circ T_\alpha)^{[2]} = (L_\alpha f)' \circ T_\alpha + \alpha^2 (L_\alpha f)^{[2]} \circ T_\alpha.$$

The result follows. □

Lemma 3.4. *Let $m \geqslant 7$ such that $m \equiv 3 \pmod 4$ and let $f = \sum_{k=0}^{m} a_{m-k} x^k$ in $\mathbb{F}_2[a_0, \dots, a_m][x]$. Consider the degree w defined by $w(\alpha) = 1$ and $w(a_i) = i$ for any i and consider also the degree $\tilde{w}$ defined by $\tilde{w}(\alpha) = 0$ and $\tilde{w}(a_i) = 1$.*

Then the resultant $\operatorname{Res}\left((D_\alpha f)', (D_\alpha f)^{[2]}\right)$ *in the variable* x*, if it is nonzero, is an homogeneous polynomial of* $\mathbb{F}_2[a_0, \dots, a_m, \alpha]$ *of degree* $m(m-3)$ *when considering the degree* w *and is an homogeneous polynomial of degree* $2(m-3)$ *when considering the degree* $\tilde{w}$*.*

Proof. As $f(x) = \sum_{k=0}^{m} a_{m-k} x^k$ and $f(x+\alpha) = \sum_{k=0}^{m} a_{m-k}(x+\alpha)^k$, these two polynomials are homogeneous of degree m for the degree w such that $w(\alpha) = 1$, $w(a_i) = i$ and $w(x) = 1$. It follows that $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ are homogeneous of degree respectively $m-1$ and $m-2$ for the degree w. Using the formulae of $D_\alpha f$ given in Section 2, we have:

$$D_\alpha f(x) = \alpha a_0 x^{m-1} + a_0\alpha^2 x^{m-2} + \left(a_0\alpha^3 + a_1\alpha^2 + a_2\alpha\right) x^{m-3} + \cdots$$

The polynomial $(D_\alpha f)'$ has degree $m-3$ in the variable x since m is odd and its leading coefficient is $a_0\alpha^2$. The polynomial $(D_\alpha f)^{[2]}$ has also degree $m-3$ in the variable x since it can be shown that $(x^k)^{[2]} = \binom{k}{2} x^{k-2}$ using the binomial theorem, the above Lucas theorem and the congruence of m. Its leading coefficient is $a_0\alpha$.

Thus we can set $(D_\alpha f)' = \sum_{i=0}^{m-3} d_i x^{m-3-i}$ and $(D_\alpha f)^{[2]} = \sum_{i=0}^{m-3} e_i x^{m-3-i}$ where $d_i, e_i \in \mathbb{F}_2[a_0, \dots, a_m, \alpha]$ are such that $w(d_i) = i+2$ and $w(e_i) = i+1$. Thus the resultant $\operatorname{Res}\left((D_\alpha f)', (D_\alpha f)^{[2]}\right)$ in the variable x, if it is nonzero, is an homogeneous polynomial of $\mathbb{F}_2[a_0, \dots, a_m, \alpha]$ of degree $m(m-3)$ for the degree w. For the second homogeneity result claimed, note that this resultant is a sum of $2(m-3)$ products of the coefficients d_i and e_i, and each one of them is a linear combination in the $a_0, \dots, a_m$. □

Lemma 3.5. *Let* $m \geqslant 7$ *such that* $m \equiv 3 \pmod 4$ *and let* $f = x^m$*. For all* $\alpha \in \mathbb{F}_q^*$ *the critical points of* $L_\alpha f$ *are non degenerate.*

Proof. Using Lemma 3.3 we look for the common roots of $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$. We compute $(D_\alpha f)' = (x+\alpha)^{m-1} + x^{m-1}$ and $(D_\alpha f)^{[2]} = (x+\alpha)^{m-2} + x^{m-2}$. Hence, if $\omega \in \overline{\mathbb{F}}_2$ was a common root of $(D_\alpha f)'$ and $(D_\alpha f)^{[2]}$ then we would have $((\omega+\alpha)/\omega)^{m-1} = ((\omega+\alpha)/\omega)^{m-2} = 1$, and so $\alpha = 0$. □

Now we are enable to prove Proposition 3.2.

Proof. Lemma 3.3 enables us to study $\tilde{R} := \operatorname{Res}\left((D_\alpha f)', (D_\alpha f)^{[2]}\right)$ rather than $\operatorname{Res}\left((L_\alpha f)', (L_\alpha f)^{[2]}\right)$. Using the homogeneity results given by Lemma 3.4 we know that there is at most one term in $\tilde{R}$ of degree at least $m(m-3)$ in α, precisely $a_0^{2(m-3)}\alpha^{m(m-3)}$. We study whether this term appears or not.

By Lemma 3.5, for nonzero α the critical points of $L_\alpha(x^m)$ are non degenerate, so $\tilde{R}(a_0 = 1, a_1 = 0, \dots, a_m = 0, \alpha = 1) \neq 0$ and this term does appear. Choosing a polynomial $f \in \mathbb{F}_q[x]$ of degree m amounts to choosing coefficients $a_0, \dots, a_m$ in $\mathbb{F}_q$ with $a_0 \neq 0$. Thus we can consider $\tilde{R}$ as a nonzero polynomial in α of degree $m(m-3)$ which has at most $m(m-3)$ roots. □

3.3. **The condition (b).** We use a similar strategy to prove that for almost every choice of α the polynomial $L_\alpha f$ has distinct critical values: we use an homogeneity result and we study the case of $L_\alpha(x^m)$. As it is a key point in our approach, we give equivalent conditions for $L_\alpha(x^m)$ to have distinct critical values. Recall that we work with $m \equiv 3 \pmod 4$ and that we set $d = (m-1)/2$.

Proposition 3.6. *Let m be an integer such that $m \geqslant 7$ and $m \equiv 3 \pmod 4$.*

(i) If there exists $\alpha \in \overline{\mathbb{F}}_2^*$ such that $L_\alpha(x^m)$ has distinct critical values then it holds true for any $\alpha \in \overline{\mathbb{F}}_2^*$.

(ii) Suppose that for any $\alpha \in \overline{\mathbb{F}}_2^*$ (or equivalently for $\alpha = 1$) the polynomial $L_\alpha(x^m)$ has distinct critical values. Let $f \in \mathbb{F}_q[x]$ be a polynomial of degree m. Then $L_\alpha f$ has distinct critical values except for at most $(5m-1)(m-3)(m-7)/64$ values of $\alpha \in \overline{\mathbb{F}}_2$.

Proof. Let $\alpha \in \overline{\mathbb{F}}_2^*$ such that $L_\alpha(x^m)$ has distinct critical values. Now let $\alpha' \in \overline{\mathbb{F}}_2^*$ and let us show that $L_{\alpha'}(x^m)$ has distinct critical values. We use the characterization given by Lemma 3.7: suppose that $(\tau, \eta) \in \left(\overline{\mathbb{F}}_2\right)^2$ are such that

$$\tau^{m-1} + (\tau + \alpha')^{m-1} = \eta^{m-1} + (\eta + \alpha')^{m-1} = 0 \tag{2}$$

and

$$\tau^m + (\tau + \alpha')^m = \eta^m + (\eta + \alpha')^m. \tag{3}$$

Mutliply Equation (2) by $\left(\frac{\alpha}{\alpha'}\right)^{m-1}$ and Equation (3) by $\left(\frac{\alpha}{\alpha'}\right)^m$, we obtain that $\frac{\alpha}{\alpha'}\eta \in \{\frac{\alpha}{\alpha'}\tau, \frac{\alpha}{\alpha'}\tau + \alpha\}$ i.e. $\eta \in \{\tau, \tau + \alpha'\}$ which gives the result.

To prove assertion (ii) we follow the strategy of the proof of Proposition 3.2. Consider $f = \sum_{i=0}^m a_{m-i}x^i \in \mathbb{F}_2[a_0, \ldots, a_m][x]$ and $L_\alpha f = \sum_{i=0}^d b_{d-i}x^i \in \mathbb{F}_2[b_0, \ldots, b_d, \alpha][x]$. By Lemma 3.8, when setting $N = d\binom{(d-1)/2}{2}$ we can see $b_0^N \times \Pi\left(L_\alpha f\right)$ as a polynomial of $\mathbb{F}_2[a_0, \ldots, a_m, \alpha]$. Now we use the homogeneity result of Lemma 3.8 to know that this last polynomial has at most one term of degree at least $(5d+2)\binom{(d-1)/2}{2}$ in α . Precisely, this term is possibly the term $a_0^{(d+2)\binom{(d-1)/2}{2}}\alpha^{(5d+2)\binom{(d-1)/2}{2}}$.

In order to know if this term appears or not, we evaluate this polynomial at $a_0 = 1$ and $a_i = 0$ for all $i > 0$ which amounts to determine if the polynomial $L_\alpha(x^m)$ has distinct critical values, which is true by hypothesis. Now fix a polynomial $f \in \mathbb{F}_q[x]$ of degree m and see $b_0^N \times \Pi(L_\alpha f)$ as a polynomial of $\mathbb{F}_2[\alpha]$. So we know its degree and thus $L_\alpha f$ has distinct critical values except for at most $(5d+2)\binom{(d-1)/2}{2}$ values of $\alpha \in \overline{\mathbb{F}}_2$. Then we conclude using the relation between m and d. □

The following lemma gives a condition on $D_\alpha f$ for $L_\alpha f$ to have distinct critical values.

Lemma 3.7. *Let $f \in \mathbb{F}_q[x]$. For all $\alpha \in \mathbb{F}_q^*$ the polynomial $L_\alpha f$ has distinct critical values if and only if for all $(\tau, \eta) \in \left(\overline{\mathbb{F}}_2\right)^2$, $(D_\alpha f)'(\tau) = (D_\alpha f)'(\eta) = 0$ and $D_\alpha f(\tau) = D_\alpha f(\eta)$ imply $\tau = \eta$ or $\tau = \eta + \alpha$.*

Proof. We have $L_\alpha f \circ T_\alpha = D_\alpha f$, so $(D_\alpha f)' = \alpha\,(L_\alpha f)' \circ T_\alpha$ where $T_\alpha(x) = x(x+\alpha)$. The result follows noticing that $T_\alpha(\tau) = T_\alpha(\eta)$ if and only if $\tau \in \{\eta, \eta+\alpha\}$. □

Lemma 3.8. *Let $m \geqslant 7$ such that $m \equiv 3 \pmod 4$ and set $N = d\binom{(d-1)/2}{2}$. We consider the polynomials $f = \sum_{k=0}^{m} a_{m-k}x^k \in \mathbb{F}_2[a_0,\dots,a_m][x]$ and $L_\alpha f = \sum_{k=0}^{d} b_{d-k}x^k \in \mathbb{F}_2[b_0,\dots,b_d,\alpha][x]$. Then $b_0^N \times \Pi\,(L_\alpha f)$ is a polynomial of $\mathbb{F}_2[a_0,\dots,a_m,\alpha]$ whose each term contains a product of $(d+2)\binom{(d-1)/2}{2}$ terms a_i. This polynomial is also homogeneous of degree $(5d+2)\binom{(d-1)/2}{2}$ when considering the weight w such that $w(\alpha)=1$ and $w(a_i)=i$.*

Proof. We set $\tau_1,\dots,\tau_{(d-1)/2}$ the double roots of the polynomial $(L_\alpha f)'$, and $\Pi\,(L_\alpha f) = \prod_{i\neq j}(L_\alpha f(\tau_i) - L_\alpha f(\tau_j))$. Then we have

$$\Pi\,(L_\alpha f) = \prod_{i<j}\left(\sum_{k=0}^{d} b_{d-k}^2(\tau_i^{2k} + \tau_j^{2k})\right).$$

So $\Pi\,(L_\alpha f)$ is an homogeneous polynomial of degree $2d\binom{(d-1)/2}{2}$ when considering the weight w such that $w(b_i)=i$ for all i and $w(\tau_j)=1$ for all j. We also have that $\Pi\,(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,\tau_1^2,\dots,\tau_{(d-1)/2}^2]$, and each term of $\Pi\,(L_\alpha f)$ contains a product of exactly $\binom{(d-1)/2}{2}$ terms b_i^2. Moreover, using the invariance under the action of $\mathfrak{S}_{(d-1)/2}$ and the theorem of symmetric functions, we obtain that $\Pi\,(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,\sigma_1,\dots,\sigma_{(d-1)/2}]$ where $\sigma_1 = \sum \tau_i^2$, $\sigma_2 = \sum_{i<j}\tau_i^2\tau_j^2$,... Using $(L_\alpha f)' = b_0\prod_{i=1}^{(d-1)/2}\left(x^2+\tau_i^2\right)$ it follows that $\Pi\,(L_\alpha f) \in \mathbb{F}_2[b_0,\dots,b_d,\frac{b_2}{b_0},\frac{b_4}{b_0},\dots,\frac{b_{d-1}}{b_0}]$. The denominator is at worst b_0^N (it happens if the τ_i are the only terms contributing to the degree, and if they only give rise to terms b_2/b_0). We deduce that $b_0^N \times \Pi\,(L_\alpha f)$ is a polynomial in the b_i, and that each term is a product of $(d+2)\binom{(d-1)/2}{2}$ indeterminates b_i. Furthermore, it is an homogeneous polynomial of degree $2d\binom{(d-1)/2}{2}$ when considering the weight w such that $w(b_i)=i$ for all i.

By Lemma 2.4, b_i is an homogeneous polynomial of $\mathbb{F}_2[a_0,\dots,a_m,\alpha]$ of degree $2i+1$ when considering the weight w such that $w(a_i)=i$ and $w(\alpha)=1$. We conclude that $b_0^N \times \Pi\,(L_\alpha f)$ is an homogeneous polynomial of degree $2\times 2d\binom{(d-1)/2}{2} + (d+2)\binom{(d-1)/2}{2}$. □

Finally we reach the goal of this section: Proposition 3.2 and Proposition 3.6 enable us to bound the number of α such that $L_\alpha f$ is Morse.

Theorem 3.9. *Let $m \geqslant 7$ such that $m \equiv 3 \pmod 4$ and such that the polynomial $L_1(x^m)$ has distinct critical values. Then for all $f \in \mathbb{F}_{2^n}[x]$ of degree m the number of elements α in $\mathbb{F}_{2^n}^*$ such that $L_\alpha f$ is Morse is at least $2^n - 1 - \frac{1}{64}(m-3)(5m^2+28m+7)$.*

Proof. Let $f \in \mathbb{F}_{2^n}[x]$ of degree m and let $\alpha \in \mathbb{F}_{2^n}^*$. The polynomial $L_\alpha f$ is Morse if the three conditions (a), (b) and (c) of Definition 3.1 hold. As $m \equiv 3 \pmod 4$ the condition (c) is satisfied. Indeed, $D_\alpha f$ has degree $m-1$ by Proposition 2.1 and thus $L_\alpha f$ has odd degree $(m-1)/2$. Moreover the condition (a) fails for at most $m(m-3)$ values of α by Proposition 3.2. Furthermore the condition (b) fails for at most $(5m-1)(m-3)(m-7)/64$ values of α by Proposition 3.6. Thus $L_\alpha f$ is not Morse for at most $m(m-3)+(5m-1)(m-3)(m-7)/64$ values of α. □

3.4. **Conditions for $L_1(x^m)$ to have distinct critical values.** The condition (b) which is essential for the proofs of our main results leads by Proposition 3.6 to study for which exponents m the polynomial $L_\alpha(x^m)$ has distinct critical values. By the first assertion of Proposition 3.6 we are reduced to consider the polynomial $L_1(x^m)$. Then it is natural to introduce the following set $\mathcal{M}$ and to look for practical characterizations.

Definition 3.10. Let $\mathcal{M}$ be the set of odd integers m such that the polynomial $L_1(x^m)$ has distinct critical values or equivalently such that for any $\alpha \in \overline{\mathbb{F}}_2^*$ the polynomial $L_\alpha(x^m)$ has distinct critical values.

Lemma 3.7 reduces the study of the critical values of $L_\alpha(x^m)$ to the study of equations involving $D_\alpha(x^m)$ and $(D_\alpha(x^m))' = x^{m-1}+(x+\alpha)^{m-1}$ for odd m.

The following proposition enables us to have a characterization of the elements of $\mathcal{M}$ in terms of roots of unity.

Proposition 3.11. *Let $m \geqslant 7$ be an odd integer. Whatever the choice of $\alpha \in \overline{\mathbb{F}}_2^*$, the polynomial $L_\alpha(x^m)$ has distinct critical values if and only if the following condition is satisfied:*

for ζ_1 and ζ_2 in $\overline{\mathbb{F}}_2 \setminus \{1\}$, the equalities $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$ imply $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$.

Proof. We use Lemma 3.7 to relate with the equations of Lemma 3.12. With the expressions of x_i and x_j obtained, we notice that $x_i = x_j + \alpha$ if and only if $\zeta_1\zeta_2 = 1$. □

Lemma 3.12. *Let $m \geqslant 7$ be an odd integer and $\alpha \in \mathbb{F}_q^*$. Two distinct elements x_i and x_j in $\overline{\mathbb{F}}_2$ satisfy*

$$x_i^{m-1} = (x_i+\alpha)^{m-1}, x_j^{m-1} = (x_j+\alpha)^{m-1} \text{ and}$$

$$x_i^m + (x_i+\alpha)^m = x_j^m + (x_j+\alpha)^m \quad (\diamond)$$

if and only if $x_i = \frac{\zeta_1(1+\zeta_2)}{\zeta_1+\zeta_2}\alpha$ and $x_j = \frac{(1+\zeta_2)}{\zeta_1+\zeta_2}\alpha$ where ζ_1 and ζ_2 are two distinct elements in $\overline{\mathbb{F}}_2 \setminus \{1\}$ such that $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$.

Proof. Suppose that x_i and x_j satisfy the first set of conditions above. We notice that they cannot be 0 neither α, so we can set $\zeta_1 = x_i/x_j$ and $\zeta_2 =$

$(x_i+\alpha)/(x_j+\alpha)$. As $x_i \neq x_j$ we have $\zeta_1 \neq \zeta_2$. Replacing $(x_i+\alpha)^{m-1}$ by x_i^{m-1} and $(x_j+\alpha)^{m-1}$ by x_j^{m-1} in $(\diamond)$ we obtain $\zeta_1^{m-1}=1$. Replacing x_i^{m-1} by $(x_i+\alpha)^{m-1}$ and x_j^{m-1} by $(x_j+\alpha)^{m-1}$ in $(\diamond)$ we obtain $\zeta_2^{m-1}=1$. Replacing x_i by $\zeta_1 x_j$ and $x_i+\alpha$ by $\zeta_2(x_j+\alpha)$ in the left hand side of $(\diamond)$, we obtain $(1+\zeta_1)x_j^m=(1+\zeta_2)(x_j+\alpha)^m$, so $(1+\zeta_1)/(1+\zeta_2)=(x_j+\alpha)/x_j$, and $((1+\zeta_1)/(1+\zeta_2))^{m-1}=1$. To obtain the claimed expressions of x_i and x_j, one can replace x_j by $\zeta_1^{-1}x_i$ in the equality $x_i+\alpha=\zeta_2(x_j+\alpha)$. The converse follows from straightforward computations. □

Example 3.13. It is straightforward to see that the integers $m=2^k+1$ for $k \geqslant 1$ belong to $\mathcal{M}$ since 1 is the only root of $x^{2^k}+1$.

Remark 3.14. As a consequence of Proposition 3.11 an odd integer m belongs to $\mathcal{M}$ if and only if $2(m-1)+1$ does. It implies that if an odd integer m belongs to $\mathcal{M}$ then for all $k \geqslant 0$ the integer $2^k(m-1)+1$ does. We also notice that if an integer m (not necessary odd) satisfy the condition of Proposition 3.11 then $2(m-1)+1$ is an element of $\mathcal{M}$.

Example 3.15. As the polynomial x^3-1 has exactly two roots ζ and ζ^{-1} different from the unity, we can deduce that $m=4$ satisfies the condition of Proposition 3.11. Thus according to the above remark, the integers $2^k 3+1$ belong to $\mathcal{M}$ for $k \geqslant 1$.

Example 3.16. Proposition 3.11 also provides us with a method to check if an odd integer m belongs to $\mathcal{M}$. For a fixed odd integer m, write $m-1=t2^s$ with t odd. Hence the $(m-1)$-th roots of unity are exactly the t-th roots of unity in characteristic two. Consider the smallest integer n such that $2^n \equiv 1 \pmod{t}$ and compute the list of the t-th roots of unity distinct from 1 in the field $\mathbb{F}_{2^n}$. Then check for ζ_1 and ζ_2 in this list if $\left(\frac{1+\zeta_1}{1+\zeta_2}\right)^t = 1$ imply $\zeta_1=\zeta_2$ or $\zeta_1=\zeta_2^{-1}$ using an exhaustive method. For example using the open source computer algebra system SAGE we have determined that the only odd integers less than 200 which do not belong to $\mathcal{M}$ are 15, 29, 31, 43, 57, 61, 63, 71, 85, 91, 99, 103, 113, 121, 125, 127, 141, 147, 151, 155, 169, 171, 179, 181, 183, 187 and 197.

We give below some infinite families of good exponents.

Example 3.17. Let us prove that for any $k \geqslant 0$ the integers $m=2^k+2$ satisfy the conditions of Proposition 3.11. First notice that if ζ is a $(m-1)$-th root of unity then $(1+\zeta)^{2^k+1}=\zeta+\zeta^{-1}$. As a consequence, if ζ_1 and ζ_2 are two $(m-1)$-th roots of unity such that $\left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1}=1$ then

$$\zeta_2\left((1+\zeta_1)^{2^k+1}+(1+\zeta_2)^{2^k+1}\right)=\zeta_2^2+(\zeta_1+\zeta_1^{-1})\zeta_2+1.$$

But this is equal to zero, so ζ_2 is equal to ζ_1 or ζ_1^{-1}.

Example 3.18. Applying Remark 3.14 to the previous example we deduce that for any k and s satisfying $k \geqslant s \geqslant 1$ the integer $2^k + 2^s + 1$ belongs to $\mathcal{M}$.

Example 3.19. In the case where $m = 2^k - 1$, with $k \geqslant 4$, we notice that for any choice of ζ_1 a $(2^{k-1}-1)$-th root of unity, we also have $(1+\zeta_1)^{2^{k-1}-1} = 1$. So any choice of a couple (ζ_1, ζ_2) of $(2^{k-1}-1)$-th roots of unity such that $\zeta_1 \neq \zeta_2$ and $\zeta_1\zeta_2 \neq 1$ will satisfy the hypothesis $\zeta_1^{m-1} = \zeta_2^{m-1} = \left(\frac{1+\zeta_1}{1+\zeta_2}\right)^{m-1} = 1$ but will not satisfy the conclusion. In this case $L_\alpha(x^m)$ does not have distinct critical values so $m \notin \mathcal{M}$.

The following result will be our main tool to obtain infinite families of good exponents with convenient congruence. Indeed this result combined with the characterization of the set $\mathcal{M}$ given in Proposition 3.11 will provide us the families of good exponents explicited in Proposition 5.2 (*iii*) and exploited in Corollaries 5.4 and 5.6.

Proposition 3.20. *Let p, ℓ be distinct primes such that $\ell \neq 2, p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ and that, if $\zeta_1, \zeta_2 \neq 1$ are ℓ-th roots of unity in characteristic p such that $(\zeta_1+1)/(\zeta_2+1)$ is also a ℓ-th root of unity, then $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$. Then, for any $k \geq 2$, if $\zeta_1, \zeta_2 \neq 1$ are ℓ^k-th roots of unity in characteristic p such that $(\zeta_1+1)/(\zeta_2+1)$ is also a ℓ^k-th root of unity, then $\zeta_1 = \zeta_2$ or $\zeta_1 = \zeta_2^{-1}$.*

Proof. Induction on k. The case $k = 1$ is the hypothesis.

Assume now that ζ_1 have order exactly $\ell^k, k \geq 2$ and let $\mathbb{F}_q = \mathbb{F}_p(\zeta_1)$. Because we assumed that $p^{\ell-1} \not\equiv 1 \pmod{\ell^2}$, we have that the order of $p \pmod{\ell^k}$ is ℓ times the order of $p \pmod{\ell^{k-1}}$. Let $\mathbb{F}_r = \mathbb{F}_p(\zeta_1^\ell)$. It follows that $[\mathbb{F}_q : \mathbb{F}_p] = \ell[\mathbb{F}_r : \mathbb{F}_p]$. Then $q = r^\ell$ and the minimal polynomial of ζ_1 over $\mathbb{F}_r$ is $x^\ell - \alpha_1$, where $\alpha_1 = \zeta_1^\ell$ has order ℓ^{k-1}. In particular $\mathrm{N}\,\zeta_1 = \alpha_1, \mathrm{Tr}\,\zeta_1 = 0$ and $\mathrm{N}(1+\zeta_1) = 1+\alpha_1$ where N, Tr are respectively the norm and trace $\mathbb{F}_q/\mathbb{F}_r$, and the last equality follows by evaluating $x^\ell - \alpha_1$ at $x = -1$.

Assume first that ζ_2 have order exactly ℓ^k also and that $\zeta_3 = (\zeta_1 + 1)/(\zeta_2 + 1)$ is also a ℓ^k-th root of unity and write $\zeta_i^\ell = \alpha_i, i = 2, 3$ so the α_i are ℓ^{k-1}-th roots of unity. As before, we get that $\mathrm{N}\,\zeta_i = \alpha_i, i = 2, 3$ and that $\mathrm{N}(1 + \zeta_2) = 1 + \alpha_2$. Taking norms, we get $\alpha_3 = (\alpha_1 + 1)/(\alpha_2 + 1)$, so by induction we get that $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_2^{-1}$.

If $\alpha_1 = \alpha_2$, then $\alpha_3 = 1$ and either $\zeta_1 = \zeta_2$ as we wanted or $\zeta_1 = \omega\zeta_2$ with ω of order ℓ. In the latter case we get $(1 + \omega\zeta_2)/(1 + \zeta_2) = \omega^j$ for some $j = 0, 1, \ldots, \ell - 1$. If $j \neq 1$, we can solve the equation for ζ_2 and get $\zeta_2 \in \mathbb{F}_p(\omega)$ which is a contradiction. If $j = 1$ we get $\omega = 1$, also a contradiction.

If $\alpha_1 = \alpha_2^{-1}$, then $\alpha_3 = \alpha_1$ and either $\zeta_1 = \zeta_2^{-1}$ as we wanted or $\zeta_1 = \omega\zeta_2^{-1}$ with ω of order ℓ. In the latter case we get $(1 + \zeta_1)/(1 + \omega\zeta_1^{-1}) = \omega^j\zeta_1$ for

some $j = 0, 1, \ldots, \ell - 1$. This gives, for $j \neq 0$, $\zeta_1 \in \mathbb{F}_p(\omega)$ which is a contradiction. For $j = 0$, this gives $\omega = 1$, also a contradiction.

Finally, assume that ζ_2 have order smaller than ℓ^k, so $\zeta_2 \in \mathbb{F}_r$. We write our equation as $(\zeta_1 + 1) = \zeta_3(\zeta_2 + 1)$. First note that ζ_3 cannot be in $\mathbb{F}_r$, since ζ_1 is not in $\mathbb{F}_r$, so $\operatorname{Tr}\zeta_1 = \operatorname{Tr}\zeta_3 = 0$, so taking trace of our equation gives $1 = 0(\zeta_2 + 1) = 0$, contradiction.

□

Example 3.21. We verified by computer calculation that the hypothesis of this proposition holds when $p = 2$ and $\ell < 200$ except for $\ell = 7, 31, 73, 89, 127$. For example the case $\ell = 3$ follows from Example 3.15. These computations will enable us to exhibit the examples of Corollaries 5.4 and 5.6.

4. Regular extensions

Let n be an integer $\geqslant 1$ and set $q = 2^n$. Let t be an element transcendental over $\mathbb{F}_q$ and K an extension field of $\mathbb{F}_q(t)$. Recall that the extension $K/\mathbb{F}_q(t)$ is said to be regular if it is separable and if $\mathbb{F}_q$ is algebraically closed in K i.e. $\mathbb{F}_q^K = \mathbb{F}_q$ where $\mathbb{F}_q^K$ is the algebraic closure of $\mathbb{F}_q$ in K.

Let $\alpha \in \mathbb{F}_q^*$, let m be an integer and $d = (m-1)/2$ if m is odd and $d = (m-2)/2$ if m is even. Fix $f \in \mathbb{F}_q[x]$ a polynomial of degree m such that the associated polynomial $L_\alpha f$ has degree exactly d. Furthermore, we suppose that d is odd which is equivalent to say that $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$.

4.1. **First floor: monodromy.** We consider the arithmetic monodromy group G of the polynomial $L_\alpha f$. It is the Galois group of the extension $F/\mathbb{F}_q(t)$ where F is the splitting field of the polynomial $L_\alpha f(x) - t$ over the field $\mathbb{F}_q(t)$. Consider also $\overline{G} := \operatorname{Gal}(F/\mathbb{F}_q^F(t))$ the geometric monodromy group of $L_\alpha f$. The groups G and $\overline{G}$ are transitive subgroups of the symmetric group $\mathfrak{S}_d$ and $\overline{G} \lhd G$.

Proposition 4.1. *Let $f \in \mathbb{F}_q[x]$ be a polynomial such that the associated polynomial $L_\alpha f$ is Morse and has (odd) degree d.*

(i) Let u be a root of $L_\alpha f(x) - t$ in F. Then, for each place $\wp$ of F above the place ∞ at infinity of $\mathbb{F}_q(t)$, we have that u has a simple pole at $\wp$.

(ii) The group $\operatorname{Gal}(F/\mathbb{F}_q(t))$ is the full symmetric group $\mathfrak{S}_d$ and the extension $F/\mathbb{F}_q(t)$ is regular.

Proof. If $v_\wp$ is the valuation at the place $\wp$, we have $v_\wp(L_\alpha f(u)) = v_\wp(t)$ and by definition of the ramification index $e(\wp|\infty)$ we have $v_\wp(t) = e(\wp|\infty)\, v_\infty(t) = -e(\wp|\infty)$. Since d is supposed to be odd, it is prime to the characteristic of $\mathbb{F}_q(t)$, and then, by the proof of Theorem 4.4.5 of [9], we have $e(\wp|\infty) = d$. Hence, we obtain $v_\wp(L_\alpha f(u)) = -d$, which implies that $v_\wp(u) = -1$ and thus u has a simple pole at $\wp$.

The analogue of the Hilbert theorem given by Serre in Theorem 4.4.5 of [9] and detailled in even characteristic in the appendix of Geyer in [5] gives that the geometric monodromy group $\mathrm{Gal}(F/\mathbb{F}_q^F(t))$ of $L_\alpha f$ is the symmetric group $\mathfrak{S}_d$. But it is contained in the arithmetic monodromy group $\mathrm{Gal}(F/\mathbb{F}_q(t))$ which is also a subgroup of $\mathfrak{S}_d$. So they are equal and $\mathbb{F}_q^F = \mathbb{F}_q$. □

A consequence of the first part of the previous proposition is that $L_\alpha f(x) - t$ has only simple roots; let us call them $u_0, \ldots, u_{d-1}$.

4.2. **Second floor.** Let x_i such that $x_i^2 + \alpha x_i = u_i$. Hence we have $D_\alpha f(x_i) = t$. Consider $\Omega = \mathbb{F}_q(x_0, \ldots, x_{d-1})$ the compositum of the fields $F(x_i)$ and $\mathbb{F}_q^\Omega F$ the compositum of F and $\mathbb{F}_q^\Omega$. Let $\Gamma = \mathrm{Gal}(\Omega/F)$ and $\overline{\Gamma} = \mathrm{Gal}(\Omega/\mathbb{F}_q^\Omega F)$.

The following statement appears in [10].

Lemma 4.2. *Suppose that $L_\alpha f$ is Morse and has degree d. If $J \subset \{0, \ldots, d-1\}$ is neither empty nor the whole set then $\sum_{j\in J} u_j$ has a pole at a place of F over the place ∞ of $\mathbb{F}_q(t)$.*

Proof. To obtain a contradiction suppose that $J \subset \{0, \ldots, d-1\}$ is such that $j_0 \in J$ whereas $j_1 \in \{0, \ldots, d-1\} \setminus J$. Suppose also that $\sum_{j\in J} u_j$ has no pole in places above ∞. Then it has no pole at all, and so it is constant. Recall that $\mathrm{Gal}\,(F/\mathbb{F}_q(t))$ is $\mathfrak{S}_d$ by Proposition 4.1. Applying to $\sum_{j\in J} u_j$ the automorphism corresponding to the transposition $(j_0 j_1) \in \mathfrak{S}_d$ one obtains $\sum_{j\in J\setminus\{j_0\}} u_j + u_{j_0} = \sum_{j\in J\setminus\{j_0\}} u_j + u_{j_1}$, which leads to $u_{j_0} = u_{j_1}$, a contradiction. □

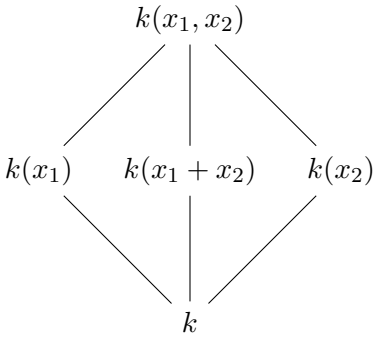
Lemma 4.3. *Suppose that $L_\alpha f$ is Morse and has degree d. Let $\widetilde{F}$ be F or $\mathbb{F}_q^\Omega F$. Let J be a non-empty subset of $\{0, \ldots, d-1\}$ different from $\{0, \ldots, d-1\}$. Then*

$$\sum_{j\in J} x_j \notin \widetilde{F}.$$

Proof. To obtain a contradiction, suppose that $\sum_{j\in J} x_j \in \widetilde{F}$. By Lemma 4.2 we know that there exists a place $\wp$ of F above ∞ such that $\sum_{j\in J} u_j$ has a pole at $\wp$. Moreover, this pole is simple as for all $j \in \{0, \ldots, d-1\}$ the root u_j has a simple pole by Proposition 4.1. Now consider $A = \left(\sum_{j\in J} x_j\right)$ and $B = \left(\sum_{j\in J} x_j + \alpha\right)$. If A (and thus B) belongs to $\widetilde{F}$, one can consider the valuation of A and B at $\wp$. As $A.B = \sum_{j\in J} u_j$ it follows that either A or B has a pole. Since A and B differ from a constant, A has a pole if and only if B has a pole. So both have a pole and the order of multiplicity is the same. Then we obtain $2v_\wp(A) = -1$, a contradiction. □

Lemma 4.4. *Let $k(x_1)$ and $k(x_2)$ be two Artin-Schreier extensions of a field k of characteristic 2. Suppose that $x_i^2 + \alpha x_i = w_i$ with α and w_i in k^*. Then $k(x_1) = k(x_2)$ if and only if $x_1 + x_2 \in k$.*

Moreover if $x_1+x_2 \notin k$ then $k(x_1,x_2)$ is a degree 4 extension of k and the three fields lying between k and $k(x_1,x_2)$ are those of the following diagram.



Proof. For the first assertion, see the proof of Lemma 4.1 in [1]. In the case where $x_1+x_2 \notin k$, we can use $[k(x_1)(x_2):k(x_1)]=2$ to prove $[k(x_1,x_2):k]=4$. We deduce that $\mathrm{Gal}(k(x_1,x_2)/k)=(\mathbb{Z}/2\mathbb{Z})^2$. The field $k(x_1+x_2)$ is a subextension since x_1+x_2 is a root of $x^2+\alpha x = w_1+w_2$. It remains to prove that $k(x_1+x_2)$ is different from $k(x_1)$ (and $k(x_2)$). According to the first statement of the lemma, it is sufficient to check that $x_1+(x_1+x_2)\notin k$. □

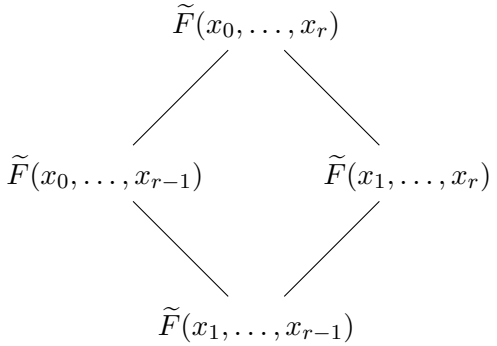
Proposition 4.5. *Suppose that $L_\alpha f$ is Morse and has degree d. Let $\widetilde{F}$ be F or $\mathbb{F}_q^\Omega F$. Let r be an integer such that $0 \leqslant r \leqslant d-2$. Then*

(i) the field $\widetilde{F}(x_0,\dots,x_r)$ is an extension of order 2^{r+1} of $\widetilde{F}$,

(ii) the Galois group $\mathrm{Gal}\left(\widetilde{F}(x_0,\dots,x_r)/\widetilde{F}\right)$ is $(\mathbb{Z}/2\mathbb{Z})^{r+1}$ and

(iii) there are $2^{r+1}-1$ quadratic extensions of $\widetilde{F}$ between $\widetilde{F}$ and $\widetilde{F}(x_0,\dots,x_r)$. Namely, these extensions are the extensions $\widetilde{F}\left(\sum_{j\in J} x_j\right)$ with non-empty $J\subset\{0,\dots,r\}$.

Proof. We proceed by induction. The case $r=0$ is trivial and the case $r=1$ is given by Lemma 4.4. Assuming that the proposition holds for $r-1$, with $1\leqslant r\leqslant d-2$, we will prove it for r. The main idea is to consider the extensions of the following diagram



and to apply Lemma 4.4. We first prove that $x_0 + x_r \notin \widetilde{F}(x_1, \dots, x_{r-1})$. Otherwise we would have the quadratic extension $\widetilde{F}(x_0 + x_r)$ between $\widetilde{F}$ and $\widetilde{F}(x_1, \dots, x_{r-1})$. By the induction hypothesis, there would exist $J \subset \{1, \dots, r-1\}$ such that $\widetilde{F}(x_0+x_r) = \widetilde{F}\left(\sum_{j\in J} x_j\right)$. By Lemma 4.4 again we would have $x_0+x_r+\sum_{j\in J} x_j \in \widetilde{F}$ and then a contradiction with Lemma 4.3. Then we can apply the conclusions of Lemma 4.4 with $k = \widetilde{F}(x_1, \dots, x_{r-1})$ to obtain that $\widetilde{F}(x_0, \dots, x_r)$ is a quadratic extension of both $\widetilde{F}(x_1, \dots, x_r)$ and $\widetilde{F}(x_0, \dots, x_{r-1})$. It follows that $[\widetilde{F}(x_0, \dots, x_r) : \widetilde{F}] = 2^{r+1}$.

Furthermore, we can define 2^{r+1} different $\widetilde{F}$-automorphisms of $\widetilde{F}(x_0, \dots, x_r)$ by sending x_i to x_i or to $x_i + \alpha$. So, all the elements of the Galois group $\mathrm{Gal}\left(\widetilde{F}(x_0, \dots, x_r)/\widetilde{F}\right)$ have order dividing 2 thus this group is certainly $(\mathbb{Z}/2\mathbb{Z})^{r+1}$.

For any non-empty subset $J \subset \{0, \dots, r\}$ we see that $\sum_{j\in J} x_j$ is a root of $x^2 + \alpha x = \sum_{j\in J} u_j$, and we know from Lemma 4.3 that $\sum_{j\in J} x_j \notin \widetilde{F}$. We obtain this way $2^{r+1} - 1$ different quadratic extensions between $\widetilde{F}$ and $\widetilde{F}(x_0, \dots, x_r)$. Indeed, we can show that these extensions are different. If $\widetilde{F}\left(\sum_{j\in J_1} x_j\right) = \widetilde{F}\left(\sum_{j\in J_2} x_j\right)$ then $\sum_{j\in J_1} x_j + \sum_{j\in J_2} x_j \in \widetilde{F}$ which leads to $J_1 = J_2$ using Lemma 4.3. Finally, these $2^{r+1}-1$ quadratic extensions are the only ones. Indeed, the quadratic extensions between $\widetilde{F}$ and $\widetilde{F}(x_0, \dots, x_r)$ are in correspondence with the subgroups of $(\mathbb{Z}/2\mathbb{Z})^{r+1}$ of index 2. These subgroups are the hyperplanes of $(\mathbb{Z}/2\mathbb{Z})^{r+1}$ and one can count $2^{r+1} - 1$ of them. □

Proposition 4.6. *Suppose that $L_\alpha f = \sum_{k=0}^{d} b_{d-k}x^k$ is Morse and has degree d. Let $\widetilde{F}$ be F or $F\mathbb{F}_q^\Omega$. If there exists $x \in \mathbb{F}_q$ such that $x^2 + \alpha x = b_1/b_0$ then $\mathrm{Gal}\left(\widetilde{F}(x_0, \dots, x_{d-1})/\widetilde{F}\right)$ is $(\mathbb{Z}/2\mathbb{Z})^{d-1}$ and thus the extensions Ω/F and $\Omega/\mathbb{F}_q(t)$ are regular.*

Proof. As Proposition 4.5 already gives $\mathrm{Gal}\left(\widetilde{F}(x_0, \dots, x_{d-2})/\widetilde{F}\right) = (\mathbb{Z}/2\mathbb{Z})^{d-1}$, it remains to study the extension $\widetilde{F}(x_0, \dots, x_{d-1})/\widetilde{F}(x_0, \dots, x_{d-2})$.

Using $\sum_{i=0}^{d-1} u_i = b_1/b_0$ and the linearity of $x \mapsto x^2 + \alpha x$, we see that in any case the equation $x^2 + \alpha x = b_1/b_0$ has two solutions in $\overline{\mathbb{F}}_q$, namely $\sum_{i=0}^{d-1} x_i$ and $\alpha+\sum_{i=0}^{d-1} x_i$. With our hypothesis we deduce that $\sum_{i=0}^{d-1} x_i \in \mathbb{F}_q$ hence $\widetilde{F}(x_0, \dots, x_{d-1}) = \widetilde{F}(x_0, \dots, x_{d-2})$ and the result about the Galois group follows. Thus we have proved that $\Gamma = \overline{\Gamma}$ and then Ω/F is regular. Proposition 4.1 shows that the extension $F/\mathbb{F}_q(t)$ is regular, hence we deduce the regularity of the extension $\Omega/\mathbb{F}_q(t)$. □

5. Main results

The main ingredient of the proof of our main results is the Chebotarev density theorem. The next proposition summarizes its contribution in our context.

Proposition 5.1. *Let $m \geqslant 7$ be an integer such that $m \equiv 3 \pmod 4$. Then there exists an integer N depending only on m such that for all $n \geqslant N$, if we set $q = 2^n$, for all $f \in \mathbb{F}_q[x]$ of degree m, and for all α in $\mathbb{F}_q^*$ such that the extension $\Omega/\mathbb{F}_q(t)$ is regular, there exists $\beta \in \mathbb{F}_q$ such that the polynomial $D_\alpha f(x) + \beta$ splits in $\mathbb{F}_q[x]$ with no repeated factors.*

Proof. As $m \equiv 3 \pmod 4$, by Proposition 2.1 the polynomial $L_\alpha f$ has degree exactly $d = (m-1)/2$, which is odd by our hypothesis on m, and thus $F/\mathbb{F}_q(t)$ is separable. Since the extension Ω/F is also separable we obtain that $\Omega/\mathbb{F}_q(t)$ is separable and thus Galois.

Since the extension $\Omega/\mathbb{F}_q(t)$ is supposed to be regular, by an application of the Chebotarev theorem (see Theorem 1 in [3] which is deduced from Proposition 4.6.8 in [8]) the number $N(S)$ of places v of $\mathbb{F}_q(t)$ of degree 1 unramified in Ω and such that the Artin symbol $\left(\frac{\Omega/\mathbb{F}_q(t)}{v}\right)$ is equal to the conjugacy class of $\mathrm{Gal}(\Omega/\mathbb{F}_q(t))$ consisting of the identity element satisfies

$$N(S) \geqslant \frac{q}{d_\Omega} - 2\Big((1+\frac{g_\Omega}{d_\Omega})q^{1/2} + q^{1/4} + 1 + \frac{g_\Omega}{d_\Omega}\Big)$$

where $d_\Omega := [\Omega : \mathbb{F}_q(t)]$ and g_Ω is the genus of Ω.

But we have seen that $G = \mathrm{Gal}(F/\mathbb{F}_q(t))$ is a subgroup of $\mathfrak{S}_d$ and $\Gamma = \mathrm{Gal}(\Omega/F)$ is a group of order bounded by 2^d, thus we have $d_\Omega \leqslant d!2^d$. Moreover, one can obtain an upper bound on g_Ω depending only on d using Lemma 14 of [7] to get that: $g_\Omega \leqslant \frac{1}{2}(\deg D_\alpha f - 3)d_\Omega + 1$ i.e.

$$g_\Omega \leqslant (d!2^d) \times (d - 3/2) + 1.$$

Then if q is sufficiently large we will have $N(S) \geqslant 1$ which concludes the proof. □

Since the methods of our proofs need the degree m of the polynomials to belong to the set $\mathcal{M}$ defined in Definition 3.10, we sum up some infinite subsets of $\mathcal{M}$ we have pointed out in Subsection 3.4.

Proposition 5.2. *The following integers m belong to the set $\mathcal{M}$:*

(i) $m = 2^k + 1$ *for* $k \geqslant 1$.
(ii) $m = 2^k + 2^s + 1$ *for* $k \geqslant s \geqslant 1$.
(iii) $m = 2^s\ell^k + 1$ *for* $k \geqslant 1$, $s \geqslant 1$ *and for* ℓ *an odd prime such that* $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ *and such that* $m' := \ell + 1$ *satisfy the condition of Proposition 3.11.*

Proof. The first two assertions are proved respectively in Example 3.13 and 3.18. If ℓ satisfy the hypothesis *(iii)* then Proposition 3.20 in the case of characteristic two tells us that $\ell^k + 1$ also satisfy the condition of Proposition

3.11. Now use Remark 3.14 to have that $2^s\ell^k+1$ satisfy the condition of Proposition 3.11. For $s \geqslant 1$ it is odd and so it belongs to $\mathcal{M}$. □

Now we can state and prove our main results which establish for some polynomials f the maximality of the differential uniformity $\delta(f)$ defined in Section 1 by $\delta(f) = \max_{(\alpha,\beta)\in\mathbb{F}_q^*\times\mathbb{F}_q} \sharp\{x \in \mathbb{F}_q \mid f(x+\alpha)+f(x)=\beta\}$.

Theorem 5.3. *Let $m \in \mathcal{M}$ such that $m \equiv 7 \pmod 8$. Then for n sufficiently large, for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m-1$.*

Proof. We fix $m \in \mathcal{M}$ such that $m \equiv 7 \pmod 8$. Let us prove that for n sufficiently large and for any polynomial $f = \sum_{i=0}^{m} a_{m-i}x^i$ in $\mathbb{F}_{2^n}[x]$ of degree m, there exists α in $\mathbb{F}_{2^n}^*$ such that:

- $L_\alpha f$ is Morse
- the equation $x^2+\alpha x = \frac{b_1}{b_0}$ has a solution in $\mathbb{F}_{2^n}$, where $L_\alpha f = \sum_{i=0}^{d} b_{d-i}x^i$.

By Theorem 3.9, for all $f \in \mathbb{F}_{2^n}[x]$ of degree m, the number of elements α in $\mathbb{F}_{2^n}^*$ such that $L_\alpha f$ is Morse is at least $2^n - \frac{1}{64}(m-3)(5m^2+28m+7)$.

Moreover, by the Hilbert'90 Theorem, the equation $x^2+\alpha x = \frac{b_1}{b_0}$ has a solution in $\mathbb{F}_{2^n}$ if and only if $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\left(\frac{b_1}{b_0\alpha^2}\right) = 0$. By Lemma 2.5 it is equivalent to $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\left(\frac{a_1^2+a_0a_2}{a_0^2\alpha^2}\right) = 0$. In the case where $a_1^2+a_0a_2 = 0$ every choice of $\alpha \in \mathbb{F}_{2^n}^*$ is convenient. Otherwise the map sending α to $\frac{a_1^2+a_0a_2}{a_0^2\alpha^2}$ is a permutation of $\mathbb{F}_{2^n}^*$ and then $2^{n-1}-1$ values of α are convenient.

Hence as soon as $2^{n-1} > \frac{1}{64}(m-3)(5m^2+28m+7)+1$ we will have for any $f \in \mathbb{F}_{2^n}[x]$ of degree m the existence of α in $\mathbb{F}_{2^n}^*$ satisfying the two conditions. Now, these conditions imply by Proposition 4.6 that the extension $\Omega/\mathbb{F}_{2^n}(t)$ is regular.

Finally we can apply Proposition 5.1 to obtain, for n sufficiently large depending only on m, the existence of $\beta \in \mathbb{F}_{2^n}$ such that the polynomial $D_\alpha f(x)+\beta$ splits in $\mathbb{F}_{2^n}[x]$ with no repeated factors. Then $\delta(f) = m-1$. □

To be concrete, using Proposition 5.2, the computations of Example 3.21 and taking into account the congruences of m we present in the following corollary some families of infinitely many integers for which Theorems 5.3 holds.

Corollary 5.4. *Let ℓ be a prime congruent to 3 modulo 4 such that $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ and $\ell+1$ satisfy the condition of Proposition 3.11 (for example, $\ell \in \{3, 11, 19, 23, 43, 47, 59, 67, 71, 79, 83, 103, 107, 131, 139, 151, 163, 167, 179, 191, 199\ldots\}$). Set $m = 2\ell^{2k+1}+1$ with $k \geqslant 0$. Then for n sufficiently large, for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m-1$.*

When m is congruent to 3 modulo 8, we also obtain some results but we have conditions on the parity of n or we have to remove some polynomials.

Theorem 5.5. *Let $m \in \mathcal{M}$ such that $m \geqslant 7$ and $m \equiv 3 \pmod 8$.*

(i) For n even and sufficiently large and for all polynomials $f \in \mathbb{F}_{2^n}[x]$ of degree m we have $\delta(f) = m - 1$.

(ii) For n sufficiently large and for all polynomials $f = \sum_{i=0}^{m} a_{m-i}x^i$ in $\mathbb{F}_{2^n}[x]$ of degree m such that $a_1^2 + a_0a_2 \neq 0$, we have $\delta(f) = m - 1$.

Proof. The proof is similar as the one of Theorem 5.3. The main difference comes from the expression of b_1/b_0 when $m \equiv 3 \pmod 8$. According to Lemma 2.5, we have $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\left(\frac{b_1}{b_0\alpha^2}\right) = 0$ if and only if $\mathrm{Tr}_{\mathbb{F}_{2^n}/\mathbb{F}_2}\left(\frac{a_1^2+a_0a_2}{a_0^2\alpha^2}\right) = n$. The arguments of the above proof apply except when $a_1^2 + a_0a_2 = 0$ and n is odd. □

We remark that one could not expect better in the case where $m \equiv 3 \pmod 8$, $a_1^2 + a_0a_2 = 0$ and n odd since Theorem 2 (iii) of [10] gives that $\delta(f) < m - 1$ in this case.

Again using Proposition 5.2 and the computations of Example 3.21 we obtain the following corollary.

Corollary 5.6. *Let ℓ be an odd prime such that $2^{\ell-1} \not\equiv 1 \pmod{\ell^2}$ and $\ell + 1$ satisfy the condition of Proposition 3.11.*

(i) If $\ell \equiv 1 \pmod 8$ then Theorem 5.5 holds for the integers $m = 2\ell^k+1$ with $k \geqslant 1$ (for example if $\ell \in \{17, 41, 97, 113, 137, 193, \ldots\}$).

(ii) If $\ell \equiv 7 \pmod 8$ then Theorem 5.5 holds for the integers $m = 2\ell^{2k+1}+1$ with $k \geqslant 0$ (for example if $\ell \in \{23, 47, 71, 79, 103, 151, 167, 191, 199, \ldots\}$).

Finally, we prove Conjecture 1.1 when $m \equiv 7 \pmod 8$.

Theorem 5.7. *For a given integer $m \in \mathcal{M}$ such that $m \equiv 7 \pmod 8$, there exists $\varepsilon_m > 0$ such that for all sufficiently large n, if f is a polynomial of degree m over $\mathbb{F}_{2^n}$, for at least $\varepsilon_m 2^{2n}$ values of $(\alpha, \beta) \in \mathbb{F}_{2^n}^* \times \mathbb{F}_{2^n}$ we have $\sharp\{x \in \mathbb{F}_q \mid f(x + \alpha) + f(x) = \beta\} = \delta(f) = m - 1$.*

Proof. We follow the strategy described in the proofs above. The point is to give lower bounds for the number of choices of α and β. We have shown the existence of a polynomial P of degree 3 such that for any n and any $f \in \mathbb{F}_{2^n}[x]$ there exist at least $2^{n-1} + P(m)$ elements α such that the extension $\Omega/\mathbb{F}_{2^n}(t)$ is regular (see the proof of Theorem 5.3). Thus for any $\gamma_m < 1/2$, for n sufficiently large, there exists $\gamma_m 2^n$ suitable choices of α. For such a choice of α, the Chebotarev theorem used in the proof of Proposition 5.1 guarantees the existence of $\frac{1}{d!2^d}2^n + Q(2^{n/4})$ elements β such that $D_\alpha f(x) + \beta$ has $\delta(f)$ solutions where Q is a polynomial of degree 2. Thus for any $\gamma'_m < 1/d!2^d$, for n sufficiently large, there exist $2^n\gamma'_m$ suitable choices of β. Hence we obtain the result for any $\varepsilon_m < 1/d!2^{d+1}$. □

Remark that the proof of Theorem 5.7 provides explicit values of ε_m, namely any ε_m between 0 and $1/d!2^{d+1}$ with $d = \frac{m-1}{2}$. Remark also that, in the case where $m \equiv 3 \pmod 8$, the same strategy leads to a proof of an analogue of this theorem for polynomials f such that $a_1^2 + a_0 a_2 \neq 0$ or a proof of another analogue for even n.

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