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Multipolar Robust Optimization

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Abstract

We consider linear programs involving uncertain parameters and propose a new tractable robust counterpart which contains and generalizes several other models including the existing *Affinely Adjustable Robust Counterpart* and the *Fully Adjustable Robust Counterpart*. It consists in selecting a set of *poles* whose convex hull contains some projection of the uncertainty set, and computing a recourse strategy for each data scenario as a convex combination of some optimized recourses (one for each pole). We show that the proposed *multipolar robust counterpart* is tractable and its complexity is controllable. Further, we show that under some mild assumptions, two sequences of upper and lower bounds converge to the optimal value of the fully adjustable robust counterpart. To illustrate the approach, a robust problem related to lobbying under some uncertain opinions of authorities is studied. Several numerical experiments are carried out showing the advantages of the proposed robustness framework and evaluating *the benefit of adaptability*.

1 Introduction

Uncertainty in optimization parameters arises in many applications due to the difficulty to measure data or because of their variability. To deal with uncertainty, there are mainly two approaches: stochastic optimization and robust optimization. In the first case, some probabilistic assumptions are made about the uncertain data [18, 26, 37]. One is then interested in computing a solution optimizing some moments of random variables depending on the data. Another variant, known as

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chance constrained programming [22], consists in imposing that some constraints are satisfied only with some probability.

Robust optimization is a more recent approach dealing with uncertainty. It does not require specifications of the exact distribution of the problem's parameters. Roughly speaking, uncertain data are assumed to belong to a known compact set, called uncertainty set, and we aim at finding a solution that is *immunized* against all possible realizations in the uncertainty set. An early contribution related to robust optimization is the work of Soyster [50] followed by intensive investigations in the last 20 years starting with [11, 29] in the context of convex optimization and the book [40] dealing with discrete optimization. Almost at the same time, and in an independent way, a lot of work was initiated in [30] and [28] on robust optimization in communication networks dealing with uncertain traffic matrix, see [7] for a survey.

Robust optimization and stochastic programming are related in numerous ways. For example, using some knowledge about the distribution of uncertain data, it is sometimes possible to define an uncertainty set in such a way that the robust solution is an approximated solution of a chance constrained problem (see, e.g., [9, 12] for details and references). An approach combining robust optimization and stochastic programming consists in computing solutions that are distributionally robust where the distribution of parameters is assumed to vary within some set (for example, when the mean and the covariance matrix are known) (see, e.g., [32, 33]).

The definition of the uncertainty set is a critical issue since a bad choice might lead to very expensive solutions. One way to alleviate overconservatism of the robust approach is to assume that a subset of the decision variables are adjustable on the realization of the uncertain data. Let us for example consider the following linear problem

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \in \mathbb{R}^n, \end{aligned}$$

involving uncertain parameters. We assume that \mathbf{x} is partitioned as $\mathbf{x} = (\mathbf{u}, \mathbf{v})$, where \mathbf{u} represents the non-adjustable and \mathbf{v} the adjustable variables. The robust counterpart of this uncertain problem under consideration reads

$$\begin{aligned} \min_{\mathbf{u}, \mathbf{v}} \quad & \mathbf{c}^T \mathbf{u} \\ \text{s.t.} \quad & \mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v}(\xi) \leq \mathbf{b}, [\mathbf{U}, \mathbf{b}] \in \Xi, \end{aligned} \tag{FARC}$$

where the uncertain parameters are $\mathbf{U} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, while \mathbf{V} and \mathbf{c} are assumed to be known. We denote by $\xi \equiv [\mathbf{U}, \mathbf{b}] \in \Xi$ the uncertain parameters belonging to the uncertainty set Ξ assumed to be compact, convex and with a non-empty interior. ξ will be considered as a vector in the rest of the paper.

The non-adjustable variables are sometimes interpreted “here and now” variables, while the adjustable ones can be seen as “wait and see” variables. This robust counterpart above is generally called *fully-adjustable robust counterpart* (FARC). FARC is sometimes called the dynamic robust counterpart since \mathbf{v} depends on ξ . FARC can be seen as a two-stage optimization problem where \mathbf{u} are the first-stage variables and \mathbf{v} are the second-stage variables.

If variables \mathbf{v} are also static, then FARC simply becomes the standard *static robust counterpart* denoted by SRC. Cases where FARC and SRC are equivalent have been pointed out in [10] where it is shown that adaptability does not lead to any improvement in the constraint-wise uncertainty case. Still, FARC is generally much less conservative than SRC. In other words, there is generally some *benefit of adaptability*. Solving FARC is, unfortunately difficult in general cases as shown by many authors [10, 23, 45]. Another concern related to FARC is the inherent difficulty of implementing the solution $\mathbf{v}(\xi)$ in a practical way.

To get a tractable optimization problem and also to alleviate some of the over-conservatism of SRC, an affinely adjustable approach was proposed in [10], where the adjustable variables \mathbf{v} are not fully adaptable (dynamic), but are assumed to depend on the uncertain data ξ in an affine way:

$$\mathbf{v}(\xi) = \mathbf{w} + \mathbf{W}\xi, \quad \xi \equiv [\mathbf{U}, \mathbf{b}] \in \Xi,$$

where \mathbf{w} and the elements of matrix \mathbf{W} are new decision variables (a.k.a. affine decision rules). The induced formulation is called *affine-adjustable robust counterpart* (AARC). An affine approach was also independently proposed for network optimization problems where the traffic matrix is supposed to be uncertain and the way how traffic is splitted through network’s paths is optimized [5, 6, 48]. Further developments appeared in [2, 47].

Applying affine decision rules naturally leads to less expensive solutions than those obtained by the static approach. The performance gap quantified by the difference between optimum of AARC and the optimum of FARC was discussed for robust linear problems with right-hand-side uncertainty in [14, 16]. One of the results of [14] states that AARC is equivalent to FARC when the uncertainty set is a simplex. Some tight approximation bounds relating the optimum of AARC to that of FARC in the right-hand-side uncertainty case are also given there.

Related investigation on problems with some special uncertainty sets (integer sublattices of the unit hypercube) are discussed in [36], where they provide sufficient conditions such that the associated affinely adjustable decision rules lead to exact optimum of FARC.

The affine approach is related to the well-known linear or first-order decision rules used in the context of multi-stage stochastic optimization [31]. Linear decisions rules were also used in [41] in the context of stochastic programming not only to get upper bounds (as done above) but also to get lower bounds by properly approximating the dual problem using linear decision rule.

As observed in [25], even though AARC has been successfully applied to several problems, its performance might be unsatisfactory under situations where the

adjustable variables exhibit high nonlinearity in terms of the uncertain parameters. This led to some extensions of the affine approach in [24, 25] (see also references therein) by reparametrizing the uncertainties and then applying the affinely adjustable approach. Roughly speaking, a new set of variables is introduced (for example the positive and the negative parts of the original uncertainties), and the adjustable variables are assumed to affinely depend on the new set of parameters. A similar idea is also proposed in [16] in the context of one-dimensional constrained multistage robust optimization.

Other extensions of affine decision rules have been proposed in literature. In [17] polynomial recourse actions are considered where \mathbf{v} is expressed as a polynomial in uncertainty parameters with degree no larger than a fixed constant. The complexity of the robust counterpart problem is then related to testing the positivity of a polynomial. Using some recent results in algebraic geometry stating that under mild conditions, a positive polynomial can be expressed as a sum of squares (not a priori bounded), the robust counterpart is approximated by considering sums of squares of degree no larger than a fixed constant. As a sum of squares can be represented by a semidefinite programming [43], the proposed robust counterpart can be efficiently handled [17].

Another robust approach dealing with uncertainty, termed as *multi-static* approach in [7], was proposed and studied in [4, 8, 52]. It consists in partitioning the uncertainty set Ξ into a finite number of subsets Ξ_1, \dots, Ξ_p and using a recourse action \mathbf{v}_i for each subset Ξ_i . In other words, if $\xi \in \Xi_i$, then we take $\mathbf{v} = \mathbf{v}_i$. The recourse actions \mathbf{v}_i are of course subject to optimization. A quite close idea is proposed in [13], where it was called *finite adaptability*. The performance of finite adaptability in a fairly general class of multi-stage stochastic and adaptive optimization problems was investigated in [15].

One can also combine finite adaptability and the affinely-adjustable approach by partitioning the uncertainty set into some subsets and considering some optimized specific affine decision rules for each subset. This was also considered in [4, 7] in the context of network design problems. This type of adaptability might also be called *piecewise-affine adaptability*. Piecewise-affine rules were also considered in several other papers such as [3, 33].

While a great number of proposals in robust optimization have appeared, there are still challenges. First, to the best of our knowledges, none are general enough to encompass static robustness, affinely adjustable robustness and fully adjustable robustness. Second, as observed in [17], there is no systematic way to influence the trade-off between the performance of the resulting policies and the computational complexity required to obtain them. Third, the uncertain parameters of an optimization problem can be sometimes difficult to observe. In several applications, only a subset of such parameters or some aggregates of them can be observed.

The objective of this paper is to provide a framework addressing those challenges at the same time. Our contributions are four-fold:

1. A novel approach. We propose a hierarchical and convergent framework of

adjustable robust optimization – *multipolar robust approach*, which generalizes notions of static robustness, affinely adjustable robustness, fully adjustable robustness and fill the gaps in-between. As a byproduct, a new way to look at the affine adaptability is proposed. The result of [14] stating that affine rules are optimal when the uncertainty set is a simplex is also obtained as a consequence of the multipolar approach.

2. A comprehensive analysis. We show that the *multipolar robust counterpart* is tractable by either a cut generation procedure or a compact formulation. Further, we prove that the multipolar approach can generate a sequence of upper bounds and a sequence of lower bounds at the same time and both sequences converge to the robust value of FARC under some mild assumptions.
3. A general constructive algorithm of *pole-sets*. The multipolar approach is based on some tools related to the uncertainty set, that we term as pole-sets. For their construction, we start with a simplex and then compute the best homothetic transformation of this simplex to allow it to enclose a given convex set. An efficient algorithm is proposed to compute such homothetic set. As a byproduct, we provide a very simple proof of the geometric results of [46] related to hypercubes. The pole-sets obtained after this homothetic transformation are then improved using a tightening procedure.
4. An application. To numerically illustrate the multipolar approach, a lobbying problem is considered where a lobby aims to minimize the budget needed to convince a set of voters taking into account a reasonable opinion dynamics model under some uncertainty. The benefit of adaptability is clearly shown for this problem.

Outline. In Section 2, we present the concept and ingredients of multipolar robust optimization and show that static robustness, affinely adjustable robustness, fully adjustable robustness are special cases of multipolar robust framework. In Section 3, we discuss the tractability, the monotonicity and the convergence of the proposed approach. A simple illustrative example is described in Section 4. In Section 5, we propose algorithms for pole-set generation. Section 6 is dedicated to a numerical example on a lobbying problem under several uncertainty scenarios. Finally, concluding remarks follow in Section 7.

Notation. Throughout this paper, we use Ξ to represent a compact convex uncertainty set and ξ to denote a member of Ξ . We use \mathbf{I} to denote the identity matrix. Vectors and matrices are marked in bold, and their scalar components are presented in italic. Given any matrix \mathbf{C} , \mathbf{C}^T denotes its transpose. We also use $[\mathbf{C}, \mathbf{D}]$ to denote the matrix where \mathbf{C} and \mathbf{D} are concatenated by columns assuming they have the same number of rows. Similarly, (\mathbf{C}, \mathbf{D}) denotes the matrix obtained by row concatenation of two matrices \mathbf{C} and \mathbf{D} when they have the same number of columns. Observe that $\mathbf{v} = (v_1, \dots, v_n)$ is then a vector and $[v_1, \dots, v_n] = \mathbf{v}^T$.

We use δ_{ij} to represent the Kronecker's delta function, where $\delta_{ij} = 1$ if $i = j$, 0 otherwise. For a set $S \in \mathbb{R}^n$, we use $\text{ext}(S)$ to represent the set of its extreme points, $\text{conv } S$ to represent its convex hull and $\dim(S)$ to denote its dimension. If S is finite, we use $|S|$ to represent its cardinality. We also use the standard notation for usual norms: $\|\cdot\|_\infty$ for the infinity norm, $\|\cdot\|_1$ for the Manhattan norm and $\|\cdot\|_2$ for the Euclidean norm.

2 The multipolar robust optimization concept

In this section, we introduce the main ingredients of multipolar robustness and then setup the multipolar robust counterpart as a novel approximation of FARC.

Shadow matrix. Like other robust approaches, multipolar approach is also based on an uncertainty set Ξ . In addition, we consider a matrix associated with certain operations on the uncertain information, which can be data aggregation, filtering, and selection. Note that these operations can either be natural or artificial. Natural operations are induced by the difficulty of measurements or shortage of data. For example, in communication networks, traffic flows are usually observed in an aggregated manner (the consequence of aggregating uncertain demands from multiple origin-destination pairs). Nevertheless, adjustable recourse actions should be implemented based the observed partial information. On the other hand, artificial operations can be certain techniques to control the complexity of the multipolar robust counterpart, as explained in the concluding remarks of this section. We call the associated matrix of an operation *shadow matrix* since the operation either reduces the size of the multipolar robust counterpart or is a direct consequence of observations. We use $\mathbf{P} \in \mathbb{R}^{n_0 \times \dim(\Xi)}$ to denote a shadow matrix, where n_0 is the dimension of the shadow (i.e., the resulting partial information) and $\dim(\Xi)$ is the dimension of the uncertainty set Ξ . The resulting partial information is defined by

$$\Xi_P := \mathbf{P}\Xi \equiv \{\mathbf{P}\xi, \xi \in \Xi\}. \quad (1)$$

When \mathbf{P} is identity matrix, we have a complete measure of uncertainty.

We will assume that \mathbf{P} is full row rank matrix. Consequently, Ξ_P is also compact, convex and has a non-empty interior.

Pole-set. A key component of the multipolar approach is a finite set of *poles*, which are given vectors in the range space of the shadow matrix. We denote by Ω such a *pole-set*. We say that Ω is a pole-set of Ξ_P iff for any $\xi \in \Xi$, $\mathbf{P}\xi$ belongs to the convex hull of Ω (a convex combination of poles) denoted by $\text{conv } \Omega$. Given a set Ξ_P , a collection of pole-sets of Ξ_P is defined as

$$\mathcal{F}_{\Xi_P} := \{\Omega : \Xi_P \subseteq \text{conv } \Omega\}. \quad (2)$$

Obviously, extreme points of Ξ_P form a pole-set, i.e., $\text{ext}(\Xi_P) \in \mathcal{F}_{\Xi_P}$.

Multipolar robust counterpart. We now setup the multipolar robust counterpart w.r.t. an uncertainty set Ξ , a shadow matrix \mathbf{P} , a pole-set $\Omega \in \mathcal{F}_{\Xi_P}$. For each

$\xi \in \Xi$, we consider a weight λ_ω^ξ for each pole ω in Ω . Then, for each scenario $\xi \in \Xi$, the following system has a solution

$$\begin{aligned} \sum_{\omega \in \Omega} \lambda_\omega^\xi \omega &= \mathbf{P}\xi, \\ \sum_{\omega \in \Omega} \lambda_\omega^\xi &= 1, \\ \lambda_\omega^\xi &\geq 0, \quad \omega \in \Omega. \end{aligned} \tag{3}$$

Let Λ_ξ be the set of weight vectors λ^ξ satisfying the above system for a given $\xi \in \Xi$. In the considered paradigm, each pole is associated with a recourse action, and the recourse action in the presence of $\xi \in \Xi$ is approximated by a convex combination of the recourse actions associated with the poles. Specifically, let vector \mathbf{v}_ω be the recourse action associated with pole ω in the above system. We require the adjustable variables $\mathbf{v}(\xi)$ to be restricted to

$$\mathbf{v}(\xi) = \sum_{\omega \in \Omega} \lambda_\omega^\xi \mathbf{v}_\omega, \tag{4}$$

where $\lambda^\xi \in \Lambda_\xi$. We can readily present the *multipolar robust counterpart* defined by

$$\begin{aligned} \Pi_\Xi(\mathbf{P}, \Omega) &= \min_{\mathbf{u}, \mathbf{v}} \mathbf{c}^T \mathbf{u} \\ \text{s.t. } \mathbf{U}\mathbf{u} + \mathbf{V} \sum_{\omega \in \Omega} \lambda_\omega^\xi \mathbf{v}_\omega &\leq \mathbf{b}, \quad \xi \in \Xi, \lambda^\xi \in \Lambda_\xi. \end{aligned} \tag{5}$$

Following the spirits of robust optimization, the multipolar robust counterpart (MRC) seeks a pair of non-adjustable solution \mathbf{u} and a set of recourse actions related to poles $\mathbf{v}_\omega, \omega \in \Omega$ such that the objective function is minimized while hedging against the uncertainty set Ξ . In brief, given Ξ , the multipolar robust approach can be seen as a set function of a pole-set Ω and a shadow matrix \mathbf{P} . We denote the function by

$$\Pi_\Xi : \mathbb{R}^{n_0 \times \dim(\Xi)} \times \mathcal{F}_{\Xi_P} \ni (\mathbf{P}, \Omega) \mapsto \Pi_\Xi(\mathbf{P}, \Omega) \in \mathbb{R}$$

and call $\Pi_\Xi(\mathbf{P}, \Omega)$ *multipolar robust value* w.r.t. (\mathbf{P}, Ω) . Also, we call (\mathbf{u}, \mathbf{v}) *multipolar solution*.

To conclude this section, we add few remarks on the concept of shadow matrix and pole-set to clarify the motivation behind these ingredients.

- Note that by (5), the solution is protected against the considered uncertainty Ξ . Neither a shadow matrix \mathbf{P} nor a pole-set Ω changes the uncertainty set, so \mathbf{P} and Ω are not used to approximate the uncertainty set.
- Observe that $1 + \dim(\Xi_P) \leq |\Omega|$, so we can use the shadow matrix \mathbf{P} to reduce the number of recourse actions and therefore the number of variables of (5). Reducing the number of poles leads to an MRC which is easier to solve as will be shown in Section 3.

- In several applications, after data is revealed, the adjustable variables should be quickly chosen and used. This is fortunately easy to do in the multipolar robust framework since the only thing to do is to find the coefficients λ_ω^ξ and use them to combine the already computed recourse vectors $\mathbf{v}_\omega, \omega \in \Omega$.

2.1 Special cases

We show in this section that MRC generalizes SRC, AARC, and FARC by different settings of pole-sets and recourse actions associated with poles.

First, we show that SRC is a special case of MRC. Imposing $\mathbf{v}_\omega = \mathbf{v}_{\omega'}$ for any pair of ω and ω' belonging to Ω leads to $\mathbf{v}(\xi) = \mathbf{v}_\omega, \forall \xi \in \Xi$, which means that the recourse action is static. Another way to get SRC is to impose that \mathbf{P} is a null matrix having one row (relaxing in this case the full row rank constraint related to P) and Ω contains just the zero vector.

Second, we show that FARC is a special case of MRC. Let Ω be the set of extreme points of Ξ and $\mathbf{P} = \mathbf{I}$. Then $\text{conv } \Omega = \Xi$, that is for $\xi \in \Xi$, there exists $\lambda^\xi \geq \mathbf{0}$ such that $\sum_{\omega \in \Omega} \lambda_\omega^\xi = 1$ and $\sum_{\omega \in \Omega} \lambda_\omega^\xi \omega = \xi$. By linearity of inequalities (5), imposing that $\mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v}_\omega \leq \mathbf{b}$ for each extreme point $\omega \in \Omega$ is necessary and sufficient to ensure the satisfaction of all inequalities (5) for each $\xi \in \Xi$. We get here the fully adjustable case representing the best that we can obtain for this problem since it is equivalent to assuming that \mathbf{v} can vary with no restrictions. Note that if the number of extreme points of Ξ is limited, then the robust optimization counterpart can be efficiently solved. However, if the number of extreme points of Ξ is non-polynomial, the problem is generally difficult (as already mentioned in Section 1, see for example [10, 23, 45]).

Third, we show that AARC can also be generalized by MRC by proving the following theorem.

Theorem 1. *Let $\Omega \in \mathcal{F}_{\Xi, p}$ such that $|\Omega| = 1 + \dim(\mathbf{P}\Xi)$. Then the optimal solution of the corresponding MRC problem is exactly the best solution that is affine in $\mathbf{P}\xi$.*

Proof. Since $\mathbf{P}\Xi$ has non-empty interior, $\mathbf{P}\Xi \subset \text{conv } \Omega$ and $|\Omega| = 1 + \dim(\mathbf{P}\Xi)$, the elements of Ω are affinely independent. Let $d = \dim(\mathbf{P}\Xi)$ and assume $\Omega = \{\omega^{(1)}, \dots, \omega^{(d+1)}\}$. The shadow matrix \mathbf{P} is here the identity matrix. Consider matrix D obtained by taking vectors $\omega^{(i)}$ as columns and adding a final line containing only coefficients equal to 1.

$$\mathbf{D} = \begin{pmatrix} \omega_1^{(1)} & \dots & \omega_1^{(d+1)} \\ \vdots & \vdots & \vdots \\ \omega_d^{(1)} & \dots & \omega_d^{(d+1)} \\ 1 & \dots & 1 \end{pmatrix}.$$

Observe that \mathbf{D} is a non-singular square matrix of size $(d+1)$.

Given any ξ , there are unique coefficients λ_ω^ξ such that $\mathbf{P}\xi = \sum_{\omega \in \Omega} \lambda_\omega^\xi \omega$ and $\sum_{\omega \in \Omega} \lambda_\omega^\xi = 1$. This can be written as $(\mathbf{P}\xi, 1) = \mathbf{D}\boldsymbol{\lambda}^\xi$ where $\boldsymbol{\lambda}^\xi$ is the vector whose components are the λ_ω^ξ for $\omega \in \Omega$. This immediately implies that $\boldsymbol{\lambda}^\xi = \mathbf{D}^{-1}(\mathbf{P}\xi, 1)$. Using \mathbf{E} to denote the matrix whose columns are the recourse vectors \mathbf{v}_ω , equation (4) becomes $\mathbf{v} = \mathbf{E}\mathbf{D}^{-1}(\mathbf{P}\xi, 1)$. This clearly implies that \mathbf{v} affinely depends on $\mathbf{P}\xi$.

Let us now consider any affine policy $\mathbf{w} + \mathbf{W}\mathbf{P}\xi$. As shown above, the recourse vector \mathbf{v} provided by the multipolar approach is given by $\mathbf{E}\mathbf{D}^{-1}(\mathbf{P}\xi, 1)$. By taking $\mathbf{E} = [\mathbf{W}, \mathbf{w}]\mathbf{D}$, we get $\mathbf{v} = \mathbf{w} + \mathbf{W}\mathbf{P}\xi$. In other words, any recourse policy that is affine in $\mathbf{P}\xi$ can be obtained through the multipolar approach. \square

When $\mathbf{P} = \mathbf{I}$, we get the desired result below.

Corollary 1. *The affinely adjustable approach is a special case of the multipolar approach. It corresponds to any set of $(\dim \Xi + 1)$ affinely independent poles, in multipolar robust optimization when $\mathbf{P} = \mathbf{I}$.*

The following corollary is also immediate.

Corollary 2. *If the uncertainty set Ξ is a simplex, then the affinely adjustable robust counterpart is equivalent to the fully adjustable robust counterpart in the sense that their objective values are equal.*

Proof. Taking all the vertices of the simplex uncertainty set as the set of poles in multipolar robust approach leads to the optimum of FARC. By Corollary 1, this pole-set corresponds to affine adjustable approach, which completes the proof. \square

Corollary 2 has been presented in [14] in the special case of right-hand-side uncertainty, so we may treat the result here as an alternative proof using the framework of multipolar approach.

3 Analysis

In this section, we first analyze the tractability of the multipolar robust counterpart MRC. Then, we show that the proposed framework can generate a monotonic sequence converging to the fully adjustable robust value of FARC. In fact, we will simultaneously generate a lower and an upper bound both converging to the optimal value of FARC under some mild assumptions.

3.1 Tractability

In this section, we show that MRC is computationally tractable. It can be solved either by cut generation or using a compact reformulation.

First, a cutting plane algorithm for solving MRC may be devised as follows. Assume that $|\Omega|$ is finite and has a reasonable size. Given a solution (\mathbf{u}, \mathbf{v}) , we

have to check if there exists a pair of $\xi \in \Xi$ and $\lambda^\xi \in \Lambda_\xi$ violating the constraints of MRC. This can be done by checking the sign of the optimum of each i^{th} problem

$$\max_{\lambda, \xi} \quad \mathbf{U}_i \mathbf{u} + \mathbf{V}_i \sum_{\omega \in \Omega} \lambda_\omega^\xi \mathbf{v}_\omega - b_i \quad (6a)$$

$$\text{s.t.} \quad \sum_{\omega \in \Omega} \lambda_\omega^\xi \omega = \mathbf{P} \xi, \quad (6b)$$

$$\sum_{\omega \in \Omega} \lambda_\omega^\xi = 1, \quad (6c)$$

$$\lambda_\omega^\xi \geq 0, \quad \omega \in \Omega, \quad (6d)$$

$$\xi \in \Xi, \quad (6e)$$

where \mathbf{U}_i and \mathbf{V}_i are the i^{th} rows of \mathbf{U} and \mathbf{V} . If it is positive, then constraint

$$\mathbf{U}_i \mathbf{u} + \mathbf{V}_i \sum_{\omega \in \Omega} \hat{\lambda}_\omega^\xi \mathbf{v}_\omega \leq b_i, \quad (7)$$

needs to be added to the restricted problem, where $(\hat{\lambda}^\xi, \hat{\xi})$ solves (6). Problem (6) can generally be solved easily when Ξ is polyhedral or ellipsoidal. In these cases, by equivalence of separation and optimization [35], the multipolar robust optimization counterpart problem can also be solved in polynomial time if the number of poles $|\Omega|$ is polynomially bounded.

Second, we may solve MRC by duality. It is sometimes possible for several kinds of convex uncertainty sets to write a strong dual of (6) leading to an extended reformulation of MRC. This holds for example if Ξ is a polytope defined by a limited number of constraints, i.e., $\Xi := \{\xi \equiv [\mathbf{U}, \mathbf{b}] : \mathbf{C} \xi \leq \mathbf{d}\}$, where $\mathbf{C} = [\mathbf{C}_1, \dots, \mathbf{C}_m]$, $\mathbf{C}_i \in \mathbb{R}^{n_d \times (n+1)}$, $d \in \mathbb{R}^{n_d}$ and ξ is expressed as a column vector of size $(n+1) \times m$. ξ contains m blocks of size $n+1$ vectors: the i^{th} block contains \mathbf{U}_i^T followed by b_i . By strong duality, the constraints of the multipolar robust counterpart MRC w.r.t. Ξ can be replaced with a polynomial number of inequalities. For each i , the inequalities $\mathbf{U}_i \mathbf{u} + \mathbf{V}_i \sum_{\omega \in \Omega} \lambda_\omega^\xi \mathbf{v}_\omega \leq b_i, \xi \in \Xi, \lambda^\xi \in \Lambda_\xi$ are replaced with

$$\begin{aligned} \mathbf{d}^T \boldsymbol{\eta}_i + \mathbf{V}_i \mathbf{v}_\omega - \omega^T \boldsymbol{\sigma}_i &\leq 0, \quad \omega \in \Omega, \\ \mathbf{C}_j^T \boldsymbol{\eta}_i - \mathbf{P}_j^T \boldsymbol{\sigma}_i &= \delta_{ij} (\mathbf{u}, -1), \quad j = 1, \dots, m, \\ \boldsymbol{\eta}_i &\in \mathbb{R}_+^{n_d}, \quad \boldsymbol{\sigma}_i \in \mathbb{R}^{n_0}, \end{aligned} \quad (8)$$

where the shadow matrix $\mathbf{P} = [\mathbf{P}_1, \dots, \mathbf{P}_j, \dots, \mathbf{P}_m]$, $\mathbf{P}_j \in \mathbb{R}^{n_0 \times (n+1)}$, $j = 1, \dots, m$, δ_{ij} is Kronecker's delta function.

When Ξ is ellipsoidal, i.e., $\Xi := \{\xi : \|\mathbf{F} \xi\|_2 \leq 1\}$, the multipolar robust counterpart can be represented by a second order cone program. Then for each i , the i^{th}

constraint of MRC is replaced with

$$\begin{aligned} \|\boldsymbol{\eta}_i\|_2 + \mathbf{V}_i \mathbf{v}_\omega - \omega^T \boldsymbol{\sigma}_i &\leq 0, \\ \mathbf{F}^T \boldsymbol{\eta}_i - \mathbf{L}_i &= \mathbf{0}, \\ \boldsymbol{\eta}_i \in \mathbb{R}^{n_q}, \boldsymbol{\sigma}_i \in \mathbb{R}^{n_0}. \end{aligned} \tag{9}$$

where $\mathbf{L}_i = (\mathbf{L}_{i1}, \dots, \mathbf{L}_{im})$, $\mathbf{L}_{ij} = \delta_{ij}(\mathbf{u}, -1) + \mathbf{P}_j^T \boldsymbol{\sigma}_i$, $j = 1, \dots, m$, n_q is number of rows of matrix \mathbf{F} . For sake of completeness, a proof of (9) is provided in Appendix.

3.2 Monotonicity

We show in this section that the function $\Pi_{\Xi}(\mathbf{P}, \cdot)$ is monotonic w.r.t. a partial order defined on \mathcal{F}_{Ξ_p} when the shadow matrix \mathbf{P} is fixed.

Given an uncertainty set Ξ , we now define a partial order over the collection of its pole-sets \mathcal{F}_{Ξ_p} denoted by $\preceq_{\mathcal{F}_{\Xi_p}}$. We set members of \mathcal{F}_{Ξ_p} ordered by the inclusion of their convex hulls, i.e., for any $\Omega', \Omega \in \mathcal{F}_{\Xi_p}$,

$$\Omega' \preceq_{\mathcal{F}_{\Xi_p}} \Omega \iff \text{conv } \Omega' \subseteq \text{conv } \Omega. \tag{10}$$

The next theorem emphasizes the fact that the function $\Pi_{\Xi}(\mathbf{P}, \cdot)$ is monotonic regarding the partial order $\preceq_{\mathcal{F}_{\Xi_p}}$ for each fixed $\mathbf{P} \in \mathbb{R}^{n_0 \times \dim(\Xi)}$. In other words, the multipolar value gets smaller when Ω is smaller w.r.t. $\preceq_{\mathcal{F}_{\Xi_p}}$.

Theorem 2. *Given $\mathbf{P} \in \mathbb{R}^{n_0 \times \dim(\Xi)}$, for any $\Omega', \Omega \in \mathcal{F}_{\Xi_p}$, if $\Omega' \preceq_{\mathcal{F}_{\Xi_p}} \Omega$, then we have $\Pi_{\Xi}(\mathbf{P}, \Omega') \leq \Pi_{\Xi}(\mathbf{P}, \Omega)$.*

Proof. If $(\mathbf{u}, (\mathbf{v}_\omega)_{\omega \in \Omega})$ is an optimal solution of MRC, then a feasible solution, when the set of poles is defined by Ω' , is given as follows. Each $\omega' \in \Omega'$ writes as a convex combination of the poles of Ω : $\omega' = \sum_{\omega \in \Omega} \lambda_{\omega'}^{\omega'} \omega$. Let $\mathbf{v}_{\omega'} = \sum_{\omega \in \Omega} \lambda_{\omega'}^{\omega'} \mathbf{v}_\omega$. The solution given by $(\mathbf{u}, (\mathbf{v}_{\omega'})_{\omega' \in \Omega'})$ is clearly feasible for MRC w.r.t. the set of poles defined by Ω' , which completes the proof. \square

Theorem 2 not only implies that the smaller the Ω w.r.t. $\preceq_{\mathcal{F}_{\Xi_p}}$, the lower the multipolar robust value, but also implies that for a given \mathbf{P} , $\Pi_{\Xi}(\mathbf{P}, \Omega)$ is minimum if $\Omega = \text{ext}(\Xi_p)$.

Given two pole-sets $\Omega, \Omega' \in \mathcal{F}_{\Xi_p}$, Theorem 2 also indicates that: first, for a fixed shadow matrix \mathbf{P} , if $|\Omega| > |\Omega'|$, then $\Pi_{\Xi}(\mathbf{P}, \Omega)$ is not necessarily less than $\Pi_{\Xi}(\mathbf{P}, \Omega')$; second, the function $\Pi_{\Xi}(\mathbf{P}, \cdot)$ is not strictly monotonically increasing. For example, let $S, S' \in \mathcal{F}_{\Xi_p}$, $S' \preceq_{\mathcal{F}_{\Xi_p}} S$ and their convex hulls are simplices. By Theorem 1, $\Pi_{\Xi}(\mathbf{P}, S') = \Pi_{\Xi}(\mathbf{P}, S)$ while by Theorem 2, $\Pi_{\Xi}(\mathbf{P}, S') \leq \Pi_{\Xi}(\mathbf{P}, S)$, which illustrates the second point. Now take any pole-set Ω whose cardinality is strictly greater than $1 + \dim(\Xi_p)$, such that $S' \preceq_{\mathcal{F}_{\Xi_p}} \Omega \preceq_{\mathcal{F}_{\Xi_p}} S$, then $\Pi_{\Xi}(\mathbf{P}, S') = \Pi_{\Xi}(\mathbf{P}, \Omega) = \Pi_{\Xi}(\mathbf{P}, S)$, which illustrates the first point.

Observe also that when $\mathbf{P} = \mathbf{I}$, any pole-set whose convex hull contains Ξ is contained in a simplex. This immediately implies that the optimal value of AARC represents the worst that can be obtained by the multipolar approach.

3.3 Convergence

The aim of this section is to show that under some mild assumptions, using the multipolar framework, one can simultaneously compute a sequence of upper bounds and a sequence of lower bounds converging to $\Pi_{\Xi}(\mathbf{I}, \text{ext}(\Xi))$, the optimal robust value of FARC. Throughout this section the shadow matrix is the identity matrix.

Definition 1. Let $\Omega \in \mathcal{F}_{\Xi}$ be a pole-set of a non-empty set Ξ , the distance function between them is defined as $d(\Omega, \Xi) = \max_{\omega \in \Omega} \min_{\xi \in \Xi} \|\omega - \xi\|_2$.

The distance function is well-defined since Ω and Ξ are closed and bounded. It characterizes the furthest distance between pole-set Ω and the uncertainty set Ξ . This distance is nothing other than the well-known Hausdorff distance.

Let $\Omega \in \mathcal{F}_{\Xi}$ such that $d(\Omega, \Xi) = \varepsilon$. For each $\omega \in \Omega$, we have $d(\Omega, \Xi) \leq \varepsilon$. Let \mathbf{z}_{ω} be the projection of ω on Ξ , i.e.,

$$\mathbf{z}_{\omega} = \arg \min_{\mathbf{x} \in \Xi} d(\omega, \mathbf{x}), \quad \mathbf{e}_{\omega} = \omega - \mathbf{z}_{\omega}, \quad (11)$$

where $\|\mathbf{e}_{\omega}\|_2 \leq \varepsilon$. For each $\xi \in \Xi$, consider convex combination coefficients (β_{ω}^{ξ}) such that

$$\xi = \sum_{\omega \in \Omega} \beta_{\omega}^{\xi} \omega, \quad \text{and let} \quad \mathbf{E} = \sum_{\omega \in \Omega} \beta_{\omega}^{\xi} \mathbf{e}_{\omega}. \quad (12)$$

Let us add subscripts to avoid confusion: $\xi \equiv [\mathbf{U}_{\xi}, \mathbf{b}_{\xi}]$, $\mathbf{E} \equiv [\mathbf{U}_E, \mathbf{b}_E]$ and $\mathbf{z}_{\omega} \equiv [\mathbf{U}_{\mathbf{z}_{\omega}}, \mathbf{b}_{\mathbf{z}_{\omega}}]$. We define the convex set

$$\Xi'_{z_{\Omega}} = \text{conv} \{\mathbf{z}_{\omega} : \omega \in \Omega\}. \quad (13)$$

We obviously have $\Xi'_{z_{\Omega}} \subseteq \Xi$. Let $(\mathbf{u}^*, (\mathbf{v}_{z_{\omega}}^*)_{\omega \in \Omega})$ be the optimal solution of the MRC problem related to $\Xi'_{z_{\Omega}}$. Due to the definition of $\Xi'_{z_{\Omega}}$, MRC and FARC are equivalent. Moreover, from $\Xi'_{z_{\Omega}} \subseteq \Xi$, we get that

$$\mathbf{c}^T \mathbf{u}^* = \Pi_{\Xi'_{z_{\Omega}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega}})) \leq \Pi_{\Xi}(\mathbf{I}, \text{ext}(\Xi)).$$

We will also assume that there is a positive number μ such that $\|(\mathbf{u}^*, \mathbf{1})\|_2 \leq \mu$. This assumption generally holds. For example, if the cost vector \mathbf{c} is positive and variables \mathbf{u} are non-negative, then $\mathbf{c}^T \mathbf{u}^* \leq \Pi_{\Xi}(\mathbf{I}, \text{ext}(\Xi))$ implies that $\|(\mathbf{u}^*, \mathbf{1})\|_2$ is upper-bounded. The number μ does not depend on ε .

Assumption 1. There exists a constant number μ such that $\|(\mathbf{u}^*, \mathbf{1})\|_2 \leq \mu$ for any $\Xi'_{z_{\Omega}} \subseteq \Xi$ and any optimal solution \mathbf{u}^* of the FARC problem related to Ξ' .

Lemma 1. Under Assumption 1, for each $\xi \in \Xi$, $(\mathbf{U}_E \mathbf{u}^* - \mathbf{b}_E)$ is bounded from above by $\varepsilon \mu \mathbf{1}$, where $\mathbf{1}$ is an all-ones vector.

Proof. The result follows from Cauchy-Schwartz inequality applied to each row of $\mathbf{E} \equiv [\mathbf{U}_E, \mathbf{b}_E]$. \square

Let δ be a small positive number and let

$$\Xi_\delta = \{ \xi \equiv [\mathbf{U}, \mathbf{b}] : \exists \xi' \equiv [\mathbf{U}, \mathbf{b}'] \in \Xi, \|\mathbf{b} - \mathbf{b}'\|_\infty \leq \delta \}. \quad (14)$$

Observe that if $\xi \equiv [\mathbf{U}, \mathbf{b}] \in \Xi$, then $[\mathbf{U}, \mathbf{b} - \delta \mathbf{1}] \in \Xi_\delta$. We will assume that for some small number δ , the static robust counterpart problem SRC is still solvable.

Assumption 2. *There exists a static robust solution $(\mathbf{u}_\delta, \mathbf{v}_\delta)$ w.r.t. uncertainty set Ξ_δ .*

Theorem 3. *Under Assumptions 1 and 2, for each pole-set $\Omega \in \mathcal{F}_\Xi$ such that $d(\Omega, \Xi) = \varepsilon \leq \frac{\delta}{\mu}$, we have*

$$\Pi_\Xi(\mathbf{I}, \Omega) \leq \left(1 - \frac{\varepsilon\mu}{\delta}\right) \mathbf{c}^T \mathbf{u}^* + \frac{\varepsilon\mu}{\delta} \mathbf{c}^T \mathbf{u}_\delta,$$

where $\mathbf{c}^T \mathbf{u}^*$ and $\mathbf{c}^T \mathbf{u}_\delta$ are respectively fully adjustable robust cost w.r.t. Ξ'_{z_Ω} and static cost w.r.t. Ξ_δ .

Proof. Assume that the optimal solution of FARC w.r.t. uncertainty set Ξ'_{z_Ω} is $(\mathbf{u}^*, (\mathbf{v}_{z_\omega}^*))$. Consider the solution

$$\hat{\mathbf{u}} = \left(1 - \frac{\varepsilon\mu}{\delta}\right) \mathbf{u}^* + \frac{\varepsilon\mu}{\delta} \mathbf{u}_\delta, \quad \hat{\mathbf{v}}_\omega = \left(1 - \frac{\varepsilon\mu}{\delta}\right) \mathbf{v}_{z_\omega}^* + \frac{\varepsilon\mu}{\delta} \mathbf{v}_\delta. \quad (15)$$

Let us show that $(\hat{\mathbf{u}}, \hat{\mathbf{v}}_\omega)$ is a feasible solution of the MRC problem related to Ξ and Ω . For any $\xi \equiv [\mathbf{U}, \mathbf{b}] \in \Xi$, by (12), one can write:

$$\begin{aligned} \mathbf{U}_\xi \hat{\mathbf{u}} + \mathbf{V} \sum_{\omega \in \Omega} \beta_\omega^\xi \hat{\mathbf{v}}_\omega &= \mathbf{U}_\xi \hat{\mathbf{u}} + \left(1 - \frac{\varepsilon\mu}{\delta}\right) \mathbf{V} \sum_{\omega \in \Omega} \beta_\omega^\xi \mathbf{v}_{z_\omega}^* + \frac{\varepsilon\mu}{\delta} \mathbf{V} \mathbf{v}_\delta \\ &\leq \mathbf{U}_\xi \hat{\mathbf{u}} + \left(1 - \frac{\varepsilon\mu}{\delta}\right) \sum_{\omega \in \Omega} \beta_\omega^\xi (\mathbf{b}_{z_\omega} - \mathbf{U}_{z_\omega} \mathbf{u}^*) \\ &\quad + \frac{\varepsilon\mu}{\delta} (\mathbf{b}_\xi - \delta \mathbf{1} - \mathbf{U}_\xi \mathbf{u}_\delta) \end{aligned} \quad (16)$$

$$\begin{aligned} &= \mathbf{U}_\xi \hat{\mathbf{u}} + \left(1 - \frac{\varepsilon\mu}{\delta}\right) (\mathbf{b}_\xi - \mathbf{b}_\mathbf{E} + \mathbf{U}_\mathbf{E} \mathbf{u}^* - \mathbf{U}_\xi \mathbf{u}^*) \\ &\quad + \frac{\varepsilon\mu}{\delta} (\mathbf{b}_\xi - \delta \mathbf{1} - \mathbf{U}_\xi \mathbf{u}_\delta) \end{aligned} \quad (17)$$

$$\begin{aligned} &= \left(1 - \frac{\varepsilon\mu}{\delta}\right) (\mathbf{U}_\mathbf{E} \mathbf{u}^* - \mathbf{b}_\mathbf{E}) - \varepsilon\mu \mathbf{1} + \mathbf{b}_\xi \\ &\leq \left(1 - \frac{\varepsilon\mu}{\delta}\right) \varepsilon\mu \mathbf{1} - \varepsilon\mu \mathbf{1} + \mathbf{b}_\xi \\ &= \mathbf{b}_\xi - \frac{\varepsilon^2 \mu^2}{\delta} \mathbf{1} \\ &\leq \mathbf{b}_\xi, \end{aligned} \quad (18)$$

where (16) follows from the fact that $(\mathbf{u}^*, \mathbf{v}_{z_\omega}^*)$ satisfies constraint $\mathbf{U}\mathbf{u} + \mathbf{V}\mathbf{v} \leq \mathbf{b}$ for $\mathbf{z}_\omega = [\mathbf{U}_{z_\omega}, \mathbf{b}_{z_\omega}]^T$ and the static solution $(\mathbf{u}_\delta, \mathbf{v}_\delta)$ satisfies $\mathbf{U}_\xi \mathbf{u} + \mathbf{V}\mathbf{v} \leq \mathbf{b}_\xi - \delta \mathbf{1}$, (17) follows from (11) and (12), and (18) is due to Lemma 1.

The robust cost incurred by $(\hat{\mathbf{u}}, (\hat{\mathbf{v}}_\omega))$ is $\left(1 - \frac{\varepsilon\mu}{\delta}\right) \mathbf{c}^T \mathbf{u}^* + \frac{\varepsilon\mu}{\delta} \mathbf{c}^T \mathbf{u}_\delta$ and is an upper bound of the optimum of MRC. \square

Corollary 3. Given any sequence of pole-sets $\Omega_i \in \mathcal{F}_\Xi$ such that $\lim_{i \rightarrow \infty} d(\Omega_i, \Xi) = 0$, then under Assumptions 1 and 2,

$$\Pi_\Xi(\mathbf{I}, \Omega_i) \geq \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)) \text{ and } \lim_{i \rightarrow \infty} \Pi_\Xi(\mathbf{I}, \Omega_i) = \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)).$$

Moreover, the corresponding sequence of sets $\Xi'_{z_{\Omega_i}}$ defined in (13) satisfies

$$\Pi_{\Xi'_{z_{\Omega_i}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega_i}})) \leq \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)) \text{ and } \lim_{i \rightarrow \infty} \Pi_{\Xi'_{z_{\Omega_i}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega_i}})) = \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)).$$

Proof. Let $\varepsilon_i = d(\Omega_i, \Xi), \forall i$. From Theorem 3, we have

$$\Pi_\Xi(\mathbf{I}, \Omega_i) \leq \left(1 - \frac{\varepsilon_i \mu}{\delta}\right) \Pi_{\Xi'_{z_{\Omega_i}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega_i}})) + \frac{\varepsilon_i \mu}{\delta} \mathbf{c}^T \mathbf{u}_\delta, \forall i,$$

and we know that

$$\Pi_{\Xi'_{z_{\Omega_i}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega_i}})) \leq \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)) \leq \Pi_\Xi(\mathbf{I}, \Omega_i).$$

Consequently,

$$\lim_{i \rightarrow \infty} \Pi_{\Xi'_{z_{\Omega_i}}}(\mathbf{I}, \text{ext}(\Xi'_{z_{\Omega_i}})) = \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi)) \text{ and } \lim_{i \rightarrow \infty} \Pi_\Xi(\mathbf{I}, \Omega_i) = \Pi_\Xi(\mathbf{I}, \text{ext}(\Xi))$$

hold in the limit at the same time. \square

4 An illustrative example

To illustrate the multipolar concept, we present a simple example, which had been previously studied in [25] and is as follows:

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & \forall \xi \in \mathbb{R}^n, \|\xi\|_1 \leq 1, \quad \exists \mathbf{v}, v_i \geq \xi_i, v_i \geq -\xi_i, i = 1, \dots, n \\ & u \geq \sum_{i=1}^n v_i. \end{aligned} \quad (19)$$

Observe that u is here the unique first-stage (non adjustable) variable. On the other hand v_i , for each $i = 1, \dots, n$, are second-stage (adjustable) variables. The uncertainty set is given by $\Xi \equiv \{\xi \in \mathbb{R}^n, \|\xi\|_1 \leq 1\}$.

As noticed in [25], an optimal fully adjustable solution is given by $u = 1$ and $v_i = \|\xi_i\|_1$, whereas the optimal affinely adjustable solution requires that $u = n$. In other words, the affine approach does not lead to any improvement compared to the static approach.

Following the paradigm of multipolar approach in Section 2, let us take $\mathbf{P}\xi = (\xi_1, \dots, \xi_{n_0})$, where $n_0 \in \mathbb{N}, n_0 \leq n$. In other words, the shadow matrix \mathbf{P} limits the dimension of Ξ to n_0 by leaving ξ_i as they are for $i \leq n_0$ and disregarding the other components for $i > n_0$. Let $\Omega \subseteq \mathbb{R}^{n_0}$ be the set of poles containing for $i = 1, \dots, n_0$,

vectors $\phi^i = (0, \dots, 0, 1, 0, \dots, 0)$ and $\bar{\phi}^i = -\phi^i$, whose components are 0 except the i^{th} component. Hence Ω contains $2n_0$ poles and $\Xi_P = \text{conv } \Omega$.

Given any $\xi \in \Xi$, let λ_{ϕ^i} and $\lambda_{\bar{\phi}^i}$ be the convex combination coefficients such that $\mathbf{P}\xi = \sum_{i=1}^{n_0} (\lambda_{\phi^i} \phi^i + \lambda_{\bar{\phi}^i} \bar{\phi}^i)$. The equation can be transformed to $\mathbf{P}\xi = \sum_{i=1}^{n_0} \phi^i (\lambda_{\phi^i} - \lambda_{\bar{\phi}^i})$; thus these coefficients should satisfy the equations $\lambda_{\phi^i} - \lambda_{\bar{\phi}^i} = \xi_i$ for $1 \leq i \leq n_0$. Let \mathbf{v}_{ϕ^i} (resp. $\mathbf{v}_{\bar{\phi}^i}$) be the recourse vector associated with pole ϕ^i (resp. $\bar{\phi}^i$). These vectors belong to \mathbb{R}^n .

In the considered example, inequalities (7) are equivalent to the following set of inequalities:

$$\sum_{i=1}^{n_0} (\lambda_{\phi^i} \mathbf{v}_{\phi^i} + \lambda_{\bar{\phi}^i} \mathbf{v}_{\bar{\phi}^i}) \geq (|\xi_1|, |\xi_2|, \dots, |\xi_n|), \quad (20a)$$

$$u \geq \left\| \sum_{i=1}^{n_0} (\lambda_{\phi^i} \mathbf{v}_{\phi^i} + \lambda_{\bar{\phi}^i} \mathbf{v}_{\bar{\phi}^i}) \right\|_1. \quad (20b)$$

Let us take $\mathbf{v}_{\phi^i} = \mathbf{v}_{\bar{\phi}^i} = (0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 1)$, where the first n_0 components are 0 except the i^{th} component, which is equal to 1, while the last $(n - n_0)$ components are equal to 1.

Observe that the last $(n - n_0)$ components of the vector $\sum_{i=1}^{n_0} (\lambda_{\phi^i} \mathbf{v}_{\phi^i} + \lambda_{\bar{\phi}^i} \mathbf{v}_{\bar{\phi}^i})$ are equal to 1. Moreover, for $1 \leq i \leq n_0$, we have $\lambda_{\phi^i} + \lambda_{\bar{\phi}^i} \geq |\lambda_{\phi^i} - \lambda_{\bar{\phi}^i}| = |\xi_i|$. This clearly implies that inequalities (20a) are satisfied. In addition, inequality (20b) leads to $u \geq \sum_{i=1}^{n_0} (\lambda_{\phi^i} + \lambda_{\bar{\phi}^i}) (1 + n - n_0)$, where $(1 + n - n_0)$ is the L^1 norm of each recourse vector \mathbf{v}_{ϕ^i} . Consequently, $u \geq 1 + n - n_0$. Since we are minimizing u , we get $u = 1 + n - n_0$. The cost decreases when n_0 increases. When n_0 is equal to 1, we get a static solution, while the optimal fully adjustable solution is obtained when $n_0 = n$. Finally, taking $1 < n_0 < n$, we obtain a compromise between the simplicity of the static approach and the efficiency of the fully adjustable solution. As mentioned earlier, such a compromise cannot be obtained for this example with the affinely adjustable approach.

Consider now a slightly changed example with the uncertainty set being the non-polyhedral set defined by $\Xi := \{\xi \in \mathbb{R}^n : \|\xi\|_2 \leq 1\}$. The rest of the problem remains as in (19); thus the new problem can be formulated as follows:

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & \forall \xi \in \mathbb{R}^n, \|\xi\|_2 \leq 1, \quad \exists \mathbf{v}, v_i \geq \xi_i, v_i \geq -\xi_i, i = 1, \dots, n, \\ & u \geq \sum_{i=1}^n v_i. \end{aligned} \quad (21)$$

Observe that since the new uncertainty set Ξ contains the previous one based on L^1 norm, the optimal value of (21) is greater than or equal to that of (19). Optimal solutions based on either the static approach or the affine approach still incur

a cost of n while the optimal fully adjustable solution has a cost of \sqrt{n} . Let us now consider the multipolar approach, where \mathbf{P} is still defined by $P\xi = (\xi_1, \dots, \xi_{n_0})$. Let us choose the following set of poles: $\Omega = \{\sqrt{n_0}\phi^i\}_{i=1}^{i=n_0} \cup \{\sqrt{n_0}\bar{\phi}^i\}_{i=1}^{i=n_0}$. One can easily show that $\Xi_P \subseteq \text{conv}(\Omega)$. Moreover, by taking $\mathbf{v}_{\sqrt{n_0}\phi^i} = \mathbf{v}_{\sqrt{n_0}\bar{\phi}^i} = (0, \dots, 0, \sqrt{n_0}, 0, \dots, 0, 1, \dots, 1)$, where the first n_0 components are 0 except the i^{th} component, which is equal to $\sqrt{n_0}$, while the last $n - n_0$ components are equal to 1, we get a solution of the multipolar robust counterpart with $u = \sqrt{n_0} + n - n_0$. Similarly to the previous case, when n_0 is equal to 1, we get a static solution, while the optimal fully adjustable solution is obtained when $n_0 = n$.

5 The construction of pole-sets

We know from Section 3 that the multipolar robust value converges to a fully adjustable robust value when the distance between Ω and Ξ_P gets close to 0, and $\mathbf{P} = \mathbf{I}$. We also proved the monotonicity of multipolar robust value w.r.t. the inclusion of $\text{conv}\Omega$. Therefore, the objective of this section is to find a pole-set $\Omega \in \mathcal{F}_P$ as close to Ξ_P as possible, while minimizing the number of poles. This is clearly related to the theory of approximation of convex sets by polytopes.

A considerable amount of work has been done in this area. A recent survey of relevant results is given in [21]. It is proved in [20, 27] that given a convex body $\Xi_P \in \mathbb{R}^{n_0}$, there exists a polytope $F_n \in \mathbb{R}^{n_0}$ having n vertices containing Ξ_P such that $d_H(\Xi_P, F_n) \leq \frac{k(\Xi_P)}{n^{2/(n_0-1)}}$ where d_H denotes the Hausdorff distance and $k(\Xi_P)$ is a constant only depending on Ξ_P . More precise approximations are obtained in dimension 2, where we can ensure the existence of $F_n \subset \mathbb{R}^2$ such that $d_H(\Xi_P, F_n) \leq \frac{l}{2n} \sin \frac{\pi}{n}$ where l is the length of the boundary of Ξ_P . Moreover, if the boundary of Ξ_P is two-times smooth, then an explicit asymptotic result is known about the distance between Ξ_P and the set of circumscribed polytopes having n vertices: the closest polytope F_n satisfies $d_H(\Xi_P, F_n) \sim \frac{k(\Xi_P)}{n^{2/(n_0-1)}}$ where $k(\Xi_P)$ is a constant depending on n_0 and the Gaussian curvature of the boundary of Ξ_P [19].

The monotonicity of multipolar robust values w.r.t. pole-sets might suggest using minimum volume circumscribed polytopes. Considering the Nikodym distance (related to volumes) instead of the Hausdorff distance, the same kind of results can be obtained [21]. One might be interested in a minimum volume simplex containing a convex set Ξ_P . We know for example that if Ξ_P is the hypercube H_{n_0} , then a minimum volume circumscribed simplex has a volume equal to $\frac{n_0^{n_0}}{n_0!}$ [42]. If Ξ_P is the unit ball, then a minimum volume simplex containing the ball is a regular simplex whose volume is $\frac{n_0^{n_0/2}(n_0+1)^{(n_0+1)/2}}{n_0!}$ [21] and whose dihedral angle is $\arccos(\frac{1}{n_0})$ [49]. It is also known that a minimum volume simplex enclosing Ξ_P satisfies the centroid property: the centroid of each facet of this simplex should be in Ξ_P [39]. A polynomial-time algorithm to find such a minimum volume simplex enclosing a set of points in \mathbb{R}^3 is given in [51]. However, it is generally unknown how to solve the problem in higher dimensions [34].

As observed by [21], most constructive algorithms were generally proposed for low dimensional cases (2 or 3). For more general cases, constructive algorithms of circumscribed polytopes such as the algorithm of [38] are generally based on the addition of inequalities without controlling the number of vertices of the circumscribed polytope. This can hardly accommodate the need of multipolar framework since we want to control the complexity of MRC by limiting the number of poles.

Note also that we are required to construct the pole-set of Ξ_P in a reasonable time. Algorithms checking whether each extreme point of Ξ_P belongs to the convex hull of Ω fail to work, since the number of extreme points of a polytope can be exponential or even infinite.

The rest of this section is organized as follows. First, we describe a general algorithm to construct a simplex enclosing Ξ_P . The resulting simplex is guaranteed to be smallest in the sense that it cannot be shrunk. Then, a project-and-cut based tightening procedure is proposed to construct pole-sets that are closer to Ξ_P .

5.1 Generation of a circumscribed simplex

In this section, we describe a general algorithm for the construction of a circumscribed simplex of Ξ_P . Specifically, we first randomly generate a set of $(n_0 + 1)$ affinely independent points, whose convex hull forms a simplex S . Then, we compute the best homothetic transformation of S such that the resulting simplex contains Ξ_P .

We denote by the $\omega^{(i)}, i = 1, \dots, (n_0 + 1)$ the $(n_0 + 1)$ affinely independent points. Then the n_0 -simplex set can be expressed as $\{\mathbf{x} : \mathbf{D}\boldsymbol{\lambda} = (\mathbf{x}, 1), \boldsymbol{\lambda} \geq \mathbf{0}\}$, where

$$\mathbf{D} = \begin{pmatrix} \omega_1^{(1)} & \dots & \omega_1^{(n_0+1)} \\ \vdots & \vdots & \vdots \\ \omega_{n_0}^{(1)} & \dots & \omega_{n_0}^{(n_0+1)} \\ 1 & \dots & 1 \end{pmatrix}.$$

Since the $(n_0 + 1)$ points are affinely independent, matrix \mathbf{D} is invertible; therefore, $\lambda_j, j = 1, \dots, (n_0 + 1)$, can be expressed as a affine function of \mathbf{x} ; the coefficients of the affine function λ_i are the components of the i th row of \mathbf{D}^{-1} , i.e.,

$$\lambda_i(x) = \sum_{j=1}^{n_0} l_{ij}x_j + l_{i(n_0+1)}, \quad i = 1, \dots, (n_0 + 1). \quad (22)$$

Note that $\lambda_i(x) \geq 0, i = 1, \dots, (n_0 + 1)$ iff \mathbf{x} belongs to the n -simplex.

Let $\boldsymbol{\Sigma}_\sigma, \mathbf{T}_\mathbf{t}$ be the associate matrices for the operations of scaling with factor $\sigma > 0$ and translation $\mathbf{t} \in \mathbb{R}^{n_0}$. Thus the associated matrix $\mathbf{D}_{\sigma, \mathbf{t}}$ of the simplex with homothetic transformation on the simplex S is $\mathbf{D}_{\sigma, \mathbf{t}} = \mathbf{T}_\mathbf{t}\boldsymbol{\Sigma}_\sigma\mathbf{D}$, where

$$\mathbf{T}_\mathbf{t} = \begin{pmatrix} \mathbf{I}_{n_0} & \mathbf{t} \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_\sigma = \begin{pmatrix} \sigma\mathbf{I}_{n_0} & 0 \\ 0 & 1 \end{pmatrix}.$$

Its corresponding inverse is then

$$\mathbf{D}_{\sigma, \mathbf{t}}^{-1} = \mathbf{D}^{-1} \begin{pmatrix} \frac{1}{\sigma} I_{n_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_{n_0} & -\mathbf{t} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sigma} l_{11} & \cdots & \frac{1}{\sigma} l_{1n_0} & l_{1(n_0+1)} - \frac{1}{\sigma} \sum_{k=1}^{n_0} t_k l_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sigma} l_{n_0 1} & \cdots & \frac{1}{\sigma} l_{n_0 n_0} & l_{n_0(n_0+1)} - \frac{1}{\sigma} \sum_{k=1}^{n_0} t_k l_{n_0 k} \\ \frac{1}{\sigma} l_{(n_0+1)1} & \cdots & \frac{1}{\sigma} l_{(n_0+1)n_0} & l_{(n_0+1)(n_0+1)} - \frac{1}{\sigma} \sum_{k=1}^{n_0} t_k l_{(n_0+1)k} \end{pmatrix}.$$

Let σ^* be the smallest scaling factor σ such that a translate of σS contains Ξ_P . The translate used when $\sigma = \sigma^*$ is denoted by \mathbf{t}^* .

Theorem 4. σ^* and \mathbf{t}^* are given by: $\mathbf{t}^* = \sum_{i=1}^{n_0+1} z_i \omega^{(i)}$ and $\sigma^* = - \sum_{i=1}^{n_0+1} z_i$, where for each $i = 1, \dots, (n_0 + 1)$, $z_i = \min \{ \sum_{j=1}^{n_0} l_{ij} x_j : \mathbf{x} \in \Xi_P \}$.

Proof. Assume that the homothetic copy of S given by $\sigma S + \mathbf{t}$ contains Ξ_P . Then the coefficients $\lambda_i(x)$ defined in (22) should be nonnegative for any point $\mathbf{x} \in \Xi_P$. Considering the matrix $\mathbf{D}_{\sigma, \mathbf{t}}^{-1}$ defined above and computing the minimum values of $\lambda_i(x)$, $i = 1, \dots, (n_0 + 1)$, we get

$$l_{i(n_0+1)} - \frac{1}{\sigma} \sum_{k=1}^{n_0} t_k l_{ik} + \frac{1}{\sigma} z_i \geq 0, \quad i = 1, \dots, (n_0 + 1),$$

For ease of notation, we express this as $l'_{i(n_0+1)} + z_i \geq 0$, $i = 1, \dots, (n_0 + 1)$, where

$$l'_{i(n_0+1)} = \sigma l_{i(n_0+1)} - \sum_{k=1}^{n_0} t_k l_{ik}, \quad i = 1, \dots, (n_0 + 1).$$

Since the matrix $(l)_{i,j=1,\dots,(n_0+1)}$ is the inverse of matrix \mathbf{D} , we have

$$\sum_{i=1}^{n_0+1} l_{ij} = 0, \quad j = 1, \dots, n_0 \quad \text{and} \quad \sum_{i=1}^{n_0+1} l_{i(n_0+1)} = 1. \quad (23)$$

Summing all $l'_{i(n_0+1)}$, we get

$$\sigma = \sum_{i=1}^{n_0+1} l'_{i(n_0+1)} \geq - \sum_{i=1}^{n_0+1} z_i. \quad (24)$$

Observe that having $l'_{i(n_0+1)}$ (and σ^* as a consequence), we can get the translate t through the linear system $\sum_{k=1}^{n_0} t_k l_{ik} = \sigma l_{i(n_0+1)} - l'_{i(n_0+1)}$, $i = 1, \dots, (n_0 + 1)$. Multiplying by \mathbf{D} , we get that $(t_1, \dots, t_{n_0}, 0) = \sigma(0, \dots, 0, 1) - \mathbf{D}(l'_{1(n_0+1)}, \dots, l'_{(n_0+1)(n_0+1)})$ which leads to $\mathbf{t} = \sum_{i=1}^{n_0+1} z_i \omega^{(i)}$.

According to (24), the smallest σ^* is $-\sum_{i=1}^{n_0+1} z_i$. We should however check if $\sigma^* \geq 0$. This holds since by considering any $x \in \Xi_P$, one can write that

$$\begin{aligned} \sum_{i=1}^{n_0+1} z_i &\leq \sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} l_{ij} x_j \\ &= \sum_{j=1}^{n_0} x_j \sum_{i=1}^{n_0+1} l_{ij} \\ &= 0, \end{aligned}$$

where the last equality is based on (23).

Since the matrix $\mathbf{D}_{\sigma, \mathbf{t}}^{-1}$ does not exist when $\sigma^* = 0$, we have to study this special case. It is clear that $\sigma^* = 0$ if and only if Ξ_P is a single point. Observe that in this case, we necessarily have $\sum_{i=1}^{n_0+1} z_i = 0$ since $z_i = \sum_{j=1}^{n_0} l_{ij} x_j$, where \mathbf{x} is the single point of Ξ_P . Then formula $\sigma^* = -\sum_{i=1}^{n_0+1} z_i$ is still valid and $\mathbf{t}^* = \mathbf{x} = \sum_{i=1}^{n_0+1} z_i \omega^{(i)}$ also occurs. \square

Note that values of $z_i, i = 1, \dots, (n_0 + 1)$ defined in Theorem 4 can easily be computed for any Ξ_P since we only have to minimize a linear function over a convex set.

As a special case, pole-sets of a hypercube are of great use in multipolar robust approach. First, hypercubes are one of the most common uncertainty sets in many applications. Second, general box sets of the form $\{\mathbf{x} : \mathbf{x} \in [\mathbf{l}, \mathbf{u}]^{n_0} \subseteq \mathbb{R}^{n_0}\}$ are simply affine transformations of a hypercube, so the pole-sets of a hypercube also apply to boxes with some simple transformations.

Corollary 4. *If Ξ_P is a hypercube, then σ^* and \mathbf{t}^* are given by:*

$$\mathbf{t}^* = \sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} \min\{0, l_{ij}\} \omega^{(i)} \text{ and } \sigma^* = \frac{1}{2} \sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} |l_{ij}|.$$

Proof. If Ξ_P is a hypercube, by Theorem 4, we have $z_i = \sum_{j=1}^{n_0} \min\{0, l_{ij}\}$, $i = 1, \dots, n_0$. By (23), we have $-\sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} \min\{0, l_{ij}\} = \frac{1}{2} \sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} |l_{ij}|$, which completes the proof. \square

According to Corollary 4, we have a closed formula for the homothetic translation for a n_0 -simplex S containing the n_0 -hypercube, i.e.,

$$\mathbf{x}_{\sigma, \mathbf{t}} = \frac{1}{2} \sum_{i=1}^{n_0+1} \sum_{j=1}^{n_0} |l_{ij}| \mathbf{x} + \sum_{i=1}^{n_0+1} \omega^{(i)} \sum_{j=1}^{n_0} \min\{0, l_{ij}\}, \forall \mathbf{x} \in S. \quad (25)$$

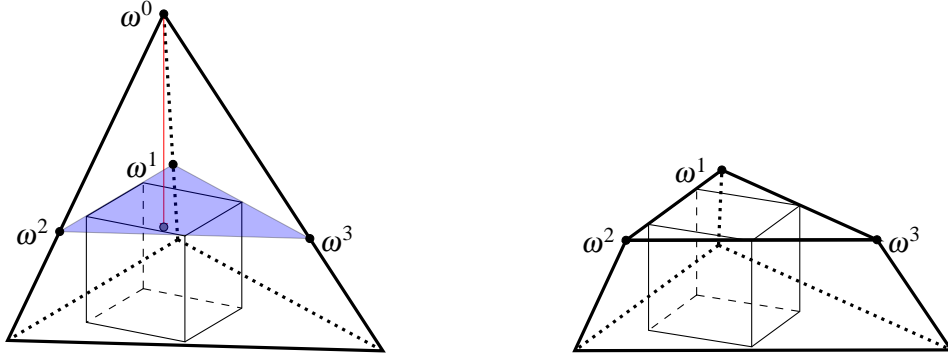


Figure 1: (Left) Pole ω^0 is replaced with poles $(\omega^i)_{i=1}^3$. (Right) The updated convex hull of the new set of poles.

Note that the value σ^* presented in Corollary 4 has been given in [46] but the proof here is much simpler.

To sum up the foregoing, we present a general algorithm for the generation of a circumscribed simplex as follows.

1. Generate $(n_0 + 1)$ affinely independent points $(\omega^{(i)})_{i=1}^{n_0+1}$.
2. Compute σ^* and \mathbf{t}^* by Theorem 4 and output the $\sigma^* \omega^{(i)} + \mathbf{t}^*, i = 1, \dots, (n_0 + 1)$.

5.2 A tightening procedure

In this section, we propose a general procedure to construct pole-sets of good quality by tightening a given pole-set.

The procedure is the following: among the vertices of Ω select the farthest one in L^2 sense from Ξ_P and compute the projection of this vertex on Ξ_P . Then we consider the hyperplane separating this vertex from Ξ_P (containing the projection) and compute the extreme points of the intersection of this hyperplane with $\text{conv } \Omega$. These extreme points are added to Ω while the vertex that has been projected is removed from Ω . Figure 1 illustrates a tightening procedure of a 3-D simplex covering H_3 . The procedure is repeated until the cardinality of Ω reaches some fixed upper bound. Details are given below:

1. Assume $\Omega = \{\omega^{(k)}, k \in I\}$. For each $k \in I$, compute the distance between $\omega^{(k)}$ and Ξ_P . Let \mathbf{z}^k be the projection of $\omega^{(k)}$ on Ξ_P . \mathbf{z}^k can be usually expressed in a closed form. For example, in the ball case, we have $\mathbf{z}^k = \frac{\omega^{(k)}}{\|\omega^{(k)}\|_2}$, while in the hypercube case we get $\mathbf{z}_i^k = \omega_i^{(k)}$ if $\omega_i^{(k)} \in [0, 1]$, $\mathbf{z}_i^k = 1$ if $\omega_i^{(k)} \geq 1$ and $\mathbf{z}_i^k = 0$ if not. The distance between $\omega^{(k)}$ and Ξ_P is then given by $\|\omega^{(k)} - \mathbf{z}^k\|_2$. Let $\omega^{(k_0)}$ be the vertex of Ω maximizing the distance from

Ξ_P :

$$\omega^{(k_0)} = \arg \max_{\omega^{(k)} \in \Omega} \|\omega^{(k)} - \mathbf{z}^{k_0}\|_2.$$

2. Let $\alpha = \omega^{(k_0)} - \mathbf{z}^{k_0}$ and let $B(\omega^{(k_0)}, \|\alpha\|_2)$ be the ball of radius $\|\alpha\|_2$ centered at $\omega^{(k_0)}$. Since $B(\omega^{(k_0)}, \|\alpha\|_2) \cap \Xi_P = \{\mathbf{z}^{k_0}\}$ and $B(\omega^{(k_0)}, \|\alpha\|_2)$ and Ξ_P are convex, there is a hyperplane separating them. This hyperplane, denoted by $h(\omega^{(k_0)})$, is here uniquely defined since it contains \mathbf{z}^{k_0} and is orthogonal to α . It is then given by $h(\omega^{(k_0)}) = \{\mathbf{x} : (\mathbf{x} - \mathbf{z}^{k_0})^T \alpha = 0\}$. We use $h^-(\omega^{(k_0)}) = \{\mathbf{x} : (\mathbf{x} - \mathbf{z}^{k_0})^T \alpha < 0\}$ and $h^+(\omega^{(k_0)}) = \{\mathbf{x} : (\mathbf{x} - \mathbf{z}^{k_0})^T \alpha \geq 0\}$ to respectively denote the inner and outer half spaces.
3. Now partition the vertices $(\omega^{(k)})_{k \in I}$ into two disjoint sets: Ω^- and Ω^+ , where $\Omega^- = \{\omega^{(k)} : \omega^{(k)} \in h^-(\omega^{(k_0)})\}$ and $\Omega^+ = \{\omega^{(k)} : \omega^{(k)} \in h^+(\omega^{(k_0)})\}$. Then consider the set of vertices Ω' obtained as intersections between the hyperplane $h(\omega^{(k_0)})$ and the set of lines $(\omega^{(i)}, \omega^{(j)})$ where $\omega^{(i)} \in \Omega^+$ and $\omega^{(j)} \in \Omega^-$: $\Omega' = \bigcup_{\omega^{(i)} \in \Omega^+, \omega^{(j)} \in \Omega^-} h(\omega^{(k_0)}) \cap (\omega^{(i)}, \omega^{(j)})$. The number of such intersections is of course less than $|\Omega^-| \times |\Omega^+|$. Also note that we need to remove redundant points from Ω' if they are convex combinations of other points of Ω' . Finally update Ω by deleting Ω^+ and adding Ω' : $\Omega = \Omega^- \cup \Omega'$. If cardinality of Ω is still under a prescribed upper bound, the procedure is repeated.

To conclude this section, we might add that it is sometimes more efficient to start with a pole-set having more than $(n_0 + 1)$ poles. Assume, for example, that Ξ_P is the unit ball $\{\mathbf{x} \in \mathbb{R}^{n_0} : \|\mathbf{x}\|_2 \leq 1\}$. Then one can consider a $2n$ -pole-set where poles are the $2n_0$ extreme points of $\{\mathbf{x} \in \mathbb{R}^{n_0} : \|\mathbf{x}\|_1 \leq \sqrt{n_0}\}$. Of course, $2n$ -pole-sets can also be easily generated for many other convex sets.

6 A numerical example: the lobbying problem

Let us consider a lobbying problem where a set of voters (for example, legislators or members of regulatory agencies) have to take some decisions. The opinion of each voter depends on the opinion of some authorities. Authority's opinions are generally uncertain. A lobby would like to ensure that an important decision will be unanimously approved by all voters. The lobby will spend some effort (energy, money, etc.) to convince each voter, while the total lobbying budget is minimized. Assume that there are m voters and n authorities. The opinion of voter i is given by $\sum_{j=1}^n Q_{ij} \xi_j$ where Q_{ij} is an estimated number belonging to $[-1, 1]$ and ξ_j represents the uncertain opinion of authority j . If Q_{ij} is close to 1, then j has a big impact on i , while $Q_{ij} = 0$ means that i does not care about j , while a negative value of Q_{ij} can be interpreted as a negative effect (i.e., when j recommends something, i is inclined to have an opposite opinion). We assume here that the lobby would be satisfied if $\sum_{j=1}^n Q_{ij} \xi_j \leq 0$ for each voter. Since this might not occur for some

voters, some effort modeled here by $v_i(\xi)$ can be made by the lobby to convince them. The total effort is quantified by $\sum_{i=1}^m r_i v_i(\xi)$ where r_i is a unit effort price corresponding with voter i . The problem can be formulated as follows,

$$\begin{aligned}
& \min \quad u \\
& \text{s.t.} \quad \sum_{i=1}^m r_i v_i(\xi) \leq u, \quad \xi \in \Xi, \\
& \quad \mathbf{Q}\xi \leq \mathbf{v}(\xi), \quad \xi \in \Xi, \\
& \quad \mathbf{v}(\xi) \geq \mathbf{0}, \quad \xi \in \Xi,
\end{aligned} \tag{26}$$

where $\mathbf{Q} \in [-\mathbf{1}, \mathbf{1}]^{m \times n}$, Ξ is the convex uncertainty set and u is the budget that has to be secured by the lobby. The lobby problem is related to opinion dynamics in social networks (see [1] and the references therein). Notice that interactions between voters are also possible since the set of authorities might include the set of voters as a subset.

To illustrate the multipolar robust approach, we consider here two different uncertainty sets: the hypercube H_n and a unit volume ball B_n . The numbers r_i are assumed to be equal to 1. Specializing (8) to H_n , we get the following formulation for MRC associated with a shadow matrix \mathbf{P} and a feasible pole-set $\Omega \in \mathcal{F}_{PH_n}$,

$$\begin{aligned}
\Pi_{H_n}(\mathbf{P}, \Omega) = \min_{u, \mathbf{v}_\omega} \quad & u \\
\text{s.t.} \quad & \sum_{i=1}^m \mathbf{v}_\omega^i + \sum_{j=1}^n \sigma_j \omega_j + \sum_{j=1}^n \beta_j \leq u, \quad \omega \in \Omega, \\
& \sum_{j=1}^n \eta_{ij}^T \omega_j + \sum_{j=1}^n \alpha_{ij} - v_\omega^i \leq 0, \quad \omega \in \Omega, i = 1, \dots, m, \\
& \sum_{j=1}^n \mu_{ij} - \mathbf{V}_\omega^i - \sum_{j=1}^n \delta_{ij} \omega_j \leq 0, \quad \omega \in \Omega, i = 1, \dots, m, \\
& \beta + \mathbf{P}^T \sigma \geq \mathbf{0}, \\
& \alpha_{ij} + \mathbf{P}^T \eta_i \geq Q_{ij}, \quad i = 1, \dots, m, j = 1, \dots, n, \\
& \mu_i - \mathbf{P}^T \tau_i \geq \mathbf{0}, \quad i = 1, \dots, m, \\
& \alpha_{ij}, \beta_j, \mu_{ij} \geq 0, \quad i = 1, \dots, m, j = 1, \dots, n, \\
& \sigma, \eta_i, \tau_i \in \mathbb{R}^{n_0}, \quad i = 1, \dots, m.
\end{aligned} \tag{27}$$

As said above, we also consider the case where the uncertainty set is a unit volume ball B_n , whose center is $\bar{\xi} = (\frac{1}{2}, \dots, \frac{1}{2}, \dots, \frac{1}{2})$ and radius $\rho = (\frac{\Gamma(n/2+1)}{\pi^{n/2}})^{\frac{1}{n}}$. Accord-

ing to (9), the dual of the multipolar robust counterpart w.r.t. B_n , writes

$$\begin{aligned} \Pi_{B_n}(\mathbf{P}, \Omega) &= \min_{u, \mathbf{v}_\omega} u \\ \text{s.t. } & \rho \|\mathbf{P}^T \boldsymbol{\sigma}\|_2 + \mathbf{1}^T \mathbf{v}_\omega - \boldsymbol{\omega}^T \boldsymbol{\sigma} \leq u, \quad \boldsymbol{\omega} \in \Omega, \\ & \rho \|\mathbf{Q}_i^T + \mathbf{P}^T \boldsymbol{\eta}_i\|_2 - \mathbf{v}_\omega^i - \boldsymbol{\omega}^T \boldsymbol{\eta}_i + \mathbf{Q}_i \bar{\xi} \leq 0, \quad \boldsymbol{\omega} \in \Omega, i = 1, \dots, m, \\ & \rho \|\mathbf{P}^T \boldsymbol{\mu}_i\|_2 - \mathbf{v}_\omega^i - \boldsymbol{\omega}^T \boldsymbol{\mu}_i \leq 0, \quad \boldsymbol{\omega} \in \Omega, i = 1, \dots, m, \\ & \boldsymbol{\sigma}, \boldsymbol{\eta}_i, \boldsymbol{\mu}_i \in \mathbb{R}^{n_0}, \quad i = 1, \dots, m, \end{aligned} \tag{28}$$

where $\Omega \in \mathcal{F}_{PB_n}$.

As stated in Section 2.1, the fully adjustable robust value w.r.t. hypercube H_n can be achieved by simply taking $\mathbf{P} = \mathbf{I}$ and $\Omega = \text{ext}(H_n)$. The problem looks more complex in the ball case since the number of extreme points is infinite. Assume again that \mathbf{P} is identity. Given $\xi \in B_n$, the optimal solution of v_i is $\max\{0, \mathbf{Q}_i \xi\}$ for each $i = 1, \dots, m$. Denote by \mathcal{P} the power set of the index set $\{1, \dots, m\}$. We partition the ball B_n into a family of disjoint subsets by a set valued mapping $S: \mathcal{P} \mapsto 2^{B_n}$, i.e., for each $J \in \mathcal{P}$,

$$S(J) := \{\xi \in B_n : \mathbf{Q}_i \xi \geq 0, i \in J, \mathbf{Q}_j \xi \leq 0, j \in \bar{J}\},$$

where $\bar{J} = \{1, \dots, m\} \setminus J$. Therefore, the fully adjustable robust program writes

$$\Pi_{B_n}^* = \max_{J \in \mathcal{P}: S(J) \neq \emptyset} \max_{\xi \in S(J)} \sum_{i \in J} \mathbf{Q}_i \xi. \tag{29}$$

Notice that (29) takes an exponential number (in the number of constraints m) of second order cone programs to obtain the fully adjustable robust value $\Pi_{B_n}^*$. We show that (29) is equivalent to a much simpler problem.

Lemma 2. *Program (29) is equivalent to*

$$\Pi_{B_n}^* = \max_{J \in \mathcal{P}} \left\{ \rho \left\| \sum_{i \in J} \mathbf{Q}_i \right\|_2 + \sum_{i \in J} \mathbf{Q}_i \bar{\xi} \right\}. \tag{30}$$

Proof. Observe first that $\max_{J \in \mathcal{P}} \left\{ \rho \left\| \sum_{i \in J} \mathbf{Q}_i \right\|_2 + \sum_{i \in J} \mathbf{Q}_i \bar{\xi} \right\}$ is an upper bound of $\Pi_{B_n}^*$ since it is obtained by relaxing the constraints $\xi \in S(J)$.

We show that it is also a lower bound of $\Pi_{B_n}^*$. Let $J_{max} \in \mathcal{P}$ be a subset for which the maximum is achieved:

$$J_{max} = \arg \max_{J \in \mathcal{P}} \left\{ \rho \left\| \sum_{i \in J} \mathbf{Q}_i \right\|_2 + \sum_{i \in J} \mathbf{Q}_i \bar{\xi} \right\}.$$

Take $\hat{\xi} \in B_n$ such that $\hat{\xi} = \bar{\xi} + \frac{\rho \cdot \sum_{i \in J_{max}} \mathbf{Q}_i}{\left\| \sum_{i \in J_{max}} \mathbf{Q}_i \right\|_2}$. Let $K \in \mathcal{P}$ such that

$$\begin{aligned} \mathbf{Q}_i \hat{\xi} &\geq 0, \quad i \in K, \\ \mathbf{Q}_i \hat{\xi} &\leq 0, \quad i \in \{1, \dots, m\} \setminus K. \end{aligned}$$

We have that

$$\begin{aligned}
\rho \left\| \sum_{i \in J_{max}} \mathbf{Q}_i \right\|_2 + \sum_{i \in J_{max}} \mathbf{Q}_i \bar{\xi} &= \sum_{i \in J_{max}} \mathbf{Q}_i \hat{\xi} \\
&\leq \sum_{i \in K} \mathbf{Q}_i \hat{\xi} \\
&\leq \max_{\xi \in S(K)} \sum_{i \in K} \mathbf{Q}_i \xi \\
&\leq \Pi_{B_n}^*
\end{aligned}$$

where the first equality follows from the choice of $\hat{\xi}$. The second inequality is due to $J_{max} \cap K \subseteq K$, while the third inequality is from the fact that $\hat{\xi}$ belongs to $S(K)$ by the definition of K . The last inequality is a direct consequence of (29). \square

Although problem (30) is easier than problem (29), it is still computationally costly when the number of constraints m is large. Notice that (30) can also be seen as a integer quadratic program that can be approximated by semidefinite programming and solved using standard quadratic programming tools. We will not elaborate more on this since this falls out of the scope of the paper.

6.1 Numerical experiments

The problem instances are randomly generated following the rules below.

1. Let $m \in \{10, 20, 30, 40, 50\}$, $n \in \{5, 9, 10, 12, 15, 20, 30\}$.
2. Generate the components of \mathbf{Q} uniformly over $[-\mathbf{1}, \mathbf{1}]$.
3. We build four different sizes of pole-sets for each considered hypercube by the circumscribed simplex generation algorithm and the tightening procedure described in Section 5. As a result, for a hypercube H_n , (Ω_i) is a monotonic sequence w.r.t. the set inclusion of their convex hulls, i.e., $\Omega_i \preceq_{H_n} \Omega_j$ for all $i > j$. Table 1 displays the cardinality of different pole-sets of H_n . The number of vertices of H_n is also provided in the last column.
4. As an illustration of multipolar robust approach for smooth convex uncertainty sets, we generate pole-sets of ball B_n as well. In Table 2, Ω_0 denotes a first pole-set whose convex hull is a simplex, while Ω_1 is the $2n$ -pole-set defined at the end of Section 5. Starting from Ω_1 and applying the tightening procedure, we get the pole-sets $\Omega_i, i = 2, 3, 4$ as outputs. The cardinality of pole-sets are shown in Table 2.

The pole-sets had been readily generated before the solution procedure. Compact formulations (27) and (28) are modeled by YALMIP [44] and all the problem instances are solved by the Linux version of CPLEX 12.5 with default settings on a Dell E6400 laptop with Intel Core(TM)2 Duo CPU clocked at 2.53 GHz and

Table 1: The pole-sets of hypercubes

Hypercube	$ \Omega_0 $	$ \Omega_1 $	$ \Omega_2 $	$ \Omega_3 $	#ext.
H_9	10	32	162	387	512
H_{10}	11	36	112	322	1,024
H_{12}	13	44	144	449	4,046
H_{15}	16	56	192	353	32,768
H_{20}	21	76	144	514	1,048,576
H_{30}	31	60	116	432	1,073,741,824

Table 2: The pole-sets of the unit volume ball

Unit ball	$ \Omega_0 $	$ \Omega_1 $	$ \Omega_2 $	$ \Omega_3 $	$ \Omega_4 $
B_5	6	10	118	218	308
B_9	10	18	62	152	352
B_{10}	11	20	72	132	374
B_{12}	13	24	88	164	478

with 4 GB of RAM. We evaluate our multipolar approach from different measuring: the impact of pole-sets, the impact of the shadow matrix \mathbf{P} , and the benefit of adaptability.

6.1.1 The influence of pole-sets

Recall that multipolar robust approach closes to some extent the gap between affine robust value and the fully adjustable value. To test the impact of pole-sets, we fix $\mathbf{P} = \mathbf{I}$. For relatively lower dimensional cases, we report the multipolar robust values w.r.t. different pole-sets, and compute the percentage of the closed gap induced by the multipolar robust approach $\frac{\Pi_{\Xi}(\Omega_0) - \Pi_{\Xi}(\Omega_i)}{\Pi_{\Xi}(\Omega_0) - \Pi_{\Xi}^*} \times 100$. For higher dimensional cases, where FARC can hardly be solved in a reasonable time, we report the multipolar robust values w.r.t. different pole-sets.

Results related to hypercubes are presented in Table 3. The closed gap percentages are given within parentheses. Overall, these results appear encouraging as indicated by the closed gaps. Observe that when the uncertainty set is fixed, the multipolar robust values in general get lower as the pole-set Ω gets smaller. Also, we report the computing time for higher dimensional instances associated with hypercube uncertainty set in Table 4. While the computational time compared with the affine robust approach scales in magnitude, the complexity of multipolar robust

Table 3: The multipolar robust values with different pole-sets (hypercube uncertainty sets)

(m,n)	static	affine/ $\Pi_{H_n}(\Omega_0)$	$\Pi_{H_n}(\Omega_1)$	$\Pi_{H_n}(\Omega_2)$	$\Pi_{H_n}(\Omega_3)$	$\Pi_{H_n}^*$
(10,9)	24.84	12.42	12.18(9.45)	10.55(73.62)	10.16(88.98)	9.88
(10,10)	25.50	12.75	11.53(58.65)	10.96(86.06)	10.70(98.56)	10.67
(10,12)	30.66	15.33	14.63(35.18)	13.71(81.41)	13.43(95.48)	13.34
(20,9)	50.75	25.37	23.82(46.69)	22.08(99.10)	22.06(99.70)	22.05
(20,10)	50.88	25.44	23.56(16.95)	20.74(42.38)	18.58(61.86)	14.35
(20,12)	59.79	29.89	27.54(23.64)	25.40(45.17)	23.58(63.48)	19.95
(10,15)	35.81	17.90	16.91	15.98	15.29	-
(10,20)	50.88	25.44	24.82	24.35	23.33	-
(10,30)	64.32	32.16	31.70	31.14	30.22	-
(20,15)	82.49	41.25	39.36	36.20	34.87	-
(20,20)	99.28	49.64	47.17	45.66	40.80	-
(20,30)	157.20	78.60	77.82	76.83	73.77	-

approach is controlled by the choice of pole-sets. In particular, in higher dimensional cases, where fully adjustable robust values are difficult to obtain, the robust multipolar approach brings lower cost (compared with affine approach) in a reasonable time.

Table 4: Computing time (in seconds)

(m,n)	static	affine/ $\Pi_{H_n}(\Omega_0)$	$\Pi_{H_n}(\Omega_1)$	$\Pi_{H_n}(\Omega_2)$	$\Pi_{H_n}(\Omega_3)$
(10,15)	0.00	0.01	0.20	1.49	6.31
(10,20)	0.00	0.03	0.87	2.66	27.54
(10,30)	0.00	0.04	1.15	4.53	39.12
(20,15)	0.00	0.03	0.68	10.43	34.98
(20,20)	0.00	0.07	4.65	16.41	152.79
(20,30)	0.00	0.14	3.91	15.40	220.48

A sequence of lower bounds can also be generated as stated in Corollary 3 of Section 3.3. All we need to do is to generate a sequence of $(\Gamma_i)_{i=0}^{i=3}$ by projecting the pole-sets $(\Omega_i)_{i=0}^{i=3}$ onto the surface of hypercubes. The obtained lower bounds are denoted by $\Pi_{H_n}(\Gamma_i)$. Note that $\text{conv } \Omega' \subseteq \Omega$ does not necessarily lead to $\text{conv } \Gamma' \subseteq \Gamma$ or $\text{conv } \Gamma' \supseteq \Gamma$. Thus it may happen that $\Pi_{H_n}(\Gamma_i) \geq \Pi_{H_n}(\Gamma_{i+1})$. The results are summarized in Table 5, where the best lower bound for each problem instance is marked in bold.

Table 5: Lower bounds related to hypercubes

(m,n)	$\Pi_{H_n}(\Gamma_0)$	$\Pi_{H_n}(\Gamma_1)$	$\Pi_{H_n}(\Gamma_2)$	$\Pi_{H_n}(\Gamma_3)$	$\Pi_{H_n}^*$
(10,9)	6.88	8.18	9.52	9.65	9.88
(10,10)	7.11	8.12	9.62	8.34	10.67
(10,12)	10.50	8.88	9.48	9.57	13.34
(20,9)	20.08	16.05	18.70	21.98	22.05
(20,10)	11.88	11.88	12.44	12.92	14.35
(20,12)	14.73	16.65	16.96	19.07	19.95
(10,15)	7.62	7.63	8.40	8.40	-
(10,20)	7.86	10.20	10.20	10.36	-
(10,30)	7.11	9.03	9.03	9.79	-
(20,15)	23.49	25.86	23.56	23.56	-
(20,20)	15.23	16.08	16.08	19.61	-
(20,30)	30.53	31.12	32.53	33.81	-

Let us now focus on the ball case. Remember that FARC is intractable in general. However, as shown in Lemma 2, we can compute the optimum of FARC by solving problem (30) when m is small. We report the robust values obtained by solving multipolar robust counterpart with different pole-sets in Table 6. The results may indicate the following. First, the approximate robust values associated with balls appear lower than those associated with hypercubes although the volume and symmetric center of balls and hypercubes are the same. Second, the closed gaps by multipolar robust approach on robust problems with ball uncertainty sets might be less significant than that with hypercube uncertainty sets. As might be expected, larger poles-sets are required for balls compared to hypercubes. Third, despite the limitations, multipolar approach closes around 30% of the optimality gap. In particular, it appears compelling when the number of constraints are large, while the dimension of the uncertainty set is small.

The lower bounds obtained in the ball case are reported in Table 7. Interestingly, the observed sequences of lower bounds associated with ball B_n are monotonically increasing and their best bounds in general are close to the fully adjustable robust value.

6.1.2 The impact of the shadow matrix

To investigate the impact of the shadow matrix on the robust value of the robust problem, we conduct some experiments on problem instances w.r.t. hypercube uncertainty sets. The shadow matrices considered here are simply projection matrices on lower subspaces. The results are displayed in Table 8, where the uncertainty set

Table 6: The multipolar robust values with different pole-sets (ball)

(m,n)	static	$\Pi_{B_n}(\Omega_0)$	$\Pi_{B_n}(\Omega_1)$	$\Pi_{B_n}(\Omega_2)$	$\Pi_{B_n}(\Omega_3)$	$\Pi_{B_n}(\Omega_4)$	$\Pi_{B_n}^*$
(10,9)	17.27	9.43	9.28(11.11)	9.21(16.30)	9.11(23.70)	9.01(31.11)	8.08
(10,10)	16.09	9.21	8.98(23.00)	8.95(26.00)	8.94(27.00)	8.89(32.00)	8.21
(10,12)	19.64	10.93	10.76(20.99)	10.76(20.99)	10.74(23.46)	10.70(28.40)	10.12
(20,9)	35.87	19.86	19.54(20.92)	19.51(22.88)	19.45(26.80)	19.36(32.68)	18.33
(20,10)	33.12	17.19	16.42(18.08)	15.62(36.85)	15.57(38.03)	15.36(42.96)	12.93
(20,12)	39.85	20.93	20.05(17.25)	19.96(19.02)	19.90(20.20)	19.81(21.96)	15.83
(30,5)	19.51	11.00	10.39	9.22	9.14	9.03	-
(40,5)	37.57	21.63	21.11	20.71	20.63	20.55	-
(50,5)	38.14	20.94	20.06	19.33	19.14	19.02	-

Table 7: The lower bounds in the ball case

(m,n)	$\Pi_{B_n}(\Gamma_0)$	$\Pi_{B_n}(\Gamma_1)$	$\Pi_{B_n}(\Gamma_2)$	$\Pi_{B_n}(\Gamma_3)$	$\Pi_{B_n}(\Gamma_4)$	$\Pi_{B_n}^*$
(10,9)	6.60	6.70	6.70	6.71	7.67	8.08
(10,10)	6.07	6.28	6.93	7.44	7.60	8.21
(10,12)	7.26	7.51	8.01	8.08	8.44	10.12
(20,9)	16.04	16.17	16.89	16.89	16.89	18.33
(20,10)	12.03	12.03	12.03	12.03	12.03	12.93
(20,12)	12.35	13.41	13.67	13.67	13.67	15.83
(30,5)	7.48	7.48	8.11	8.11	8.34	-
(40,5)	16.54	17.31	19.27	19.27	19.87	-
(50,5)	15.53	15.53	17.22	17.22	17.80	-

is a hypercube and several projections are considered (on H_5 , H_7 , H_{10} , and H_{12}). The pole-set considered is the set of extreme points of the projected set. As might be expected, the robust value decreases as more information is employed in MRC.

6.1.3 The benefit of adaptability

To illustrate the concept of *benefit of adaptability* in the framework of multipolar robust approach, we compute the multipolar robust values of problem (26) with different proportions of adjustable variables \mathbf{v} . We allow the first $\lfloor \theta m \rfloor$ components of \mathbf{v} to be adaptable to the realization of ξ , while keeping the remaining $m - \lfloor \theta m \rfloor$ variables independent of the realization of ξ , where $\theta \in [0, 1]$. Note that when $\theta = 0$, we get the static case SRC. The results are summarized in Table 9 and we emphasize here two observations:

Table 8: Impact of the shadow matrix

(m,n)	H_5	H_7	H_{10}	H_{12}
(10,30)	56.95	51.50	45.59	43.29
(10,50)	109.84	105.68	100.69	96.11
(10,70)	162.14	159.52	156.12	154.53
(10,100)	246.98	244.13	239.34	237.43
(20,30)	133.35	126.59	117.63	111.53
(20,50)	233.73	224.82	213.04	203.89
(20,70)	345.56	340.41	328.49	319.79
(20,100)	462.51	453.26	439.71	431.39

1. For each problem instance, as the adaptability ratio θ increases, the robust value decreases significantly, which is reasonable both in theory and practice.
2. As the adaptability ratio θ increases, the influence of pole-sets on the robust value increases. For example, when the adaptability ratio $\theta = 0.25$, the multipolar robust values of all problem instances remain the same with different pole-sets except instance (20,20). When the adaptability increases, the robust values of more instances improve as the better pole-sets are used, which can be clearly seen when $\theta = 0.75$ and $\theta = 1$.

7 Conclusion

In this paper, we have presented a novel approach to handle uncertainty in optimization problems called the multipolar robust approach, which is based on a set of poles that are used to approximate the fully adjustable policy by a set of associated recourse decisions at poles. The approach generalizes the static approach, the affinely adjustable approach, and the fully adjustable approach, still we can control its complexity by using the concept of the shadow matrix and considering a reasonable number of poles. Several algorithms are proposed for the construction of proper pole-sets for hypercubes and balls. Comprehensive numerical experiments are carried out to evaluate the performance of the proposed approach in terms of the robust values, the complexity, and the benefit of adaptability. In general, the results appear encouraging.

It would be interesting to investigate further the performance of the multipolar robust approach on other problems. A systematic study of good approximations of convex bodies by enclosing polytopes with a limited number of extreme points should help to alleviate overconservatism and get closer to the optimal fully adaptable robust value. One can also put more focus on the approximation of convex

Table 9: The benefit of adaptability (hypercube)

$(m,n), \Omega_1$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
(10,9)	23.42	19.88	16.56	12.18
(10,15)	32.64	24.83	21.83	16.91
(10,20)	46.13	37.06	32.78	24.82
(10,30)	59.72	47.14	41.59	31.70
(20,9)	45.01	36.57	28.99	23.82
(20,15)	73.99	58.66	46.80	39.36
(20,20)	85.18	73.46	59.54	47.17
(20,30)	137.78	118.51	99.94	77.82
$(m,n), \Omega_2$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
(10,9)	23.42	19.88	15.91	10.55
(10,15)	32.64	24.55	21.21	15.98
(10,20)	46.13	36.86	32.46	24.35
(10,30)	59.72	46.95	41.32	31.14
(20,9)	45.01	36.52	27.24	22.08
(20,15)	73.99	58.16	45.06	36.20
(20,20)	84.98	72.98	58.77	45.66
(20,30)	137.78	117.96	99.13	76.83
$(m,n), \Omega_3$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
(10,9)	23.42	19.88	15.88	10.16
(10,15)	32.64	24.54	20.96	15.29
(10,20)	46.13	36.60	31.81	23.33
(10,30)	59.72	46.70	40.74	30.22
(20,9)	45.01	36.52	27.06	22.06
(20,15)	73.99	58.14	44.89	34.87
(20,20)	84.78	72.42	57.23	40.80
(20,30)	137.78	116.78	96.99	73.77

bodies from inside using, for example, maximum volume inscribed polytopes to get better lower bounds of the fully adjustable robust value.

While the approach was proposed in the context of a two-stage optimization problem, it can be adapted to multistage optimization. Multipolar decision rules can also be considered in stochastic programming. The multipolar approach might

also be combined with finite adaptability or multi-static robustness by partitioning the uncertainty set into several subsets and considering some multipolar decision rules for each subset.

Appendix: the derivation of (9)

We derive the compact formulation (9) of Section 3.1 w.r.t. an ellipsoidal uncertainty set defined by $\Xi := \{\xi : \|\mathbf{F}\xi\|_2 \leq 1\}$.

For each i^{th} constraint, MRC requires the optimum of the following problem non-positive.

$$\begin{aligned} \max_{\xi, \mathbf{s}, \lambda \geq 0} \quad & \mathbf{U}_i \mathbf{u} - b_i + \sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} \mathbf{V}_i \mathbf{v}_{\omega} \\ \text{s.t.} \quad & \|\mathbf{s}\|_2 \leq 1, \\ & \mathbf{F}\xi = \mathbf{s}, & \boldsymbol{\eta}_i \in \mathbb{R}^{n_q} \\ & \sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} \boldsymbol{\omega} = \mathbf{P}\xi, & \boldsymbol{\sigma}_i \in \mathbb{R}^{n_0} \\ & \sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} = 1, & \tau_i \in \mathbb{R} \end{aligned}$$

where $\xi \equiv [\mathbf{U}, \mathbf{b}]$ and $\boldsymbol{\eta}_i, \boldsymbol{\sigma}_i, \tau_i$ are dual multipliers corresponding to each group of constraints. Consider the corresponding Lagrangian

$$\begin{aligned} \mathcal{L}(\lambda, \xi, \mathbf{s}, \boldsymbol{\eta}_i, \tau_i, \boldsymbol{\sigma}_i) = & \mathbf{U}_i \mathbf{u} - b_i + \sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} \mathbf{V}_i \mathbf{v}_{\omega} + \boldsymbol{\eta}_i^T (\mathbf{s} - \mathbf{F}\xi) \\ & + \boldsymbol{\sigma}_i^T \left(\sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} \boldsymbol{\omega} - \mathbf{P}\xi \right) + \tau_i \left(\sum_{\omega \in \Omega} \lambda_{\omega}^{\xi} - 1 \right). \end{aligned}$$

The dual function is then $\max_{\lambda, \xi, \|\mathbf{s}\|_2 \leq 1} \mathcal{L}(\lambda, \xi, \boldsymbol{\eta}_i, \tau_i, \boldsymbol{\sigma}_i)$. Setting the derivative w.r.t. λ, ξ leads to the dual constraints

$$\begin{aligned} \mathbf{V}_i \mathbf{v}_{\omega} + \tau_i - \omega^T \boldsymbol{\sigma}_i &\leq 0, \\ \mathbf{F}^T \boldsymbol{\eta}_i - \mathbf{L}_i &= \mathbf{0}, \end{aligned} \tag{31}$$

where $\mathbf{P} = [\mathbf{P}_1, \dots, \mathbf{P}_k, \dots, \mathbf{P}_m]$, $\mathbf{L}_i = (\mathbf{L}_{i1}, \dots, \mathbf{L}_{im})$, $\mathbf{L}_{ij} = \delta_{ij}(\mathbf{u}, -1) + \mathbf{P}_j^T \boldsymbol{\sigma}_i$, $j = 1, \dots, m$. The dual objective is

$$\min_{\boldsymbol{\eta}_i, \tau_i, \boldsymbol{\sigma}_i} \|\boldsymbol{\eta}_i\|_2 - \tau_i. \tag{32}$$

By duality, the optimum of the above dual problem is equal to the optimum of the problem. Thus restricting the non-positivity of the primal optimum can be equivalently represented as

$$\begin{aligned} \|\boldsymbol{\eta}_i\|_2 + \mathbf{V}_i \mathbf{v}_{\omega} - \omega^T \boldsymbol{\sigma}_i &\leq 0, \\ \mathbf{F}^T \boldsymbol{\eta}_i - \mathbf{L}_i &= \mathbf{0}, \\ \boldsymbol{\eta}_i \in \mathbb{R}^{n_q}, \boldsymbol{\sigma}_i \in \mathbb{R}^{n_0}. \end{aligned} \tag{33}$$

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