On a quantum Hamiltonian in a unitary magnetic field with axisymmetric potential

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Abstract

We study a magnetic Schrödinger Hamiltonian, with axisymmetric potential in any dimension. The associated magnetic field is unitary and non constant. The problem reduces to a 1D family of singular Sturm-Liouville operators on the half-line indexed by a quantum number. We study the associated band functions. They have finite limits that are the Landau levels. These limits play the role of thresholds in the spectrum of the Hamiltonian. We provide an asymptotic expansion of the band functions at infinity. Each Landau level concerns an infinity of band functions and each energy level is intersected by an infinity of band functions. We show that among the band functions that intersect a fixed energy level, the derivative can be arbitrary small. We apply this result to prove that even if they are localized in energy away from the thresholds, quantum states possess a bulk component. A similar result is also true in classical mechanics.

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Introduction

General context

The motion of a spinless quantum particle in $\mathbb{R}^n$ is described by the spectral properties of the associated Hamiltonian. When the particle moves in a magnetic field, it is the magnetic Laplacian $(-i\nabla - \mathbf{A})^2$ acting on $\mathbf{L}^2(\mathbb{R}^n)$, where $\mathbf{A}$ is a magnetic potential.

One of the simplest example of magnetic field is the constant one. In the case $n \in \{2,3\}$, this model has been studied from the beginning of quantum mechanics [LL77] and also more recently for the general case $n \geqslant 2$ [HM96, RD01].

The variations of a non constant field can induce transport properties for the particle. In this context, we focus on magnetic fields that are translationally invariant along one direction. For such fields, the Hamiltonian has a band structure and transport properties in the direction of invariance are linked to the study of band functions (also called dispersion curves) that are the eigenvalues of the fibered operators. Moreover, the propagation of the particle in this direction is determined by the derivatives of these band functions that play the role of group velocities [Yaf08, EJK99].

In the case $n = 2$, one of the studied models of this class is the Iwatsuka model [Iwa85, MP97]. For $n = 3$, similar models are the planar translationally invariant magnetic fields [Yaf08, Rai08]. Let (r, θ, z) denote the cylindrical coordinates of $\mathbb{R}^3$. The potential takes the form $\mathbf{A}(r, \theta, z) = (0, 0, a(r))$, where $a : \mathbb{R} \to \mathbb{R}$ is the intensity of the potential. The associated magnetic field is therefore given by

$$\mathbf{B}(r, \theta, z) = a'(r)(-\sin(\theta), \cos(\theta), 0). \tag{1}$$

Thus this field is planar and its norm is $\|\mathbf{B}(r, \theta, z)\| = |a'(r)|$. Moreover the associated field lines are circles contained in planes $\{z = \mathrm{cst}\}$ with center on the invariant axis (see Figure 1).

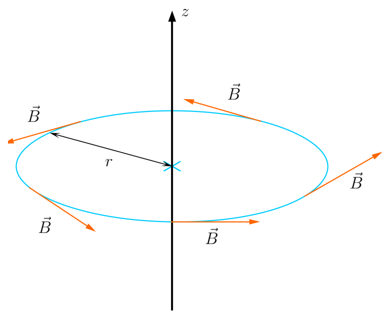


Figure 1: Schematic of translationally invariant magnetic fields.

In view of the form of the magnetic field (1), two specific cases are relevant. The first model consists of a magnetic field generated by an infinite rectilinear wire bearing a constant current [Yaf03, BP15]. If we assume that the wire coincides with the Oz axis, then the Biot & Savard law states that the generated magnetic fields writes as the field (1) for the intensity $a(r) = \ln(r)$. Here all the band functions are decreasing from $+\infty$ to 0. Hence the spectrum of H is $\sigma(H) = \mathbb{R}_+$. The band functions tend exponentially to 0 and it provides a reaction of the ground state energy of H under an electric perturbation [BP15]. Moreover the particle has a preferable direction of propagation along the Oz axis [Yaf03].

It is also natural to consider the case of a unitary magnetic field. For the field (1), it corresponds to the intensity $a(r) = r$. In this case the band functions tend to finite limits that are the Landau levels [Yaf08, Proposition 3.6]. Therefore the bottom of the spectrum of H is positive. An approximated value has been calculated and used to compare the energy on a wedge in a magnetic model and the one coming from the regular part of the wedge [Pop12, Pop15].

In this article we continue to study this magnetic field in the case $a(r) = r$ and we generalize the framework to any dimension $n \geqslant 3$. In particular we will show that the derivatives of the band functions possess an new type of behavior.

Spectral decomposition of the Hamiltonian and description of the model

For every $x \in \mathbb{R}^n$, we set $r := \|(x_1, \cdots, x_{n-1})\|_2$ and we define the magnetic potential $\mathbf{A}$ by

$$\mathbf{A}(x_1, \cdots, x_n) := (0, \cdots, 0, r). \tag{2}$$

We define the Hamiltonian as the following operator, self-adjoint in $\mathbf{L}^2(\mathbb{R}^n)$:

$$H := (-i\nabla - \mathbf{A})^2. \tag{3}$$

In order to define the magnetic field we consider, we identify this potential with the 1-differential form $r\mathrm{d}x_n$. We define the magnetic field $\mathbf{B}$ as $\mathbf{B} = \mathrm{d}\mathbf{A}$. We calculate $B_{j,k} = (\delta_{j,n} - \delta_{n,k}) x_j r^{-1}$, $(i,j) \in [\![1,n]\!]^2$. Therefore $\mathbf{B}$ is unitary since $2^{-1}\mathrm{Tr}(\mathbf{B}^*\mathbf{B}) = 2^{-1}\mathrm{Tr}^+(\mathbf{B}) = 1$ [HM96, Section 1].

After a partial Fourier transform in the x_n variable, H is unitarily equivalent to the direct integral in $\mathbf{L}^2\left(\mathbb{R}_\xi; \mathbf{L}^2\left(\mathbb{R}^{n-1}\right)\right)$ of the family of operators $H(\xi)$, self-adjoint in $\mathbf{L}^2\left(\mathbb{R}^{n-1}\right)$ and defined by

$$H(\xi) := -\Delta_{\mathbb{R}^{n-1}} + (r - \xi)^2. \tag{4}$$

Moreover for any frequency $\xi \in \mathbb{R}$, $H(\xi)$ reduces to the orthogonal sum over $m \in \mathbb{Z}_+$ (called the magnetic quantum numbers) of operators $H_m(\xi)$ self-adjoint in $\mathbf{L}^2\left(\mathbb{R}_+; r^{n-1}dr\right)$ and defined by

$$H_m(\xi) := -\frac{1}{r^{n-2}}\partial_r\left(r^{n-2}\partial_r\right) + \frac{m(m+n-3)}{r^2} + (r-\xi)^2.$$

The spectrum of each $H_m(\xi)$ is discrete. Let $\lambda_{m,p}(\xi)$, $p \in \mathbb{N}$ be the increasing sequence of its eigenvalues. The $\lambda_{m,p}$ are the band functions (also called dispersion curves).

We say that an operator A is fibered [RS78, Section XIII.16] if it can be written as

$$A := \int_M^{\oplus} A(\xi)\, d\xi,$$

with $(M, d\xi)$ a σ-finite measure space. An important class of fibered operators is the one of analytically fibered operators introduced in [GN98]. In this framework, M is a real analytic manifold and thresholds can be defined [GN98, Section 3]. They form a discrete set and their definition remains to stratification. Moreover away from them, some spectral results are rather standard. For example a limiting absorption principle as well as propagation estimates hold [GN98, Theorem 3.3] and it is tied to Mourre estimates. For a fibered operator A, we define the energy-momentum set Σ as

$$\Sigma := \{(\lambda, \xi) \in \mathbb{R} \times M, \quad \lambda \in \sigma(A(\xi))\}.$$

One of the conditions for the operator A to be analytically fibered in this sense is that the projection $\pi : \Sigma \to \mathbb{R}$ defined as $\pi((\lambda, \xi)) = \lambda$ is proper. Finally, notice that if M is a 1-dimensional manifold, then these thresholds correspond to the critical points of the band functions and can be referred to as attained thresholds [GS97, HM01, Soc01, BHRS09].

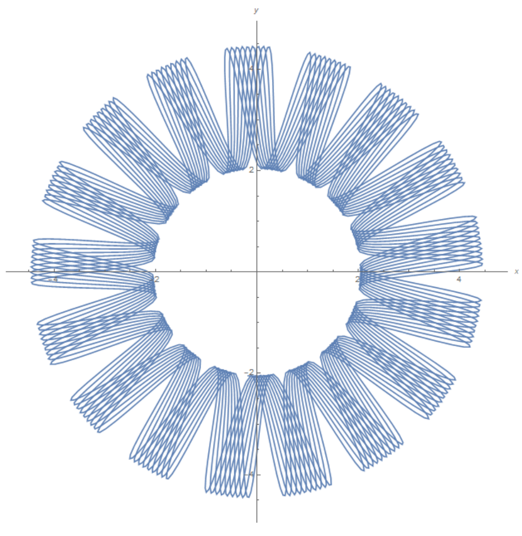
Other examples of fibered magnetic models can be found in the literature, in dimension 2 [Iwa85], on the half-plane [BMR14] or in dimension 3 [Yaf08]. In these models, the considered Hamiltonian is also fibered along $\mathbb{R}$ and the band functions tend to finite limits. The sets of frequencies associated with the energy levels concentrated in the neighborhood of these limits are unbounded. Hence the previous projection, π, is not proper. So these magnetic models are not contained in the class of analytically fibered operators that we described above. Nevertheless thresholds can still be defined as the limits of the band functions.

The model described in this article remains in this case. Indeed it is already known that the band functions tend to the Landau levels E_p [Yaf08, Proposition 3.6]. Our first goal is to precise the convergence of the band functions to these levels. To that aim we provide an asymptotic expansion for $\lambda_{m,p}(\xi)$ as $\xi \to +\infty$ (see Theorem 3.1). The method used to prove this theorem is inspired by the method of quasi-modes [DS99] that has already been used in the proof of similar result [BP15, HPS16].

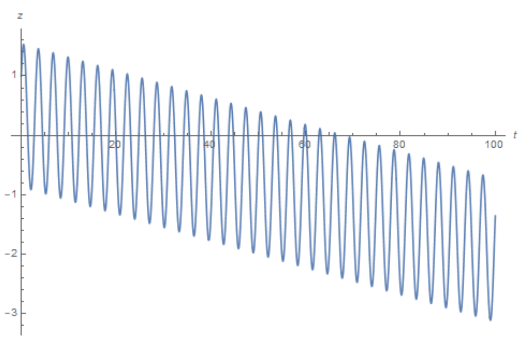
For the previous magnetic models, some studies of classical spectral problems already exists [MP97, DBP99, HS15, HPS16, PS16]. Our model contains one additional challenge. Actually for the Iwatsuka model and for half-plane model, each limit corresponds to a finite number of band functions. In this article, each threshold E_p is the limit of all the band functions $\lambda_{m,p}$ for $m \geqslant 0$. Therefore any interval of energy $I \subset \sigma(H)$ is intersected by an infinity of band functions (see equation (3.15)) and the set of frequencies $\left\{\lambda_{m,p}^{-1}(I), m \geqslant 0, p \in \mathbb{N}\right\}$ associated with I (even if $\bar{I}$ is away from the Landau levels) is unbounded (see proposition 3.2). Furthermore we will prove in Theorem 3.2 that even if $\bar{I}$ is away from the Landau levels, the group velocity tends to 0 at any energy as $m \to +\infty$. Therefore it is not clear at first sight that the Mourre estimates used in the case of the analytically fibered operators still hold. The proof of Theorem 3.2 use a convenient formula for the derivative $\lambda'_{m,p}$ (see Proposition 2.2) which links it to the normalized eigenfunctions of operators $H_m(\xi)$ as well as an exponential decay of these eigenfunctions that is uniform with respect to m and relies on Agmon estimates.

These properties have consequences on the transport properties associated with the magnetic field that we consider. Indeed these propreties are determined from the behavior of the velocity operator (see Definition (4.3)) that is linked to the multiplication operator by the family of the derivatives of the band functions (see formula (4.4)). In the previous models [Iwa85, BMR14], the velocity operator is bounded from below by a positive constant in any energy interval I, away from the thresholds. Hence every quantum state localized in energy away from them carries a non trivial current [MP97, DBP99] and is called edge state. Such an estimate does not hold anymore if the infimum of the derivatives of the band functions on I is 0. This situation occurs in the previous cases when there is a threshold in $\bar{I}$ [HPS16] and according to Theorem 3.2, everywhere in the model of our work. In this sense the definition of "thresholds" as the Landau levels seems not to be relevant. Indeed for the Iwatsuka model and for the Half-plane model, only the quantum state that are localized in energy around the thresholds possess a bulk component [MP97, HPS16]. In the case of the model considered here, we show in Theorem 4.1 that any quantum state, even if it is localized in energy away from the Landau levels can be decomposed into one state that carries an arbitrary small current and one that carries a non trivial one.

In classical mechanics, such a magnetic fields also induces transport properties. Indeed a charged particle follows the Newton law $m\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \wedge \mathbf{B}$. This equation can be integrated [Yaf03, Section 4] and we ploted the classical trajectories (Figure 2) in the case $a(r) = r$. We can observe that the particle propagate in the Oz direction and one can show that it has an effective velocity v_z in this direction: there is a constant v_z such that $z(t) = v_z t + O(1)$ [Yaf03, Theorem 4.2]. Furthermore, denote by E the total energy of the particle and by σ its areal velocity that are constants fixed by the initial conditions. These quantities satisfy the estimate $|v_z| \leqslant E^3\sigma^{-1}$. Moreover, for $(E,\sigma) \in \mathbb{R}_+ \times \mathbb{R}$, with $\sigma \neq 0$, one can find initial conditions such that E is the energy of the particle and σ its areal velocity. Therefore one can find initial conditions such that v_z is arbitrarily small, namely such that the particle propagate arbitrarily slowly along the Oz axis.



(a) Projection in the plan xOy.



(b) Plot of z in function of the time.

Figure 2: Trajectories of a charged particle moving in the magnetic field shown by Figure 1.

Organization

In Sections 1, the Hamiltonian is reduced to a family of 1D singular Sturm-Liouville operators. The band functions are introduced and described in Section 2. Section 3 presents the results concerning the asymptotic behaviors of these band functions. In Subsection 3.1, we prove Theorem 3.1 that provides an asymptotic expansion of $\lambda_{m,p}(\xi)$ as ξ gets large. Subsection 3.2 presents the asymptotic study of the derivative. In particular, Theorem 3.2 provides the asymptotic behavior of $\lambda'_{m,p}(\xi)$ as $m \to +\infty$ and as $\lambda_{m,p}(\xi)$ is fixed far from the Landau level E_p. In Section 4, we analyze the current carried by quantum states that are localized in energy far from the thresholds.

1 Reduction to one-dimensional Hamiltonians

In this section we define precisely the operators that we consider and we explain how H is reduced to 1 dimensional operators.

Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ be the magnetic potential given by definition (2) and let H be the self-adjoint Schrödinger operator (3). This operator is defined via its quadratic form

$$q(u) := \int_{\mathbb{R}^n} |-i\nabla u(x) - A(x)u(x)|^2 \, dx.$$

This form, initially defined on $\mathscr{C}_0^\infty(\mathbb{R}^n)$, is semi-bounded from below. Thus it admits a Friedrichs extension: H. Let q_ξ be the quadratic form defined by

$$q_\xi(u) := \int_{\mathbb{R}^{n-1}} \left(|(\nabla u)(x)|^2 + (-\xi + \|(x_1, \cdots, x_{n-1})\|_2)^2 |u(x)|^2 \right) dx, \quad \xi \in \mathbb{R}.$$

This form, initially defined on $\mathscr{C}_0^\infty\left(\mathbb{R}^{n-1}\right)$ and then closed in $\mathbf{L}^2\left(\mathbb{R}^{n-1}\right)$, is the quadratic form associated with the operator (4). Denotes by $\mathscr{F}$ the Fourier-transform with respect to x_n, which is defined by

$$(\mathscr{F}u)(\tilde{x}, \xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_n} u(\tilde{x}, x_n) \, dx_n, \quad (\tilde{x}, \xi) \in \mathbb{R}^n.$$

The forms q and q_ξ are related through the relation

$$q(u) = \int_{\mathbb{R}} q_\xi\left((\mathscr{F}(u))(\xi)\right) d\xi.$$

Therefore the operator H is decomposed as follows:

$$H = \mathscr{F}^{-1} \left(\int_{\mathbb{R}}^{\oplus} H(\xi) \, d\xi \right) \mathscr{F}.$$

We now reduce the problem to a 1-dimensional one using both the cylindrical symmetry and the following Laplace-Beltrami formula:

$$\Delta_{\mathbb{R}^{n-1}} = \frac{1}{r^{n-2}} \partial_r \left(r^{n-2} \partial_r \right) + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-2}}.$$

Recall that $-\Delta_{\mathbb{S}^{n-2}}$ is essentially self-adjoint on $\mathbf{L}^2\left(\mathbb{S}^{n-2}\right)$ and that its spectrum is discrete. Its eigenvalues are $\mu_m := m(m+n-3)$, $m \in \mathbb{Z}_+$. Denote by X_m the corresponding eigenspaces. Remember that X_m has finite dimension: N_m. The spaces $\mathbf{L}^2\left(\mathbb{R}_+; r^{n-2}dr\right) \otimes X_m$ are invariant by $H(\xi)$. In addition, the restrictions of the operator $H(\xi)$ on these spaces are identified with the operators

$$H_m(\xi) := -\frac{1}{r^{n-2}} \partial_r \left(r^{n-2} \partial_r \right) + \frac{\mu_m}{r^2} + (r - \xi)^2.$$

These operators act on $\mathbf{L}^2\left(r^{n-2}dr\right)$. They are associated with the bilinear forms

$$h_m(u, v) := \int_0^{+\infty} \left(u'(r) v'(r) + \frac{\mu_m}{r^2} u(r) v(r) + (r - \xi)^2 u(r) v(r) \right) r^{n-2} dr. \tag{1.1}$$

Denote by Φ the angular Fourier transform. The operator $H(\xi)$ is decomposed as:

$$H(\xi) = \Phi^{-1}\left(\bigoplus_{m\in\mathbb{N}} H_m(\xi)\right)\Phi.$$

Finally, it is more convenient to consider operators acting on the Hilbert space $\mathbf{L}^2(\mathbb{R}_+)$. To proceed we use the isometry $\phi : \mathbf{L}^2\left(\mathbb{R}_+; r^{n-2}dr\right) \to \mathbf{L}^2(\mathbb{R}_+; dr)$ defined by $(\phi u)(r) = r^{(n-2)/2}u(r)$. We define k_m as

$$k_m := \mu_m + \frac{n-2}{2}\left(\frac{n-2}{2} - 1\right) = \frac{(2m+n-3)^2 - 1}{4} \tag{1.2}$$

and the functions V_m as

$$V_m(r,\xi) := \frac{k_m}{r^2} + (r-\xi)^2, \quad (r,\xi) \in \mathbb{R}_+\setminus\{0\}\times\mathbb{R}. \tag{1.3}$$

So $H_m(\xi) = \phi^{-1}L_m(\xi)\,\phi$ where $L_m(\xi)$ is defined by

$$L_m(\xi) := -\partial_r^2 + V_m(r,\xi). \tag{1.4}$$

This operator acts on $\mathbf{L}^2(\mathbb{R}_+)$ with domain $\mathscr{D}(L_m(\xi)) = \phi(\mathscr{D}(H_m(\xi)))$. It is associated with the quadratic form

$$l_m(u,\xi) := \int_0^{+\infty}\left(\left|u'(r)\right|^2 + V_m(r,\xi)\,|u(r)|^2\right)dr. \tag{1.5}$$

2 Basics about the eigenpairs of the fiber operator

In this section we prove that the dispersion curves are analytic functions, we calculate their derivative and we investigate the behavior of the eigenfunctions at 0 .

2.1 Behavior of the eigenfunctions at 0

First we investigate the behavior of the functions of $\mathscr{D}(L_m(\xi))$ at 0, namely:

Lemma 2.1
Let $n \geqslant 3$, $m \in \mathbb{Z}_+$ and $\xi \in \mathbb{R}$.

$$\forall\varepsilon > 0, \quad \mathscr{D}(L_m(\xi)) \subset \left\{u \in \mathbf{L}^2(\mathbb{R}_+), \quad u \underset{r\to 0}{=} o\left(r^{\frac{1}{2}-\varepsilon}\right)\right\}. \tag{2.1}$$

Moreover

$$\begin{aligned} &\textit{if } n = 3, \quad \mathscr{D}(L_m(\xi)) \subset \left\{u \in \mathbf{L}^2(\mathbb{R}_+), \quad u(r) \underset{r\to 0}{=} O\left(\sqrt{r}\right)\right\};\\ &\textit{if } n = 4, \quad \mathscr{D}(L_m(\xi)) \subset \left\{u \in \mathbf{L}^2(\mathbb{R}_+), \quad u(r) \underset{r\to 0}{=} O(r)\right\}. \end{aligned} \tag{2.2}$$

of (2.1). The bilinear form associated with $H_m(\xi)$ is given by relation (1.1). For every $u \in \mathscr{D}(H_m(\xi))$ and every $v \in \mathscr{D}(h_m)$, we have $\langle H_m(\xi)u, v\rangle = h_m(u,v)$. Notice that $\mathscr{D}(h_m) \subset \mathbf{H}^1(\mathbb{R}_+)$. We integrate by part the first term of the form h_m which yields:

$$\lim_{r\to 0} u'(r)\,v(r)\,r^{n-2} = 0, \quad u \in \mathscr{D}(H_m(\xi)),\, v \in \mathscr{D}(h_m).$$

We apply this formula to an arbitrary function $u \in \mathscr{D}(H_m(\xi))$ and to functions $v_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}_+) \cap \mathscr{D}(h_m)$ that satisfy for any $\varepsilon > 0$

$$\begin{aligned} &v_\varepsilon(r) = r^{\frac{3-n}{2}+\varepsilon}, && \text{if } r \in (0,1);\\ &v_\varepsilon(r) = 0, && \text{if } r \geqslant 2. \end{aligned}$$

We deduce that

$$u'(r) \underset{r\to 0}{=} o\left(\frac{1}{r^{\frac{n-1}{2}+\varepsilon}}\right), \quad \varepsilon > 0,\, u \in \mathscr{D}(H_m(\xi)).$$

Therefore integrating this condition, we deduce that

$$u(r) \underset{r\to 0}{=} o\left(r^{\frac{3-n}{2}-\varepsilon}\right), \quad \varepsilon > 0, u \in \mathscr{D}(H_m(\xi)).$$

Thus remembering that $\mathscr{D}(L_m(\xi)) = \phi(\mathscr{D}(H_m(\xi)))$, we conclude that relation (2.1) holds. □

of (2.2). Note that $\mathscr{D}(H(\xi)) \subset \mathbf{H}^2(\mathbb{R}^{n-1})$. So if $n \in \{3,4\}$ then awing to a Sobolev embedding, $\mathbf{H}^2(\mathbb{R}^{n-1}) \subset \mathbf{L}^\infty(\mathbb{R}^{n-1})$. Hence $\mathscr{D}(H(\xi)) \subset \mathbf{L}^\infty(\mathbb{R}^{n-1})$. Thus if $u \in \mathscr{D}(H_m(\xi))$, then $u(r)$ is bounded as $r \to 0$. Combine it with the fact that $\mathscr{D}(L_m(\xi)) = \phi(\mathscr{D}(H_m(\xi)))$ and it provides the embedding (2.2).

□

Notice that $V_m(r,\xi) \to +\infty$ as $r \to +\infty$. Therefore the operator $L_m(\xi)$ has compact resolvent. So for every $\xi \in \mathbb{R}$ and for every $m \in \mathbb{Z}_+$ the spectrum of $L_m(\xi)$ is an increasing sequence of positive eigenvalues $\lambda_{m,p}(\xi)$, $p \in \mathbb{N}$. We conclude this subsection by proving the following proposition.

Proposition 2.1 (Behavior of the eigenfunctions at 0**)**
Let $\xi \in \mathbb{R}$, $m \in \mathbb{Z}_+$ and $p \in \mathbb{N}$. The eigenvalue $\lambda_{m,p}(\xi)$ is non-degenerate. Let $u_{m,p}(\cdot,\xi)$ be the normalized eigenfunction associated with it. There exists an analytic function f such that $f(0) \neq 0$ and such that in a neighborhood of 0*,*

$$u_{m,p}(r,\xi) = r^{\frac{1+|2m+n-3|}{2}} f(r). \tag{2.3}$$

Proof. First, consider the differential equation

$$r^2 u''(r) + \left(r^2\left(\lambda_{m,p}(\xi) - (r-\xi)^2\right) - k_m\right) u(r) = 0, \quad r > 0. \tag{2.4}$$

We look for solutions that admit a series expansion in a neighborhood of 0. By the Frobenius method, if a solution u is given by $u(r) = r^\nu f(r)$ where f is an analytic function such that $f(0) \neq 0$, then ν satisfies the indicial equation

$$\nu(\nu-1) = k_m.$$

This equation has $\nu_\pm := (1 \pm (2m+n-3))/2$ as solutions. Thus the equation (2.4) admits a solution of the form $u_+(r) = r^{\nu_+} f(r)$ with f an analytic function such that $f(0) = 1$. In order to have a basis of solutions for equation (2.4) we look for a solution of the form $u_- = hu_+$. By straightforward calculations we find that $h'(r) = Ku_+^{-2}(r) \sim Kr^{-1-|2m+n-3|}$ as $r \to 0$, so

- if $(n,m) = (3,0)$, then $u_-(r) \underset{r\to 0}{\sim} K\log(r)\sqrt{r}$,
- in the other cases, $u_-(r) \underset{r\to 0}{\sim} Kr^{\nu_-}$.

Finally, we deduce from Lemma 2.1 that in both cases $u_- \notin \mathscr{D}(L_m(\xi))$. Hence $\ker(L_m(\xi) - \lambda_{m,p}(\xi)) = \operatorname{span}(u_+)$. This concludes the proof since $\lambda_{m,p}(\xi)$ is an eigenvalue of $L_m(\xi)$. □

Remark 2.1: We deduce from this proposition that the embedding (2.2) is optimal.

According to Proposition 2.1, the eigenvalues $\lambda_{m,p}(\xi)$ are non degenerate. Moreover, $L_m(\xi)$ is a Kato analytic family [Kat66, Chapter VII]. Therefore it follows from Proposition 2.1 that $\lambda_{m,p}$ are real analytic functions that are called band functions.

2.2 Derivative of the band functions

Here we give a formula for the derivative of the band functions.

Proposition 2.2
Let, for $(\xi,m,p) \in \mathbb{R}\times\mathbb{Z}_+\times\mathbb{N}$, $K_{m,p}(\xi) := \lim_{r\to 0} \frac{u_{m,p}(r,\xi)^2}{r}$. The derivative $\lambda'_{m,p}(\xi)$ is given by:

$$\lambda'_{m,p}(\xi) = \begin{cases} -\int_0^{+\infty} \frac{1}{r^2}\left[\frac{u_{m,p}(r,\xi)^2}{r} - K_{0,p}\right] dr & \textit{if } n=3 \textit{ and } m=0, \\ -\left|u'_{m,p}(0,\xi)\right|^2 & \textit{if } n=4 \textit{ and } m=0, \\ -2k_m \int_0^{+\infty} \frac{|u_{m,p}(r,\xi)|^2}{r^3} dr & \textit{in the other cases.} \end{cases}$$

Proof. In the case $n=3$, this proposition has already been proved [Yaf08, Theorem 4.3]. The way to prove it in the general case is the same as in this particular case so we refer to this proof for more details. We still present the main ideas of the proof.

The Feynman-Hellmann formula [MR88] yields that

$$\lambda'_{m,p}(\xi) = \int_{\mathbb{R}_+} \partial_\xi \left((r-\xi)^2\right) |u_{m,p}(r,\xi)|^2 dr = -\int_{\mathbb{R}_+} \partial_r \left((r-\xi)^2\right) |u_{m,p}(r,\xi)|^2 dr. \tag{2.5}$$

We apply integrations by parts to get the result. We use the super-exponential decay of eigenfunctions $u_{m,p}(\cdot,\xi)$ for handling the non-integral terms corresponding to $r \to +\infty$ [Shn57, Olv97] and Proposition 2.1 for handling the non-integral term at $r=0$. In the particular case $(n,m)=(3,0)$, the result of Proposition 2.1 is not sharp enough. In order to improve it, we inject the identity (2.3) into the following eigenvalue equation:

$$L_0(\xi)\, u_{0,p}(r,\xi) = \lambda_{0,p}(\xi)\, u_{0,p}(r,\xi).$$

Therefore we obtain that $u_{m,p}(r,\xi)^2 r^{-1} - K_{0,p} = O\left(r^2\right)$ as $r \to 0$ and we use it for handling non-integral term at $r=0$. □

2.3 Global behavior of the band functions

The min-max principle implies that

$$\lambda_{m,p}(\xi) \underset{\xi\to-\infty}{\sim} \xi^2.$$

Indeed first note that if $\xi \leqslant 0$, then $L_m(\xi) \geqslant \xi^2$. Therefore

$$\lambda_{m,p}(\xi) \geqslant \xi^2, \quad \xi \leqslant 0.$$

On the other hand, we define for $\varepsilon > 0$ the operator $G(\varepsilon)$, self-adjoint on $\mathbf{L}^2(\mathbb{R}_+)$,

$$G(\varepsilon) := -\partial_r^2 + \frac{k_m}{r^2} + \left(1+\frac{1}{\varepsilon}\right) r^2.$$

This operator has compact resolvent, therefore its spectrum is discrete. Let $(\nu_q(\varepsilon))_{q\in\mathbb{N}}$ be the increasing sequence of its eigenvalues. Note that $L_m(\xi) \leqslant G(\varepsilon) + (1+\varepsilon)\xi^2$. Hence, for any $p \in \mathbb{N}$, $\lambda_{m,p}(\xi) \leqslant \nu_p(\varepsilon) + (1+\varepsilon)\xi^2$. Thus,

$$\forall \varepsilon > 0, \quad \limsup_{\xi\to-\infty} \frac{\lambda_{m,p}(\xi)}{\xi^2} \leqslant 1+\varepsilon. \tag{2.6}$$

From Proposition 2.2 we deduce that if $(n,m) \neq (3,0)$, then for every $p \in \mathbb{N}$, $\lambda'_{m,p}$ is negative on $\mathbb{R}$. Therefore in this case the band functions are decreasing. So these functions admit finite limits at $+\infty$. In the case $n=3$ the min-max principle yields that these limits are the Landau levels [Yaf08, Proposition 3.6], namely

$$\lim_{\xi\to+\infty} \lambda_{m,p}(\xi) = E_p := 2p-1, \quad p \in \mathbb{N}. \tag{2.7}$$

This proof is still valid if $n>3$ and Subsection 3.1 provides an asymptotic expansion of $\lambda_{m,p}(\xi)$ when ξ tends to $+\infty$. In the case $n=3$ then $k_0 = -4^{-1} < 0$. Therefore we will deduce from Theorem 3.1 (see remark 2) that for every $p \in \mathbb{N}$, $\lambda_{0,p}$ admits local minima (the question of the number of minima stays open). In the other cases, according to Proposition 2.2, for every $p \in \mathbb{N}$, $\lambda_{m,p}$ is decreasing from $+\infty$ to E_p.

Numerical approximation. We use a finite difference method to compute numerical approximations of the band function $\lambda_{m,p}(\xi)$ with $n = 5$, $m \in [\![0,6]\!]$ and $p \in [\![1,3]\!]$. We compute for $\xi \in [-1,6]$ on the interval $[0,20]$ with an artificial Dirichlet boundary condition at $r = 20$.

On Figure 3, we have ploted the numerical approximation of $\lambda_{m,p}(\xi)$ for $\xi \in [-1,6]$, $m \in [\![0,3]\!]$ and $p \in [\![1,3]\!]$. According to the theory, $\lambda_{m,p}$ decrease from $+\infty$ to $E_p = 2p-1$. We also ploted this level. Note that different band function may intersect for different values of m.

Figure 4 presents a zoom on the first level: $p = 1$ for $\xi \in [-1,6]$ and $m \in [\![0,6]\!]$.

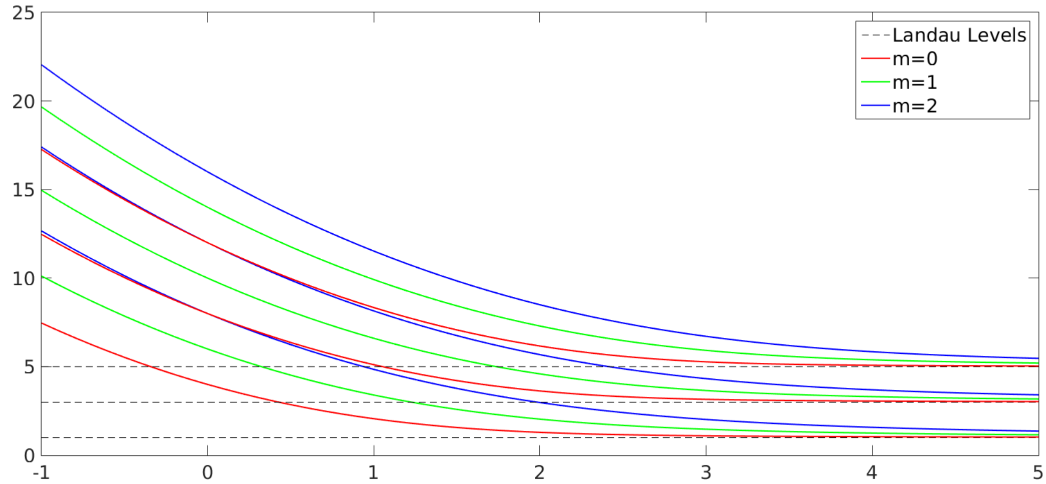


Figure 3: Plot of the band functions $\lambda_{m,p}(\xi)$ for $n = 5$, $0 \leqslant m \leqslant 3$, $1 \leqslant p \leqslant 3$ and $\xi \in [-1,5]$.

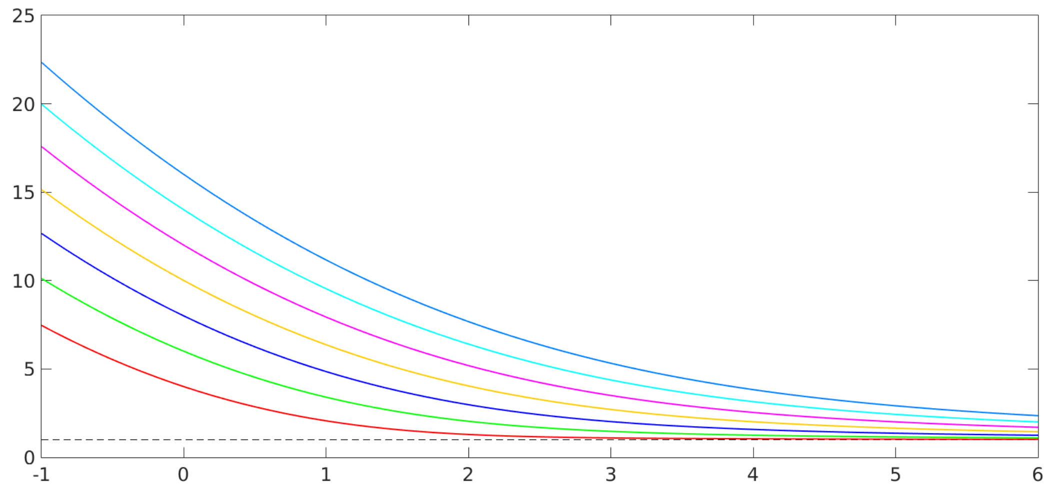


Figure 4: Plot of the band functions $\lambda_{m,1}(\xi)$ for $n = 5$, $0 \leqslant m \leqslant 6$ and $\xi \in [-1,6]$.

3 Asymptotic behaviour of the band functions

In this section we provide an asymptotic expansion for the band functions and their derivative. First we provide an asymptotic expansion for $\lambda_{m,p}(\xi)$ as $\xi \to +\infty$ with m and p fixed. In a second time we estimate the behavior of $\lambda'_{m,p}(\xi)$ as p is fixed and as m and ξ tend to $+\infty$ and are related to eachother by the condition $\lambda_{m,p}(\xi)=E$ where E is a constant.

3.1 Near thresholds: high frequency

In this subsection we study the behaviour of the spectrum of H near the thresholds. Namely we describe the behaviour of $\lambda_{m,p}(\xi)$ when m and p are fixed and $\xi\to+\infty$. More precisely, this subsection is devoted to the proof of the following theorem.

Theorem 3.1 (Asymptotic expansion of the band functions)
Let $m\in\mathbb{Z}_+$ and $p\in\mathbb{N}$. There is a sequence of real numbers $\left(\alpha_q^p\right)_{q\in\mathbb{N}}$ such that

$$\forall N\geqslant 0,\ \exists C>0,\ \exists\xi_0>0,\ \forall\xi\geqslant\xi_0,\qquad \left|\lambda_{m,p}(\xi)-E_p-k_m\sum_{q=1}^{N}\frac{\alpha_q^p}{\xi^q}\right|\leqslant\frac{C}{\xi^{N+1}}.$$

To prove this theorem we consider the operators $L_m(\xi)$ defined by relation (1.4) and we apply the method of the harmonic approximation [Hel88, DS99] to derive an asymptotic expansion of its eigenvalues.

Remark 3.1: In the case $k_m=0$, that is $(n,m)=(4,0)$, Theorem 3.1 states that $\lambda_{m,p}(\xi)=E_p+O(\xi^{-\infty})$, as $\xi\to+\infty$. In this case, the operator is $-\partial_r^2+(r-\xi)^2$ with Dirichlet boundary condition at 0. This operator has already been studied and we know [HPS16, Theorem 1.4] [Ivr18, Section 15.A] that there are some constant $\gamma_p>0$ such that

$$\lambda_{0,p}(\xi)\underset{\xi\to+\infty}{=}E_p+\gamma_p\xi^{2p-1}e^{-\xi^2}\left(1+O\left(\xi^{-2}\right)\right).$$

So we focus on the proof in the particular case $k_m\neq 0$.

Remark 3.2: We compute that $\alpha_1=0$ and $\alpha_2=1$. Therefore for $N=2$, Theorem 3.1 yields

$$\lambda_{m,p}(\xi)=E_p+\frac{k_m}{\xi^2}+O\left(\frac{1}{\xi^3}\right).$$

In the case $n=3$ and $m=0$, $k_m=-4^{-1}<0$. Therefore for every $p\in\mathbb{N}$, $\lambda_{0,p}(\xi)$ tend to E_p from below. Hence the $\lambda_{0,p}$ have local minima.

Canonical transformation and asymptotic expansion of the operator
For $\xi\in\mathbb{R}_+$ we apply the change of variable $s=r-\xi$. It shows that $L_m(\xi)$ is unitarily equivalent to the following operator acting on $\mathbf{L}^2(-\xi,+\infty)$:

$$\tilde{L}_m(\xi)=-\partial_s^2+\frac{k_m}{(s+\xi)^2}+s^2=-\partial_s^2+s^2+\frac{k_m}{\xi^2}\frac{1}{\left(1+\frac{s}{\xi}\right)^2}.$$

A Taylor expansion of the potential for large ξ provides

$$\frac{k_m}{(s+\xi)^2}=\frac{k_m}{\xi^2}\sum_{q=0}^{N}(q+1)\left(\frac{-s}{\xi}\right)^q+R_N(s,\xi),\quad N\geqslant 0. \tag{3.1}$$

Estimation on the remainder term $R_N(s,\xi)$ will be written later (see equation (3.8)). We define a sequence of formal operators by

$$\begin{cases} H_0:=-\partial_s^2+s^2,\\ A_1:=0,\\ \forall q\geqslant 2, A_q:=(q-1)(-s)^{q-2}.\end{cases}$$

For every $N\in\mathbb{N}$, we set

$$\tilde{L}_m^N(\xi):=H_0+k_m\sum_{q=1}^{N}\frac{A_q}{\xi^q}, \tag{3.2}$$

with the convention $\tilde{L}_m^0 = H_0$. We set $R_{-2}(s,\xi) = R_{-1}(s,\xi) := k_m (s+\xi)^{-2}$. For every $N \geqslant 0$, the operator $\tilde{L}_m(\xi)$ can be formally decomposed into:

$$\tilde{L}_m(\xi) = \tilde{L}_m^N(\xi) + R_{N-2}(s,\xi).$$

First we look for quasi-modes for the formal operator $\tilde{L}_m^N(\xi)$ acting on $\mathbf{L}^2(\mathbb{R})$. This formal procedure provides functions defined on $\mathbb{R}$ and we use a suitable cut-off function in $(-\xi,+\infty)$ to derive quasi-modes for $\tilde{L}_m(\xi)$.

Calculation of the quasi-modes

We look for quasi-eigenpairs $\left(\lambda_m^N(\xi), f_m^N(\cdot,\xi)\right)$ of $\tilde{L}_m^N(\xi)$ of the form

$$\left(\lambda_m^N(\xi), f_m^N(\cdot,\xi)\right) = \left(\alpha_0 + k_m \sum_{q=1}^{N} \frac{\alpha_q}{\xi^q}, \sum_{q=0}^{N} \frac{g_q}{\xi^q}\right),$$

where the functions g_q are mutually orthogonal in $\mathbf{L}^2(\mathbb{R})$. Note that the functions g_q may depend on k_m. We are led to solve the system

$$\begin{aligned}
&(H_0 - \alpha_0) g_0 = 0, \\
&(H_0 - \alpha_0) g_q + k_m \sum_{j=1}^{q} (A_j - \alpha_j) g_{q-j} = 0, \quad q \in [\![1,N]\!].
\end{aligned} \tag{3.3}$$

We solve it by induction:

- $\mathbf{q = 0}$
  Note that H_0 is the quantum harmonic oscillator. Hence we choose for (α_0, g_0) a couple (E_p, Ψ_p) for $p \in \mathbb{N}$ where $E_p = 2p-1$ is a Landau level, and Ψ_p is the corresponding normalised Hermite function with the convention that $\Psi_1(s) = (2\pi)^{-1/4} e^{-t^2/2}$. So from now on we set $(\alpha_0, g_0) = \left(\alpha_0^p, g_0^p\right) = (E_p, \Psi_p)$ for a certain $p \in \mathbb{N}$, fixed. All the quantities considered in what follows may depend on the choice of p. We simplify the notations with ommiting this index.

- **Induction**
  We assume that there exists $q_0 \in [\![1,N]\!]$ such that for every $q \leqslant q_0 - 1$, α_q and g_q have been constructed.

  The scalar product of the second equation of the system (3.3) with g_0 provides the value of α_{q_0}:

  $$\alpha_{q_0} = \left\langle A_{q_0} g_0, g_0 \right\rangle + \left\langle \sum_{q=1}^{q_0-1} (A_q - \alpha_q) g_{q_0-q}, g_0 \right\rangle.$$

  So α_{q_0} is known, therefore the Fredholm alternative provides a unique value for g_{q_0} such that $\left\langle g_{q_0}, g_q \right\rangle = 0$ for every $q < q_0$.

The quasi-modes $f_m^N(\cdot,\xi)$ can be computed using the Hermite functions. The Hermite functions satisfy the following results

$$\begin{aligned}
&\forall q \geqslant 1, \exists P \in \mathbb{R}[X], \forall s \in \mathbb{R}, \quad \Psi_q(s) = e^{-\frac{s^2}{2}} P(s), \\
&\forall q \geqslant 1, \quad s\Psi_q(s) = \sqrt{\frac{q-1}{2}} \Psi_{q-1}(s) + \sqrt{\frac{q}{2}} \Psi_{q+1}(s).
\end{aligned}$$

Combining them with the system (3.3) we infer that for every $N \geqslant 0$, there exist polynomial functions $P_0, \cdots, P_N$ such that

$$f_m^N(s,\xi) = e^{-\frac{s^2}{2}} \sum_{q=0}^{N} \frac{P_q(s)}{\xi^q}, \quad \xi > 0, s \in \mathbb{R}. \tag{3.4}$$

Evaluation of the quasi-mode

Previously we have obtained quasi-eigenpairs $\left(\lambda_m^N(\xi), f_m^N(\cdot,\xi)\right)$ for $\tilde{L}_m^N(\xi)$. The functions f_m^N are defined on $\mathbb{R}$. We now use a suitable cut-off function to get quasi-modes $u_m^N(\cdot,\xi)$ for $\tilde{L}_m(\xi)$.

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R};[0,1])$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1/2; \\ 0 & \text{if } |x| \geqslant 1. \end{cases}$$

For $\xi \in \mathbb{R}_+$, we define the cut-off function χ_ξ on $\mathbb{R}$ by

$$\chi_\xi(t) := \chi\left(\frac{2t}{\xi}\right), \quad t \in \mathbb{R}. \tag{3.5}$$

Note that this function is supported in $(-\xi/2, \xi/2)$ and is equal to 1 on $(-\xi/4, \xi/4)$. Let, for $N \geqslant 0$, u_m^N be defined by

$$u_m^N(r,\xi) := \chi_\xi(r) f_m^N(r,\xi), \quad \xi > 0, r \in \mathbb{R}. \tag{3.6}$$

Since $\operatorname{supp}\left(u_m^N(\cdot,\xi)\right) \subset \operatorname{supp}\left(\chi_\xi\right) \subset (-\xi/2, \xi/2)$, u_m^N can be used as a quasi-mode for $\tilde{L}_m(\xi)$.

Lemma 3.1 (Control of the quasi-mode)

Let $N \in \mathbb{Z}_+$. Recalling that m, N and p are fixed, there is a constant $K \geqslant 0$ such that

$$\exists \xi_0 > 0, \forall \xi \geqslant \xi_0, \quad \left\| \left(\tilde{L}_m(\xi) - \lambda_m^N(\xi)\right) u_m^N(\cdot,\xi) \right\|_2 \leqslant \frac{K}{\xi^{N+1}}$$

Proof. First, observe that

$$\begin{aligned} \left\| \left(\tilde{L}_m(\xi) - \lambda_m^N(\xi)\right) u_m^N(\cdot,\xi) \right\|_2 \leqslant & \left\| \chi_\xi \left(\tilde{L}_m^N(\xi) - \lambda_m^N(\xi)\right) f_m^N(\cdot,\xi) \right\|_2 + \left\| R_{N-2}(\cdot,\xi) u_m^N(\cdot,\xi) \right\|_2 \\ & + \left\| \left[\tilde{L}_m^N(\xi), \chi_\xi\right] f_m^N(\cdot,\xi) \right\|_2 . \end{aligned} \tag{3.7}$$

We proceed to control the right hand side term by term:

- We use the definition of f_m^N to compute the first term:

$$\left(\tilde{L}_m^N(\xi) - \lambda_m^N(\xi)\right) f_m^N(\cdot,\xi) = \sum_{q=N+1}^{2N} \frac{k_m}{\xi^q} \sum_{i+j=q} (A_i - \alpha_i) g_j .$$

  Thus we deduce that

$$\exists K > 0, \quad \left\| \chi_\xi \left(\tilde{L}_m^N(\xi) - \lambda_m^N(\xi)\right) f_m^N(\cdot,\xi) \right\|_2 \leqslant \frac{K k_m}{\xi^{N+1}} .$$

  Note that K may depend on m.

- Remind that $R_N(s,\xi)$ is defined by relation (3.1), the localization of $\operatorname{supp}\left(\chi_\xi\right)$ provides the following estimate:

$$\begin{aligned} & \exists C > 0, \forall s \in \operatorname{supp}\left(\chi_\xi\right), \quad \left|R_{N-2}(s,\xi)\right| \leqslant \frac{C s^{N+1}}{\xi^{N+1}}, \quad \text{if } N \geqslant 2; \\ & \forall s \in \operatorname{supp}\left(\chi_\xi\right), \quad \left|R_{N-2}(s,\xi)\right| \leqslant 4\frac{k_m}{\xi^2}, \quad \text{if } N \in \{0,1\}. \end{aligned}$$

  Hence using the exponential decay of Hermite functions, we deduce from the definition of u_m^N and from relation (3.4) that

$$\begin{aligned} \left\| R_{N-2}(\cdot,\xi) u_m^N(\cdot,\xi) \right\|_2 & \leqslant \frac{K}{\xi^{N+1}} \quad \text{if } N \geqslant 2, \\ \left\| R_{N-2}(\cdot,\xi) u_m^N(\cdot,\xi) \right\|_2 & \leqslant 4\frac{k_m}{\xi^2} \quad \text{if } N \in \{0,1\}. \end{aligned} \tag{3.8}$$

- Finally notice that $\left[\tilde{L}_m^N(\xi), \chi_\xi\right] f_m^N(\cdot,\xi) = 2\chi_\xi' \left(f_m^N\right)'(\cdot,\xi) + \chi_\xi'' f_m^N(\cdot,\xi)$. Moreover, χ_ξ' and χ_ξ'' are supported in $\{t \in \mathbb{R}, \quad \xi/4 < |t| < \xi/2\}$. Therefore we deduce from formula (3.4) that

$$\left\| \left[\tilde{L}_m^N(\xi), \chi_\xi\right] f_m^N(\cdot,\xi) \right\|_2 = O\left(\frac{1}{\xi^\infty}\right).$$

$\square$

Proof of Theorem 3.1
We deduce from the spectral theorem and from Lemma 3.1 that

$$d\left(\lambda_m^N(\xi),\sigma(L_m(\xi))\right)\left\|u_m^N(\cdot,\xi)\right\|_{\mathbf{L}^2(-\xi,+\infty)}\leqslant\frac{K}{\xi^{N+1}}.$$

Moreover $\left\|u_m^N(\cdot,\xi)\right\|_{\mathbf{L}^2(-\xi,+\infty)}=\left\|f_m^N(\cdot,\xi)\right\|_{\mathbf{L}^2(\mathbb{R})}+O(\xi^{-\infty})$ and $\left\|f_m^N(\cdot,\xi)\right\|_{\mathbf{L}^2(\mathbb{R})}=1+O\left(\xi^{-2}\right)$. Therefore

$$\lim_{\xi\to+\infty}\left\|u_m^N(\cdot,\xi)\right\|_{\mathbf{L}^2(-\xi,+\infty)}=1 \tag{3.9}$$

Hence for ξ large enough

$$d\left(\lambda_m^N(\xi),\sigma(L_m(\xi))\right)\leqslant\frac{\tilde{K}}{\xi^{N+1}}.$$

Finally we observe that $\lambda_m^N(\xi)\to E_p$, as $\xi\to+\infty$. We combine it with the identity (2.7) that provides the statement of the theorem.

3.2 Near other energy levels: high frequency and high angular momentum

We are now interested in the behaviour of the spectrum of H near other energy levels. First if ξ is fixed, then $\lambda_{m,p}(\xi)$ tends to $+\infty$ as $m\to+\infty$. In a second time we study the behaviour of the band functions when m and ξ tends to $+\infty$ together. More precisely we fix an integer p and an energy level $E>E_p$ and we study the behaviour of $\lambda'_{m,p}(\xi)$ when $\lambda_{m,p}(\xi)=E$.

Remember that the quadratic form defined by equation (1.5) is associated to $L_m(\xi)$ and that $u_{m,p}(\cdot,\xi)$ denotes the normalized eigenfunction of $L_m(\xi)$ associated with the eigenvalue $\lambda_{m,p}(\xi)$. Therefore,

$$\lambda_{m,p}(\xi)=\int_0^{+\infty}\left|u'_{m,p}(r,\xi)\right|^2+V_m(r,\xi)\,u_{m,p}(r,\xi)^2\,dr. \tag{3.10}$$

Moreover k_m (resp. V_m) is defined by relation (1.2) (resp. relation (1.3)). Hence for every $m>0$, $k_m\geqslant 0$. Therefore the following useful estimates are valid for every $m>0$:

$$\lambda_{m,p}(\xi)\geqslant k_m\int_0^{+\infty}\frac{|u_{m,p}(r,\xi)|^2}{r^2}dr, \tag{3.11}$$

$$\lambda_{m,p}(\xi)\geqslant\int_0^{+\infty}(r-\xi)^2\,|u_{m,p}(r,\xi)|^2\,dr. \tag{3.12}$$

Proposition 3.1 (Limit of the band functions)
For every $p\in\mathbb{N}$ and every $\xi\in\mathbb{R}$,

$$\lim_{m\to+\infty}\lambda_{m,p}(\xi)=+\infty.$$

Proof. We simplify the notations by omitting the index p. According to estimate (3.11),

$$\lambda_m(\xi)\geqslant\frac{k_m}{R_0^2}\int_0^{R_0}|u_m(r,\xi)|^2\,dr=\frac{k_m}{R_0^2}\left(1-\int_{R_0}^{+\infty}|u_m(r,\xi)|^2\,dr\right),\quad R_0>0. \tag{3.13}$$

Moreover, if $R_0\geqslant\xi$, then

$$\int_0^{+\infty}(r-\xi)^2\,|u_m(r,\xi)|^2\geqslant(R_0-\xi)^2\int_{R_0}^{+\infty}|u_m(r,\xi)|^2\,dr.$$

Therefore, from estimate (3.12) we deduce that

$$\lambda_m(\xi)\geqslant(R_0-\xi)^2\int_{R_0}^{+\infty}|u_m(r,\xi)|^2\,dr,\quad R_0\geqslant\xi. \tag{3.14}$$

Therefore, combining estimates (3.13) and (3.14) we obtain

$$\lambda_m(\xi) \geqslant \frac{k_m}{R_0^2}\left(1-(R_0-\xi)^{-2}\lambda_m(\xi)\right), \quad R_0 \geqslant \xi.$$

Hence, recalling that $k_m \to +\infty$ as $m \to +\infty$, we deduce that

$$\exists M \in \mathbb{N}, \forall m \geqslant M, \quad \lambda_m(\xi) \geqslant \frac{k_m}{R_0^2}\left(1+k_m\left(R_0(R_0-\xi)\right)^{-2}\right)^{-1} \geqslant \frac{(R_0-\xi)^2}{2}.$$

This is true for all $R_0 > \xi$. So letting R_0 tend to $+\infty$ provides the result. □

We now study $\lambda'_{m,p}(\xi)$. Remember that for any $m \in \mathbb{N}$ and for any $p \in \mathbb{N}$, $\lambda_{m,p}$ is decreasing from $+\infty$ to E_p. Therefore

$$\forall m \in \mathbb{N}, \forall p \in \mathbb{N}, \forall E > E_p, \quad \exists! \xi_m \in \mathbb{R}, \quad E = \lambda_{m,p}(\xi_m). \tag{3.15}$$

3.2.1 Preliminary results: some localization properties

First we look for the behaviour of ξ_m when m tends to $+\infty$.

Proposition 3.2 (Control of ξ_m)
There exist constants $K_\pm > 0$ and $M > 0$ such that as m gets large,

$$K_- \sqrt{k_m} \leqslant \xi_m \leqslant K_+ \sqrt{k_m}.$$

To get the lower bound, we use formula (3.10) and we localize the normalized eigenfunctions $u_m := u_m(\cdot, \xi_m)$ of $L_m := L_m(\xi_m)$.

of the lower bound. Let $\alpha \in [0,1)$ and let $R_m(\alpha) := \sqrt{k_m \alpha E^{-1}}$. We inject $\lambda_m(\xi_m) = E$ into estimate (3.11). It yields

$$\frac{E}{k_m} \geqslant \int_0^{+\infty} \frac{|u_m(r)|^2}{r^2} dr \geqslant \int_0^{R_m(\alpha)} \frac{|u_m(r)|^2}{r^2} dr \geqslant \frac{1}{R_m(\alpha)^2} \int_0^{R_m(\alpha)} |u_m(r)|^2 dr.$$

So

$$\int_0^{R_m(\alpha)} |u_m(r)|^2 dr \leqslant \alpha. \tag{3.16}$$

Let $\varepsilon > 0$ and let $C(\varepsilon) := \sqrt{E\varepsilon^{-1}}$. We make use of estimate (3.12) to prove in the same way that,

$$\int_{\{|r-\xi_m| \leqslant C(\varepsilon)\}} |u_m(r,\xi_m)|^2 dr \geqslant 1-\varepsilon. \tag{3.17}$$

We combine these estimates to derive an upper bound for ξ_m. Let $(\varepsilon, \alpha) \in (0,1)^2$ such that $1-\varepsilon > \alpha$. We assume that for some $m \in \mathbb{N}$,

$$(\xi_m - C(\varepsilon), \xi_m + C(\varepsilon)) \subset (0, R_m(\alpha)). \tag{3.18}$$

We deduce from estimates (3.16) and (3.17) that

$$1-\varepsilon \leqslant \int_{-C(\varepsilon)+\xi_m}^{C(\varepsilon)+\xi_m} |u_m(r,\xi_m)|^2 dr \leqslant \int_0^{R_m(\alpha)} |u_m(r,\xi_m)|^2 dr \leqslant \alpha.$$

So hypothesis (3.18) can not hold. Moreover according to Proposition 3.1, $\xi_m \to +\infty$ as $m \to +\infty$. Therefore for m large enough $\xi_m - C(\varepsilon) \geqslant 0$. Hence,

$$\exists M > 0, \forall m \geqslant M, \quad C(\varepsilon) + \xi_m \geqslant R_m(\alpha).$$

Thus we deduce the existence of K_-. □

of the upper bound. We now examine the second part of Proposition 3.2: we show that $\left(\xi_m k_m^{-1/2}\right)_{m\in\mathbb{N}}$ admits an upper bound. The key argument is $E \neq E_p$. Indeed we prove that if ξ_m tends too fast to $+\infty$, the limit operator is a quantum harmonic oscillator whose eigenvalues are the Landau levels. Let's assume that the sequence $\left(\xi_m k_m^{-1/2}\right)_{m\in\mathbb{N}}$ admits no upper bounds. Up to an extraction, one can assume that

$$\lim_{m\to+\infty} \frac{\xi_m}{\sqrt{k_m}} = +\infty. \tag{3.19}$$

Recall (see Subsection 3.1) that H_0 is the quantum harmonic oscillator acting on $\mathbf{L}^2(\mathbb{R})$ and that the operator L_m is unitarily equivalent to the following operator acting on $\mathbf{L}^2(-\xi_m, +\infty)$

$$H_0 + \left(\frac{\sqrt{k_m}}{\xi_m}\right)^2 \frac{1}{\left(1+\frac{s}{\xi_m}\right)^2}.$$

Let $(E_q, \Psi_q)_{q\in\mathbb{N}}$ be the eigenpairs of H_0. For any $m \in \mathbb{N}$, $q \in \mathbb{N}$, we use the functions χ_{ξ_m} and $u_m^1(\cdot, \xi_m)$ defined by formulas (3.5) and (3.6). Note that $\chi_{\xi_m}(H_0 - E_p)\Psi_q = 0$, therefore according to estimates (3.7) and (3.8),

$$\left\|(L_m - E_q)\, u_{m,q}^1\right\|_2 \leqslant \left\|[H_0, \chi_m]\,\Psi_q\right\|_2 + 4\left(\frac{\sqrt{k_m}}{\xi_m}\right)^2, \quad q \in \mathbb{N}.$$

Moreover,

$$\left\|[H_0, \chi_m]\,\Psi_q\right\|_2 = O\left(\frac{1}{\xi_m^\infty}\right).$$

Recall that $\left\|u_m^1(\cdot, \xi_m)\right\| \to 1$ as $m \to +\infty$ (remember that $\xi_m \to +\infty$ as $m \to +\infty$ and see the identity (3.9)) and that we have assumed that $\sqrt{k_m}\xi_m^{-1} \to 0$ as $m \to +\infty$. We thus conclude from the spectral theorem that

$$\lim_{m\to+\infty} d\left(\sigma(L_m), E_q\right) = 0.$$

It implies that for every $q \in \mathbb{N}$, $d\left(\{\lambda_{m,s}(\xi_m), s \geqslant 1\}, E_q\right) \to 0$ as $m \to +\infty$. So for every $q \in \mathbb{N}$, $\lambda_{m,q}(\xi_m) \to E_q$ as $m \to +\infty$, therefore $E = E_p$. But we have assumed that $E \neq E_p$, hence the hypothesis (3.19) can not hold and we get the upper-bound.

□

We now study the potential V_m, defined by formula (1.3). Note that V_m is strictly convex and that it verifies $V_m(r) \to +\infty$ as $r \to 0$ or $r \to +\infty$. Therefore V_m admits an unique minimum on $\mathbb{R}_+$, $V_m^{\min}$, reached at the single critical point of V_m: r_m. In Lemma 3.2, we use Proposition 3.2 to localize the quantities r_m and $V_m^{\min}$.

Lemma 3.2 (Localization of extrema)
There are constants $M \in \mathbb{N}$, $R_\pm > 0$ and $V_\pm > 0$ such that for every $m \geqslant M$,

1. $R_-\sqrt{k_m} \leqslant r_m \leqslant R_+\sqrt{k_m}$;
2. $V_- \leqslant V_m^{\min} \leqslant V_+$.

Moreover, for any $y > V_m^{\min}$, the two solutions $r_\pm$ of $V_m(r) = y$ satisfy:

$$\exists K_\pm > 0,\ \exists M \in \mathbb{N},\ \forall m \geqslant M, \quad K_-\sqrt{k_m} \leqslant r_- < r_m < r_+ \leqslant K_+\sqrt{k_m}.$$

Proof.

1. First, recall that $r_m \in \mathbb{R}_+$ is the single critical point of V_m. Therefore, $V_m'(r_m) = 0$ provides

$$\frac{k_m}{r_m^3} = r_m - \xi_m.$$

Since $r_m > 0$, we deduce that $r_m - \xi_m > 0$. So according to Proposition 3.2

$$\exists K_+ > 0, \exists M \in \mathbb{N}, \forall m \geqslant M, \quad r_m > \xi_m \geqslant K_+\sqrt{k_m}.$$

Moreover $0 < r_m - \xi_m \leqslant k_m \left(K_+\sqrt{k_m}\right)^{-3} = K_+^{-3}k_m^{-1/2}$. So using $k_m \to +\infty$ as $m \to +\infty$, we deduce that

$$\lim_{m\to+\infty} r_m - \xi_m = 0. \tag{3.20}$$

Thus Proposition 3.2 provides the result.

2. Recall that $V_m^{\min} = V_m(r_m)$. Hence, according to equation (3.20), $V_m^{\min} - k_m r_m^{-2} \to 0$ as $m \to +\infty$. So the first point provides the result.

3. According to the variations of V_m, $r_\pm$ exists and is solution of the equation $k_m r^{-2} + (r-\xi_m)^2 = y$. Thus $(r_\pm - \xi_m)^2 \leqslant y$ and therefore $|r_\pm - \xi_m| \leqslant \sqrt{y}$. The result follows from Proposition 3.2.

□

Remark 3.3: We do not know if the limits $\lim\limits_{m\to+\infty} \dfrac{\xi_m}{\sqrt{k_m}}$ and $\lim\limits_{m\to+\infty} V_m^{\min}$ exist.

3.2.2 Exponential decay of the eigenfunctions

Here we introduce some tools to estimate the exponential decay of the eigenfunctions. This is an application of the well-known Agmon estimates for 1D Schrödinger operators with confining potential. In our case we would like to take into account the dependance on m. Therefore we are led to perturb the Agmon distance to get some uniform estimates.

We define the Agmon distance by:

$$d_m(r_1,r_2) = \left| \int_{r_1}^{r_2} \sqrt{(V_m(r)-E)_+}\,dr \right|, \quad (r_1,r_2) \in \mathbb{R}_+^2.$$

For $\alpha > 3/2$ and for every $m \in \mathbb{N}$, we define δ_m by

$$\delta_m = \delta_m(\alpha) := \frac{\alpha}{\sqrt{k_m}}.$$

Let I_m be defined by

$$I_m = I_m(E) := \{r > 0, \quad V_m(r) < E\}. \tag{3.21}$$

We recall that we have chosen $E > E_p$, therfore $I_m \neq \emptyset$. Indeed,

$$E = l_m(u_m) > \int_{\mathbb{R}_+} V_m |u_m|^2 \geqslant V_m^{\min} \|u_m\|_2^2 = V_m^{\min}. \tag{3.22}$$

Furthermore, remember that V_m is strictly convex and that $V_m(r) \to +\infty$ as $r \to 0$. Therefore I_m is an open bounded interval of $\mathbb{R}_+$. Recall that the distance between $x \in \mathbb{R}$ and a set $X \subset \mathbb{R}$ is defined as $d(x,X) := \inf(d(x,y), y \in X)$. For every $m \in \mathbb{N}$, we define the function Φ_m on $\mathbb{R}_+$ by

$$\Phi_m = \Phi_m(\cdot,\delta_m) := \delta_m d_m(\cdot, I_m). \tag{3.23}$$

The function Φ_m is decreasing on $(0, \inf(I_m))$, zero on I_m and increasing on $(\sup(I_m), +\infty)$. Moreover, it satisfies the eikonal equation:

$$\left|\Phi_m'\right|^2 = \delta_m^2 (V_m(r) - E)_+. \tag{3.24}$$

Notice that Φ_m is a perturbated Agmon distance and that $\delta_m \to 0$ as $m \to +\infty$. We use this fact to prove the following proposition that provides a uniform control for $e^{\Phi_m} u_m$.

First of all we use the definition of Φ_m given by equation (3.34) and a Taylor expansion at 0 and at $+\infty$ to get the following lemma.

Lemma 3.3
Let Φ_m be the function defined by definition (3.34)*. The behavior of $\Phi_m(r)$ as $r \to \partial\mathbb{R}_+$ is given by:*

- $\Phi_m(r) = -\alpha \ln(r) + O(1)$ *as* $r \to 0$;
- $\Phi_m(r) = \dfrac{\delta_m r^2}{2} + O(r)$ *as* $r \to +\infty$.

The following proposition is a well known Agmon estimate result [Agm82]. Here we are interested in the uniformity with respect to m. To that aim we adapt the classical proof of the result [Hel88].

Proposition 3.3
There exist a constant K and an integer M such that

$$\forall m \geqslant M, \quad \left\| e^{\Phi_m} u_m \right\|_2 \leqslant K.$$

Proof. According to Lemma 3.4, there is a constant $\beta \in \mathbb{R}$ such that,

$$\begin{array}{ll} e^{2\Phi_m(r)} = O\left(r^{-2\alpha}\right) & r \to 0; \\ e^{2\Phi_m(r)} = O\left(e^{\frac{\delta_m}{2}r^2+\beta r}\right) & r \to +\infty. \end{array}$$

Hence according to Proposition 2.1,

$$e^{2\Phi_m(r)} u_m(r) = O\left(r^{m+\frac{n}{2}-1-2\alpha}\right), \quad r \to 0.$$

Therefore for m large enough, $e^{2\Phi_m} u_m \in \mathbf{L}^2(0,1)$. Moreover according to the Liouville-Green approximation [Olv97, Chapter 6],

$$u_m(r) \sim (V_m(r) - E)^{-\frac{1}{4}} e^{-\int \sqrt{V_m(r)-E}dr}, \quad r \to +\infty.$$

Remember that $\int \sqrt{V_m(r) - E}dr \sim r^2/2$ as $r \to +\infty$, we deduce that for m large enough, $e^{\phi_m} u_m \in \mathbf{L}^2(1,+\infty)$, therefore $e^{2\phi_m} u_m \in \mathbf{L}^2(\mathbb{R}_+)$. Moreover an integration by parts yields

$$\left\langle -u_m'', e^{2\phi_m} u_m \right\rangle = \int_{\mathbb{R}_+} \left(\left|u_m'\right|^2 + 2\phi_m' u_m u_m' \right) e^{2\phi_m} - \left[e^{2\phi_m} u_m u_m' \right]_0^{+\infty}.$$

According to what preceds, $\left[e^{2\phi_m} u_m u_m' \right]_0^{+\infty} = 0$, thus $e^{\phi_m} u_m \in \mathscr{D}(h_m)$. Moreover by combining it with the relations (3.10) and (3.15) we obtain

$$\int_{\mathbb{R}_+} \left| \left(e^{\Phi_m} u_m\right)' \right|^2 + \int_{\mathbb{R}_+} e^{2\Phi_m} \left(V_m - E - \left|\phi_m'\right|^2 \right) |u_m|^2 = 0. \tag{3.25}$$

Furthermore, according to estimate (3.33), $E - V_m^{\min} > 0$. Let for every $m \in \mathbb{N}$, $\varepsilon_m := 2^{-1}\left(E - V_m^{\min}\right) > 0$. Recall that I_m is given by definition (3.32). We define $I_\pm$ as

$$\begin{array}{l} I_- := I_m(E+\varepsilon_m) = \left\{ r \in \mathbb{R}_+, V_m(r) < E + \varepsilon_m \right\}, \\ I_+ := \mathbb{R}_+ \backslash I_- = \left\{ r \in \mathbb{R}_+, V_m(r) \geqslant E + \varepsilon_m \right\}. \end{array}$$

By injecting $\mathbb{R}_+ = I_+ \sqcup I_-$ into equation (3.36), we prove that

$$\begin{array}{c} \displaystyle\int_{\mathbb{R}_+} \left| \left(e^{\phi_m} u_m\right)' \right|^2 + \int_{I_+} e^{2\phi_m} \left(V_m - E - \left|\phi_m'\right|^2 \right) |u_m|^2 = \\ \displaystyle -\int_{I_-} e^{2\phi_m} \left(V_m - E - \left|\phi_m'\right|^2 \right) |u_m|^2 \leqslant \left\| V_m - E - \left|\phi_m'\right|^2 \right\|_{\mathbf{L}^\infty(I_-)} \int_{I_-} e^{2\phi_m} |u_m|^2.. \end{array}$$

Let for m large enough such that $\delta_m \leqslant 1$, $C_m := \left(1 - \delta_m^2\right) \varepsilon_m > 0$. We combine equation (3.35) with the definition of $I_\pm$ to get

$$\begin{array}{ll} V_m(r) - E - \left|\Phi_m'(r)\right|^2 \geqslant C_m, & \text{if } r \in I_+; \\ V_m^{min} - E \leqslant V_m(r) - E - \left|\Phi_m'(r)\right|^2 < C_m, & \text{if } r \in I_-. \end{array}$$

So remembering that $\varepsilon_m = \left(E - V_m^{\min}\right)/2$, we get $\left\| V_m - E - |\Phi_m'|^2 \right\|_{\mathbf{L}^\infty(I_-)} \leqslant E - V_m^{\min}$ and we deduce that

$$\int_{\mathbb{R}_+} \left| \left(e^{\Phi_m} u_m\right)' \right|^2 + C_m \int_{I_+} e^{2\Phi_m} |u_m|^2 \leqslant \left(E - V_m^{\min}\right) \int_{I_-} e^{2\Phi_m(r)} |u_m|^2.$$

We recall that u_m is normalized that provides

$$\int_{\mathbb{R}_+} \left| \left(e^{\Phi_m} u_m\right)' \right|^2 + C_m \int_{\mathbb{R}_+} e^{2\Phi_m} |u_m|^2 \leqslant \left(E - V_m^{\min} + C_m\right) \int_{I_-} e^{2\Phi_m} |u_m|^2 \leqslant \left(E - V_m^{\min} + C_m\right) e^{2\|\Phi_m\|_{\mathbf{L}^\infty(I_-)}}.$$

Finally we deduce the following estimate

$$\int_{\mathbb{R}_+} e^{2\Phi_m(r)} |u_m(r)|^2 dr \leqslant \frac{E - V_m^{\min} + C_m}{C_m} e^{2\|\Phi_m\|_{\mathbf{L}^\infty(I_-)}}. \tag{3.26}$$

The choices of δ_m and ε_m yield $\left(E - V_m^{\min} + C_m\right) C_m^{-1} = \left(3 - \delta_m^2\right)\left(1 - \delta_m^2\right)^{-1}$. Thus $\left(E - V_m^{\min} + C_m\right) C_m^{-1}$ is bounded as $m \to +\infty$. Moreover the variations of Φ_m ensure that $\|\Phi_m\|_{\mathbf{L}^\infty(I_-)} = \|\Phi_m\|_{\mathbf{L}^\infty(\partial I_-)}$. Therefore Lemma 3.2 provides the following control

$$\exists K > 0,\, \exists M_0 \in \mathbb{N},\, \forall m \geqslant M_0, \quad \|\Phi_m\|_{\mathbf{L}^\infty(I_-)} \leqslant K \delta_m \sqrt{k_m} = K\alpha.$$

We conclude the proof by combining it with estimate (3.37) that provides the expected result. □

3.2.3 Asymptotic expansion of the derivative

Here we prove the following theorem.

Theorem 3.2 (Asymptotic behaviour of the derivative)
Recall that ξ_m is defined by relation (3.15). *There are constants $K_\pm > 0$ and there exists $M \in \mathbb{N}$ such that*

$$\forall m \geqslant M, \quad \frac{K_-}{\sqrt{k_m}} \leqslant \left|\lambda_m'(\xi_m)\right| \leqslant \frac{K_+}{\sqrt{k_m}}.$$

Remark 3.4: For further use note that this theorem can be adapted to the case where the energy level is an interval J. Namely, if $J \subset \mathbb{R}$ denotes an interval such that $\overline{J} \cap \{E_p,\, p \in \mathbb{N}\} = \emptyset$, then

$$\exists M \in \mathbb{N}, \forall m \geqslant M, \forall \xi \in \lambda_m^{-1}(J), \quad \frac{K_-(J)}{\sqrt{k_m}} \leqslant \left|\lambda_m'(\xi)\right| \leqslant \frac{K_+(J)}{\sqrt{k_m}}.$$

Remark 3.5: If J is on the form $(E_p, E_p + \eta)$, them the combinaison of Theorem 3.1 and of Proposition 3.2 states that there is a constant $C > 0$ such that if $\lambda_m(\xi) \in J$ then $\xi \geqslant C\sqrt{k_m \eta^{-1}}$. Therefore one could prove that

$$\exists C > 0, \exists M \in \mathbb{N}, \forall m \geqslant M, \forall \xi \in \lambda_m^{-1}(J), \quad \left|\lambda_m'(\xi)\right| \leqslant C\sqrt{\frac{\eta}{k_m}}.$$

Lower bound
According to Proposition 2.2,

$$\left|\lambda_m'(\xi)\right| \geqslant \frac{2k_m}{R^3} \int_0^R |u_m(r,\xi)|^2 dr, \quad \xi \in \mathbb{R}, R > 0, m \geqslant 1.$$

Let's combine it with estimate (3.14) and with $\|u_m(\cdot,\xi)\| = 1$. We deduce that

$$\left|\lambda_m'(\xi)\right| \geqslant \frac{2k_m}{R^3}\left(1 - \int_R^{+\infty} |u_m(r,\xi)|^2 dr\right) \geqslant \frac{2k_m}{R^3}\left(1 - \frac{\lambda_m(\xi)}{(R-\xi)^2}\right), \quad R > \xi > 0.$$

Remembering that $\lambda_{m,p}(\xi_m) = E$, we get

$$\left|\lambda_m'(\xi_m)\right| \geqslant \frac{2k_m}{R^3}\left[1 - \frac{E}{(R-\xi_m)^2}\right], \quad R > \xi_m.$$

Let us choose $R = R_m := \xi_m + \sqrt{2E} > \xi_m$. in order to obtain $E(R_m - \xi_m)^{-2} = 1/2$. This implies that $\left|\lambda_m'(\xi_m)\right| \geqslant k_m R_m^{-3}$. Observe that $R_m \sim \xi_m$ as $m \to +\infty$. Therefore Proposition 3.2 provides

$$\exists K > 0, \exists M \in \mathbb{N}, \forall m \geqslant M, \quad \left|\lambda_m'(\xi_m)\right| \geqslant \frac{K}{\sqrt{k_m}}.$$

Upper bound

Recall that Φ_m is defined by the formula (3.34). Let us define the function Ψ_m by

$$\Psi_m(r) = \frac{e^{-2\Phi_m(r)}}{r^3}, \quad r > 0.$$

Let's combine Propositions 2.2 and 3.4. We get that for m large enough

$$\left|\lambda_m'(\xi_m)\right| \leqslant 2Kk_m \sup_{r\in\mathbb{R}_+} \Psi_m(r). \tag{3.27}$$

Therefore it is enough to prove that there is an integer M and a constant $K > 0$ such that

$$\forall m \geqslant M, \quad \|\Psi_m\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \leqslant \frac{K}{k_m\sqrt{k_m}}. \tag{3.28}$$

First note that $\Phi_m \geqslant 0$. Therefore for any $r \in \mathbb{R}_+$, $\Psi_m(r) \leqslant r^{-1}$, meaning that $\Psi_m(r) \to 0$ as $r \to +\infty$. Moreover, according to Lemma 3.4, $-2\phi_m(r) = 2\alpha\ln(r) + O(1)$ as $r \to 0$. By combining it with the definition of Ψ_m, we deduce that $\Psi_m(r) = O\left(r^{2\alpha-3}\right)$ as $r \to 0$. Remembering that $\alpha > 3/2$, we conclude that $\Psi_m(r) \to 0$ as $r \to 0$. Hence we deduce that

$$\exists \tilde{r}_m > 0, \quad \|\Psi_m\|_{\mathbf{L}^\infty(\mathbb{R})} = \Psi_m(\tilde{r}_m).$$

Furthermore, $\tilde{r}_m$ is a critical point of Ψ_m. Therefore $\Psi_m'(\tilde{r}_m) = 0$ implies that

$$\Phi_m'(\tilde{r}_m) = -\frac{3}{2\tilde{r}_m}. \tag{3.29}$$

Observing that $\tilde{r}_m > 0$, we get $\Phi_m'(\tilde{r}_m) < 0$. Remembering that Φ_m is non decreasing on $(\inf(I_m), +\infty)$, we deduce that $\tilde{r}_m < \inf(I_m)$. Note that $\inf(I_m)$ is solution of $V_m(r) = E$. Therefore Proposition 3.2 provides a constant $K_+ > 0$ such that

$$\tilde{r}_m \leqslant \inf(I_m) \leqslant K_+\sqrt{k_m}.$$

Now combine equations (3.29) and (3.35). It yields

$$\delta_m^2\left(\frac{k_m}{\tilde{r}_m^2} + (\tilde{r}_m - \xi_m)^2 - E\right) = \frac{9}{4\tilde{r}_m^2}.$$

Hence we get

$$\frac{\delta_m^2 k_m - \frac{9}{4}}{\tilde{r}_m^2} = \delta_m^2\left(E - (\tilde{r}_m - \xi_m)^2\right) \leqslant \delta_m^2 E. \tag{3.30}$$

Remembering that $\delta_m\sqrt{k_m} = \alpha$, estimate (3.30) can be written as $\left(\alpha^2 - 9/4\right)\tilde{r}_m^{-2} \leqslant E\alpha^2 k_m^{-1}$. Moreover $\alpha > 3/2$, so there is a constant $K_- > 0$ such that for m large enough,

$$\tilde{r}_m \geqslant K_-\sqrt{k_m}. \tag{3.31}$$

Recall that $\Phi_m(\tilde{r}_m) \geqslant 0$ meaning that $\|\Psi_m\|_{\mathbf{L}^\infty(\mathbb{R}_+)} = \Psi(\tilde{r}_m) \leqslant \tilde{r}_m^{-3}$. Combine it with the estimate (3.31) and recall that $\Psi_m(r) \to 0$ as $r \to 0$ and as $r \to +\infty$. It provides the estimate (3.28). Finally we combine the estimates (3.27) and (3.28) that proves the upper bound.

3.2.4 Exponential decay of the eigenfunctions

Here we introduce some tools to estimate the exponential decay of the eigenfunctions. This is an application of the well-known Agmon estimates for 1D Schrödinger operators with confining potential. In our case we would like to take into account the dependance on m. Therefore we are led to perturb the Agmon distance to get some uniform estimates.

We define the Agmon distance by:

$$d_m(r_1,r_2) = \left| \int_{r_1}^{r_2} \sqrt{(V_m(r)-E)_+}\,dr \right|, \quad (r_1,r_2) \in \mathbb{R}_+^2.$$

For $\alpha > 3/2$ and for every $m \in \mathbb{N}$, we define δ_m by

$$\delta_m = \delta_m(\alpha) := \frac{\alpha}{\sqrt{k_m}}.$$

Let I_m be defined by

$$I_m = I_m(E) := \{r > 0, \quad V_m(r) < E\}. \tag{3.32}$$

We recall that we have chosen $E > E_p$, therfore $I_m \neq \emptyset$. Indeed,

$$E = l_m(u_m) > \int_{\mathbb{R}_+} V_m |u_m|^2 \geqslant V_m^{\min} \|u_m\|_2^2 = V_m^{\min}. \tag{3.33}$$

Furthermore, remember that V_m is strictly convex and that $V_m(r) \to +\infty$ as $r \to 0$. Therefore I_m is an open bounded interval of $\mathbb{R}_+$. Recall that the distance between $x \in \mathbb{R}$ and a set $X \subset \mathbb{R}$ is defined as $d(x,X) := \inf(d(x,y), y \in X)$. For every $m \in \mathbb{N}$, we define the function Φ_m on $\mathbb{R}_+$ by

$$\Phi_m = \Phi_m(\cdot,\delta_m) := \delta_m d_m(\cdot, I_m). \tag{3.34}$$

The function Φ_m is decreasing on $(0,\inf(I_m))$, zero on I_m and increasing on $(\sup(I_m),+\infty)$. Moreover, it satisfies the eikonal equation:

$$\left|\Phi_m'\right|^2 = \delta_m^2 (V_m(r)-E)_+. \tag{3.35}$$

Notice that Φ_m is a perturbated Agmon distance and that $\delta_m \to 0$ as $m \to +\infty$. We use this fact to prove the following proposition that provides a uniform control for $e^{\Phi_m} u_m$.

First of all we use the definition of Φ_m given by equation (3.34) and a Taylor expansion at 0 and at $+\infty$ to get the following lemma.

Lemma 3.4
Let Φ_m be the function defined by definition (3.34). *The behavior of $\Phi_m(r)$ as $r \to \partial\mathbb{R}_+$ is given by:*

- $\Phi_m(r) = -\alpha \ln(r) + O(1)$ *as* $r \to 0$;
- $\Phi_m(r) = \dfrac{\delta_m r^2}{2} + O(r)$ *as* $r \to +\infty$.

The following proposition is a well known Agmon estimate result [Agm82]. Here we are interested in the uniformity with respect to m. To that aim we adapt the classical proof of the result [Hel88].

Proposition 3.4
There exist a constant K and an integer M such that

$$\forall m \geqslant M, \quad \left\|e^{\Phi_m} u_m\right\|_2 \leqslant K.$$

Proof. According to Lemma 3.4, there is a constant $\beta \in \mathbb{R}$ such that,

$$\begin{aligned} e^{2\Phi_m(r)} &= O\left(r^{-2\alpha}\right) && r \to 0; \\ e^{2\Phi_m(r)} &= O\left(e^{\frac{\delta_m}{2}r^2+\beta r}\right) && r \to +\infty. \end{aligned}$$

Hence according to Proposition 2.1,

$$e^{2\Phi_m(r)}u_m(r) = O\left(r^{m+\frac{n}{2}-1-2\alpha}\right), \quad r \to 0.$$

Therefore for m large enough, $e^{2\Phi_m}u_m \in \mathbf{L}^2(0,1)$. Moreover according to the Liouville-Green approximation [Olv97, Chapter 6],

$$u_m(r) \sim (V_m(r) - E)^{-\frac{1}{4}} e^{-\int \sqrt{V_m(r)-E}dr}, \quad r \to +\infty.$$

Remember that $\int \sqrt{V_m(r) - E}dr \sim r^2/2$ as $r \to +\infty$, we deduce that for m large enough, $e^{\phi_m}u_m \in \mathbf{L}^2(1,+\infty)$, therefore $e^{2\phi_m}u_m \in \mathbf{L}^2(\mathbb{R}_+)$. Moreover an integration by parts yields

$$\left\langle -u_m'', e^{2\phi_m}u_m\right\rangle = \int_{\mathbb{R}_+}\left(\left|u_m'\right|^2 + 2\phi_m' u_m u_m'\right)e^{2\phi_m} - \left[e^{2\phi_m}u_m u_m'\right]_0^{+\infty}.$$

According to what preceds, $\left[e^{2\phi_m}u_m u_m'\right]_0^{+\infty} = 0$, thus $e^{\phi_m}u_m \in \mathscr{D}(h_m)$. Moreover by combining it with the relations (3.10) and (3.15) we obtain

$$\int_{\mathbb{R}_+}\left|\left(e^{\Phi_m}u_m\right)'\right|^2 + \int_{\mathbb{R}_+} e^{2\Phi_m}\left(V_m - E - \left|\phi_m'\right|^2\right)|u_m|^2 = 0. \tag{3.36}$$

Furthermore, according to estimate (3.33), $E - V_m^{\min} > 0$. Let for every $m \in \mathbb{N}$, $\varepsilon_m := 2^{-1}\left(E - V_m^{\min}\right) > 0$. Recall that I_m is given by definition (3.32). We define $I_\pm$ as

$$\begin{aligned} I_- &:= I_m(E+\varepsilon_m) = \{r \in \mathbb{R}_+, V_m(r) < E + \varepsilon_m\}, \\ I_+ &:= \mathbb{R}_+ \backslash I_- = \{r \in \mathbb{R}_+, V_m(r) \geqslant E + \varepsilon_m\}. \end{aligned}$$

By injecting $\mathbb{R}_+ = I_+ \sqcup I_-$ into equation (3.36), we prove that

$$\begin{aligned} &\int_{\mathbb{R}_+}\left|\left(e^{\phi_m}u_m\right)'\right|^2 + \int_{I_+} e^{2\phi_m}\left(V_m - E - \left|\phi_m'\right|^2\right)|u_m|^2 = \\ -&\int_{I_-} e^{2\phi_m}\left(V_m - E - \left|\phi_m'\right|^2\right)|u_m|^2 \leqslant \left\|V_m - E - \left|\phi_m'\right|^2\right\|_{\mathbf{L}^\infty(I_-)}\int_{I_-} e^{2\phi_m}|u_m|^2.. \end{aligned}$$

Let for m large enough such that $\delta_m \leqslant 1$, $C_m := \left(1 - \delta_m^2\right)\varepsilon_m > 0$. We combine equation (3.35) with the definition of $I_\pm$ to get

$$\begin{aligned} &V_m(r) - E - |\Phi_m'(r)|^2 \geqslant C_m, && \text{if } r \in I_+; \\ &V_m^{min} - E \leqslant V_m(r) - E - |\Phi_m'(r)|^2 < C_m, && \text{if } r \in I_-. \end{aligned}$$

So remembering that $\varepsilon_m = \left(E - V_m^{\min}\right)/2$, we get $\left\|V_m - E - |\Phi_m'|^2\right\|_{\mathbf{L}^\infty(I_-)} \leqslant E - V_m^{\min}$ and we deduce that

$$\int_{\mathbb{R}_+}\left|\left(e^{\Phi_m}u_m\right)'\right|^2 + C_m\int_{I_+} e^{2\Phi_m}|u_m|^2 \leqslant \left(E - V_m^{\min}\right)\int_{I_-} e^{2\Phi_m(r)}|u_m|^2.$$

We recall that u_m is normalized that provides

$$\int_{\mathbb{R}_+}\left|\left(e^{\Phi_m}u_m\right)'\right|^2 + C_m\int_{\mathbb{R}_+} e^{2\Phi_m}|u_m|^2 \leqslant \left(E - V_m^{\min} + C_m\right)\int_{I_-} e^{2\Phi_m}|u_m|^2 \leqslant \left(E - V_m^{\min} + C_m\right)e^{2\|\Phi_m\|_{\mathbf{L}^\infty(I_-)}}.$$

Finally we deduce the following estimate

$$\int_{\mathbb{R}_+} e^{2\Phi_m(r)}|u_m(r)|^2 dr \leqslant \frac{E - V_m^{\min} + C_m}{C_m}e^{2\|\Phi_m\|_{\mathbf{L}^\infty(I_-)}}. \tag{3.37}$$

The choices of δ_m and ε_m yield $\left(E - V_m^{\min} + C_m\right)C_m^{-1} = \left(3 - \delta_m^2\right)\left(1 - \delta_m^2\right)^{-1}$. Thus $\left(E - V_m^{\min} + C_m\right)C_m^{-1}$ is bounded as $m \to +\infty$. Moreover the variations of Φ_m ensure that $\|\Phi_m\|_{\mathbf{L}^\infty(I_-)} = \|\Phi_m\|_{\mathbf{L}^\infty(\partial I_-)}$. Therefore Lemma 3.2 provides the following control

$$\exists K > 0, \exists M_0 \in \mathbb{N}, \forall m \geqslant M_0, \quad \|\Phi_m\|_{\mathbf{L}^\infty(I_-)} \leqslant K\delta_m\sqrt{k_m} = K\alpha.$$

We conclude the proof by combining it with estimate (3.37) that provides the expected result. □

4 Velocity operator

In this section we assume that $n \geqslant 4$. The case $n=3$ could also been studied but according to remark 2, attained thresholds arrise in that case. We apply the results of the previous section to derive some properties about the velocity operator.

We refer to Section 1 for the notations. Remember that $\mathscr{F}$ denotes the partial Fourier transform with respect to x_n. Let (r,ω) be the cylindrical coordinates of $\mathbb{R}^{n-1}$, namely, for any $x \in \mathbb{R}^{n-1}\backslash\{0\}$, $r = \|x\|_2$ and $\omega = r^{-1}x \in \mathbb{S}^{n-2}$. In terms of these variables, $\mathbf{L}^2\left(\mathbb{R}^{n-1}\right) = \mathbf{L}^2\left(\mathbb{R}_+ \times \mathbb{S}^{n-2}; r^{n-2}dr\right)$. Let $Y_{m,j}$, $m \geqslant 0$, $j \in [\![1,N_m]\!]$ be the spherical Harmonics that are an orthonormal basis of solutions for the equation $-\Delta_{\mathbb{S}^{n-2}}u = \mu_m u$, $u \in \mathbf{L}^2\left(\mathbb{S}^{n-2}\right)$ and denote by $v_{m,p}(\cdot,\xi)$ the eigenfunctions of $H_m(\xi)$, $m \geqslant 0$ and $\xi \in \mathbb{R}$.

We define the (m,j,p)-th generalized Fourier coefficient of $\varphi \in \mathbf{L}^2(\mathbb{R}^n)$ as

$$\varphi_{m,j,p}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^{n-1}} \widehat{\varphi}(r,\omega,\xi)\overline{Y_{m,j}(\omega)\,v_{m,p}(r,\xi)}r^{n-2}drd\omega, \quad \varphi \in \mathbf{L}^2(\mathbb{R}^n).$$

Moreover for every $m \geqslant 0$, $j \in [\![1,N_m]\!]$ and $p \in \mathbb{N}$, denote by $\pi_{m,j,p}$ the orthogonal projection associated with the (m,j,p)-th harmonic and by π_p, the projection associated with all the harmonic that have p as level:

$$\pi_{m,j,p}(\varphi)(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\xi x_n}\varphi_{m,j,p}(\xi)Y_{m,j}(\omega)\,v_{m,p}(r,\xi)\,d\xi, \quad x \in \mathbb{R}^n,$$

$$\pi_p := \sum_{m=0}^{+\infty}\sum_{j=1}^{N_m}\pi_{m,j,p}.$$

In light of Section 1, every $\varphi \in \mathbf{L}^2(\mathbb{R}^n)$ is decomposed as

$$\varphi = \sum_{m=0}^{+\infty}\sum_{j=1}^{N_m}\sum_{p=1}^{+\infty}\pi_{m,j,p}(\varphi) = \sum_{p=1}^{+\infty}\pi_p(\varphi).$$

Moreover the Parseval theorem yields

$$\|\varphi\|_2^2 = \sum_{m=0}^{+\infty}\sum_{j=1}^{N_m}\sum_{p=1}^{+\infty}\left\|\varphi_{m,j,p}\right\|_2^2. \tag{4.1}$$

Finally for any non-empty interval $I \subset \mathbb{R}$, denote by $\mathbb{P}_I$ the spectral projection of H associated with I. A quantum state $\varphi \in \mathbf{L}^2(\mathbb{R}^n)$ is said concentrated in I if $\mathbb{P}_I\varphi = \varphi$. With reference to Section 1, this condition can be written as

$$\forall m \geqslant 0, \forall j \in [\![1,N_m]\!], \forall p \geqslant 1, \quad \operatorname{supp}(\varphi_{m,j,p}) \subset \lambda_{m,p}^{-1}(I). \tag{4.2}$$

Let x_n be the position operator defined as the multiplier by coordinate x_n in $\mathbf{L}^2(\mathbb{R}^n)$:

$$(x_n f)x = x_n f(x), \quad x = (x_1,\cdots,x_n) \in \mathbb{R}^n$$

and let $x_n(t)$ be the Heisenberg variable defined as

$$x_n(t) := e^{itH}x_n e^{-itH}.$$

A quantum state φ is a solution of the Schrödinger equation $i\partial_t\varphi = H\varphi$. Thus $\varphi(x,t) = e^{-itH}\varphi(x,0)$ and we deduce by a straightforward calculation that

$$\langle x_n\varphi(\cdot,t),\varphi(\cdot,t)\rangle = \langle x_n(t)\varphi(\cdot,0),\varphi(\cdot,0)\rangle, \quad t \in \mathbb{R}.$$

Therefore the time evolution of the position operator x_n is $x_n(t)$ and its time derivative is the velocity, given by $\partial_t x_n(t) = ie^{itH}[H,x_n]e^{-itH}$. We define the velocity operator J as the following self-adjoint operator acting on $\mathscr{D}(H)\cap\mathscr{D}(x_n)$ such that the current carried by a state φ is $\langle J\varphi,\varphi\rangle$ [Ens83].

$$J := -i[H,x_n] = -2(i\partial_{x_n} + r). \tag{4.3}$$

Since $\mathscr{F}$, is an isometry, we observe that $\mathscr{F}J\mathscr{F}^{-1} = -2(r-\xi)$. Therefore the Feynman-Hellman formula (see equation (2.5)) yields

$$\langle J\pi_p\varphi, \pi_p\varphi\rangle = \sum_{m=0}^{+\infty}\sum_{j=1}^{N_m}\int_{\mathbb{R}} \lambda'_{m,p}(\xi)\left|\varphi_{m,j,p}(\xi)\right|^2 d\xi, \quad p\in\mathbb{N}. \tag{4.4}$$

In Theorem (4.1), we will combine this identity with Theorem 3.2 to control the current operator. We define for each $M\in\mathbb{N}$ and each $p\in\mathbb{N}$:

$$\begin{aligned}
X^-_{I,M,p} &:= \left\{\varphi\in\operatorname{Ran}(\mathbb{P}_I)\cap\operatorname{Ran}(\pi_p), \forall m\geqslant M+1, \forall j\in[\![1,N_m]\!], \varphi_{m,j,p}=0\right\},\\
X^+_{I,M,p} &:= \left\{\varphi\in\operatorname{Ran}(\mathbb{P}_I)\cap\operatorname{Ran}(\pi_p), \forall m\leqslant M, \forall j\in[\![1,N_m]\!], \varphi_{m,j,p}=0\right\},
\end{aligned}$$

$$X^-_{I,M} := \bigoplus_{p=1}^{+\infty} X^-_{I,M,p};$$

$$X^+_{I,M} := \bigoplus_{p=1}^{+\infty} X^+_{I,M,p}.$$

Note that $\operatorname{Ran}(\mathbb{P}_I) = X^-_{I,M}\oplus X^+_{I,M}$ and that these spaces are H invariant.

Theorem 4.1
Let $I\subset\sigma(H)$ be a non-empty interval such that $I\cap\{E_p,\ p\geqslant 1\}=\emptyset$.

1. $\forall M\geqslant 0$, $\exists C^->0$, $\forall\varphi\in X^-_{I,M}$, $\quad |\langle J\varphi,\varphi\rangle|\geqslant C^-\|\varphi\|_2^2$
2. $\exists C_+$, $\exists M_0\geqslant 0$, $\forall M\geqslant M_0$, $\quad\forall\varphi\in X^+_{I,M}$, $\quad |\langle J\varphi,\varphi\rangle|\leqslant \dfrac{C^+}{\sqrt{k_{M+1}}}\|\varphi\|_2^2$.

Proof. First of all, observe that I is bounded and recall that $E_p\to+\infty$ as $p\to+\infty$. Therefore $P_I := \{p\in\mathbb{N},\quad E_p\geqslant\sup(I)\}$ is a finite set. Moreover, remembering that for every $m\in\mathbb{N}$ and $p\in\mathbb{N}$, $\inf\{\lambda_{m,p}(\xi)\in\mathbb{R},\quad \xi\in\mathbb{R}\}=E_p$, we get that $P_I=\{p\in\mathbb{N}, \exists m\geqslant 0,\quad I\cap\lambda_{m,p}(\mathbb{R})\neq\emptyset\}$. Moreover notice that for every $p\in P_I$ and every $m\in\mathbb{N}$, $I\cap\lambda_{m,p}(\mathbb{R})\neq\emptyset$. Therefore it is enough to prove the theorem for $\varphi\in X^\pm_{I,m,p}$ for a certain $p\in P_I$ fixed. We simplify the notations by ommiting the index p.

Proof of the first part. Let $M\geqslant 0$ and let $\varphi\in X^-_{I,M}$. Note that $\varphi=\pi_p\varphi$. Therefore, according to the embedding (4.2) and to the identity (4.4),

$$\langle J\varphi,\varphi\rangle = \sum_{m=0}^{M}\sum_{j=1}^{N_m}\int_{\lambda_m^{-1}(I)}\lambda'_m(\xi)\left|\varphi_{m,j}(\xi)\right|^2 d\xi.$$

Moreover $E_p\notin\bar{I}$, thus for every $m\in[\![0,M]\!]$, $\lambda_m^{-1}(I)$ is bounded. Therefore, Proposition 2.2 states that for every $m\in[\![0,M]\!]$, $D_m := \inf\{\lambda'_m(\xi), \lambda_m(\xi)\in I\}>0$. Thus $C^- := \inf\{D_m, m\in[\![0,M]\!]\}>0$. Hence

$$|\langle J\varphi,\varphi\rangle|\geqslant C^-\sum_{m=0}^{M}\sum_{j=1}^{N_m}\int_{\lambda_m^{-1}(I)}\left|\varphi_{m,j}(\xi)\right|^2 d\xi. \tag{4.5}$$

Remember that $\varphi_{m,j}$ is localized into $\lambda_m^{-1}(I)$ (see the embedding (4.2)). Therefore,

$$\int_{\lambda_m^{-1}(I)}\left|\varphi_{m,j}(\xi)\right|^2 d\xi = \int_{\mathbb{R}}\left|\varphi_{m,j}(\xi)\right|^2 d\xi = \left\|\varphi_{m,j}\right\|_2^2.$$

Hence according to the Parseval's identity (4.1),

$$\sum_{m=0}^{M}\sum_{j=1}^{N_m}\int_{\lambda_m^{-1}(I)}\left|\varphi_{m,j}(\xi)\right|^2 d\xi = \|\varphi\|_2^2. \tag{4.6}$$

We combine it with the estimate (4.5) that provides the first statment of the Theorem.

Proof of the second part. Let $\varphi \in X_{I,M}^{+}$. We prove in the same way as for the first part that

$$\langle J\varphi, \varphi\rangle = \sum_{m=M+1}^{+\infty} \sum_{j=1}^{N_m} \int_{\lambda_m^{-1}(I)} \lambda_m'(\xi) \left|\varphi_{m,j}(\xi)\right|^2 d\xi.$$

Therefore, according to Theorem 3.2, there exist $M_0 \geqslant 0$ and $C_+ > 0$ such that for every $M \geqslant M_0$,

$$|\langle J\varphi, \varphi\rangle| \leqslant \sum_{m=M+1}^{+\infty} \sum_{j=1}^{N_m} \frac{C_+}{\sqrt{k_m}} \int_{\lambda_m^{-1}(I)} \left|\varphi_{m,j}(\xi)\right|^2 d\xi. \tag{4.7}$$

Observe that for $m \geqslant M+1$, $C_+ k_m^{-1/2} \leqslant C_+ k_{M+1}^{-1/2}$. We combine it with the estimate (4.7) and with the Parseval's identity (4.1) that yields

$$|\langle J\varphi, \varphi\rangle| \leqslant \frac{C_+}{\sqrt{k_{M+1}}} \|\varphi\|_2^2.$$

□

Remark 4.1: Remember that $k_M \to +\infty$ as $M \to +\infty$. According to Theorem 4.1, for every $\varepsilon > 0$ and for any bounded energy interval $I \subset \sigma(H)$, there are some quantum state $\varphi_\varepsilon \in \mathrm{Ran}(\mathbb{P}_I)$ such that $\|\varphi_\varepsilon\| = 1$ and $|\langle J\varphi_\varepsilon, \varphi_\varepsilon\rangle| \leqslant \varepsilon$, even if $\bar{I}$ is away from the Landau levels.

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