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ON ω -CATEGORICAL GROUPS AND RINGS OF FINITE BURDEN

JAN DOBROWOLSKI AND FRANK O. WAGNER

ABSTRACT. An ω -categorical group of finite burden is virtually finite-by-abelian; an ω -categorical ring of finite burden is virtually finite-by-null; an ω -categorical NTP_2 ring is nilpotent-by-finite.

1. INTRODUCTION

A structure \mathfrak{M} is ω -categorical if its theory has a unique countable model up to isomorphism. Basic examples include the pure set, the dense linear order, the random graph, and vector spaces over a finite field. A fundamental theorem by Ryll-Nardzewski [25] (proven independently by Svenonius [27] and Engeler [11]) states that a structure is ω -categorical if and only if in any arity there are only finitely many parameter-free definable sets, up to equivalence.

There is a long history of study of ω -categorical groups. In the general case, the main result is Wilson's classification of characteristically simple ω -categorical groups as either elementary abelian, certain groups of functions from Cantor space to some finite simple group, or perfect p -groups (see Fact 8.1); he conjectured that the third possibility is impossible (but this is still open). While a complete classification of all ω -categorical groups (and rings) appears out of reach at present, the question seems accessible under some model-theoretic tameness assumptions, giving rise to the following meta-conjecture (where a ring is associative, but need not be commutative or have an identity):

- Meta-Conjecture.** (1) *A tame ω -categorical group or ring is virtually nilpotent.*
(2) *A supertame ω -categorical group is virtually finite-by-abelian; a supertame ω -categorical ring is virtually finite-by-null.*

(Recall that a group/ring is *virtually P* if it has a finite index subgroup/-ring which is P ; it is *finite-by- P* if it has a finite normal subgroup/ideal I such that it is P modulo I ; and a ring is *null* if multiplication is trivial: $a \cdot b = 0$ for all a, b .) Of course, one has to specify the precise meaning of tame.

We shall prove a general theorem about ω -categorical bilinear quasi-forms of *finite burden*, and deduce Conjecture (2) in the finite burden case; moreover, we show (1) for rings with NTP_2 . Here, NTP_2 is a combinatorially defined very general model-theoretic tameness condition currently under intense investigation in neostability theory, and burden, also called *inp-rank*, is a cardinal-valued rank well defined (i.e. not assuming value ∞) precisely on the class of NTP_2 theories, thus providing a hierarchy inside of this class (see Definition 2.1). The principal examples of structures of burden 1 are real closed fields (and

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expansions thereof with (weakly) \mathcal{o} -minimal theories), the valued fields of p -adic numbers for any prime p , valued algebraically closed fields, Presburger arithmetic $(\mathbb{Z}, 0, +, <)$, as well as the random graph (and any other weight one simple theory, by [1, Proposition 8]). By sub-multiplicativity of burden [8, Theorem 2.5], finite burden structures include all structures interpretable in inp-minimal ones, e.g. algebraic groups over the fields of real, complex and p -adic numbers. For more details on burden and related topics see [8] or [1].

History of results. If tame is read as *stable*, then (1) has been shown for groups by Felgner [13] and for rings by Baldwin and Rose [5]; (2) has been shown by Baur, Cherlin and Macintyre [7]; if tame means *simple*, then the group case of (2) has been shown by Evans and Wagner [12], and if tame means *NSOP* (so, in particular, if it means simple), the group case of (1) has been shown by Macpherson [21]. If tame is taken as *dependent*, then (1) has been shown by Krupiński [20], assuming in addition *finitely satisfiable generics* for the group case. Moreover, building on work of Baginski [4], Krupiński proves that the versions of (2) for *nilpotent* groups and for rings are equivalent [19] (in fact he does not explicitly cover the case with finite normal subgroups/ideals, but his proof adapts), and the group version of (1) implies that for rings. In particular, (1) also holds for NSOP rings. Finally, Kaplan, Levi and Simon [17] show (1) for dependent groups of burden 1. Note that extraspecial p -groups [14] yield an example showing that the finite normal subgroup cannot be avoided in (2), unless one assumes the existence of connected components (which holds, for instance, in dependent theories [6]).

An earlier version of this paper [10] obtained the same results under the stronger hypothesis of burden 1; virtually the only consequence used was that any two definable groups are comparable with respect to almost inclusion. For the generalisation to the finite burden case, we use essentially the same proof; considerable work is being spent to show that all the relevant groups are still comparable with respect to almost inclusion in a minimal counterexample of finite burden.

The paper is organized as follows: In Section 2, we recall the definition of burden, and deduce some algebraic consequences when the burden is finite. In section 3, we introduce additive relations and the ring of quasi-endomorphisms; in Section 4, we study the properties of quasi-homomorphisms under the assumption of ω -categoricity. In Section 5, we generalize the notion of a bilinear form using quasi-homomorphisms instead of homomorphisms, and in Section 6 we define a the notion of a *principal subgroup-generic sequence* which generalizes a Morley sequence in a principal generic type. In section 7, we prove our Main Theorem, Theorem 7.5, about virtual almost triviality of bilinear quasi-forms. This is applied in Section 8 to obtain the results about groups and rings. In Section 9, we state some questions and we prove that ω -categorical rings with NTP_2 are virtually nilpotent.

In line with common model-theoretic practice, we shall write $x \in \bar{x}$ if x is one of the coordinates of the tuple \bar{x} , and $\bar{x} \in X$ if $x \in X$ for all $x \in \bar{x}$. We would like to thank the anonymous referee for his careful reading, and for pointing out a missing assumption in what is now Lemma 4.2.

2. BURDEN

Throughout the paper we will work in a monster model of the relevant complete theory (i.e. a $\bar{\kappa}$ -saturated, $\bar{\kappa}$ -homogeneous model, where $\bar{\kappa}$ is a sufficiently big cardinal number). Definability of a set is with parameters, and includes imaginary sets, i.e. definable sets modulo definable equivalence relations (as we shall want to talk about the quotient of a

definable group by a definable normal subgroup). For the basic notions of model theory, the reader may want to consult [16], [24] or [28].

Definition 2.1. (1) Let κ be a cardinal number. An *inp-pattern of depth κ in a partial type $\pi(\bar{x})$* is a sequence $\langle \varphi_i(\bar{x}, \bar{y}_i) : i < \kappa \rangle$ of formulas and an array $\langle \bar{a}_{i,j} : i < \kappa, j < \omega \rangle$ of parameters such that:

- (a) For each $i < \kappa$, there is some $k_i < \omega$ such that $\{\varphi_i(\bar{x}, \bar{a}_{i,j}) : j < \omega\}$ is k_i -inconsistent; and
- (b) For each $\eta : \kappa \rightarrow \omega$, the partial type

$$\pi(\bar{x}) \cup \{\varphi_i(\bar{x}, \bar{a}_{i,\eta(i)}) : i < \kappa\}$$

is consistent.

- (2) The *burden* (or *inp-rank*) of a partial type $\pi(\bar{x})$ is the maximal κ such that there is an inp-pattern of depth κ in $\pi(\bar{x})$, if such a maximum exists. In case there are inp-patterns of depth λ in $\pi(\bar{x})$ for every cardinal $\lambda < \kappa$ but no inp-pattern of depth κ , we say that the burden of $\pi(\bar{x})$ is κ_- . We will denote the burden of $\pi(\bar{x})$ by $\text{bdn}(\pi(\bar{x}))$. By the burden of a type-definable set we mean the burden of a type defining this set (this, of course, does not depend on the choice of the type). A theory T is called *strong* if the burden of any partial type in finitely many variables is bounded by $(\aleph_0)_-$; it is NTP_2 if the burden of any partial type in finitely many variables is bounded by $|T|^+$ (or equivalently, if it is bounded by some cardinal).

Note that the formulas φ_i can be taken parameter-free, as we may incorporate eventual parameters into the $\bar{a}_{i,j}$. Clearly, burden does not depend on the base parameters; if V and W are type-definable and $f : V \rightarrow W$ is a definable surjective map, then $\text{bdn}(W) \leq \text{bdn}(V)$.

Remark 2.2. Suppose $k = \text{bdn}(\pi(\bar{x}))$ and $l = \text{bdn}(\rho(\bar{y}))$ are finite, where \bar{x} and \bar{y} are disjoint. Then $\text{bdn}(\pi(\bar{x}) \cup \rho(\bar{y})) \geq k + l$. In other words, for type-definable sets V and W of finite burden we have: $\text{bdn}(V \times W) \geq \text{bdn}(V) + \text{bdn}(W)$.

Proof. This is clear, as the concatenation of an inp-pattern in $\pi(\bar{x})$ with an inp-pattern in $\rho(\bar{y})$ is an inp-pattern in $\pi(\bar{x}) \cup \rho(\bar{y})$. \square

Remark 2.3. Suppose $f : V \rightarrow W$ is definable and all fibres of f have size at most k , where $k < \omega$. Then $\text{bdn}(V) \leq \text{bdn}(W)$.

Proof. Suppose $\langle \varphi_i(v, \bar{y}_i) : i < \kappa \rangle$ together with $\langle \bar{a}_{i,j} : i < \kappa, j < \omega \rangle$ form an inp-pattern in V . We may assume that $\langle \bar{a}_{i,j} : j < \omega \rangle$ are pairwise distinct for any $i < \kappa$. Put $\psi_i(w, \bar{y}_i) := (\exists v)(\varphi_i(v, \bar{y}_i) \wedge f(v) = w)$ for $i < \kappa$. We claim that these form an inp-pattern in W (with the same parameters). Indeed, for any $i < \kappa$, if ℓ_i is such that $\{\varphi_i(v, \bar{a}_{i,j}) : j < \omega\}$ is ℓ_i -inconsistent, then by the pigeonhole principle $\{\psi_i(w, \bar{a}_{i,j}) : j < \omega\}$ is $(\ell_i - 1)k + 1$ inconsistent. Also, for each $\eta : \kappa \rightarrow \omega$, if $v_0 \in V$ satisfies $\varphi_i(v, \bar{a}_{i,\eta(i)})$ for each $i < \kappa$, then $f(v_0) \in W$ satisfies $\psi_i(w, \bar{a}_{i,\eta(i)})$ for each $i < \kappa$. \square

For the next results we introduce some notation for subgroups H and K of a group G . We say that H is *almost contained* in K , denoted $H \lesssim K$, if $H \cap K$ has finite index in H . If $H \lesssim K$ and $K \lesssim H$, the two groups are *commensurable*, denoted $H \sim K$. The *almost centraliser* of H in G is defined as

$$\tilde{C}_G(H) = \{g \in G : H \lesssim C_H(g)\},$$

and the *almost centre* of G is $\tilde{Z}(G) = \tilde{C}_G(G)$. Note that if $K \lesssim H$, then $\tilde{C}_G(H) \leq \tilde{C}_G(K)$.

The following fact is a special case of [15, Theorem 2.10]. Recall that the ambient model should be sufficiently saturated. So we cannot just add predicates for H and K .

Fact 2.4. *If H and K are definable, then $H \lesssim \tilde{C}_G(K)$ if and only if $K \lesssim \tilde{C}_G(H)$.*

In particular $H \lesssim \tilde{C}_H(\tilde{C}_G(H))$, as clearly $\tilde{C}_G(H) \lesssim \tilde{C}_G(H)$.

We now turn to the consequences of finite burden we use.

Fact 2.5 ([9, Corollary 2.3]). *Let G be an abelian group with NTP_2 and $\langle H_i : i \in I \rangle$ a family of uniformly definable subgroups. Then there is n such that for all $I_0 \subseteq I$ of size at least n there is $i_0 \in I_0$ with $\bigcap_{i \in I_0 \setminus \{i_0\}} H_i \lesssim H_{i_0}$. In particular, this holds if G has finite burden.*

Thus any irreducible intersection $\bigcap_{i < n} H_i$ (meaning that $\bigcap_{j \neq i} H_j \not\lesssim H_i$ for all $i < n$) of uniformly definable groups has its size n bounded as a function of the formula used to define the H_i .

Lemma 2.6. *Let G be an abelian group of finite burden, and $\langle H_i : i < n \rangle$ definable subgroups of G . If the sum $\sum_{i < n} H_i$ is irreducible (meaning that $H_i \not\lesssim \sum_{j \neq i} H_j$ for all $i < n$), then $n \leq \text{bdn}(G)$.*

Proof. Let $\varphi_i(x, y)$ be the formula $x - y \in \sum_{j \neq i} H_j$, and choose $\langle a_{i,j} : j < \omega \rangle$ to be representatives in H_i for distinct cosets of $H_i \cap \sum_{j \neq i} H_j$. Then $\langle \varphi(x, a_{i,j}) : j < \omega \rangle$ is 2-inconsistent, and consistency of any path $\sigma \in \omega^n$ is witnessed by $\sum_{i < n} a_{i, \sigma(i)}$. So we obtain an inp-pattern of depth n . \square

3. ADDITIVE RELATIONS AND QUASI-ENDOMORPHISMS

We extend the construction of the definable quasi-endomorphisms ring from [7, Section 3.2] to non-connected groups.

Definition 3.1. Let G and H be abelian groups. An *additive relation* between G and H is a subgroup $R \leq G \times H$. We call $\pi_1(R)$, the projection to the first coordinate, the *domain* $\text{dom } R$ and $\pi_2(R)$ the image $\text{im } R$ of R ; the subgroup $\{g \in G : (g, 0) \in R\}$ is the *kernel* $\ker R$, and $\{h \in H : (0, h) \in R\}$ is the *cokernel* $\text{coker } R$. If $\text{dom } R$ has finite index in G and $\text{coker } R$ is finite, the additive relation R is a *quasi-homomorphism* from G to H (not to be confused with quasi-homomorphism in the sense of metric groups). A quasi-homomorphism R induces a homomorphism $\text{dom } R \rightarrow H/\text{coker } R$. If $G = H$ we call R a *quasi-endomorphism* or *endogeny*. Particular additive relations are $\text{id}_G = \{(g, g) : g \in G\}$ and $0_G = G \times \{0\}$.

Remark 3.2. Let $g \leq G \times H$ be a quasi-homomorphism. Then $|G : \ker g|$ is finite if and only if $\text{im } g$ is finite. More precisely, $|G : \ker g| + |\text{coker } g| = |G : \text{dom } g| + |\text{im } g|$.

Proof. Clearly g induces an isomorphism $\text{dom } g / \ker g \cong \text{im } g / \text{coker } g$. The result follows. \square

Definition 3.3. • If $R \leq G \times H$ is an additive relation, $g \in G$ and $K \leq G$, put $R(g) = \{h \in H : (g, h) \in R\}$ and $R[K] = \bigcup_{g \in K} R(g)$.

• If $R, R' \leq G \times H$ are additive relations, put

$$R + R' = \{(a, b + b') \in G \times H : (a, b) \in R, (a, b') \in R'\}.$$

This is again an additive relation. If moreover R and R' are quasi-homomorphisms from G to H , so is $R + R'$. Note that $R + R'$ (as additive relations) is different from the sum when R and R' are considered as subgroups. Clearly $\text{dom}(R + R') = \text{dom } R \cap \text{dom } R'$.

- We call $R, R' \leq G \times H$ *equivalent*, denoted $R \equiv R'$, if there is a subgroup G_1 of finite index in G and a finite group $F \leq H$ such that

$$R + (G_1 \times F) = R' + (G_1 \times F).$$

This is clearly an equivalence relation. Note that it implies $G_1 \cap \text{dom } R = G_1 \cap \text{dom } R'$.

- If $R \leq G \times H$ and $R' \leq H \times K$ are additive relations, define a multiplication \circ by

$$R' \circ R = \{(a, c) \in G \times K : \exists b [(a, b) \in R \text{ and } (b, c) \in R']\}.$$

This is again an additive relation between G and K . If R and R' are quasi-homomorphisms, so is $R' \circ R$. We shall usually just write $R'R$ instead of $R' \circ R$.

- For an additive relation $R \leq G \times H$ put

$$-R = \{(g, -h) : (g, h) \in R\} \quad \text{and} \quad R^{-1} = \{(h, g) \in H \times G : (g, h) \in R\}.$$

Note that $-R$ is again an additive relation between G and H , and R^{-1} is an additive relation between H and G .

Remark 3.4. If $\text{im } R$ has finite index in H and $\ker R$ is finite, then R^{-1} is a quasi-homomorphism from H to G . In general,

- $R + (-R) = 0_{\text{dom } R} + (G \times \text{coker } R)$,
- $R^{-1}R = \text{id}_{\text{dom } R} + (G \times \ker R)$, and
- $RR^{-1} = \text{id}_{\text{im } R} + (H \times \text{coker } R)$.

In particular, if $\text{coker } R$ is finite, then $R + (-R) \equiv 0_{\text{dom } R}$; if both $\ker R$ and $\text{coker } R$ are finite, then $RR^{-1} \equiv \text{id}_{\text{im } R}$ and $R^{-1}R \equiv \text{id}_{\text{dom } R}$.

Lemma 3.5. *Addition is associative and commutative, multiplication is associative, 0_G and id_G are additive and multiplicative identity elements, respectively (for the appropriate G). Moreover, addition is well-defined modulo equivalence. The distributive laws hold for quasi-homomorphism modulo equivalence, and multiplication is well-defined modulo equivalence for quasi-homomorphisms. The collection of (definable) quasi-endomorphisms forms an associative ring with unit.*

Proof. The first two sentences are basically [7, Lemmas 27 and 29] — note that the product in [7] is defined the opposite way. For the distributive laws, let R, R' be quasi-homomorphisms from G to H , and S, S' be quasi-homomorphisms from H to K . Let $H_0 = \text{dom } S \cap \text{dom } S'$ and $G_0 = R^{-1}[H_0] \cap R'^{-1}[H_0]$. Then the indices $|G : G_0|$ and $|H : H_0|$ are finite, and R, R' restrict to quasi-homomorphisms \bar{R}, \bar{R}' from G_0 to H_0 in the sense of [7] (i.e. defined on all of G_0), and S, S' restrict to quasi-homomorphisms \bar{S} and \bar{S}' from H_0 to K in the sense of [7] (i.e. defined on all of H_0). By [7, Lemma 31], we have

$$\bar{S}\bar{R} + \bar{S}'\bar{R} \equiv (\bar{S} + \bar{S}')\bar{R} \quad \text{and} \quad \bar{S}\bar{R} + \bar{S}\bar{R}' \equiv \bar{S}(\bar{R} + \bar{R}'),$$

which implies $SR + S'R \equiv (S + S')R$ and $SR + SR' \equiv S(R + R')$. Thus the distributive laws hold modulo equivalence. Finally, multiplication is well-defined modulo equivalence as in [7, Lemma 32]; it follows that (definable) quasi-endorpisms form an associative ring with unit. \square

4. QUASI-HOMOMORPHISMS OF ω -CATEGORICAL GROUPS

Recall that a complete first order theory in a countable language is said to be ω -categorical if it has only one countable model up to isomorphism, and a structure M is ω -categorical if $Th(M)$ is. By the Ryll-Nardzewski Theorem, this is equivalent to the following statement: for every $n < \omega$ there are only finitely many complete n -types over \emptyset . Hence, for any finite set A in an ω -categorical structure M there are only finitely many definable sets over A , and ω -categorical structures are uniformly locally finite (i.e. there is a function $f : \omega \rightarrow \omega$ such that, for any $n \in \omega$, each substructure of M generated by n elements has at most $f(n)$ elements) [16, Corollary 7.3.2].

Lemma 4.1. *Let G and H be abelian groups, and let $g \leq G \times H$ be an additive relation.*

- (1) *If $\text{coker } g$ is finite, $|H : \text{im } g|$ is finite, and $H_1 \leq H$ has infinite index in H , then $|\text{dom } g : g^{-1}[H_1]|$ is infinite.*
- (2) *If $\text{ker } g$ is finite, $|G : \text{dom } g|$ is finite, and $G_1 \leq G$ has infinite index, then $|\text{im } g : g[G_1]|$ is infinite.*
- (3) *If $H_1 \leq H$, then $|\text{dom } g : g^{-1}[H_1]| \leq |\text{im } g : \text{im } g \cap H_1|$.*

Proof. (1) Let $\langle h_i : i < \omega \rangle$ be such that $h_i - h_j \notin H_1 + \text{coker } g$ for $i \neq j$. Since $|H : \text{im } g|$ is finite, we may assume that all g_i are in the same coset of $\text{im } g$, so without loss of generality they are all in $\text{im } g$. For each i let $g_i \in G$ be such that $h_i \in g(g_i)$. If $g_i - g_j \in g^{-1}[H_1]$ for $i \neq j$, then there is $h \in H_1$ such that $h \in g(g_i - g_j)$, so $h - (h_i - h_j) \in \text{coker } g$, a contradiction. Hence all g_i are in pairwise distinct cosets modulo $g^{-1}[H_1]$.

(2) Follows from (1) applied to g^{-1} .

(3) If the elements $\langle g_i : i \in I \rangle$ in $\text{dom } g$ are pairwise distinct modulo $g^{-1}[H_1]$ and $h_i \in g(g_i)$, then the elements $\langle h_i : i \in I \rangle$ are in pairwise distinct cosets modulo H_1 . \square

Lemma 4.2. *Let G and H be definable abelian groups in an ω -categorical structure, and $f, g \leq G \times H$ definable additive relations such that $\text{ker } f$ and $\text{coker } g$ are finite, $\text{im } g$ has finite index in H , and $\text{dom } f$ has finite index in G . Then $\text{ker } g$ and $\text{coker } f$ are finite, $\text{im } f$ has finite index in H and $\text{dom } g$ has finite index in G .*

Proof. Let A be a finite set over which all the above objects are definable.

Claim. *Suppose that $H_1 < H_2 \leq H$ are such that H_1 has infinite index in H_2 . Then $f[g^{-1}[H_1]]$ has infinite index in $f[g^{-1}[H_2]]$.*

Proof. As $\text{im } g$ has finite index in H , the index of $H_1 \cap \text{im } g$ in $H_2 \cap \text{im } g$ is infinite. Now, $g^{-1}[H_1]$ has infinite index in $g^{-1}[H_2]$ by Lemma 4.1(1) applied to $g \cap (g^{-1}[H_2] \times H_2)$, so $f[g^{-1}[H_1]]$ is a subgroup of infinite index in $f[g^{-1}[H_2]]$ by Lemma 4.1(2) applied to $f \cap (g^{-1}[H_2] \times f[g^{-1}[H_2]])$. \square

Suppose for a contradiction that $\text{ker } g$ or $\text{coker } f$ is infinite. Put $K_0 = \{0\} \leq H$ and define inductively $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 is infinite; by the claim K_n is a subgroup of infinite index in K_{n+1} for all $n < \omega$, contradicting ω -categoricity (as this implies there are infinitely many disjoint definable sets $\langle K_{n+1} \setminus K_n : n < \omega \rangle$ over A in H).

Now suppose that $\text{im } f$ has infinite index in H or $\text{dom } g$ has infinite index in G . Put $K_0 = H$ and define as before $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 has infinite index in K_0 ; by the claim K_{n+1} is a subgroup of infinite index in K_n for all $n < \omega$, again contradicting ω -categoricity. \square

Remark 4.3. Note that commutativity was not used in the proof. An analogous lemma holds for arbitrary groups, and *multiplicative* relations (with the obvious definition adapting Definition 3.1 to non-commutative groups).

Lemma 4.4. *Let G and H be abelian groups definable in an ω -categorical structure, and $f, g \leq G \times H$ definable quasi-homomorphisms. If $\ker f \lesssim \ker g$ and $\operatorname{im} f \lesssim \operatorname{im} g$, then $\operatorname{im} g \sim \operatorname{im} f$ and $\ker g \sim \ker f$.*

Proof. Suppose $\ker f \lesssim \ker g$ and $\operatorname{im} f \lesssim \operatorname{im} g$. Let $f_1, g_1 \leq G/(\ker f \cap \ker g) \times \operatorname{im} g$ be the additive relations induced by f and g , namely

$$f_1(x + (\ker f \cap \ker g), y) \iff f(x', y) \text{ for some/all } x' \in x + (\ker f \cap \ker g),$$

and likewise for g_1 . Then $\ker f_1$ is finite since $\ker f \lesssim \ker g$, and $\operatorname{coker} g_1 = g[\ker f \cap \ker g] = \operatorname{coker} g$ is finite, too.

Now $\operatorname{im} f \cap \operatorname{im} g$ has finite index in $\operatorname{im} f$, so $f^{-1}[\operatorname{im} g \cap \operatorname{im} f]$ has finite index in G by Lemma 4.1(3); it follows that $\operatorname{dom} f_1 = f^{-1}[\operatorname{im} g \cap \operatorname{im} f]/(\ker f \cap \ker g)$ has finite index in $G/(\ker f \cap \ker g)$. Moreover $\operatorname{im} g_1 = \operatorname{im} g$. Thus $\operatorname{im} f_1 = \operatorname{im} f \cap \operatorname{im} g$ has finite index in $\operatorname{im} g$ and $\ker g_1 = \ker g/(\ker f \cap \ker g)$ is finite by Lemma 4.2. Thus $\operatorname{im} f \sim \operatorname{im} g$ and $\ker f \sim \ker g$. \square

Corollary 4.5. *Let G and H be abelian groups definable in an ω -categorical theory, $f \leq G \times G$ a definable quasi-endomorphism of G , and $g \leq G \times H$ a definable quasi-homomorphism.*

- (1) $\ker f$ is finite if and only if $|G : \operatorname{im} f|$ is finite.
- (2) If $G \leq H$ and $|H : \operatorname{im} g|$ is finite, then $|H : G|$ and $\ker g$ are finite.

Proof. For (1), apply Lemma 4.4 to f and id_G for the implication, and to id_G and f for the converse. For (2) consider the inclusion $i \leq G \times H$. As the assumptions imply $\operatorname{im} i \lesssim \operatorname{im} g$ we may apply Lemma 4.4 and obtain $H \lesssim \operatorname{im} g \sim \operatorname{im} i = G$ and $\ker g \sim \ker i = \{0\}$. \square

Lemma 4.6. *Let G be an ω -categorical abelian group and f a definable quasi-endomorphism of G . Then there is $n < \omega$ such that G decomposes as an almost direct sum of $\operatorname{im} f^{\circ n}$ and $\ker f^{\circ n}$ (i.e. $G \sim \operatorname{im} f^{\circ n} + \ker f^{\circ n}$ and $\operatorname{im} f^{\circ n} \cap \ker f^{\circ n}$ is finite).*

Proof. The $f^{\circ n}[G]$ form a descending chain of subgroups, all definable over the same finite set of parameters. By ω -categoricity there is some n such that $f^{\circ n}[G] = f^{\circ n+1}[G] = f^{\circ 2n}[G]$. Consider $g \in \operatorname{dom} f^{\circ n}$. There is $h \in f^{\circ n}[G]$ such that $f^{\circ n}(g) \cap f^{\circ n}(h) \neq \emptyset$. But this means $g - h \in \ker f^{\circ n}$, so

$$G \lesssim \operatorname{dom} f^{\circ n} \leq \operatorname{im} f^{\circ n} + \ker f^{\circ n}.$$

As $f^{\circ n}[\operatorname{im} f^{\circ n}] = \operatorname{im} f^{\circ 2n} = \operatorname{im} f^{\circ n}$, the intersection $\operatorname{im} f^{\circ n} \cap \ker f^{\circ n}$ must be finite by applying Corollary 4.5(2) to $\operatorname{im} f^{\circ n} \leq \operatorname{im} f^{\circ n}$ and $g = f^{\circ n}$. \square

5. BILINEAR QUASI-FORMS

We shall now introduce a generalization of the notion of a bilinear form. As before, definability will be with parameters in a monster model.

Definition 5.1. Let G, H and K be abelian groups. A *bilinear quasi-form* is a partial function $\lambda : G \times H \rightarrow K$ such that for every $g \in G$ and $h \in H$ the partial functions $\lambda_g : H \rightarrow K$ given by $x \mapsto \lambda(g, x)$ and $\lambda'_h : G \rightarrow K$ given by $\lambda'_h(y) = \lambda(y, h)$ are quasi-homomorphisms with trivial cokernel (i.e. partial homomorphisms defined on a subgroup of finite index).

We shall call λ *definable* if G , H , K and λ are definable.

Remark 5.2. In Definition 5.1 we require λ_g and λ'_h to be quasi-homomorphisms for *all* $g \in G$ and $h \in H$. An alternative (and more general) definition would only require them to be quasi-homomorphisms for g and h in some subgroups of finite index in G and H , respectively. We shall not need this added generality, though.

Definition 5.3. Let $\lambda : G \times H \rightarrow K$ be a bilinear quasi-form. For $g \in G$ (or $h \in H$) the *annihilator* of g (or of h) is the subgroup

$$\begin{aligned} \text{ann}_H(g) &= \{h \in H : \lambda(g, h) = 0\} = \ker \lambda_g \leq H \quad \text{and} \\ \text{ann}_G(h) &= \{g \in G : \lambda(g, h) = 0\} = \ker \lambda'_h \leq G. \end{aligned}$$

If $\bar{g} \in G$ and $\bar{h} \in H$ are finite tuples, put

$$\text{ann}_G(\bar{h}) = \bigcap_{h \in \bar{h}} \text{ann}_G(h) \quad \text{and} \quad \text{ann}_H(\bar{g}) = \bigcap_{g \in \bar{g}} \text{ann}_H(g).$$

Remark 5.4. Of course the annihilators depend on the bilinear quasi-form; if it is not obvious from the context, we shall indicate this by a superscript: ann^λ .

Suppose $\lambda : G \times H \rightarrow K$ is a bilinear quasi-form. For any $g, g' \in G$, we shall consider the additive relation $\lambda_{g,g'} = \lambda_{g'}^{-1} \circ \lambda_g \leq H \times H$ given by $\{(h, h') \in H \times H : \lambda(g, h) = \lambda(g', h')\}$. Clearly $\ker \lambda_{g,g'} = \text{ann}_H(g)$ and $\text{coker } \lambda_{g,g'} = \text{ann}_H(g')$.

Lemma 5.5. *If $\text{ann}_H(g') \lesssim \text{ann}_H(g)$ and $\text{im } \lambda_g \lesssim \text{im } \lambda_{g'}$, then $\lambda_{g,g'}$ induces a quasi-endomorphism $\bar{\lambda}_{g,g'}$ of $H/\text{ann}_H(g')$ given by*

$$\bar{\lambda}_{g,g'}(x + \text{ann}_H(g'), y + \text{ann}_H(g')) \Leftrightarrow \lambda_{g,g'}(x', y) \text{ for some } x' \in x + \text{ann}_H(g').$$

Proof. Note first that this does not depend on the choice of y in the coset $y + \text{ann}_H(g')$, as $\text{ann}_H(g') = \text{coker } \lambda_{g,g'}$. Second, $\text{dom } \lambda_g$ has finite index in H , so

$$\text{dom } \bar{\lambda}_{g,g'} = \text{dom } \lambda_{g,g'} / \text{ann}_H(g') = \lambda_g^{-1}[\text{im } \lambda_g \cap \text{im } \lambda_{g'}] / \text{ann}_H(g')$$

has finite index in $H/\text{ann}_H(g')$ by Lemma 4.1(3). Third,

$$\begin{aligned} \text{coker } \bar{\lambda}_{g,g'} &= \{h \in H : \exists h' \in \text{ann}_H(g') \lambda(g, h') = \lambda(g', h)\} / \text{ann}_H(g') \\ &= \lambda_{g'}^{-1}[\lambda_g[\text{ann}_H(g')]] / \text{ann}_H(g') \end{aligned}$$

is finite, as $\lambda_g[\text{ann}_H(g')]$ is finite due to $\text{ann}_H(g') \lesssim \text{ann}_H(g)$. So $\bar{\lambda}_{g,g'}$ is indeed a quasi-endomorphism. \square

Definition 5.6. For $A \leq G$ and $B \leq H$ put

$$\widetilde{\text{ann}}_H(A) = \{h \in H : A \lesssim \text{ann}_G(h)\} \quad \text{and} \quad \widetilde{\text{ann}}_G(B) = \{g \in G : B \lesssim \text{ann}_H(g)\},$$

the *almost* annihilators of A and B .

As for the annihilators, the almost annihilators depend on the bilinear quasi-form λ , which will be indicated as a superscript if needed: $\widetilde{\text{ann}}^\lambda$.

Remark 5.7. We have

$$\widetilde{\text{ann}}_H(A) = \{h \in H : \lambda'_h[A] \text{ is finite}\} \quad \text{and} \quad \widetilde{\text{ann}}_G(B) = \{g \in G : \lambda_g[B] \text{ is finite}\}.$$

Proof. This follows from Remark 3.2. \square

If $A \lesssim \text{ann}_G(h)$ and $A \lesssim \text{ann}_G(h')$ then $A \lesssim \text{ann}_G(h) \cap \text{ann}_G(h') \leq \text{ann}_G(h \pm h')$; if $B \lesssim \text{ann}_H(g)$ and $B \lesssim \text{ann}_H(g')$ then $B \lesssim \text{ann}_H(g) \cap \text{ann}_H(g') \leq \text{ann}_H(g \pm g')$. Thus the almost annihilators are subgroups of G and of H . Moreover, if G, H, λ, A and B are definable, the almost annihilators are given as a countable increasing union of sets definable over the same parameters, and will be definable in an ω -categorical theory.

The next proposition is an adaptation of Fact 2.4 to bilinear quasi-forms.

Proposition 5.8. *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form, and $A \leq G$ and $B \leq H$ be definable subgroups. Then $B \lesssim \widetilde{\text{ann}}_H(A)$ if and only if $A \lesssim \widetilde{\text{ann}}_G(B)$.*

Proof. We may assume that G, H, A and B are defined over \emptyset . Suppose that $B \not\lesssim \widetilde{\text{ann}}_H(A)$. Consider a sequence $\langle h_i : i < \omega \rangle$ in B representing different cosets of $\widetilde{\text{ann}}_H(A)$. Then $h_i - h_j \notin \widetilde{\text{ann}}_H(A)$ for $i \neq j$, so the index $|A : \text{ann}_A(h_i - h_j)|$ is infinite. By Neumann's Lemma [22] no finite union of cosets of the various $\text{ann}_A(h_i - h_j)$ can cover A . By compactness and sufficient saturation of the monster model, there is an infinite sequence $\langle g_k : k < \omega \rangle$ in A such that $\lambda(g_k - g_\ell, h_i - h_j) \neq 0$ for all $i \neq j$ and $k \neq \ell$. It follows that $|B : \text{ann}_B(g_k - g_\ell)|$ is infinite, whence $g_k - g_\ell \notin \widetilde{\text{ann}}_G(B)$ for all $k \neq \ell$. Thus $A \not\lesssim \widetilde{\text{ann}}_G(B)$.

The other direction follows by symmetry. \square

Definition 5.9. A bilinear quasi-form λ is *almost trivial* if there is a finite subgroup of K containing $\text{im}\lambda$. It is *virtually almost trivial* if there are finite index subgroups G_0 in G and H_0 in H such that the restriction of λ to $G_0 \times H_0$ is almost trivial.

Proposition 5.10. *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form. Then λ is almost trivial if and only if there is a finite integer which bounds the indices of $\text{ann}_H(g)$ and $\text{ann}_G(h)$ in H and G , respectively, for all $g \in G$ and $h \in H$.*

Proof. Suppose $\text{im}\lambda$ generates a finite group K_0 . Since λ is a definable bilinear quasi-form, by compactness there is a finite bound on the indices of $\text{dom}\lambda_g$ in H and of $\text{dom}\lambda'_h$ in G . As the indices $|\text{dom}\lambda_g : \text{ann}_H(g)|$ and $|\text{dom}\lambda'_h : \text{ann}_G(h)|$ are bounded by $|K_0|$, the implication follows.

Conversely, suppose there is a finite bound ℓ for the indices of $\text{ann}_H(g)$ and $\text{ann}_G(h)$ in H and in G for all $g \in G$ and $h \in H$. Then ℓ bounds the size of $\lambda_g[H]$ and of $\lambda'_h[G]$, for all $g \in G$ and $h \in H$, by Remark 3.2. Note that this implies that $\langle \text{im}\lambda \rangle$ has finite exponent, as $n\lambda(g, h) = \lambda(g, nh) \in \lambda_g[H]$ for all $g \in G, h \in H$ and $n < \omega$. So it is enough to show that $\text{im}\lambda$ is finite.

Consider $g \in G$ with $\lambda_g[H]$ maximal, and choose $h_0, \dots, h_n \in H$ with $\lambda_g[H] = \{\lambda(g, h_i) : i \leq n\}$. Then for $g' \in g + \bigcap_{i \leq n} \text{ann}_G(h_i)$ we have $\lambda(g', h_i) = \lambda(g, h_i)$, whence $\lambda_{g'}[H] \supseteq \lambda_g[H]$, and $\lambda_{g'}[H] = \lambda_g[H]$ by maximality. Note that $\bigcap_{i \leq n} \text{ann}_G(h_i)$ is a subgroup of boundedly finite index in G (i.e. bounded independently from g). It follows that there can only be finitely many maximal sets of the form $\lambda_g[H]$ for $g \in G$, and $\text{im}\lambda$ is finite. \square

Corollary 5.11. *Let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form. The following are equivalent:*

- (1) $G \lesssim \widetilde{\text{ann}}_G(H)$.
- (2) $H \lesssim \widetilde{\text{ann}}_H(G)$.
- (3) λ is virtually almost trivial.

Moreover, in this case $\widetilde{\text{ann}}_G(H)$ and $\widetilde{\text{ann}}_H(G)$ are definable.

Proof. Conditions (1) and (2) are equivalent by Proposition 5.8.

Suppose (1) and (2) hold. Put $A_n = \{g \in G : |H : \text{ann}_H(g)| \leq n\}$. Then each A_n is definable, and $\widetilde{\text{ann}}_G(H) = \bigcup_{n < \omega} A_n$. By compactness and (1), there are $n, k < \omega$ such that there are no k disjoint translates of A_n by elements in G . Let $A = \bigcup_i a_i + A_n$ be a maximal union of disjoint translates of A_n by elements of $\widetilde{\text{ann}}_G(H)$. So for any $a \in \widetilde{\text{ann}}_G(H)$ we have $(a + A) \cap A \neq \emptyset$, whence $a \in A - A$. Thus $\widetilde{\text{ann}}_G(H) = A - A$ is definable; it follows that there is a finite bound on $|H : \text{ann}_H(a)|$ for all $a \in \widetilde{\text{ann}}_G(H)$, as otherwise by sufficient saturation we could find $a \in \widetilde{\text{ann}}_G(H)$ such that $|H : \text{ann}_H(a)|$ is infinite, a contradiction. By symmetry, the same holds for $\widetilde{\text{ann}}_H(G)$. Proposition 5.10 now implies that λ restricted to $\widetilde{\text{ann}}_G(H) \times \widetilde{\text{ann}}_H(G)$ is almost trivial, so λ is virtually almost trivial.

Conversely, if λ is virtually almost trivial as witnessed by G_0 and H_0 , then $\text{ann}_{H_0}(g)$ has finite index in H_0 for $g \in G_0$, so $\text{ann}_H(G)$ has finite index in H . Thus $G_0 \leq \widetilde{\text{ann}}_G(H)$, whence $G \lesssim \widetilde{\text{ann}}_G(H)$. Similarly $H \lesssim \widetilde{\text{ann}}_H(G)$. \square

6. GENERICITY AND PRINCIPALITY

In the following, cosets could be left or right cosets.

Definition 6.1. Let G be an infinite definable group and A a set of parameters. A complete type $p \in S_G(A)$ is *subgroup-generic* if p is in no definable coset of a subgroup of infinite index in G which has only finitely many images under automorphisms fixing A (so it is $\text{acl}^{eq}(A)$ -definable). A sequence $\langle (g_i, \bar{a}_i) : i \in I \rangle$ is *subgroup-generic* over A if $\text{tp}(g_i/A, \{g_j, \bar{a}_j : j < i\})$ is subgroup-generic for all $i \in I$.

Remark 6.2. As a left coset of a subgroup H of finite index is a right coset of a conjugate of H , it is sufficient to verify subgroup-genericity for left or for right cosets. Thus, if $\text{tp}(g/A)$ is subgroup-generic, so are $\text{tp}(g^{-1}/A)$, as well as $\text{tp}(hg/A)$ and $\text{tp}(gh/A)$ for any $h \in \text{acl}(A) \cap G$.

Note that by Neumann's Lemma [22] the group G is not in the ideal of definable sets covered by finitely many cosets of definable subgroups of G of infinite index. By a standard construction (as, for example, in [18, Fact 2.1.3]), G has a subgroup-generic complete type over any set of parameters. By Ramsey's Theorem and compactness, indiscernible subgroup-generic sequences of any order type exist.

Recall that if G is a type-definable group over some parameters A , the *A -connected component* G_A^0 is the intersection of all relatively type-definable subgroups of finite index in G . If G_A^0 does not depend on A (for instance in a dependent theory [6]), it is just called the connected component G^0 of G . The following notion provides a useful replacement for principal generic types in a context where connected components need not exist.

Definition 6.3. Let G be an infinite definable group and A a set of parameters. An A -indiscernible sequence $\langle (g_i, \bar{a}_i) : i \in I \rangle$ with $g_i \in G$ for all $i \in I$ is *principal indiscernible* if for any $i \in I$ and $A_i = A \cup \{g_j, \bar{a}_j : j \neq i\}$, whenever C is an A_i -definable coset of some subgroup H and $g_i \in C$, then $g_i \in H_{A_i}^0$. The sequence is *principal subgroup-generic* if moreover $\text{tp}(g_i/A, \{g_j, \bar{a}_j : j < i\})$ is subgroup-generic for all $i \in I$.

Note that if C is a left coset, then $H = C^{-1}C$; if C is a right coset, $H = CC^{-1}$. In either case H is A_i -definable.

Proposition 6.4. *Principal subgroup-generic sequences exist. More precisely, let $\epsilon > 0$ be infinitesimal, let $\langle (g_i, \bar{a}_i) : i \in \mathbb{Q} \cup (\mathbb{Q} + \epsilon) \rangle$ be an A -indiscernible sequence, and put*

$h_i = g_{i+\epsilon}^{-1}g_i$. Then $\langle (h_i, \bar{a}_i \bar{a}_{i+\epsilon}) : i \in \mathbb{Q} \rangle$ is principal indiscernible over A ; if moreover $\text{tp}(g_i/A, \{g_j, \bar{a}_j : j < i\})$ is subgroup-generic for all i , then $\langle (h_i, \bar{a}_i \bar{a}_{i+\epsilon}) : i \in \mathbb{Q} \rangle$ is principal subgroup-generic over A .

It follows by compactness and indiscernibility that there are principal subgroup-generic sequences of any order-type.

Proof. Let C be an $(A \cup \{\bar{a}_j \bar{a}_{j+\epsilon} : j \neq i\})$ -definable coset of some subgroup H , such that $h_i \in C$. Choose a finite set $J \subset \mathbb{Q} \setminus \{i\}$ such that C is definable over $A_J = A \cup \{g_j, g_{j+\epsilon}, \bar{a}_j, \bar{a}_{j+\epsilon} : j \in J\}$; take $m < i < M$ in \mathbb{Q} with $]m, M[\cap J = \emptyset$ and put $I =]m, M[\cap (\mathbb{Q} \cup (\mathbb{Q} + \epsilon))$. Let H_0 be an A_J -definable subgroup of finite index in H . Since $h_i = g_{i+\epsilon}^{-1}g_i \in C$ we have $g_k^{-1}g_j \in C$ for all $j < k$ in I by indiscernibility of $\langle g_j : j \in I \rangle$ over A_J . By Ramsey's Theorem there is an infinite set of indices $I' \subseteq I$ such that all $g_k^{-1}g_j$ with $j < k$ in I' are in the same coset C_0 modulo H_0 . Choose $j < k < \ell$ in I' for left cosets, and $\ell < j < k$ in I' for right cosets. Then

$$\begin{aligned} g_k^{-1}g_j &= g_k^{-1}g_\ell g_\ell^{-1}g_j = (g_\ell^{-1}g_k)^{-1}g_\ell^{-1}g_j \in C_0^{-1}C_0 = H_0 \quad \text{for left cosets} \\ &= g_k^{-1}g_\ell (g_j^{-1}g_\ell)^{-1} \in C_0 C_0^{-1} = H_0 \quad \text{for right cosets.} \end{aligned}$$

By indiscernibility again $g_k^{-1}g_j \in H_0$ for all $j < k$ in I . In particular $h_i = g_{i+\epsilon}^{-1}g_i \in H_0$. As this is true for all finite $J \subset \mathbb{Q} \setminus \{i\}$ and all A_J -definable subgroups H_0 of H of finite index, and since $A_i \subseteq \text{dcl}(A_J : J \subset \mathbb{Q} \setminus \{i\} \text{ finite})$, we get $h_i \in H_A^0$.

Moreover, if $\text{tp}(g_{i+\epsilon}/A, \{g_j, \bar{a}_j : j \leq i\})$ is subgroup-generic, so are $\text{tp}(g_{i+\epsilon}^{-1}/A, \{g_j, \bar{a}_j : j \leq i\})$ and $\text{tp}(g_{i+\epsilon}^{-1}g_i/A, \{g_j, \bar{a}_j : j \leq i\})$. Thus $\text{tp}(h_i/A, \{g_j, \bar{a}_j : j \leq i\})$ is subgroup-generic, and $\langle (h_i, \bar{a}_i \bar{a}_{i+\epsilon}) : i \in \mathbb{Q} \rangle$ is subgroup-generic over A . \square

Remark 6.5. Definitions 6.1 and 6.3 as well as Proposition 6.4 could also have been formulated using the A -type-definable connected component H_A^{00} or the A -invariant connected component H_A^{000} instead of H_A^0 , with similar proofs but using a more highly saturated dense linear order instead of \mathbb{Q} , and the Erdős-Rado theorem instead of Ramsey. As we work in the ω -categorical context, we do not need this added generality.

7. VIRTUAL ALMOST TRIVIALITY

In this section, we shall consider an \emptyset -definable bilinear quasi-form $\lambda : G \times H \rightarrow K$ in an ω -categorical theory. Clearly, if $A \leq G$, $B \leq H$ and $C \leq K$ are definable, then the restriction of λ to $A \times B$ composed with the quotient map $K \rightarrow K/C$ is a definable bilinear quasi-form $A \times B \rightarrow K/C$.

Definition 7.1. The *reduced burden* of G and of H with respect to λ is

$$\text{rbdn}_\lambda(G) = \sup_{\bar{h} \in H \text{ finite}} \text{bdn}(G/\text{ann}_G(\bar{h})) \quad \text{and} \quad \text{rbdn}_\lambda(H) = \sup_{\bar{g} \in G \text{ finite}} \text{bdn}(H/\text{ann}_H(\bar{g})).$$

We define the *burden* of λ to be $\text{bdn}(\lambda) = \text{rbdn}_\lambda(G) + \text{rbdn}_\lambda(H) + \text{bdn}(K)$.

Definition 7.2. A definable bilinear quasi-form $\bar{\lambda} : A \times B \rightarrow D/C$ is *induced* from λ if $A \leq G$, $B \leq H$ and $C \leq D \leq K$ are definable subgroups such that the map $\bar{\lambda}$, obtained by composing the restriction of λ to $(A \times B) \cap \lambda^{-1}[D]$ with the quotient map $D \rightarrow D/C$, is still a bilinear quasi-form.

Remark 7.3. It follows from Lemma 4.1 that $\bar{\lambda}$ is an induced bilinear form if $\lambda_a[B] \lesssim D$ for all $a \in A$ and $\lambda'_b[A] \lesssim D$ for all $b \in B$.

Lemma 7.4. *If $\bar{\lambda} : A \times B \rightarrow D/C$ is induced from λ , then $\text{rbdn}_{\bar{\lambda}}(A) \leq \text{rbdn}_{\lambda}(G)$, $\text{rbdn}_{\bar{\lambda}}(B) \leq \text{rbdn}_{\lambda}(H)$ and $\text{bdn}(D/C) \leq \text{bdn}(K)$.*

Proof. Clearly $\text{bdn}(D/C) \leq \text{bdn}(D) \leq \text{bdn}(K)$, as D/C is the image of D under a definable function. Moreover,

$$\begin{aligned} \text{rbdn}_{\lambda}(G) &= \sup_{\bar{h} \in H \text{ finite}} \text{bdn}(G/\text{ann}_G(\bar{h})) \geq \sup_{\bar{h} \in B \text{ finite}} \text{bdn}(G/\text{ann}_G(\bar{h})) \\ &\geq \sup_{\bar{h} \in B \text{ finite}} \text{bdn}(A/\text{ann}_A(\bar{h})) = \text{rbdn}_{\bar{\lambda}}(A), \end{aligned}$$

similarly $\text{rbdn}_{\lambda}(H) \geq \text{rbdn}_{\bar{\lambda}}(B)$. \square

The aim of this section is to prove:

Theorem 7.5. *Let $\lambda : G \times H \rightarrow K$ be a bilinear quasi-form of finite burden definable in an ω -categorical theory. Then λ is virtually almost trivial. If G and H are connected, then λ is trivial.*

The following preliminary lemmas are trivially true if λ itself is virtually almost trivial. Of course they hold for any dense linear order as index set; we chose \mathbb{Q} for convenience.

Lemma 7.6. *Suppose $\text{bdn}(\lambda)$ is finite, and any induced bilinear quasi-form of strictly smaller burden is virtually almost trivial. Let $\langle y_i : i \in \mathbb{Q} \rangle$ be a principal indiscernible sequence in G over \emptyset . Then $\text{im}\lambda_{y_i}$ and $\text{im}\lambda_{y_j}$ are \lesssim -comparable for all $i < j$, as are $\text{ann}_H(y_i)$ and $\text{ann}_H(y_j)$.*

Proof. By Lemma 2.6 there is a minimal $1 \leq \ell \leq \text{bdn}(K)$ such that the sum $\sum_{i=0}^{\ell} \text{im}\lambda_{y_i}$ is reducible (in the sense of Lemma 2.6). Note that if $\ell = 1$ we are done. In any case there is $i_0 \leq \ell$ such that with $I = \{0, 1, \dots, \ell\} \setminus \{i_0\}$ we have

$$\text{im}\lambda_{y_{i_0}} \lesssim \sum_{i \in I} \text{im}\lambda_{y_i} =: C.$$

Consider induced bilinear quasi-form $\bar{\lambda} : G \times H \rightarrow K/C$, and put

$$\begin{aligned} A &= \{g \in G : \text{im}\lambda_g \lesssim C\} = \{g \in G : \bar{\lambda}_g[H] \lesssim \{\bar{0}\}\} = \widetilde{\text{ann}}_{\bar{\lambda}}^G(H) \quad \text{and} \\ B &= \{h \in H : \lambda'_h[A] \lesssim C\} = \{h \in H : \bar{\lambda}'_h[A] \lesssim \{\bar{0}\}\} = \widetilde{\text{ann}}_{\bar{\lambda}}^H(A). \end{aligned}$$

They are $\{y_j : j \in I\}$ -definable by ω -categoricity, and $H \lesssim B$ by Lemma 5.8. Thus B has finite index in H .

By definition of A and B we have $\lambda_g[B] \leq \lambda_g[H] \lesssim C$ and $\lambda'_h[A] \lesssim C$ for all $g \in A$ and $h \in B$, the restriction $\lambda : A \times B \rightarrow C$ is an induced $\{y_j : j \in I\}$ -definable bilinear quasi-form by Remark 7.3, as is

$$\lambda_i : A \times B \rightarrow C/\text{im}\lambda_{y_i}$$

for every $i \in I$.

By irreducibility of the sum $\sum_{j \in I} \text{im}\lambda_{y_j}$, the quotient $\text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})$ is infinite for all $i \in I$. Hence, as

$$(\text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})) \times (\sum_{j \in I, j \neq i} \text{im}\lambda_{y_j}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j}))$$

embeds definably into $C/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})$, Remark 2.2 implies the following strict inequality:

$$\begin{aligned} \text{bdn}(C/\text{im}\lambda_i) &= \text{bdn}\left(\sum_{j \in I, j \neq i} \text{im}\lambda_{y_j} / (\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})\right) \\ &< \text{bdn}(C/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})) \leq \text{bdn}(K). \end{aligned}$$

Thus $\text{bdn}(\lambda_i) < \text{bdn}(\lambda)$ by Lemma 7.4, so λ_i is virtually almost trivial by assumption. By Corollary 5.11 the almost annihilator $\widetilde{\text{ann}}_A^{\lambda_i}(B)$ of B with respect to the quasi-form λ_i is an $\{y_j : j \in I\}$ -definable subgroup of A of finite index. Now $\text{im}\lambda_{y_{i_0}} \lesssim C$ by choice of i_0 , so $y_{i_0} \in A$ by definition of A and

$$y_{i_0} \in A_{\{y_j : j \in I\}}^0 \leq \widetilde{\text{ann}}_A^{\lambda_i}(B)$$

by principal indiscernibility. Thus $\lambda_{y_{i_0}}[B] \lesssim \text{im}\lambda_{y_i}$; as B has finite index in H we also have $\text{im}\lambda_{y_{i_0}} \lesssim \text{im}\lambda_{y_i}$. By indiscernibility, $\text{im}\lambda_{y_j}$ and $\text{im}\lambda_{y_i}$ are \lesssim -comparable for all $i \neq j$.

For the second assertion, Lemma 2.5 yields a minimal $\ell \geq 1$ such that the intersection $\bigcap_{i=0}^{\ell} \text{ann}_H(y_i)$ is reducible (and again we are done if $\ell = 1$). So there is $i_0 \leq \ell$ such that

$$B := \bigcap_{i \in I} \text{ann}_H(y_i) \lesssim \text{ann}_H(y_{i_0}),$$

where $I = \{0, 1, \dots, \ell\} \setminus \{i_0\}$. By ω -categoricity, the subgroup

$$A := \{g \in G : B \lesssim \text{ann}_H(g)\}$$

is $\{y_j : j \in I\}$ -definable. For every $i \in I$ consider the restricted bilinear quasi-form $\lambda_i : A \times \text{ann}_H(y_i) \rightarrow K$. Note that for any finite $\bar{g} \in A$ we have $B \lesssim \text{ann}_H(\bar{g})$, so by Remark 2.3

$$\begin{aligned} \text{rbdn}_{\lambda_i}(\text{ann}_H(y_i)) &= \sup_{\bar{g} \in A \text{ finite}} \text{bdn}(\text{ann}_H(y_i) / \text{ann}_{\text{ann}_H(y_i)}(\bar{g})) \leq \text{bdn}(\text{ann}_H(y_i) / B) \\ &< \text{bdn}(\text{ann}_H(y_i) / B) + \text{bdn}\left(\bigcap_{j \in I, j \neq i} \text{ann}_H(y_j)\right) / B \\ &\leq \text{bdn}\left(\text{ann}_H(y_i) + \bigcap_{j \in I, j \neq i} \text{ann}_H(y_j)\right) / B \\ &\leq \text{bdn}(H/B) = \text{bdn}(H/\text{ann}(y_i : i \in I)) \leq \text{rbdn}_{\lambda}(H), \end{aligned}$$

where the strict inequality holds as $(\bigcap_{j \in I, j \neq i} \text{ann}_H(y_j)) / B$ is infinite by minimality of ℓ , and the second weak inequality follows from Remark 2.2.

Thus $\text{bdn}(\lambda_i) < \text{bdn}(\lambda)$; by assumption λ_i is virtually almost trivial. By Corollary 5.11 the almost annihilator $\widetilde{\text{ann}}_A^{\lambda_i}(\text{ann}_H(y_i))$ is a subgroup of A of finite index definable over $\{y_j : j \in I\}$. Since $y_{i_0} \in A$ by definition of A and B , we have

$$y_{i_0} \in A_{\{y_j : j \in I\}}^0 \leq \widetilde{\text{ann}}_A^{\lambda_i}(\text{ann}_H(y_i))$$

by principal indiscernibility, whence $\text{ann}_H(y_i) \lesssim \text{ann}_H(y_{i_0})$. By indiscernibility $\text{ann}_H(y_j)$ and $\text{ann}_H(y_i)$ are \lesssim -comparable for all $i \neq j$. \square

Lemma 7.7. *Suppose $\text{bdn}(\lambda)$ is finite, and any induced bilinear quasi-form of strictly smaller burden is virtually almost trivial. Let $\langle y_i : i \in \mathbb{Q} \rangle$ be a principal indiscernible sequence in G over \emptyset such that $\text{tp}(y_i)$ is subgroup-generic. Then, for any $j \in \mathbb{Q}$, any definable quasi-endomorphism f of $H/\text{ann}_H(y_j)$ is invertible or nilpotent.*

Proof. By Lemma 4.6, we have an almost direct decomposition $H/\text{ann}_H(y_j) \sim \text{im} f^{\text{on}} + \ker f^{\text{on}}$ for some $n < \omega$. Put $A = \{g \in G : \text{ann}_H(y_j) \lesssim \text{ann}_H(g)\}$, a definable subgroup by ω -categoricity, and let $B_1, B_2 \leq H$ be the preimages of $\text{im} f^{\text{on}}$ and $\ker f^{\text{on}}$, respectively, under the quotient map $H \rightarrow H/\text{ann}_H(y_j)$. Then $\text{ann}_H(y_j)$ has finite index in $B_1 \cap B_2$, so $\text{ann}_H(y_j) = \text{ann}_{B_i}(y_j)$.

If f were neither invertible nor nilpotent, then both summands are infinite. For $i = 1, 2$ consider the restriction λ_i of λ to $A \times B_i$, again a bilinear quasi-form. By the definition of A we have $\text{ann}_H(y_j) \lesssim \text{ann}_H(\bar{g})$ for all finite $\bar{g} \in A$, so by Remarks 2.3 and 2.2 we obtain

$$\begin{aligned} \text{rbdn}_{\lambda_i}(B_i) &= \sup_{\bar{g} \in A \text{ finite}} \text{bdn}(B_i/\text{ann}_{B_i}(\bar{g})) \leq \text{bdn}(B_i/\text{ann}_{B_i}(y_j)) = \text{bdn}(B_i/(B_1 \cap B_2)) \\ &< \text{bdn}((B_1 + B_2)/(B_1 \cap B_2)) = \text{bdn}(H/\text{ann}_H(y_j)) \leq \text{rbdn}_{\lambda}(H). \end{aligned}$$

Thus $\text{bdn}(\lambda_1), \text{bdn}(\lambda_2) < \text{bdn}(\lambda)$; by assumption λ_1 and λ_2 are almost trivial, and so is the restriction

$$\lambda' : A \times H = A \times (B_1 + B_2) \rightarrow K.$$

By Lemma 7.6 we have $y_i \in A$ either for $i < j$ or for $i > j$, so $y_i \in A_{y_j}^0 \leq \widetilde{\text{ann}}_A(H) \leq \widetilde{\text{ann}}_G(H)$ by principal indiscernibility. By indiscernibility $y_j \in \widetilde{\text{ann}}_G(H)$, so $H/\text{ann}_H(y_j)$ is finite, and any quasi-endomorphism of it is equivalent to zero. \square

Proof of Theorem 7.5. Let $\lambda : G \times H \rightarrow K$ be a counter-example with $\text{bdn}(\lambda)$ minimal possible. Adding finitely many parameters to the language, we may assume that everything is \emptyset -definable. Let $\epsilon > 0$ be infinitesimal, let $\langle (x_i, x'_i) : i \in \mathbb{Q} \cup (\mathbb{Q} + \epsilon) \rangle$ be an \emptyset -indiscernible subgroup-generic sequence in $G \times H$, and put $y_i = x_i - x_{i+\epsilon}$ and $y'_i = x'_i - x'_{i+\epsilon}$. Then $\langle (y_i, y'_i) : i \in I \rangle$ is principal indiscernible over $\{y_i, y'_i : i \in \mathbb{Q} \setminus I\}$ for any interval $I \subseteq \mathbb{Q}$ by Proposition 6.4 (in G for the first coordinate, in H for the second one).

Claim 1. *For $i < j$ we have $\text{ann}_H(y_j) \lesssim \text{ann}_H(y_i)$, and if $B \leq H$ is definable over $\{y_k, y'_k : k \notin [i, j]\}$, then $\lambda_{y_i}[B] \lesssim \lambda_{y_j}[B]$. In particular $\text{im} \lambda_{y_i} \lesssim \text{im} \lambda_{y_j}$.*

Proof of Claim. By ω -categoricity $\widetilde{\text{ann}}_H(G)$ is definable, and the index $|G : \text{ann}_G(h)|$ takes only finitely many finite values; let n be the maximal one. Choose h realizing a subgroup-generic type for the group $\widetilde{\text{ann}}_H(G)$ over x_0, \dots, x_n . Then $x_i - x_j \in \text{ann}_G(h)$ for some $0 \leq i < j \leq n$, whence $h \in \text{ann}_H(x_i - x_j)$. By subgroup-genericity of h over x_0, \dots, x_n , the group $\text{ann}_H(x_i - x_j) \cap \widetilde{\text{ann}}_H(G)$ must have a finite index in $\widetilde{\text{ann}}_H(G)$. Thus $\widetilde{\text{ann}}_H(G) \lesssim \text{ann}_H(x_i - x_j)$; by indiscernibility $\widetilde{\text{ann}}_H(G) \lesssim \text{ann}_H(x_0 - x_\epsilon) = \text{ann}_H(y_0)$.

Suppose $\text{ann}_H(y_0) \lesssim \text{ann}_H(y_1)$. Then $y_1 \in \widetilde{\text{ann}}_G(\text{ann}_H(y_0))$; as y_1 is subgroup-generic over y_0 we have $G \lesssim \widetilde{\text{ann}}_G(\text{ann}_H(y_0))$. By Proposition 5.8 we obtain $\text{ann}_H(y_0) \lesssim \widetilde{\text{ann}}_H(G)$. It follows that $\text{ann}_H(y_i) \sim \widetilde{\text{ann}}_H(G)$ for all $i \in \omega$, and $\text{ann}_H(y_1) \sim \text{ann}_H(y_0)$.

If $\text{ann}_H(y_0) \not\lesssim \text{ann}_H(y_1)$, then $\text{ann}_H(y_1) \lesssim \text{ann}_H(y_0)$ by Lemma 7.6. In either case $\text{ann}_H(y_j) \lesssim \text{ann}_H(y_i)$ for all $i < j$ by indiscernibility.

For the second assertion, let $J \subset \mathbb{Q} \setminus [i, j]$ be a finite set such that B is definable over $\{y_k, y'_k : k \in J\}$, and $I \subset \mathbb{Q} \setminus J$ an open interval containing $[i, j]$. If λ' is the restriction of λ to $G \times B$, then $\text{bdn}(\lambda') \leq \text{bdn}(\lambda)$, and the sequence $\langle y_k : k \in I \rangle$ is principal indiscernible over $\{y_k, y'_k : k \in J\}$. Hence $\lambda_{y_i}[B] \lesssim \lambda_{y_j}[B]$ or $\lambda_{y_j}[B] \lesssim \lambda_{y_i}[B]$ by Lemma 7.6 applied to λ' . In the first case we are done; in the second case, since $\text{ann}_B(y_j) = \text{ann}_H(y_j) \cap B \lesssim \text{ann}_H(y_i) \cap B = \text{ann}_B(y_i)$ by the first part of the claim, we obtain $\lambda_{y_i}[B] \sim \lambda_{y_j}[B]$ by Lemma 4.4, and we are done again. \square

Claim 2. *If $i < j$ and $C \leq K$ is definable over $\{y_k, y'_k : k \notin [i, j]\}$, then $\lambda_{y_j}^{-1}[C] \lesssim \lambda_{y_i}^{-1}[C]$.*

Proof of Claim. Let $J \subset \mathbb{Q} \setminus [i, j]$ be a finite set such that C is definable over $\{y_k, y'_k : k \in I\}$, and $I \subset \mathbb{Q} \setminus J$ an open interval containing $[i, j]$. The induced bilinear quasi-form $\bar{\lambda} : G \times H \rightarrow K/C$ satisfies $\text{bdn}(\bar{\lambda}) \leq \text{bdn}(\lambda)$, and $\langle y_k : k \in I \rangle$ is principal indiscernible over $\{y_k, y'_k : k \in J\}$. As $\lambda_y^{-1}[C] = \text{ann}_H^{\bar{\lambda}}(y)$, Lemma 7.6 yields \lesssim -comparability of $\lambda_{y_i}^{-1}[C]$ and $\lambda_{y_j}^{-1}[C]$.

If $\lambda_{y_j}^{-1}[C] \lesssim \lambda_{y_i}^{-1}[C]$ we are done. So suppose $\lambda_{y_i}^{-1}[C] \lesssim \lambda_{y_j}^{-1}[C]$. Then λ_{y_i} and λ_{y_j} induce quasi-homomorphisms from $B := \lambda_{y_i}^{-1}[C]$ to C . As $\text{ann}_H(y_j) \lesssim \text{ann}_H(y_i)$ by Claim 1 and $\lambda_{y_j}[B] \lesssim \lambda_{y_i}[B] = C$, Lemma 4.4 implies $\lambda_{y_j}[B] \sim C$. Thus

$$\lambda_{y_j}^{-1}[C] \sim B + \text{ann}_H(y_j) \lesssim B + \text{ann}_H(y_i) = B = \lambda_{y_i}^{-1}[C]. \quad \square$$

We shall now study λ_{y_i, y_j} for $i < j$. By Claim 1 and Lemma 5.5 it induces a quasi-endomorphism $\bar{\lambda}_{y_i, y_j}$ of $H/\text{ann}_H(y_j)$.

Claim 3. For $i \neq j$ we have $\text{ann}_H(y_i) \not\sim \text{ann}_H(y_j)$.

Proof of Claim. Suppose otherwise, and put $\bar{H} = H/\text{ann}_H(y_0)$. Note that \bar{H} is infinite: if $\text{ann}_H(y_0)$ had finite index in H , then $\widetilde{\text{ann}}_G(H)$ would contain y_0 and have finite index in G by subgroup-genericity of y_0 , so λ would be virtually almost trivial by Corollary 5.11.

Now by indiscernibility, for all $i, j \in \mathbb{Q}$ and $k < \ell$ in $\mathbb{Q} \cup (\mathbb{Q} + \epsilon)$ we have

$$\text{ann}_H(y_i) \sim \text{ann}_H(y_j) \sim \text{ann}_H(x_k - x_\ell).$$

By Lemma 4.4, Claim 1, and indiscernibility again,

$$\text{im} \lambda_{y_i} \sim \text{im} \lambda_{y_j} \sim \text{im} \lambda_{x_k - x_\ell},$$

so any $\lambda_{x_k - x_\ell}$ induces a quasi-isomorphism between \bar{H} and $\text{im} \lambda_{y_0}$. It follows that for $g, g', g'' \in \{x_i - x_j : i < j\}$ we have $\bar{\lambda}_{g, g''} \equiv \bar{\lambda}_{g', g''} \circ \bar{\lambda}_{g, g'}$, and this is a quasi-automorphism of \bar{H} .

Let R be the ring of definable quasi-endomorphisms of \bar{H} modulo equivalence. By Lemma 7.7 any $r \in R$ is nilpotent or invertible. It follows that the subset $I \subseteq R$ of nilpotent definable quasi-endomorphisms (modulo equivalence) of \bar{H} is an ideal: it is clearly invariant under left and right multiplication; if f and g are nilpotent but $f + g$ is not nilpotent, there is invertible h with $h(f + g) = hf + hg = \text{id}$. So $hf = \text{id} - hg$ is nilpotent. But

$$(\text{id} - hg)(\text{id} + hg + (hg)^2 + (hg)^3 + \cdots) = \text{id}$$

(note that the sum is finite, as hg is nilpotent), so $hf = \text{id} - hg$ is invertible, a contradiction. Thus R/I is a division ring.

If \bar{r} is a finite tuple of quasi-endomorphisms definable over a finite set A of parameters, then by ω -categoricity there are only finitely many A -definable additive relations on \bar{H}^2 ; it follows that \bar{r} generates a finite subring of R . Thus R is locally finite, as is R/I ; by Wedderburn's Little Theorem R/I is a locally finite field.

Let $\hat{\lambda}_{y, y'} \in R$ be the equivalence class of $\bar{\lambda}_{y, y'}$. By local finiteness and indiscernibility, $\hat{\lambda}_{y_i, y_j} + I \in R/I$ has a fixed finite multiplicative order N independent of $i < j$, and must be one of the finitely many primitive N^{th} roots of unity. By the pigeonhole principle and indiscernibility, this root $\zeta = \hat{\lambda}_{y_i, y_j} + I$ does not depend on $i < j$. But then $\zeta^2 = \hat{\lambda}_{y_j, y_k} \cdot \hat{\lambda}_{y_i, y_j} = \hat{\lambda}_{y_i, y_k} = \zeta$ for $i < j < k$, whence $\hat{\lambda}_{y_i, y_j} + I = \zeta = \text{id}_{\bar{H}} + I$. By indiscernibility, $\hat{\lambda}_{x_i - x_j, x_k - x_\ell} \in \text{id}_{\bar{H}} + I$ for all $0 < i < j < k < \ell$ in ω .

Consider

$$\hat{\lambda}_{x_1 - x_3, x_2 - x_3} = \hat{\lambda}_{x_2 - x_3, x_4 - x_5}^{-1} \cdot \hat{\lambda}_{x_1 - x_3, x_4 - x_5} \in \text{id}_{\bar{H}} + I.$$

Since $\hat{\lambda}_{x_1-x_3, x_2-x_3} - \text{id}_{\bar{H}} \in I$ is nilpotent, $B := \text{im}(\lambda_{x_1-x_3, x_2-x_3} - \text{id}_H)$ is a definable subgroup of infinite index in H almost containing $\text{ann}_H(y_0)$. Then for all

$$h \in H_{x_1, x_2, x_3}^0 \leq \text{dom } \lambda_{x_1-x_3, x_2-x_3}$$

there is $b \in B$ with $h + b \in \lambda_{x_1-x_3, x_2-x_3}(h)$. Hence $\lambda_{x_2-x_3}(h + b) = \lambda_{x_1-x_3}(h)$, that is

$$\lambda(x_1 - x_3, h) = \lambda(x_2 - x_3, h + b) = \lambda(x_2 - x_3, h) + \lambda(x_2 - x_3, b),$$

whence

$$\lambda(x_1 - x_2, h) = \lambda((x_1 - x_3) - (x_2 - x_3), h) = \lambda(x_1 - x_3, h) - \lambda(x_2 - x_3, h) = \lambda(x_2 - x_3, b).$$

But this means that $\text{im } \lambda_{x_2-x_3} \sim \text{im } \lambda_{x_1-x_2} \lesssim \lambda_{x_2-x_3}[B]$. But B has infinite index in H and $\ker \lambda_{x_2-x_3} \lesssim B$, so $\lambda_{x_2-x_3}[B]$ has infinite index in $\lambda_{x_2-x_3}[H] = \text{im } \lambda_{x_2-x_3}$, a contradiction. \square

Claim 4. For $i < j < k < \ell$ and $B \leq H$ definable over $\{y_s, y'_s : s \notin [j, k]\}$ we have $\lambda_{y_j, y_\ell}[B] \lesssim \lambda_{y_k, y_\ell}[B]$ and $\lambda_{y_i, y_k}[B] \lesssim \lambda_{y_i, y_j}[B]$.

Proof of Claim. We have $\lambda_{y_j}[B] \lesssim \lambda_{y_k}[B]$ by Claim 1, whence

$$\lambda_{y_j, y_\ell}[B] = \lambda_{y_\ell}^{-1}[\lambda_{y_j}[B]] \lesssim \lambda_{y_\ell}^{-1}[\lambda_{y_k}[B]] = \lambda_{y_k, y_\ell}[B].$$

Moreover, $\lambda_{y_i, y_k}[B] = \lambda_{y_k}^{-1}[\lambda_{y_i}[B]] \lesssim \lambda_{y_j}^{-1}[\lambda_{y_i}[B]] = \lambda_{y_i, y_j}[B]$ by Claim 2. \square

Claim 5. If $i < j < k < \ell$ and $B \leq H$ is definable over $\{y_s, y'_s : s \notin \{i, j\} \cup [k, \ell]\}$ then $\lambda_{y_i, y_j}[B]$ and $\lambda_{y_k, y_\ell}[B]$ are \lesssim -comparable.

Proof of Claim. Suppose $\lambda_{y_i, y_j}[B] \not\lesssim \lambda_{y_k, y_\ell}[B]$. Put

$$A = \{g \in G : \lambda_g[B] \lesssim \lambda_{y_k}[B]\} \quad \text{and} \quad B' = \{h \in B : \lambda'_h[A] \lesssim \lambda_{y_k}[B]\}.$$

Then $A = \widetilde{\text{ann}}_G^{\bar{\lambda}}(B)$ and $B' = \widetilde{\text{ann}}_B^{\bar{\lambda}}(A)$, where $\bar{\lambda} : G \times B \rightarrow K/\lambda_{y_k}[B]$ is the induced bilinear quasi-form. Then $B \lesssim B'$ by Lemma 5.8, so B' has finite index in B .

Consider the induced bilinear quasi-form

$$\tilde{\lambda} : A \times B' \rightarrow \lambda_{y_k}[B]/(\lambda_{y_k}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B]).$$

As $\lambda_k[B] \lesssim \lambda_\ell[B]$ and

$$\text{ann}_H(y_\ell) \lesssim \text{ann}_H(y_j) \cap \text{ann}_H(y_\ell) \leq \lambda_{y_i, y_j}[B] \cap \lambda_{y_k, y_\ell}[B]$$

by Claim 1, we have

$$\begin{aligned} \text{bdn}(\lambda_{y_k}[B]/(\lambda_{y_k}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B])) &= \text{bdn}(\lambda_{y_\ell} \lambda_{y_k, y_\ell}[B]/(\lambda_{y_\ell} \lambda_{y_k, y_\ell}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B])) \\ &\leq \text{bdn}(\lambda_{y_k, y_\ell}[B]/(\lambda_{y_i, y_j}[B] \cap \lambda_{y_k, y_\ell}[B])) \\ &< \text{bdn}((\lambda_{y_i, y_j}[B] + \lambda_{y_k, y_\ell}[B])/(\lambda_{y_i, y_j}[B] \cap \lambda_{y_k, y_\ell}[B])) \\ &\leq \text{bdn}(H/\text{ann}_H(y_\ell)) = \text{bdn}(\text{im } \lambda_{y_\ell}) \leq \text{bdn}(K). \end{aligned}$$

Hence $\tilde{\lambda}$ is virtually almost trivial by induction. Since $y_{k'} \in A_{y_i, y_j, y_k, y_\ell}^0$ for all $j < k' < k$ such that B is definable over $\{y_s, y'_s : s \notin [k', \ell] \cup \{i, j\}\}$ by Claim 1 and principal indiscernibility, it follows that

$$\lambda_{y_{k'}}[B] \lesssim \lambda_{y_\ell} \lambda_{y_i, y_j}[B].$$

Hence $\lambda_{y_{k'}, y_\ell}[B] \lesssim \lambda_{y_i, y_j}[B] + \text{ann}_B(y_\ell) \lesssim \lambda_{y_i, y_j}[B]$, and $\lambda_{y_k, y_\ell}[B] \lesssim \lambda_{y_i, y_j}[B]$ by indiscernibility. \square

Claim 6. $\ker \bar{\lambda}_{y_i, y_j}$ and $\ker \bar{\lambda}_{y_k, y_\ell}$ are \lesssim -comparable for all $i < j < k < \ell$, where $\bar{\lambda}_{y, y'}$ is the quasi-homomorphism from H to $H/\text{ann}_H(y_j)$ induced by $\lambda_{y, y'}$.

Proof of Claim. We have $\ker \bar{\lambda}_{y_i, y_j} = \text{ann}_H(y_i)$; put $B = \ker \bar{\lambda}_{y_k, y_\ell} = \lambda_{y_k, y_\ell}^{-1}[\text{ann}_H(y_j)]$ and suppose $\text{ann}_H(y_i) \not\lesssim B$. Let $A = \{g \in G : \text{ann}_H(y_i) \lesssim \text{ann}_H(g)\}$ and consider the restricted bilinear quasi-form

$$\bar{\lambda} : A \times B \rightarrow K.$$

Since $\text{ann}_B(y_i) \lesssim \text{ann}_B(\bar{g})$ for all finite $\bar{g} \in A$ and $\text{ann}_H(y_k) \lesssim B \cap \text{ann}_H(y_i)$, we have

$$\begin{aligned} \text{rbdn}_{\bar{\lambda}}(B) &= \max_{\bar{g} \in A \text{ finite}} \text{bdn}(B/\text{ann}_B(\bar{g})) \leq \text{bdn}(B/\text{ann}_B(y_i)) \\ &< \text{bdn}((B + \text{ann}_H(y_i))/(\text{ann}_H(y_i))) \leq \text{bdn}(H/\text{ann}_H(y_k)) \leq \text{rbdn}_\lambda(H). \end{aligned}$$

By induction, $\bar{\lambda}$ must be virtually almost trivial. Since $y_s \in A_{y_i, y_j, y_k, y_\ell}^0$ for $s < i$, we have $\ker \bar{\lambda}_{y_k, y_\ell} = B \lesssim \text{ann}_H(y_s)$ for all $s < i$; the claim now follows from indiscernibility. \square

Claim 7. For $i < j < k$ we have $\lambda_{y_i, y_k} \equiv \lambda_{y_j, y_k} \circ \lambda_{y_i, y_j}$.

Proof of Claim. Since $\text{coker } \lambda_g$ is trivial, $\lambda_g \circ \lambda_g^{-1} = \text{id}_{\text{im } \lambda_g}$ for any $g \in G$ by Remark 3.4. As $\text{im } \lambda_{y_i} \lesssim \text{im } \lambda_{y_j}$ by Claim 1,

$$\lambda_{y_j, y_k} \circ \lambda_{y_i, y_j} = \lambda_{y_k}^{-1} \circ \lambda_{y_j} \circ \lambda_{y_j}^{-1} \circ \lambda_{y_i} = \lambda_{y_k}^{-1} \circ \text{id}_{\text{im } \lambda_{y_j}} \circ \lambda_{y_i}$$

is a restriction of

$$\lambda_{y_i, y_k} = \lambda_{y_k}^{-1} \circ \lambda_{y_i} = \lambda_{y_k}^{-1} \circ \text{id}_{\text{im } \lambda_{y_i}} \circ \lambda_{y_i}$$

to a subgroup of H of finite index. The claim follows. \square

Claim 8. If $\text{im } \lambda_{y_0, y_1} \lesssim \text{im } \lambda_{y_2, y_3}$ then $\text{im } \lambda_{y_0, y_j} \lesssim \text{im } \lambda_{y_i, y_j}^{\circ n}$ for all $0 < i < j$ and $1 \leq n < \omega$.

Proof of Claim. We proceed by induction on n . For $n = 1$ this is clear, as $\text{im } \lambda_{y_0, y_j} \lesssim \text{im } \lambda_{y_i, y_j}$ by Claim 4. Assume it holds for some n . Choose $0 < k < \ell < i$. Then

$$\begin{aligned} \text{im } \lambda_{y_0, y_j} &\sim \text{im}(\lambda_{y_i, y_j} \circ \lambda_{y_0, y_i}) = \lambda_{y_i, y_j}[\text{im } \lambda_{y_0, y_i}] \lesssim \lambda_{y_i, y_j}[\text{im } \lambda_{y_0, y_k}] \\ &\lesssim \lambda_{y_i, y_j}[\text{im } \lambda_{y_\ell, y_j}] \lesssim \lambda_{y_i, y_j}[\text{im } \lambda_{y_i, y_j}^{\circ n}] = \text{im}(\lambda_{y_i, y_j} \circ \lambda_{y_i, y_j}^{\circ n}) = \text{im } \lambda_{y_i, y_j}^{\circ(n+1)} \end{aligned}$$

(the first inequality follows by Claim 4, the second inequality follows by the assumption of the claim, and the last one by the inductive assumption). \square

Claim 9. If $\text{im } \lambda_{y_2, y_3} \lesssim \text{im } \lambda_{y_0, y_1}$ then $\text{im } \lambda_{y_0, y_k} \lesssim \text{im } \lambda_{y_i, y_j}^{\circ n}$ for all $0 < i < j < k$ and $1 \leq n < \omega$.

Proof of Claim. The case $n = 1$ follows from Claim 4, so assume the statement holds for some n . Choose $0 < i < j < \ell < m < k$. Let $\bar{\lambda}_{y, y'}$ be the quasi-homomorphism from H to $H/\text{ann}_H(y_j)$ induced by $\lambda_{y, y'}$. By the assumption and indiscernibility we have $\text{im } \bar{\lambda}_{y_m, y_k} \lesssim \text{im } \bar{\lambda}_{y_i, y_j}$. Hence, Claim 6 and Lemma 4.4 imply $\ker \bar{\lambda}_{y_i, y_j} \lesssim \ker \bar{\lambda}_{y_m, y_k}$, so the same holds for the restrictions to $B := \text{im } \lambda_{y_0, y_\ell}$. But now by Lemma 4.4 and Claim 5 we have $\lambda_{y_m, y_k}[B] \lesssim \lambda_{y_i, y_j}[B]$. Then

$$\begin{aligned} \text{im } \lambda_{y_0, y_k} &\sim \text{im}(\lambda_{y_m, y_k} \circ \lambda_{y_0, y_m}) = \lambda_{y_m, y_k}[\text{im } \lambda_{y_0, y_m}] \lesssim \lambda_{y_m, y_k}[\text{im } \lambda_{y_0, y_\ell}] \\ &\lesssim \lambda_{y_i, y_j}[\text{im } \lambda_{y_0, y_\ell}] \lesssim \lambda_{y_i, y_j}[\text{im } \lambda_{y_i, y_j}^{\circ n}] = \text{im}(\lambda_{y_i, y_j} \circ \lambda_{y_i, y_j}^{\circ n}) = \text{im } \lambda_{y_i, y_j}^{\circ(n+1)} \end{aligned}$$

(last inequality follows by inductive hypothesis and indiscernibility). \square

Claim 10. $\text{im } \lambda_{y_i, y_k} \lesssim \text{ann}_H(y_j)$ for all $i, j < k$.

Proof of Claim. By Claim 3, the quasi-endomorphism $\bar{\lambda}_{y_i, y_j}$ of $H/\text{ann}_H(y_j)$ induced by λ_{y_i, y_j} is not invertible, so it must be nilpotent by Lemma 7.7. The assertion now follows from Claims 5, 8 and 9. \square

Of course, all of the previous claims also hold with the roles of G and H exchanged.

Claim 11. *For any $i \neq j$ we have $\text{im}\lambda_{y_i} \lesssim \text{im}\lambda'_{y'_j}$ or $\text{im}\lambda'_{y'_j} \lesssim \text{im}\lambda_{y_i}$.*

Proof of Claim. Suppose $\text{im}\lambda'_{y'_j} \not\lesssim \text{im}\lambda_{y_i}$. Put $A = \{g \in G : \text{im}\lambda_g \lesssim \text{im}\lambda_{y_i}\}$, and consider the induced bilinear quasi-form

$$\bar{\lambda} : A \times H \rightarrow \text{im}\lambda_{y_i} / (\text{im}\lambda_{y_i} \cap \text{im}\lambda'_{y'_j}).$$

As

$$\text{bdn}(\text{im}\lambda_{y_i} / (\text{im}\lambda_{y_i} \cap \text{im}\lambda'_{y'_j})) < \text{bdn}((\text{im}\lambda_{y_i} + \text{im}\lambda'_{y'_j}) / (\text{im}\lambda_{y_i} \cap \text{im}\lambda'_{y'_j})) \leq \text{bdn}(K),$$

the quasi-form $\bar{\lambda}$ must be virtually almost trivial. But $y_k \in A^0_{y_i, y'_j}$ for $j \neq k < i$ by Claim 1 and principal indiscernibility. Hence $\text{im}\lambda_{y_k} \lesssim \text{im}\lambda'_{y'_j}$, a contradiction, as there is $j \neq k < i$ with $y_k \equiv_{y'_j} y_i$. \square

By Claim 11 and symmetry we may assume that $\text{im}\lambda'_{y'_i} \lesssim \text{im}\lambda_{y_j}$ for all $i < j$. Fix $i \neq k$, and choose $i < j < \ell$ and $k < \ell < m$ with $k \notin \{i, j\}$. Then $\text{im}\lambda_{y_j, y_\ell} \lesssim \text{ann}_H(y_k)$ by Claim 10, whence

$$\text{im}\lambda_{y_j} \lesssim \lambda_{y_\ell}[\text{ann}_H(y_k)].$$

Then, as $y_m \in G^0_{y'_i, y_j, y_k, y_\ell}$, we have

$$\lambda(y_m, y'_i) \in (\text{im}\lambda'_{y'_i})^0_{y'_i, y_j, y_k, y_\ell} \leq (\text{im}\lambda_{y_j})^0_{y'_i, y_j, y_k, y_\ell} \leq \lambda_{y_\ell}[\text{ann}_H(y_k)].$$

Moreover, $\lambda_{y_\ell}[\text{ann}_H(y_k)] \lesssim \lambda_{y_m}[\text{ann}_H(y_k)]$ by Claim 1. By principal indiscernibility,

$$y'_i \in \lambda_{y_m}^{-1}[\lambda_{y_\ell}[\text{ann}_H(y_k)]]^0_{y_k, y_\ell, y_m} \leq \lambda_{y_m}^{-1}[\lambda_{y_\ell}[\text{ann}_H(y_k)]]^0_{y_k, y_\ell, y_m} \leq \lambda_{y_m}^{-1}[\lambda_{y_m}[\text{ann}_H(y_k)]].$$

Thus, by principal indiscernibility,

$$y'_i \in (\text{ann}_H(y_k) + \text{ann}_H(y_m))^0_{y_k, y_m} \leq \text{ann}_H(y_k),$$

and $\lambda(y_k, y'_i) = 0$ for all $i \neq k$. Hence, as $\langle (y_i, y'_i) : i \in \mathbb{Q} \rangle$ is a subgroup-generic sequence, $\text{ann}_G(y'_0)$ has finite index in G and $\text{ann}_H(y_0)$ has finite index in H . Since (y_0, y'_0) is subgroup-generic, $\widetilde{\text{ann}}_H(G)$ and $\widetilde{\text{ann}}_G(H)$ have finite index in H and G respectively. So λ is virtually almost trivial by Corollary 5.11.

Finally, if G and H are connected, then λ is almost trivial. But then for every $g \in G$ and $h \in H$ the annihilators $\text{ann}_H(g)$ and $\text{ann}_G(h)$ have finite index in H and in G , and must be equal to H and G , respectively, by connectedness. Thus λ is trivial. \square

8. ON GROUPS AND RINGS

We shall use standard notation $x^y = y^{-1}xy$ for conjugation, and $[x, y] = x^{-1}y^{-1}xy = y^{-x}y = x^{-1}x^y$ for the commutator. As in the introduction, rings are supposed to be associative, but need not be commutative or have an identity element.

Since any characteristic (i.e. invariant under the automorphism group) subgroup of a countable ω -categorical group is \emptyset -definable, each countable, ω -categorical group has a finite series of characteristic subgroups in which all successive quotients are characteristically simple groups (i.e. they do not have non-trivial, proper characteristic subgroups). On the other hand, Wilson [29] proved (see also [2] for an exposition of the proof):

Fact 8.1. *For each infinite, countable, ω -categorical, characteristically simple group H , one of the following holds.*

- (i) *For some prime number p , H is an elementary abelian p -group (i.e. an abelian group, in which every nontrivial element has order p).*
- (ii) *$H \cong B(F)$ or $H \cong B^-(F)$ for some non-abelian, finite, simple group F , where $B(F)$ is the group of all continuous functions from the Cantor space C to F , and $B^-(F)$ is the subgroup of $B(F)$ consisting of the functions f such that $f(x_0) = e$ for a fixed element $x_0 \in C$.*
- (iii) *H is a perfect p -group for some prime number p (perfect means that H equals its commutator subgroup).*

It remains a difficult open question whether there exist infinite, ω -categorical, perfect p -groups.

Remark 8.2. The groups $B(F)$ and $B^-(F)$ above have TP_2 (in particular, they do not have finite burden).

Proof. Let $f \in F$ be a non-central element, and let $\langle A_i : i < \omega \rangle$ be pairwise disjoint clopen sets in C . Let $g_i \in B(F)$ be given by $g_i[A_i] = \{f\}$ and $g_i[C \setminus A_i] = \{0\}$ for each i . Then the centralizers of the g_i do not satisfy the conclusion of [9, Theorem 2.4], hence $B(F)$ has TP_2 . The argument for $B^-(F)$ is analogous. \square

Fact 8.3 ([23, Theorem 3.1]). *There is a finite bound of the size of conjugacy classes in a group G if and only if the derived subgroup G' is finite.*

This implies in particular that if the almost centre $\tilde{Z}(G)$ of a group G is definable (in an \aleph_1 -saturated model), then it is finite-by-abelian, as definability plus saturation yields a finite bound on the index $|G : C_G(a)|$ for $a \in \tilde{Z}(G)$.

- Lemma 8.4.**
- (1) *If G is a virtually finite-by-abelian group, then $\tilde{Z}(G)$ is characteristic, definable, finite-by-abelian and of finite index in G .*
 - (2) *A virtually finite-by-soluble/nilpotent group has a characteristic definable soluble/nilpotent subgroup of finite index.*
 - (3) *A virtually finite-by-null ring has a definable subring R_0 of finite index which is finite-by-null.*

Proof. (1) Let G be a virtually finite-by-abelian group, and consider a subgroup H of finite index whose derived subgroup H' is finite. Then for any $g \in H$,

$$|G : C_G(g)| \leq |G : H| \cdot |H : C_H(g)| \leq |G : H| \cdot |H'| =: n.$$

Put $X = \{g \in G : |G : C_G(g)| \leq n\}$. Then X is definable and $H \subseteq X \subseteq \tilde{Z}(G)$; in particular, $|G : \tilde{Z}(G)|$ is finite. Let \bar{g} be a finite set of representatives of all cosets from $\tilde{Z}(G)/H$. Then $\tilde{Z}(G) = \bigcup_{g \in \bar{g}} gX$. Thus $\tilde{Z}(G)$ is definable; if $m = \max\{|G : C_G(g)| : g \in \bar{g}\}$, then $|G : C_G(g)| \leq nm$ for all $g \in \tilde{Z}(G)$. Hence also $|\tilde{Z}(G) : C_{\tilde{Z}(G)}(g)| \leq nm$ for all $g \in \tilde{Z}(G)$, so $\tilde{Z}(G)'$ is finite by Fact 8.3. Clearly $\tilde{Z}(G)$ and $\tilde{Z}(G)'$ are characteristic in G .

- (2) If G is virtually finite-by-soluble/nilpotent, let H be a finite-by-soluble/nilpotent subgroup of finite index, and let $F \leq H$ be a normal finite subgroup of H such that H/F is soluble/nilpotent. Replacing H by the intersection of its conjugates, which still has finite index in G , and replacing F by its intersection with the group obtained in this way, we may assume that H is normal in G .

Suppose that the derived length/nilpotency class of H/F is n . We may then replace F by the n^{th} derived subgroup $H^{(n)}$, or the n^{th} lower central subgroup H^n , which must be contained in F , and assume that F is characteristic in H , whence normal in G . Now F is finite normal in G , so $C_G(F)$ has finite index in G and is still normal in G ; moreover $F \cap C_G(F)$ is central in $C_G(F)$. It follows that $F \cap C_G(F) \cap H$ is central in $C_G(F) \cap H$, so $C_G(F) \cap H$ is soluble/nilpotent of derived length/nilpotency class $n + 1$. Thus G has a normal soluble/nilpotent subgroup of finite index; as the product of two normal soluble/nilpotent subgroups is still soluble/nilpotent, G has a unique maximal normal soluble/nilpotent subgroup N of finite index, which must be characteristic. If m is its derived length/nilpotency class, N is definable by the formula

$$\begin{aligned} [\dots [[x^G, x^G], [x^G, x^G]], \dots, [[x^G, x^G], [x^G, x^G]] \dots] &= \{1\} \quad \text{in the soluble case} \\ [x^G, x^G, \dots, x^G] &= \{1\} \quad \text{in the nilpotent case,} \end{aligned}$$

with 2^m occurrences of x^G in the soluble case and $m + 1$ in the nilpotent case: clearly, any $x \in N$ satisfies the formula; conversely, any x satisfying the formula generates a soluble/nilpotent normal subgroup and must be contained in N .

- (3) If R is a virtually finite-by-null ring, let S_0 be a finite-by-null subring of finite (additive) index, and I a finite ideal of S_0 containing $S_0 \cdot S_0$. Then $S := \bigcap_{s \in S_0} \{r \in R : rs \in I\}$ is an additive subgroup of R containing S_0 , with $S \cdot S_0 \subseteq I$; it is definable as a finite conjunction of size at most $|R : S| \leq |R : S_0|$ of formulas $xs \in I$ for suitable $s \in S_0$. Now $R_0 := S \cap \bigcap_{s \in S} \{r \in R : sr \in I\}$ is again a definable additive subgroup of finite index containing S_0 . Since $R_0 \cdot R_0 \subseteq I \leq R_0$, this is as required. □

Fact 8.5. *An atomless boolean algebra has TP_2 .*

Proof. Let B be a monster model of the theory of atomless boolean algebras. We can easily construct a Boolean algebra with elements $(a_{i,j})_{i,j < \omega}$ such that $a_{i,j_1} \wedge a_{i,j_2} = 0$ for each i and $j_1 \neq j_2$, and for any $\eta : \omega \rightarrow \omega$ there is $x \neq 0$ such that $x \leq a_{i,\eta(i)}$ for each $i < \omega$. As every Boolean algebra embeds in an atomless Boolean algebra, we may find such $a_{i,j}$ in B . Then putting $\phi_i(x, y) = \phi(x, y) = (x \neq 0 \wedge x \leq y)$, we get that $\langle \phi_i(x, y) : i < \omega \rangle$ together with parameters $(a_{i,j})_{i,j < \omega}$ is an inp-pattern of depth \aleph_0 . As we use only the formula ϕ in it, by compactness we can obtain an inp-pattern of arbitrarily large depth in a single variable x . □

We will use the following variant of Proposition 2.5 from [17]. As in our context we cannot use connected components, we have to modify the proof slightly.

Lemma 8.6. *Let \mathcal{C} be a class of countable, ω -categorical NTP_2 (pure) groups, closed under taking definable subgroups and quotients by definable normal subgroups. Suppose that every infinite, characteristically simple group in \mathcal{C} is soluble. Then every group in \mathcal{C} is virtually nilpotent.*

Proof. Let $G \in \mathcal{C}$. Let us first show that G is virtually soluble.

As stated in the beginning of the section, there is a maximal chain $\{1\} = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$ of \emptyset -definable subgroups of G , each normal in its successor. Then every quotient G_{i+1}/G_i for $i < n$ is characteristically simple, whence either finite or soluble by assumption. By Lemma 8.4(1), if G_i is virtually finite-by-soluble, it has a definable

characteristic soluble subgroup S_i of finite index. So S_i is normal in G_{i+1} , and G_{i+1}/S_i is virtually finite-by-soluble, whence virtually soluble by Lemma 8.4(2). It follows that G_{i+1} is virtually soluble. As G_0 is trivial, this shows that $G = G_n$ is virtually soluble, and has a definable soluble characteristic subgroup N of finite index.

Now since N is NTP₂, it does not interpret the atomless boolean algebra by Fact 8.5, so by ω -categoricity and [3, Theorem 1.2] it is virtually nilpotent, and so is G . \square

Proposition 8.7. *A nilpotent ω -categorical group of finite burden is virtually finite-by-abelian.*

Proof. Let G be a counter-example; we may assume it is nilpotent of minimal class possible. Then $G/Z(G)$ is virtually finite-by-abelian; as G is not virtually finite-by-abelian, $Z(G)$ must be infinite. By Lemma 8.4(1) we may replace G by the preimage of $\tilde{Z}(G/Z(G))$, a definable subgroup of finite index in G , and assume that $F = G'Z(G)$ is a finite extension of $Z(G)$. As any orbit g^F has finite size $\leq |F : Z(G)|$ for $g \in F \leq G$, the derived subgroup F' is finite by Fact 8.3; further replacing G by G/F' we may assume that F' is trivial.

Consider a definable subgroup G_0 of finite index in G such that $|G'_0 : G'_0 \cap Z(G)|$ is minimal possible. Replacing G by G_0 , we obtain that $|G' : G' \cap Z(G)| = |H' : H' \cap Z(G)|$ for any definable subgroup H of finite index (\dagger).

As $F/Z(G)$ is finite, $C_G(F/Z(G)) = \{g \in G : [g, F] \subseteq Z(G)\}$ has finite index in G , and we can replace G by $C_G(F/Z(G))$. Then for any $g \in G$ the map $x \mapsto [x, g]Z(G)$ is a definable homomorphism from G to $G'/Z(G)$: Indeed, for any $x, y \in G$ we have that $[x, g] \in F$, so $[x, g]Z(G) = [x, g]^y Z(G)$ and

$$[xy, g]Z(G) = [x, g]^y [y, g]Z(G) = [x, g][y, g]Z(G).$$

As the image of the map $x \mapsto [x, g]Z(G)$ is finite, its kernel must have finite index in G . Then $x \mapsto [x, g]$ is a definable homomorphism from H to $Z(G)$ with abelian image; its kernel must hence contain $H'Z(G)$. As $H'Z(G) = G'Z(G)$ by (\dagger), we see that $G' \leq C_G(g)$. This holds for all $g \in G$, so $G' \leq Z(G)$.

Now commutation is a definable bilinear form from $G/Z(G)$ to $Z(G)$. By Theorem 7.5 it is virtually almost trivial. But this means that G is virtually finite-by-abelian, contradicting our assumption. \square

Proposition 8.8. *An ω -categorical group of finite burden is virtually nilpotent.*

Proof. If the proposition does not hold, there is an infinite non-soluble ω -categorical characteristically simple group G of finite burden by Lemma 8.6; it must be a perfect p -group for some prime p by Fact 8.1 and Remark 8.2. We choose such a G of minimal possible burden k . Then every interpretable section (subquotient) of strictly smaller burden is soluble, as its characteristically simple sections are soluble by minimality of k (and the fact that finite p -groups are soluble).

Note that $\tilde{Z}(G)$ is characteristic; by ω -categoricity it is definable and there is a finite bound on the indices $|G : C_G(g)|$ for $g \in \tilde{Z}(G)$. Hence $\tilde{Z}(G)$ is finite-by-abelian by Fact 8.3, whence soluble (as it is a p -group). Thus it cannot be the whole of G and must be trivial. As any finite normal subgroup is contained in $\tilde{Z}(G)$, there are no non-trivial finite normal subgroups, and all non-trivial conjugacy classes are infinite.

Claim 1. *The soluble radical $R(G)$ of G is trivial.*

Proof of Claim. For $1 \neq a \in G$ consider the normal subgroup A_a generated by the conjugacy class a^G . Then the A_a are infinite, and uniformly definable by ω -categoricity. If $R(G)$ is non-trivial, we can choose a in the last non-trivial derived subgroup of some normal soluble subgroup of G . Then A_a is abelian. Let $\mathfrak{A} = \{A_a : a \in G, A_a \text{ abelian}\}$. This is a definable invariant non-empty collection of uniformly definable abelian normal subgroups.

For $A, B \in \mathfrak{A}$ the product AB is definable and nilpotent, whence virtually finite-by-abelian by Proposition 8.7. Then $\tilde{Z}(AB)$ has finite index in AB by Lemma 8.4(1), and $\tilde{Z}(AB)'$ is finite characteristic in AB , whence finite normal in G , and therefore trivial. Thus

$$A \lesssim \tilde{Z}(AB) \leq \tilde{C}_G(AB) \leq \tilde{C}_G(B),$$

whence

$$B \lesssim \tilde{C}_B(\tilde{C}_G(B)) \leq \tilde{C}_B(A) = \tilde{C}_B(AB) \leq \tilde{Z}(AB).$$

So

$$\tilde{C}_A(\tilde{C}_G(A)) \cdot \tilde{C}_B(\tilde{C}_G(B)) \leq \tilde{Z}(AB),$$

which is abelian. Thus $\mathfrak{A}' = \{\tilde{C}_A(\tilde{C}_G(A)) : A \in \mathfrak{A}\}$ is an invariant family of pairwise commuting abelian groups, and generates a characteristic abelian subgroup, which must be the whole of G . This contradiction finishes the proof of the claim. \square

Suppose every centraliser of a non-trivial element is soluble. Then by compactness (or ω -categoricity) there is a bound s on the derived length of any proper centraliser. Since G is a locally finite p -group, every finite subset of G generates a finite nilpotent group, and must be contained in the centralizer of a nontrivial element. But then G must be soluble of derived length s , a contradiction. Hence there is $1 \neq n \in G$ such that $H := C_G(n)$ is non-soluble. Put $N = \langle n^G \rangle$, an infinite normal subgroup, which is definable by ω -categoricity.

Since $\tilde{C}_G(N) \cap N$ is normal and finite-by-abelian (by Fact 8.3 and ω -categoricity), whence soluble, it must be trivial.

Claim 2. $\tilde{C}_G(N) = \{1\}$.

Proof. Suppose $\tilde{C}_G(N)$ is nontrivial, whence infinite, and $\text{bdn}(\tilde{C}_G(N)) > 0$. Then

$$k = \text{bdn}(G) \geq \text{bdn}(N) + \text{bdn}(\tilde{C}_G(N)) > \text{bdn}(N)$$

by Remark 2.2. So N is soluble by minimality of k , a contradiction. \square

Claim 3. Any definable normal subgroup M of G with $M \lesssim H$ is trivial.

Proof. If $M \lesssim H = C_G(n)$, then $n \in \tilde{C}_G(M)$; by normality of M we get $n^G \subseteq \tilde{C}_G(M)$, and hence $N \leq \tilde{C}_G(M)$. Then $M \lesssim \tilde{C}_G(N) = \{1\}$ by Fact 2.4. \square

Consider a nontrivial definable normal subgroup M of G . Since $M \cap H$ is normalized by H , we have a definable injection

$$M/(M \cap H) \times H/(M \cap H) \rightarrow G/(M \cap H)$$

given by multiplication. As $M/M \cap H$ is infinite by the claim, we conclude that

$$\text{bdn}(H/M \cap H) < \text{bdn}(G/(M \cap H)) \leq \text{bdn}(G) = k.$$

So $H/(M \cap H)$ is soluble by minimality of k . If M runs through the family \mathcal{M} of 1-generated normal subgroups, the family $\{H/(M \cap H) : M \in \mathcal{M}\}$ is uniformly definable by ω -categoricity, and by compactness there is $d < \omega$ such that $H/(M \cap H)$ has derived

length at most d for all $M \in \mathcal{M}$. But this means that $H^{(d)}$ is contained in M for all $M \in \mathcal{M}$, and thus is contained in all nontrivial normal subgroups.

Since H is not soluble, $H^{(d)}$ generates a non-abelian minimal normal subgroup L . But then L is finite by [2, Theorem D], a contradiction. This completes the proof. \square

Theorem 8.9. *An ω -categorical group of finite burden is virtually finite-by-abelian.*

Proof. This follows immediately from Propositions 8.7 and 8.8. \square

Corollary 8.10. *An ω -categorical NIP group of finite burden is virtually abelian.*

Proof. Let G be an ω -categorical NIP group of finite burden. As G is NIP, the connected component G^0 , i.e. the intersection of all definable subgroups of finite index, exists and is an intersection of \emptyset -definable groups [6, 26, Section 8.1.2]. By ω -categoricity it is itself \emptyset -definable and has finite index in G , so we may assume that G is connected. Then G is finite-by-abelian by Lemma 8.4(1). Thus, the centralizer of any element in G has finite index in G , hence, by connectedness, is equal to G . This means that G is abelian. \square

Theorem 8.11. *An ω -categorical ring of finite burden is virtually finite-by-null.*

Proof. This is immediate from Theorem 7.5, as multiplication is a definable bilinear map. \square

As in the group case, we get a corollary for NIP rings:

Corollary 8.12. *An ω -categorical NIP ring of finite burden is virtually null.*

Proof. Let R be such a ring. We may again assume that R is connected (in the sense of the additive group). Then R is finite-by-null by Lemma 8.4(3). Hence, the left annihilator of any element in R has finite index in R , and must be equal to R by connectedness. This shows that R is null. \square

9. QUESTIONS AND CONCLUDING REMARKS

One can ask various questions about generalizations of the above results to more general contexts, such as strong or NTP₂ theories. For example, one can ask:

Question 9.1. Are ω -categorical strong groups

- (1) virtually nilpotent-by-finite?
- (2) virtually abelian-by-finite?

An analogue of Question 9.1(1) for rings has positive answer by Theorem 9.3 below. As to the stronger version, we do not know:

Question 9.2. Are ω -categorical strong rings null-by-finite?

The proof below is a modification of the proof of Theorem 2.1 from [20], generalizing that result from the NIP to the NTP₂ context.

Theorem 9.3. *Every ω -categorical NTP₂ ring is nilpotent-by-finite.*

Proof. As in [20], it is enough to show that a semisimple ω -categorical NTP₂ ring R is finite, and we can assume that R is a subring of $\prod_{i \in I} R_i$, where each R_i is finite, and $|\{R_i : i \in I\}| < \omega$. Let π_i be the projection onto the i -th coordinate. For $i_0, \dots, i_n \in I$ and $r_0 \in R_{i_0}, \dots, r_n \in R_{i_n}$, we define

$$R_{i_0, \dots, i_n}^{r_0, \dots, r_n} = \left\{ r \in R : \bigwedge_{j=0}^n \pi_{i_j}(r) = r_j \right\}.$$

Suppose for a contradiction that R is infinite. Again as in [20], we get the following claim:

Claim 1. *For any $N \in \omega$ there are pairwise distinct $i(0), \dots, i(N-1) \in I$ and non-nilpotent elements $r_i \in R_i$ for $i < N$ such that the sets*

$$R_{i_0, \dots, i_{N-1}}^{r_0, 0, \dots, 0}, R_{i_0, \dots, i_n}^{0, r_1, \dots, 0}, \dots, R_{i_0, \dots, i_n}^{0, 0, \dots, r_{N-1}}$$

are all non-empty.

Notice that, by ω -categoricity, the principal two-sided ideals RxR for $x \in R$ are uniformly definable. Hence, by [9, Theorem 2.4] and compactness, we obtain in particular that in order to contradict NTP_2 it is enough to find for any $n, m < \omega$ elements b_0, \dots, b_{n-1} such that

$$(*) \quad \left| \bigcap_{j \in n \setminus \{j_0\}} Rb_jR : \bigcap_{j \in n} Rb_jR \right| \geq m$$

for any $j_0 < n$ (where $n = \{0, 1, \dots, n-1\}$). So fix any $n, m < \omega$, and for $N = nm$ choose i_j and r_j as in the claim. Let $(i_{j,k})_{j < n, k < m}$ be another enumeration of $(i_j)_{j < N}$, and let $(r_{j,k})_{j < n, k < m}$ be the corresponding enumeration of $(r_j)_{j < N}$ and $(\pi_{j,k})_{j < n, k < m}$ the corresponding enumeration of $(\pi_j)_{j < N}$. For any $j_0 < n, k_0 < m$ let $s_{j_0, k_0} \in R$ be such that $\pi_{j,k}(s_{j_0, k_0}) = 0$ for $(j, k) \neq (j_0, k_0)$ and $\pi_{j_0, k_0}(s_{j_0, k_0}) = r_{j_0, k_0}$. Put $b_j = \sum_{j' \neq j, k < m} s_{j', k}$ for all $j < n$.

Claim 2. $\left| \bigcap_{j \in n \setminus \{j_0\}} Rb_jR : \bigcap_{j \in n} Rb_jR \right| \geq m$ for any $j_0 < n$.

Proof. Fix any $j_0 < n$ and put $b = b_0 b_1 \dots b_{j_0-1} b_{j_0+1} b_{j_0+2} \dots b_{n-1}$. Notice that for any $r \in \bigcap_{j \in n} Rb_jR$ and $k < m$ we have that $\pi_{j_0, k}(r) = 0$. On the other hand, for distinct $k_1, k_2 < m$ we have that

$$\pi_{j_0, k_1}(s_{j_0, k_1} b - s_{j_0, k_2} b) = \pi_{j_0, k_1}(s_{j_0, k_1} b) = \pi_{j_0, k_1}(s_{j_0, k_1}) \pi_{j_0, k_1}(b) = r_{j_0, k_1} r_{j_0, k_1}^{n-1} = r_{j_0, k_1}^n \neq 0.$$

Hence the elements

$$s_{j_0, 0} b, s_{j_0, 1} b, \dots, s_{j_0, m-1} b \in \bigcap_{j \in n \setminus \{j_0\}} Rb_jR$$

are in pairwise distinct cosets of $\bigcap_{j \in n} Rb_jR$. □

By the claim and (*) we obtain a contradiction. □

REFERENCES

- [1] H. Adler. *Strong theories, burden, and weight*, <http://www.logic.univie.ac.at/~adler/docs/strong.pdf>
- [2] A. Apps. *On the structure of \aleph_0 -categorical groups*, J. Alg. 81, 320–39, 1983.
- [3] R. Archer, D. Macpherson. *Soluble ω -categorical groups*, Math. Proc. Cambridge Philos. Soc., 121(2):219–227, 1997.
- [4] P. Baginski. *Stable \aleph_0 -categorical algebraic structures*, PhD thesis, University of California at Berkeley, 2009.
- [5] J.T. Baldwin and B. Rose. *\aleph_0 -categoricity and stability of rings*, J. Alg. 45(1), 1–16, 1977.
- [6] J.T. Baldwin and J. Saxl. *Logical stability in group theory*, J. Austral. Math. Soc. (Series A) 21(3), 267–276, 1976.
- [7] W. Baur, G. Cherlin, A. Macintyre. *Totally categorical groups and rings*, J. Alg. 57, 407–440, 1979.
- [8] A. Chernikov. *Theories without the tree property of the second kind*, Ann. Pure Appl. Logic 165 (2), 695–723, 2014.
- [9] A. Chernikov, I. Kaplan and P. Simon. *Groups and fields with NTP_2* , Proc. Am. Math. Soc. 143, 395–406, 2015.
- [10] J. Dobrowolski and F.O. Wagner. *On ω -categorical inp-minimal groups and rings*, preprint, 2018, arxiv:1806.10462v1

- [11] E. Engeler. *Äquivalenzklassen von n -Tupeln*, Z. Math. Logik Grundl. Math. 5, 340–345, 1959.
- [12] D. Evans and F.O. Wagner. *Supersimple w -categorical groups and theories*, J. Symb. Log. 65(2), 767–776, 2000.
- [13] U. Felgner. *\aleph_0 -categorical stable groups*, Math. Z. 160, 27–49, 1978.
- [14] U. Felgner. *On \aleph_0 -categorical extra-special p -groups*, Logique Et Anal. 18 (71), 407–428, 1975.
- [15] N. Hempel. *Almost group theory*, ArXiv 1509.09087
- [16] W. Hodges. *Model Theory*, Cambridge University Press, 1993.
- [17] I. Kaplan, E. Levi and P. Simon. *Some remarks on dp -minimal groups*, Proceedings of the Muelheim conference - New Pathways between Group Theory and Model Theory, accepted, 2016.
- [18] Byunghan Kim. *Simplicity Theory*, Oxford University Press, 2014.
- [19] K. Krupiński. *On relationships between algebraic properties of groups and rings in some model-theoretic contexts*, J. Symb. Logic 76, 1403–1417, 2011.
- [20] K. Krupiński. *On ω -categorical groups and rings with NIP*, Proc. Am. Math. Soc. 140, 2501–2512, 2012.
- [21] D. MacPherson. *Absolutely ubiquitous structures and \aleph_0 -categorical groups*, Quart. J. Math. (Oxford) (2), 39, 483–500, 1988.
- [22] B. H. Neumann. *Groups covered by finitely many cosets*, Publ. Math. Debrecen 3, 227–242, 1954.
- [23] B. H. Neumann. *Groups covered with permutable subsets*, J. London Math. Soc. 29, 236–248, 1954.
- [24] B. Poizat. *Cours de théorie des modèles (A Course in Model Theory)*, Nūr al-mantiq wa-al-marifah, 1985 and Springer Verlag, 2000.
- [25] C. Ryll-Nardzewski. *On category in power $\leq \aleph_0$* , Bull. Acad. Poi. Sér. Math. Astr. Phys. 7, 545–548, 1959.
- [26] P. Simon. *A Guide to NIP Theories (Lecture Notes in Logic)*, Cambridge University Press, 2015.
- [27] L. Svenonius. *\aleph_0 -categoricity in first-order predicate calculus*, Theoria 25, 82–94, 1959.
- [28] K. Tent and M. Ziegler. *A Course in Model Theory*, Cambridge University Press, 2012.
- [29] J. Wilson. *The algebraic structure of ω -categorical groups*, in: Groups-St. Andrews, Ed. C. M. Campbell, E. F. Robertson, London Math. Soc. Lecture Notes 71, Cambridge, 345–358, 1981.
- [30] M. Ziegler. *Introduction to the Lascar group*, in *Tits buildings and the model theory of groups*, London Math. Soc. Lecture Note Series, 291 (Cambridge University Press), 279–298, 2002.

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