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ON ω -CATEGORICAL GROUPS AND RINGS OF FINITE BURDEN

JAN DOBROWOLSKI AND FRANK O. WAGNER

ABSTRACT. An ω -categorical group of finite burden is virtually finite-by-abelian; an ω -categorical ring of finite burden is virtually finite-by-null; an ω -categorical NTP_2 ring is nilpotent-by-finite.

1. INTRODUCTION

The NIP property has recently been one of the most intensely studied subjects in model theory (see for example [20] for an overview of the subject). Dp-rank is a cardinal-valued rank well defined (i.e. not assuming value ∞) on the class of NIP theories, thus providing a hierarchy inside of this class. The simplest in the sense of this hierarchy of NIP structures are *dp-minimal structures*, that is, the structures of dp-rank one, which include the field of real numbers (and its (weakly) o-minimal expansions), the valued fields of p -adic numbers for any prime p , valued algebraically closed fields, Presburger arithmetic $(\mathbb{Z}, 0, +, <)$, as well as any strongly minimal structure. An analogous rank for NTP_2 structures (a generalization of NIP, including all simple structures) is burden (called also inp-rank). In fact, both ranks coincide as long as the dp-rank is well-defined, i.e. under the NIP hypothesis. An example of an inp-minimal (i.e. burden one) structure which is not dp-minimal is the random graph (and any other weight one simple theory, by [1, Proposition 8]). By sub-multiplicativity of burden [6, Theorem 2.5], finite burden structures include all structures interpretable in inp-minimal ones, e.g. algebraic groups over the fields of real, complex and p -adic numbers. For more details on burden and related topics see [6] or [1].

There is a long history of study of ω -categorical groups. In the general case, the main result is the theorem of Wilson on characteristically simple ω -categorical groups (Fact 7.1). It is a hard open problem whether the possibility (iii) from that theorem can be eliminated. A positive answer would yield a complete classification of characteristically simple ω -categorical groups, but a complete classification of all ω -categorical groups (and rings) seems to be out of reach at present. Yet the situation is clear under some model-theoretic assumptions:

Fact 1.1 ([4]). *Every ω -categorical superstable group is abelian-by-finite.*

Fact 1.2 ([4]). *Every ω -categorical superstable ring is null-by-finite (i.e. it has a null ideal of finite index).*

Fact 1.3 ([10]). *Every ω -categorical supersimple group is virtually finite-by-abelian*

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Extraspecial p -groups yield an example showing that the finite normal subgroup cannot be avoided in the last fact ([11]). Many variants of these now-classical results have been proven. In [16], it was shown that ω -categorical NIP rings are nilpotent-by-finite (we generalize this to NTP_2 setting in Theorem 8.3), and it was conjectured that ω -categorical groups are nilpotent-by-finite as well. The latter conjecture remains open, yet a special case of it was obtained in [14]: dp-minimal ω -categorical groups are virtually nilpotent. It was asked in [14, Problem 2.18] whether they are actually virtually abelian-by-finite. In this paper we answer the question affirmatively more generally for groups of finite burden (Theorem 7.8 and Corollary 7.9). This also solves [14, Problem 2.21] in this higher generality. To get the result, we prove that ω -categorical bilinear quasi-forms of finite burden are virtually almost trivial, and, as a by-product, we obtain analogous results for rings (Theorem 7.10, Corollary 7.11).

An earlier version of this paper [9] obtained the same results under the stronger hypothesis of inp-minimality; virtually the only consequence of inp-minimality used was that any two definable groups are comparable with respect to almost inclusion. For the generalisation to the finite burden case, we use essentially the same proof; considerable work is being spent to show that all the relevant groups are still comparable with respect to almost inclusion in a minimal counterexample of finite burden.

The paper is organized as follows: In Section 2, we recall the definition of burden, and deduce some algebraic consequences when the burden is finite. In section 3, we introduce additive relations and the ring of quasi-endomorphisms; in Section 4, we study the properties of quasi-homomorphisms under the assumption of ω -categoricity. In Section 5, we generalize the notion of a bilinear form using quasi-homomorphisms instead of homomorphisms. In section 6, we prove our Main Theorem, Theorem 6.2, about virtual almost triviality of bilinear quasi-forms. This is applied in Section 7 to obtain the results about groups and rings. In Section 8, we state some questions and we prove that ω -categorical rings with NTP_2 are virtually nilpotent.

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2. BURDEN

Throughout the paper we will work in a monster model of the relevant complete theory (i.e. a $\bar{\kappa}$ -saturated, $\bar{\kappa}$ -homogeneous model, where $\bar{\kappa}$ is a sufficiently big cardinal number).

Definition 2.1. (1) Let κ be a cardinal number. An *inp-pattern of depth κ in a partial type $\pi(\bar{x})$* is a sequence $\langle \varphi_i(\bar{x}; \bar{y}_i) : i < \kappa \rangle$ of formulas and an array $\langle \bar{a}_{i,j} : i < \kappa, j < \omega \rangle$ of parameters (from some model of T) such that:

- (a) For each $i < \kappa$, there is some $k_i < \omega$ such that $\{\varphi_i(\bar{x}; \bar{a}_{i,j}) : j < \omega\}$ is k_i -inconsistent; and
- (b) For each $\eta : \kappa \rightarrow \omega$, the partial type

$$\pi(\bar{x}) \cup \{\varphi_i(\bar{x}; \bar{a}_{i,\eta(i)}) : i < \kappa\}$$

is consistent.

- (2) The *burden* (or *inp-rank*) of a partial type $\pi(\bar{x})$ is the maximal κ such that there is an inp-pattern of depth κ in $\pi(\bar{x})$, if such a maximum exists. In case there are inp-patterns of depth λ in $\pi(\bar{x})$ for every cardinal $\lambda < \kappa$ but no inp-pattern of depth κ , we say that the inp-rank of $\pi(\bar{x})$ is κ_- . We will denote the burden of $\pi(\bar{x})$ by $\text{bdn}(\pi(\bar{x}))$. By the burden of a type-definable set we mean the burden of

a type defining this set. A theory T is called *strong*, if burden of any partial type in finitely many variables is bounded by $(\aleph_0)_-$.

Remark 2.2. Suppose $k = \text{bdn}(\pi(\bar{x}))$ and $l = \text{bdn}(\rho(\bar{y}))$ are finite, where \bar{x} and \bar{y} are disjoint. Then $\text{bdn}(\pi(\bar{x}) \cup \rho(\bar{y})) \geq k + l$. In other words, for type-definable sets V and W of finite burden we have: $\text{bdn}(V \times W) \geq \text{bdn}(V) + \text{bdn}(W)$.

Proof. This is clear, as the concatenation of an inp-pattern in $\pi(\bar{x})$ with an inp-pattern in $\rho(\bar{y})$ is an inp-pattern in $\pi(\bar{x}) \cup \rho(\bar{y})$. \square

Remark 2.3. Suppose $f : V \rightarrow W$ is definable and all fibres of f have size at most k , where $k < \omega$. Then $\text{bdn}(V) \leq \text{bdn}(W)$.

Proof. Suppose $\langle \varphi_i(v; y_i) : i < \kappa \rangle$ together with $\langle a_{ij} : i < \kappa, j < \omega \rangle$ form an inp-pattern in V . We claim that $\langle \psi_i(w, y) := (\exists v)(\varphi_i(v; y_i) \wedge f(v) = w), i < \kappa \rangle$ together with $\langle a_{ij} : i < \kappa, j < \omega \rangle$ form an inp-pattern in W . Indeed, for any $i < \kappa$, if l_i is such that $\{\varphi_i(v, a_{i,j}) : j < \omega\}$ is l_i -inconsistent, then, by the pigeonhole principle, $\{\psi_i(w, a_{i,j}) : j < \omega\}$ is $(k-1)l_i + 1$ inconsistent. Also, for each $\eta : \kappa \rightarrow \omega$, if $v_0 \in V$ satisfies $\varphi_i(v, a_{i,\eta(i)})$ for each $i < \kappa$, then $f(v_0) \in W$ satisfies $\psi_i(w, a_{i,\eta(i)})$ for each $i < \kappa$. \square

For the next lemma we introduce some notation for subgroups H and K of a group G . We say that H is *almost contained* in K , denoted $H \lesssim K$, if $H \cap K$ has finite index in H . If $H \lesssim K$ and $K \lesssim H$, the two groups are *commensurable*, denoted $H \sim K$. The *almost centraliser* of H is defined as

$$\tilde{C}_G(H) = \{g \in G : H \lesssim C_H(g)\},$$

and the *almost centre* of G is $\tilde{C}_G(G)$.

Theorem 2.4 ([12, Theorem 2.10]). *If H and K are definable, then $H \lesssim \tilde{C}_G(K)$ iff $K \lesssim \tilde{C}_G(H)$.*

The following lemma is a special case of well-known facts (Proposition 3.3 from [8] and Proposition 4.5 from [7]); we give a brief proof for convenience. It is the only consequence of finiteness of burden we shall use.

Lemma 2.5. *Let G be an abelian group of finite burden, and $(H_i : i < n)$ definable subgroups of G . If the sum $\sum_{i < n} H_i$ is irreducible (meaning that $H_i \not\lesssim \sum_{j \neq i} H_j$ for all $i < n$), then $n \leq \text{bdn}(G)$. Similarly, if the intersection $\bigcap_{i < n} H_i$ is irreducible (meaning that $\bigcap_{j \neq i} H_j \not\lesssim H_i$ for all $i < n$), then $n \leq \text{bdn}(G)$.*

Proof. For the first assertion, let $\phi_i(x, y)$ be the formula $x - y \in \sum_{j \neq i} H_j$, and choose $(a_{i,j} : j < \omega)$ to be representatives in H_i for cosets of $\sum_{j \neq i} H_j$.

For the second assertion, let $\phi_i(x, y)$ be the formula $x - y \in H_i$, and choose $(a_{i,j} : j < \omega)$ to be representatives in $\bigcap_{j \neq i} H_j$ for cosets of H_i .

In either case we obtain an inp-pattern of depth n . \square

3. ADDITIVE RELATIONS AND QUASI-ENDOMORPHISMS

We slightly modify the construction of the definable quasi-endomorphisms ring from [4] in order to be able to apply it to non-connected definable groups.

Definition 3.1. Let G and H be abelian groups. An *additive relation* between G and H is a subgroup $R \leq G \times H$. We call $\pi_1(R)$, the projection to the first coordinate, the

domain $\text{dom } R$ and $\pi_2(R)$ the image $\text{im } R$ of R ; the subgroup $\{g \in G : (g, 0) \in R\}$ is the *kernel* $\ker R$, and $\{h \in H : (0, h) \in R\}$ is the *cokernel* $\text{coker } R$. If $\text{dom } R$ has finite index in G and $\text{coker } R$ is finite, the additive relation R is a *quasi-homomorphism* from G to H (not to be confused with quasi-homomorphism in the sense of metric groups). A quasi-homomorphism R induces a homomorphism $\text{dom } R \rightarrow H/\text{coker } R$. If $G = H$ we call R a *quasi-endomorphism*. Particular additive relations are $\text{id}_G = \{(g, g) : g \in G\}$ and $0_G = G \times \{0\}$.

Definition 3.2. • If $R \leq G \times H$ is an additive relation, $g \in G$ and $K \leq G$, put $R(g) = \{h \in H : (g, h) \in R\}$ and $R[K] = \bigcup_{g \in K} R(g)$.

- If $R, R' \leq G \times H$ are additive relations, put

$$R + R' = \{(a, b + b') \in G \times H : (a, b) \in R, (a, b') \in R'\}.$$

This is again an additive relation. If moreover R and R' are quasi-homomorphisms from G to H , so is $R + R'$. Note that $R + R'$ (as additive relations) is different from the sum when R and R' are considered as subgroups.

- We call $R, R' \leq G \times H$ *equivalent*, denoted $R \equiv R'$, if there is a subgroup G_1 of finite index in G and a finite group $F \leq H$ such that

$$(R \cap (G_1 \times H)) + (G \times F) = (R' \cap (G_1 \times H)) + (G \times F).$$

This is clearly an equivalence relation.

- If $R \leq G \times H$ and $R' \leq H \times K$ are additive relations, put

$$R' \circ R = \{(a, c) \in G \times K : \exists b [(a, b) \in R \text{ and } (b, c) \in R']\}.$$

This is again an additive relation between G and K . If R and R' are quasi-homomorphisms, so is $R' \circ R$. We denote the n -fold composition of R with itself by R^{cn} .

- For an additive relation $R \leq G \times H$ put

$$R^{-1} = \{(h, g) \in H \times G : (g, h) \in R\}.$$

Note that this is also an additive relation between H and G .

Remark 3.3. Note that

$$R^{-1} \circ R = \text{id}_{\text{dom } R} + (\text{dom } R \times \ker R) \quad \text{and} \quad R \circ R^{-1} = \text{id}_{\text{im } R} + (\text{im } R \times \text{coker } R).$$

If $\text{im } R$ has finite index in H and $\ker R$ is finite, then R^{-1} is a quasi-homomorphism from H to G . If moreover R is a quasi-homomorphism, then $R \circ R^{-1} \equiv \text{id}_H$ and $R^{-1} \circ R \equiv \text{id}_G$.

Lemma 3.4. *Let G be an abelian group. The sum, difference and product of definable quasi-endomorphisms of G is again a definable quasi-endomorphism. The set of definable quasi-endomorphisms of G modulo equivalence forms an associative ring.*

Proof. As in [4], taking subgroups of finite index where needed. □

4. QUASI-HOMOMORPHISMS OF ω -CATEGORICAL GROUPS

Recall that a complete first order theory in a countable language is said to be ω -categorical if it has only one countable model up to isomorphism, and a structure M is ω -categorical if $\text{Th}(M)$ is. By the Ryll-Nardzewski Theorem, this is equivalent to the following statement: for every $n < \omega$ there are only finitely many complete n -types over \emptyset . Hence, for any finite set A in an ω -categorical structure M there are only finitely many definable sets over A , and ω -categorical structures are uniformly locally finite (i.e. there

if a function $f : \omega \rightarrow \omega$ such that, for any $n \in \omega$, each substructure of M generated by n elements has at most $f(n)$ elements).

Lemma 4.1. *Let G and H be abelian groups, and let $g \leq G \times H$ be an additive relation.*

- (1) *If $\text{coker}(g)$ is finite, $|H : \text{im}(g)|$ is finite, and $H_1 \leq H$ has infinite index in H , then $|G : g^{-1}[H_1]|$ is infinite.*
- (2) *If $\text{ker}(g)$ is finite, $|G : \text{dom}(g)|$ is finite, and $G_1 \leq G$ has infinite index, then $|H : g[G_1]|$ is infinite.*
- (3) *If $H_1 \leq H$, then $|\text{dom}(g) : g^{-1}[H_1]| \leq |H : H_1|$.*

Proof. (1) Let $(h_i : i < \omega)$ be such that $h_i - h_j \notin H_1 + \text{coker } g$ for $i \neq j$. We may assume they are all in the same coset of $\text{im } g$, so without loss of generality they are all in $\text{im } g$. For each i let $g_i \in G$ be such that $g(g_i, h_i)$. If $g_i - g_j \in g^{-1}[H_1]$ for $i \neq j$, then there is $h \in H_1$ such that $g(g_i - g_j, h)$, so $h - (h_i - h_j) \in \text{coker } g$, a contradiction. Hence all g_i are in pairwise distinct cosets modulo $g^{-1}[H_1]$.

(2) Follows from (1) applied to g^{-1} .

(3) If elements $(g_i : i \in I)$ are pairwise distinct modulo $g^{-1}[H_1]$ elements in $\text{dom } g$, and $h_i \in g(g_i)$, then the elements $(h_i : i \in I)$ are in pairwise distinct cosets modulo H_1 . \square

Lemma 4.2. *Let G and H be definable abelian groups in an ω -categorical structure, and $f, g \leq G \times H$ definable additive relations such that $\text{ker } f$ and $\text{coker } g$ are finite, $\text{im } g$ has finite index in H , and $\text{dom } f$ has finite index in G . Then $\text{ker } g$ and $\text{coker } f$ are finite, $\text{im } f$ has finite index in H and $\text{dom } g$ has finite index in G .*

Proof. Let A be a finite set over which all the above objects are definable.

Claim. *Suppose that $H_1 < H_2 \leq H$ are such that H_1 has infinite index in H_2 . Then $f[g^{-1}[H_1]]$ has infinite index in $f[g^{-1}[H_2]]$.*

Proof. As $\text{im } g$ has finite index in H , the index of $H_1 \cap \text{im } g$ in $H_2 \cap \text{im } g$ is infinite. Now, $g^{-1}[H_1]$ has infinite index in $g^{-1}[H_2]$ by Lemma 4.1(1), so $f[g^{-1}[H_1]]$ is a subgroup of infinite index in $f[g^{-1}[H_2]]$ by 4.1(2). \square

Suppose for a contradiction that $\text{ker}(g)$ or $\text{coker } f$ is infinite. Put $K_0 = \{e\} \leq H$ and define inductively $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 is infinite; by the claim K_n is a subgroup of infinite index in K_{n+1} for all $n < \omega$, contradicting ω -categoricity (as this implies there are infinitely many types over A in H).

Now suppose that $\text{im } f$ has infinite index in H or $\text{dom}(g)$ has infinite index in G . Put $K_0 = H$ and define as before $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 has infinite index in K_0 ; by the claim K_{n+1} is a subgroup of infinite index in K_n for all $n < \omega$, again contradicting ω -categoricity. \square

Remark 4.3. Note that commutativity was not used in the proof. An analogous lemma holds for arbitrary groups, and *multiplicative* relations (with the obvious definition adapting Definition 3.1 to non-commutative groups).

Lemma 4.4. *Let G and H be abelian groups definable in an ω -categorical structure, and $f, g \leq G \times H$ definable quasi-homomorphisms. If $\text{ker } f \lesssim \text{ker } g$ and $\text{im } f \lesssim \text{im } g$, then $\text{im } g \sim \text{im } f$ and $\text{ker } g \sim \text{ker } f$.*

Proof. Suppose $\text{ker } f \lesssim \text{ker } g$ and $\text{im } f \lesssim \text{im } g$. Let $f_1, g_1 \leq G/(\text{ker } f \cap \text{ker } g) \times \text{im } g$ be the additive relations induced by f and g , namely

$$f_1(x + (\text{ker } f \cap \text{ker } g), y) \iff f(x', y) \text{ for some/all } x' \in x + (\text{ker } f \cap \text{ker } g),$$

and likewise for g_1 . Then $\ker f_1$ is finite since $\ker f \lesssim \ker g$, and $\text{coker } g_1 = g[\ker f \cap \ker g] = \text{coker } g$ is finite, too.

Also, $\text{im } g_1 = \text{im } g$, and $\text{dom } f_1 = f^{-1}[\text{im } g \cap \text{im } f]$ has finite index in G by Lemma 4.1(3) and the assumption. Thus, by Lemma 4.2, $\text{im } f_1 = \text{im } f \cap \text{im } g$ has finite index in $\text{im } g$, i.e. $\text{im } f \sim \text{im } g$ and $\ker g_1 = \ker g / (\ker f \cap \ker g)$ is finite, i.e. $\ker f \sim \ker g$. \square

Corollary 4.5. *Let G and H be abelian groups definable in an ω -categorical theory, $f \leq G \times G$ a definable quasi-endomorphism of G , and $g \leq G \times H$ a definable quasi-homomorphism.*

- (1) $\ker(f)$ is finite iff $|G : \text{im}(f)|$ is finite.
- (2) $|G : \ker(g)|$ is finite iff $\text{im}(g)$ is finite.
- (3) If $G \leq H$ and $|H : \text{im}(g)|$ is finite, then $|H : G|$ and $\ker g$ are finite.

Proof. For (1), apply Lemma 4.4 to f and id_G , and for (2), apply it to g and 0_G (both ways, in order to obtain both directions of the equivalence). For (3), apply Lemma 4.4 to the inclusion $i \leq G \times H$ and g ; we obtain $H \lesssim \text{im } g \sim \text{im } i = G$ and $\ker g \sim \ker i = \{e\}$. \square

Lemma 4.6. *Let G be an ω -categorical abelian group and f a definable quasi-endomorphism of G . Then there is $n < \omega$ such that G decomposes as an almost direct sum of $\text{im } f^{on}$ and $\ker f^{on}$ (i.e. $G \sim \text{im } f^{on} + \ker f^{on}$ and $\text{im } f^{on} \cap \ker f^{on}$ is finite).*

Proof. The $f^{on}[G]$ form a descending chain of subgroups, all definable over the same finite set of parameters. By ω -categoricity there is some n such that $f^{on}[G] = f^{on+1}[G] = f^{o2n}[G]$. Consider $g \in \text{dom}(f^{on})$. There is $h \in f^{on}[G]$ such that $f^{on}(g) \cap f^{on}(h) \neq \emptyset$. But this means $g - h \in \ker f^{on}$, so $G \lesssim \text{dom}(f^{on}) \leq \text{im } f^{on} + \ker f^{on}$. As $f^{on}[\text{im } f^{on}] = \text{im } f^{o2n} = \text{im } f^{on}$, the intersection $\text{im } f^{on} \cap \ker f^{on}$ must be finite by Corollary 4.5(3), applied to $G = H = \text{im } f^{on}$ and $g = f^{on}$ (where G, H and g are in the sense of Corollary 4.5). \square

5. BILINEAR QUASI-FORMS

We shall now introduce a generalization of the notion of bilinear form.

Definition 5.1. Let G, H and K be abelian groups. A *bilinear quasi-form* is a multivalued function $\lambda : G \times H \rightarrow K$ (i.e. a subset of $G \times H \times K$) such that for every $g \in G$ and $h \in H$ the multivalued functions $\lambda_g : H \rightarrow K$ given by $x \mapsto \lambda(g, x)$ and $\lambda'_h : G \rightarrow K$ given by $\lambda'_h(y) = \lambda(y, h)$ are quasi-homomorphisms.

For $g \in G$ and $h \in H$ the *annihilator* of g (or of h) are the subgroups

$$\begin{aligned} \text{ann}_H(g) &= \{h \in H : \lambda(g, h) = 0\} = \ker \lambda_g \leq G \quad \text{and} \\ \text{ann}_G(h) &= \{g \in G : \lambda(g, h) = 0\} = \ker \lambda'_h \leq H. \end{aligned}$$

Remark 5.2. If $\lambda : G \times H \rightarrow K$ is a bilinear quasi-form and $\lambda(g, h)$ is defined, then $\lambda(g, h)$ is a coset of $\text{coker } \lambda_g = \lambda(g, 0)$ and of $\text{coker } \lambda'_h = \lambda(0, h)$. So

$$\text{coker } \lambda_g = \lambda(g, h) - \lambda(g, h) = \text{coker } \lambda'_h.$$

As for any $g, g' \in G$ there is $h \in H$ such that $\lambda(g, h)$ and $\lambda(g', h)$ are defined, this means

$$\text{coker } \lambda_g = \text{coker } \lambda'_h = \text{coker } \lambda_{g'}.$$

It follows by symmetry that all cokernels are equal to $K_0 = \lambda(0, 0)$; dividing by K_0 we can always assume that λ has trivial cokernels.

Suppose $\lambda : G \times H \rightarrow K$ is a bilinear quasi-form with $\lambda(0, 0)$ trivial (hence we can treat it as a single-valued partial function). For any $g, g' \in G$, we shall consider the additive relation $\lambda_{g,g'} = \lambda_{g'}^{-1} \circ \lambda_g \leq H \times H$ given by $\{(h, h') \in H \times H : \lambda(g, h) = \lambda(g', h')\}$. Clearly $\ker \lambda_{g,g'} = \text{ann}_H(g)$ and $\text{coker } \lambda_{g,g'} = \text{ann}_H(g')$. If $\text{ann}_H(g') \lesssim \text{ann}_H(g)$ and $\lambda(g, H) \lesssim \lambda(g', H)$, then $\lambda_{g,g'}$ induces a quasi-endomorphism $\bar{\lambda}_{g,g'}$ of $H/\text{ann}_H(g')$ given by

$$\bar{\lambda}_{g,g'}(x + \text{ann}_H(g'), y + \text{ann}_H(g')) \Leftrightarrow \lambda_{g,g'}(x', y) \text{ for some } x' \in x + \text{ann}_H(g').$$

Note that this does not depend on y inside $y + \text{ann}_H(g')$, as $\text{ann}_H(g') = \text{coker } \lambda_{g,g'}$.

This is indeed a quasi-endomorphism, as

$$\text{dom } \bar{\lambda}_{g,g'} = \text{dom } \lambda_{g,g'}/\text{ann}_H(g') = \lambda_g^{-1}[\text{im } \lambda_g \cap \text{im } \lambda_{g'}]/\text{ann}_H(g')$$

has finite index in $H/\text{ann}_H(g')$ by Lemma 4.1(3), and

$$\begin{aligned} \text{coker } \bar{\lambda}_{g,g'} &= \{h \in H : \exists h' \in \text{ann}_H(g') \lambda(g, h') = \lambda(g', h)\}/\text{ann}_H(g') \\ &= \lambda_{g'}^{-1}[\lambda_g[\text{ann}_H(g')]]/\text{ann}_H(g') \end{aligned}$$

is finite, as $\lambda_g[\text{ann}_H(g')]$ is finite and $\ker \lambda_{g'} = \text{ann}_H(g')$.

Definition 5.3. For $A \leq G$ and $B \leq H$ definable put

$$\begin{aligned} \widetilde{\text{ann}}_H(A) &= \{h \in H : A \lesssim \text{ann}_G(h)\} = \{h \in H : \lambda(A, h) \text{ is finite}\} \quad \text{and} \\ \widetilde{\text{ann}}_G(B) &= \{g \in G : B \lesssim \text{ann}_H(g)\} = \{g \in G : \lambda(g, B) \text{ is finite}\}, \end{aligned}$$

the *almost annihilators* of A and B (the equalities above follow from Corollary 4.5(2)).

The almost annihilators are subgroups of G and H ; they are given as a countable increasing union of definable sets (over the same parameters as G , H and A (respectively B)). Thus, in an ω -categorical theory, they are definable.

The next proposition is an adaptation of [12, Theorem 2.10] to bilinear quasi-forms.

Proposition 5.4. *Let $A \leq G$ and $B \leq H$ be definable subgroups. Then $B \lesssim \widetilde{\text{ann}}_H(A)$ if and only if $A \lesssim \widetilde{\text{ann}}_G(B)$.*

Proof. We may assume that G , H , A and B are defined over \emptyset . Suppose that $B \not\lesssim \widetilde{\text{ann}}_H(A)$. Consider a sequence $(h_i : i < \omega)$ in B representing different cosets of $\widetilde{\text{ann}}_H(A)$. So $h_i - h_j \notin \widetilde{\text{ann}}_H(A)$ for $i \neq j$, and the index $|A : \text{ann}_A(h_i - h_j)|$ is infinite. By Neumann's Lemma no finite union of cosets of the various $\text{ann}_A(h_i - h_j)$ can cover A . Hence, by the compactness theorem (and by the saturation of the monster model), there is an infinite sequence $(g_k : k < \omega)$ in A such that $\lambda(g_k - g_\ell, h_i - h_j) \neq 0$ for all $i \neq j$ and $k \neq \ell$. It follows that $|B : \text{ann}_B(g_k - g_\ell)|$ is infinite and $g_k - g_\ell \notin \widetilde{\text{ann}}_G(B)$ for all $k \neq \ell$, whence $A \not\lesssim \widetilde{\text{ann}}_G(B)$.

The other direction follows by symmetry. \square

Definition 5.5. A bilinear quasi-form λ is *almost trivial* if there is a finite subgroup of K containing $\text{im } \lambda$. It is *virtually almost trivial* if there are subgroups G_0 of finite index in G and H_0 in H such that the restriction of λ to $G_0 \times H_0$ is almost trivial.

Proposition 5.6. *Let G , H and K be abelian groups and $\lambda : G \times H \rightarrow K$ a bilinear quasi-form. Then λ is almost trivial iff $\text{ann}_H(g)$ and $\text{ann}_G(h)$ have uniformly finite index in H and G , respectively, for all $g \in G$ and $h \in H$.*

Proof. Clearly left implies right. So suppose $\text{ann}_H(g)$ and $\text{ann}_G(h)$ have uniformly finite index in G and H for all $g \in G$ and $h \in H$. Note that this implies that $\langle \text{im } \lambda \rangle$ has finite

exponent. So it is enough to show that $\text{im}\lambda$ is finite. Note that $\lambda(g, H)$ and $\lambda(G, h)$ are uniformly finite for all $g \in G$ and $h \in H$.

Consider $g \in G$ with $\lambda(g, G)$ maximal, and choose $h_0, \dots, h_n \in H$ with $\lambda(g, H) = \{\lambda(g, h_i) : i \leq n\}$. Then for $g' \in g + \bigcap_{i \leq n} \text{ann}_G(h_i)$ we have $\lambda(g', h_i) = \lambda(g, h_i)$, whence $\lambda(g', H) \supseteq \lambda(g, H)$, and $\lambda(g', H) = \lambda(g, H)$ by maximality. Note that $\bigcap_{i \leq n} \text{ann}_G(h_i)$ is a subgroup of boundedly finite index in G (i.e. bounded independently from g). It follows that there can only be finitely many maximal sets of the form $\lambda(g, H)$ for $g \in G$, and $\text{im}\lambda$ is finite. \square

Corollary 5.7. *Let $\lambda : G \times H \rightarrow K$ be a bilinear quasi-form. The following are equivalent:*

- (1) $G \lesssim \widetilde{\text{ann}}_G(H)$.
- (2) $H \lesssim \widetilde{\text{ann}}_H(G)$.
- (3) λ is virtually almost trivial.

Moreover, in this case $\widetilde{\text{ann}}_G(H)$ and $\widetilde{\text{ann}}_H(G)$ are definable.

Proof. Conditions (1) and (2) are equivalent by Proposition 5.4.

Suppose (1) and (2) hold. Put $A_n = \{g \in G : |H : \text{ann}_H(g)| \leq n\}$. Then each A_n is definable, and $\widetilde{\text{ann}}_G(H) = \bigcup_{n < \omega} A_n$. By compactness and (1), there are $n, k < \omega$ such that there are no k disjoint translates of A_n by elements in G . Let $A = \bigcup_i a_i + A_n$ be a maximal union of disjoint translates of A_n by elements of $\widetilde{\text{ann}}_G(H)$. So for any $a \in \widetilde{\text{ann}}_G(H)$ we have $(a + A) \cap A \neq \emptyset$, whence $a \in A - A$. Thus $\widetilde{\text{ann}}_G(H) = A - A$ is definable; it follows by compactness that $|H : \text{ann}_H(a)|$ is bounded for all $a \in \widetilde{\text{ann}}_G(H)$. By symmetry, the same holds for $\widetilde{\text{ann}}_H(G)$. Proposition 5.6 now implies that λ restricted to $\widetilde{\text{ann}}_G(H) \times \widetilde{\text{ann}}_H(G)$ is almost trivial, so λ is virtually almost trivial.

Conversely, if λ is virtually almost trivial as witnessed by G_0 and H_0 , then $G_0 \leq \widetilde{\text{ann}}_G(H)$ and $H_0 \leq \widetilde{\text{ann}}_H(G)$. \square

6. VIRTUAL ALMOST TRIVIALITY

Definition 6.1. Let G be an infinite definable group. We shall say that a type $p \in S_G(A)$ is *subgroup-generic* if p is in no $\text{acl}^{\text{eq}}(A)$ -definable coset of a subgroup of infinite index in G (where a coset is $\text{acl}^{\text{eq}}(A)$ -definable if it has only finitely many images under automorphisms fixing A). A sequence $(g_i : i \in I)$ is *subgroup-generic* over A if $\text{tp}(g_i/A, g_j : j < i)$ is subgroup-generic for all $i \in I$.

Note that by Neumann's Lemma the group G is not in the filter of definable sets covered by finitely many cosets of definable subgroups of G of infinite index. By a standard inductive construction G has a complete subgroup-generic type over any set of parameters. By Ramsey's Theorem and compactness, indiscernible subgroup-generic sequences of any order type exist.

Theorem 6.2. *Let G, H and K be abelian groups of finite burden definable in some ω -categorical theory, and let $\lambda : G \times H \rightarrow K$ be a definable bilinear quasi-form. Then λ is virtually almost trivial. If G and H are connected, then λ is trivial.*

Proof. Let $\lambda : G \times H \rightarrow K$ be a counter-example with $\text{bdn}(G) + \text{bdn}(H) + \text{bdn}(K)$ minimal possible. We may assume that $\lambda(0, 0)$ is trivial, and hence we can treat λ as a partial function $G \times H \rightarrow K$.

Let $\epsilon > 0$ be infinitesimal, let $((x_i, x'_i) : i \in \mathbb{Q} \cup (\mathbb{Q} + \epsilon))$ be an indiscernible subgroup-generic sequence in $G \times H$, and put $y_i = x_i - x_{i+\epsilon}$ and $y'_i = x'_i - x'_{i+\epsilon}$. Then $(y_i, y'_i : i < \omega)$ is still an indiscernible subgroup-generic sequence.

Claim 1. *If $A \leq G$ is Y -definable where $Y \subseteq (y_j, y'_j : j \neq i)$, then $y_i \in A$ implies $y_i \in A_Y^0$, where A_Y^0 is the intersection of all subgroups of A of finite index which are definable over Y .*

Proof of Claim. Let X be a finite subset of $\{x_j, x'_j : j \neq i, i + \epsilon\}$ such that A is definable over X . Put

$$m = \max\{j < i : x_j \in X\} \quad \text{and} \quad M = \min\{j > i + \epsilon : x_j \in X\},$$

where $m = -\infty$ or $M = \infty$ if the relevant set is empty. Let A_0 be an X -definable subgroup of finite index in A . Since $y_i = x_i - x_{i+\epsilon} \in A$, by indiscernibility of $(x_i : m < i < M)$ over X we obtain $x_j - x_k \in A$, so by Ramsey's Theorem there is an infinite set of indices $I \subseteq (m, M)$ such that all $x_j - x_k$ with $j \neq k$ in I are in the same coset modulo A_0 . This together with indiscernibility implies that $x_j - x_k \in A_0$ for all $m < j < k < M$. In particular $y_i = x_i - x_{i+\epsilon} \in A_0$. As this is true for all finite $X \subseteq \{x_j, x'_j : j \neq i, i + \epsilon\}$, we get $y_i \in A_{\{x_j, x'_j : j \neq i, i + \epsilon\}}^0$. Since $Y \in \text{dcl}(x_j, x'_j : j \neq i, i + \epsilon)$, we obtain $y_i \in A_Y^0$. \square

Claim 2. *$\text{im}\lambda_{y_i}$ and $\text{im}\lambda_{y_j}$ are \lesssim -comparable for all $i < j$.*

Proof of Claim. Suppose not. By Lemma 2.5 there is a minimal $2 \leq \ell \leq \text{bdn}(K)$ such that the sum $\sum_{i=0}^{\ell} \text{im}\lambda_{y_i}$ is reducible (in the sense of Lemma 2.5). So there is $i_0 \leq \ell$ such that with $I = \{0, 1, \dots, \ell\} \setminus \{i_0\}$ we have

$$\text{im}\lambda_{y_{i_0}} \lesssim \sum_{i \in I} \text{im}\lambda_{y_i} =: C.$$

Consider the $\{y_j : j \in I\}$ -definable subgroups

$$A = \{g \in G : \text{im}\lambda_g \lesssim C\} \quad \text{and} \quad B = \{h \in H : \lambda'_h[A] \lesssim C\},$$

and note that $A = \widetilde{\text{ann}}_{\bar{\lambda}}(H)$ and $B = \widetilde{\text{ann}}_{\bar{\lambda}}(A)$, where the almost annihilators are taken with respect to the bilinear quasi-form $\bar{\lambda} : G \times H \rightarrow K/C$ induced by λ . Then $H \lesssim B$ by Lemma 5.4, so B has finite index in H .

For every $i \in I$ there is a $\{y_j : j \in I\}$ -definable induced bilinear quasi-form

$$\lambda_i : A \times B \rightarrow C/\text{im}\lambda_{y_i}$$

given by $\lambda_i(a, b) = \lambda(a, b) + \text{im}\lambda_{y_i}$ if $\lambda(a, b)$ is defined and belongs to C (otherwise $\lambda_i(a, b)$ is not defined). Note that for $a \in A$ the domain $\text{dom}(\lambda_i)_a = \lambda_a^{-1}[C \cap \text{im}\lambda_a] \cap A$ has finite index in A by Lemma 4.1(3), and likewise for $(\lambda_i)'_b$ with $b \in B$, so λ_i is indeed a bilinear quasi-form.

By irreducibility of the sum $\sum_{j \in I} \text{im}\lambda_{y_j}$ the quotient $\text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})$ is infinite. Hence, as

$$(\text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})) \times (\sum_{j \in I, j \neq i} \text{im}\lambda_{y_j}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j}))$$

embeds definably into $C/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})$, Remark 2.2 implies the following strict inequality:

$$\begin{aligned} \text{bdn}(C/\text{im}\lambda_i) &= \text{bdn}(\sum_{j \in I, j \neq i} \text{im}\lambda_{y_j}/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})) \\ &< \text{bdn}(C/(\text{im}\lambda_{y_i} \cap \sum_{j \in I, j \neq i} \text{im}\lambda_{y_j})) \leq \text{bdn}(K). \end{aligned}$$

Hence, the bilinear quasi-form λ_i is virtually almost trivial by induction, so the almost annihilator $\widetilde{\text{ann}}_A^{\lambda_i}(B)$ of B with respect to the quasi-form λ_i is an $\{y_j : j \in I\}$ -definable subgroup of A of finite index. Since $y_{i_0} \in A_{\{y_j : j \in I\}}^0$ by Claim 1, we obtain that $y_{i_0} \in \widetilde{\text{ann}}_A^{\lambda_i}(B)$. Thus $\lambda_{y_{i_0}}[B] \lesssim \text{im} \lambda_{y_i}$; as B has finite index in H we also have $\text{im} \lambda_{y_{i_0}} \lesssim \text{im} \lambda_{y_i}$. The claim now follows from indiscernibility. \square

Claim 3. $\text{ann}_H(y_i)$ and $\text{ann}_H(y_j)$ are \lesssim -comparable for all $i < j$.

Proof of Claim. Suppose not. By Lemma 2.5 there is a minimal $2 \leq \ell \leq \text{bdn}(H)$ such that the intersection $\bigcap_{i=0}^{\ell} \text{ann}_H(y_i)$ is reducible. So there is $i_0 \leq \ell$ such that

$$B := \bigcap_{i \in I} \text{ann}_H(y_i) \lesssim \text{ann}_H(y_{i_0}),$$

where $I = \{0, 1, \dots, \ell\} \setminus \{i_0\}$. Consider the $\{y_j : j \in I\}$ -definable subgroup

$$A = \{g \in G : B \lesssim \text{ann}_H(g)\}.$$

For every $i \in I$ there is an induced bilinear quasi-form

$$\lambda_i : A \times \text{ann}_H(y_i)/B \rightarrow K$$

given by $\lambda_i(a, h) := \lambda_a[h + B]$ (note that $\lambda_a[h + B]$ is finite, as $a \in \widetilde{\text{ann}}_G(B)$). As $(\bigcap_{j \in I, j \neq i} \text{ann}_H(y_j))/B$ is infinite by minimality of ℓ , we get by Remark 2.2

$$\text{bdn}(\text{ann}_H(y_i)/B) < \text{bdn}((\text{ann}_H(y_i) + \bigcap_{j \in I, j \neq i} \text{ann}_H(y_j))/B) \leq \text{bdn}(H).$$

Hence, the bilinear quasi-form λ_i is virtually almost trivial by induction, so

$$\widetilde{\text{ann}}_A^{\lambda_i}(\text{ann}_H(y_i)/B)$$

is a subgroup of A of finite index definable over $\{y_j : j \in I\}$. Since $y_{i_0} \in A_{\{y_j : j \in I\}}^0$ by Claim 1, we get that $\text{ann}_H(y_i) \lesssim \text{ann}_H(y_{i_0})$. The claim now follows from indiscernibility. \square

Note that Claims 1.–3. do not use subgroup-genericity of the sequence, only indiscernibility. We will use this observation to apply (the proofs of) these claims below to certain forms induced by λ .

Claim 4. For $i < j$ we have $\text{ann}_H(y_j) \lesssim \text{ann}_H(y_i)$, and if $B \leq H$ is definable over $\{y_k, y'_k : k \notin [i, j]\}$, then $\lambda_{y_i}[B] \lesssim \lambda_{y_j}[B]$. In particular $\text{im} \lambda_{y_i} \lesssim \text{im} \lambda_{y_j}$.

Proof of Claim. By ω -categoricity there is a bound n on the index of $\text{ann}_G(h)$ in G for $h \in \widetilde{\text{ann}}_H(G)$. Choose $h \in \widetilde{\text{ann}}_H(G)$ subgroup-generic over x_0, \dots, x_n . Then $x_i - x_j \in \text{ann}_G(h)$ for some $0 \leq i < j \leq n$, whence $h \in \text{ann}_H(x_i - x_j)$. By subgroup-genericity of h over x_0, \dots, x_n , the group $\text{ann}_H(x_i - x_j) \cap \widetilde{\text{ann}}_H(G)$ must have a finite index in $\widetilde{\text{ann}}_H(G)$, i.e. $\widetilde{\text{ann}}_H(G) \lesssim \text{ann}_H(x_i - x_j)$; by indiscernibility $\widetilde{\text{ann}}_H(G) \lesssim \text{ann}_H(x_0 - x_\epsilon) = \text{ann}_H(y_0)$.

Suppose $\text{ann}_H(y_0) \lesssim \text{ann}_H(y_1)$. Then $y_1 \in \widetilde{\text{ann}}_G(\text{ann}_H(y_0))$; as y_1 is subgroup-generic over y_0 we have $G \lesssim \widetilde{\text{ann}}_G(\text{ann}_H(y_0))$. By Proposition 5.4, $\text{ann}_H(y_0) \lesssim \widetilde{\text{ann}}_H(G)$. It follows that $\text{ann}_H(y_i) \sim \widetilde{\text{ann}}_H(G)$ for all $i \in \omega$, and $\text{ann}_H(y_1) \sim \text{ann}_H(y_0)$.

The first assertion now follows from Claim 3.

For the second assertion, let $\{y_k, y'_k : k \in I\}$ be the finitely many parameters needed to define B . Put $m = \max I \cap (-\infty, i)$ and $M = \min I \cap (j, \infty)$. We can apply Claim 2 to the restriction of λ to $G \times B$ and the sequences $(x_k : m + \epsilon < k < M)$ and $(y_k : m < k < M)$. Hence $\lambda_{y_i}[B] \lesssim \lambda_{y_j}[B]$ or $\lambda_{y_j}[B] \lesssim \lambda_{y_i}[B]$. But $\text{ann}_B(y_j) \lesssim \text{ann}_B(y_i)$ by the first part, so if the second option holds, then Lemma 4.4 yields $\lambda_{y_0}[B] \sim \lambda_{y_1}[B]$. \square

Claim 5. *If $i < j$ and $C \leq K$ is definable over $\{y_k, y'_k : k \notin [i, j]\}$, then $\lambda_{y_j}^{-1}[C] \lesssim \lambda_{y_i}^{-1}[C]$.*

Proof of Claim. Consider the bilinear quasi-form $\bar{\lambda} : G \times H \rightarrow K/C$, and note that $\lambda_y^{-1}[C] = \ker \bar{\lambda}_y$. Then $\lambda_{y_i}^{-1}[C]$ and $\lambda_{y_j}^{-1}[C]$ are \lesssim -comparable by Claim 3; suppose $\lambda_{y_i}^{-1}[C] \lesssim \lambda_{y_j}^{-1}[C]$. Then λ_{y_i} and λ_{y_j} induce quasi-homomorphisms from $B := \lambda_{y_i}^{-1}[C]$ to C . As $\text{ann}_B(y_j) \lesssim \text{ann}_B(y_i)$ by Claim 4 and $\lambda_{y_j}[B] \lesssim \lambda_{y_i}[B] = C$, Proposition 4.4 implies $\lambda_{y_j}[B] \sim C$. Thus

$$\lambda_{y_j}^{-1}[C] \sim B + \text{ann}_H(y_j) \lesssim B + \text{ann}_H(y_i) = B = \lambda_{y_i}^{-1}[C]. \quad \square$$

We shall now study λ_{y_i, y_j} for $i < j$. By Claim 4 it induces a quasi-endomorphism of $H/\text{ann}_H(y_j)$ (see the discussion after Remark 5.2). By Corollary 4.5, any definable quasi-endomorphism of $H/\text{ann}_H(y_j)$ with finite kernel must be almost surjective, and any definable almost surjective quasi-endomorphism must have finite kernel; these are precisely the invertible quasi-endomorphisms.

Claim 6. *If f is a definable quasi-endomorphism of $H/\text{ann}_H(y_j)$, then f is invertible or nilpotent.*

Proof of Claim. By Lemma 4.6, we have an almost direct decomposition $H/\text{ann}_H(y_j) \sim \text{im } f^{on} + \ker f^{on}$ for some $n < \omega$. Put $A = \{g \in G : \text{ann}_H(y_j) \lesssim \text{ann}_H(g)\}$. Then $y_i \in G$ for $i < j$ by Claim 1. If f were neither invertible nor nilpotent, then both summands are infinite. Hence, by Remarks 2.2 and 2.3

$$\text{bdn}(\text{im } f^{on}) = \text{bdn}(\text{im } f^{on} / (\text{im } f^{on} \cap \ker f^{on})) < \text{bdn}(H)$$

and

$$\text{bdn}(\ker f^{on}) = \text{bdn}(\ker f^{on} / (\text{im } f^{on} \cap \ker f^{on})) < \text{bdn}(H),$$

so the induced bilinear quasi-form

$$\bar{\lambda} : A \times (\text{im } f^{on} + \ker f^{on}) \rightarrow K$$

is virtually almost trivial on $A \times \text{im } f^{on}$ and on $A \times \ker f^{on}$, whence virtually almost trivial. So for $i < j$ the image $\text{im } \lambda_{y_i}$ is finite and $y_i \in \widetilde{\text{ann}}_G(H)$. By subgroup-genericity of y_i we get that $\widetilde{\text{ann}}_G(H) \sim G$, and λ is virtually almost trivial by Corollary 5.7, a contradiction. \square

Claim 7. *For $i < j < k$ we have $\lambda_{y_i, y_k} \equiv \lambda_{y_j, y_k} \circ \lambda_{y_i, y_j}$.*

Proof of Claim. Note that $\lambda_g \circ \lambda_g^{-1} \equiv \text{id}_{\text{im } \lambda_g}$ for any $g \in G$ by Remark 3.3.

$$\begin{aligned} \lambda_{y_0, y_j} &= \lambda_{y_j}^{-1} \circ \lambda_{y_0} = \lambda_{y_j}^{-1} \circ \text{id}_{\text{im } \lambda_{y_0}} \circ \lambda_{y_0} \\ &\equiv \lambda_{y_j}^{-1} \circ \text{id}_{\text{im } \lambda_{y_i}} \circ \lambda_{y_0} \equiv \lambda_{y_j}^{-1} \circ \lambda_{y_i} \circ \lambda_{y_i}^{-1} \circ \lambda_{y_0} = \lambda_{y_i, y_j} \circ \lambda_{y_0, y_i}. \quad \square \end{aligned}$$

Claim 8. *For $i \neq j$ we have $\text{ann}_H(y_i) \not\sim \text{ann}_H(y_j)$.*

Proof of Claim. Suppose otherwise. Then $\text{ann}_H(y_i) \sim \text{ann}_H(y_j)$ for all $i, j \in \mathbb{Q}$. Let R be the ring of definable quasi-endomorphisms of $\bar{H} = H/\text{ann}_H(y_0)$. Note that \bar{H} is infinite, as $\text{ann}_H(y_0)$ has infinite index in H by assumption (otherwise, as above, we get that λ is virtually almost trivial by Corollary 5.7).

It follows from Claim 6 that the set of nilpotent quasi-endomorphisms of \bar{H} is an ideal: it is clearly invariant under left and right multiplication; if f and g are nilpotent but $f + g$ is not nilpotent, there is invertible h with $h(f + g) = hf + hg = \text{id}$. So $hf = \text{id} - hg$ is nilpotent. But $(\text{id} - hg)(\text{id} + hg + (hg)^2 + (hg)^3 + \dots) = \text{id}$ (note that the sum is finite,

as hg is nilpotent), so $hf = \text{id} - hg$ is invertible, a contradiction. Thus R/I is a division ring, which is locally finite by ω -categoricity, whence a locally finite field by Wedderburn's Theorem.

Consider $0 < i < j$. As $\text{ann}_H(y_i) \sim \text{ann}_H(y_j)$, the quasi-endomorphism λ_{y_i, y_j} has finite kernel, and must be invertible. By local finiteness and indiscernibility, it has a fixed finite multiplicative order N modulo I . Hence there are only finitely many possibilities for $\bar{\lambda}_{y_i, y_j} + I$ (where $\bar{\lambda}$ is the equivalence class of λ in R). So, by indiscernibility, $\bar{\lambda}_{y_i, y_j} + I$ does not depend on i, j . But $\bar{\lambda}_{y_j, y_k} \cdot \bar{\lambda}_{y_i, y_j} = \bar{\lambda}_{y_i, y_k}$ for $i < j < k$, so $\bar{\lambda}_{y_i, y_j} = (\bar{\lambda}_{y_i, y_{i+(j-i)/N}})^{\circ N} \in \text{id}_{\bar{H}} + I$. By indiscernibility, $\bar{\lambda}_{x_i - x_j, x_k - x_\ell} \in \text{id}_{\bar{H}} + I$ for all $0 < i < j < k < \ell$ in ω . Now

$$\bar{\lambda}_{x_1 - x_3, x_2 - x_3} = \bar{\lambda}_{x_2 - x_3, x_4 - x_5}^{-1} \cdot \bar{\lambda}_{x_1 - x_3, x_4 - x_5} \in \text{id}_{\bar{H}} + I.$$

Let $B = \text{im}(\lambda_{x_1 - x_3, x_2 - x_3} - \text{id}_H)$, a definable subgroup of infinite index in H almost containing $\text{ann}_H(y_0)$. Then for all $h \in H_{x_1, x_2, x_3}^0$ there is $b \in B$ with $(h, h + b) \in \lambda_{x_1 - x_3, x_2 - x_3}$ (as $\text{dom} \lambda_{x_1 - x_3, x_2 - x_3}$ is a $\{x_1, x_2, x_3\}$ -definable subgroup of H of finite index). Hence $\lambda_{x_2 - x_3}(h + b) = \lambda_{x_1 - x_3}(h)$, that is

$$\lambda(x_1 - x_3, h) = \lambda(x_2 - x_3, h + b) = \lambda(x_2 - x_3, h) + \lambda(x_2 - x_3, b),$$

whence

$$\lambda(x_1 - x_2, h) = \lambda((x_1 - x_3) - (x_2 - x_3), h) = \lambda(x_2 - x_3, b).$$

But this means that $\text{im} \lambda_{x_1 - x_2} \lesssim \lambda_{x_2 - x_3}[B]$. On the other hand, as λ_{y_i, y_j} is a quasi-isomorphism of \bar{H} for $i < j$, so is $\lambda_{x_i - x_j, x_k - x_\ell}$ for all $i < j < k < \ell$. In particular

$$\text{im} \lambda_{x_2 - x_3} \sim \text{im} \lambda_{x_4 - x_5} \sim \text{im} \lambda_{x_1 - x_2} \lesssim \lambda_{x_2 - x_3}[B].$$

But B has infinite index in H and $\ker \lambda_{x_2 - x_3} \lesssim B$, so $B/(\ker \lambda_{x_2 - x_3} \cap B)$ has infinite index in $H/(\ker \lambda_{x_2 - x_3} \cap B)$, and both of them are definably quasi-isomorphic to $\text{im} \lambda_{x_2 - x_3}$, hence definably quasi-isomorphic to each other. This contradicts Corollary 4.5(3). \square

Note that for $i < j < k < \ell$ and $B \leq H$ definable over $\{y_s, y'_s : s \notin [j, k]\}$ we have $\lambda_{y_j, y_\ell}[B] \lesssim \lambda_{y_k, y_\ell}[B]$ by Claim 4, and $\lambda_{y_i, y_k}[B] \lesssim \lambda_{y_i, y_j}[B]$ by Claim 5.

Claim 9. *If $i < j < k < \ell$ and $B \leq H$ is definable over $\{y_s, y'_s : s \neq i, j, k, \ell\}$ then $\lambda_{y_i, y_j}[B]$ and $\lambda_{y_k, y_\ell}[B]$ are \lesssim -comparable.*

Proof of Claim. Suppose not, and put

$$A = \{g \in G : \lambda_g[B] \lesssim \lambda_{y_k}[B]\} \quad \text{and} \quad B' = \{h \in B : \lambda'_h[A] \lesssim \lambda_{y_k}[B]\}.$$

Then $A = \widetilde{\text{ann}}_{\bar{G}}^{\bar{\lambda}}(B)$ and $B' = \widetilde{\text{ann}}_{\bar{B}}^{\bar{\lambda}}(A)$, where the almost annihilators are taken with respect to the bilinear quasi-form $\bar{\lambda} : G \times B \rightarrow K/\lambda_{y_k}[B]$ induced by λ . Then $B \lesssim B'$ by Lemma 5.4, so B' has finite index in B .

Consider the induced bilinear quasi-form

$$\tilde{\lambda} : A \times B' \rightarrow \lambda_{y_k}[B]/(\lambda_{y_k}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B]).$$

As

$$\begin{aligned} \text{bdn}(\lambda_{y_k}[B]/(\lambda_{y_k}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B])) &= \text{bdn}(\lambda_{y_\ell} \lambda_{y_k, y_\ell}[B]/(\lambda_{y_\ell} \lambda_{y_k, y_\ell}[B] \cap \lambda_{y_\ell} \lambda_{y_i, y_j}[B])) \\ &\leq \text{bdn}(\lambda_{y_k, y_\ell}[B]/(\lambda_{y_i, y_j}[B] \cap \lambda_{y_k, y_\ell}[B])) \\ &< \text{bdn}((\lambda_{y_i, y_j}[B] + \lambda_{y_k, y_\ell}[B])/(\lambda_{y_i, y_j}[B] \cap \lambda_{y_k, y_\ell}[B])) \\ &\leq \text{bdn}(H/\text{ann}_H(y_\ell)) = \text{bdn}(\text{im} \lambda_{y_\ell}) \leq \text{bdn}(K), \end{aligned}$$

the bilinear quasi-form $\tilde{\lambda}$ is virtually almost trivial. Since $y_{k'} \in A_{y_i, y_j, y_k, y_\ell}^0$ for some $j < k' < k$ such that B is definable over $\{y_s, y'_s : s \notin [k', k] \cup \{i, j, \ell\}\}$ by Claims 1 and 4, it follows that

$$\lambda_{y_{k'}}[B] \lesssim \lambda_{y_\ell} \lambda_{y_i, y_j}[B].$$

Hence $\lambda_{y_{k'}, y_\ell}[B] \lesssim \lambda_{y_i, y_j}[B] + \text{ann}_B(y_\ell) \lesssim \lambda_{y_i, y_j}[B]$. This contradicts indiscernibility. \square

Claim 10. $\ker \bar{\lambda}_{y_i, y_j}$ and $\ker \bar{\lambda}_{y_k, y_\ell}$ are \lesssim -comparable for all $i < j < k < \ell$, where $\bar{\lambda}_{y, y'}$ is the quasi-homomorphism from H to $H/\text{ann}_H(y_j)$ induced by $\lambda_{y, y'}$.

Proof of Claim. We have $\ker \bar{\lambda}_{y_i, y_j} = \text{ann}_H(y_i)$; put $B = \ker \bar{\lambda}_{y_k, y_\ell}$ and suppose that they are not \lesssim -comparable. Let $A = \{g \in G : \text{ann}_H(y_i) \lesssim \text{ann}_H(g)\}$ and consider the induced bilinear quasi-form

$$\bar{\lambda} : A \times B / (B \cap \text{ann}_H(y_i)) \rightarrow K.$$

Since

$$\text{bdn}(B / (B \cap \text{ann}_H(y_i))) < \text{bdn}((B + \text{ann}_H(y_i)) / (B \cap \text{ann}_H(y_i))) \leq \text{bdn}(H),$$

the bilinear quasi-form $\bar{\lambda}$ must be virtually trivial. Since $y_s \in A_{y_i, y_j, y_k, y_\ell}^0$ for $s < i$, we have $\ker \lambda_{y_k, y_\ell} = B \lesssim \text{ann}_H(y_s)$ for all $s < i$; the claim now follows from indiscernibility. \square

Claim 11. If $\text{im} \lambda_{y_0, y_1} \lesssim \text{im} \lambda_{y_2, y_3}$ then $\text{im} \lambda_{y_0, y_j} \lesssim \text{im} \lambda_{y_i, y_j}^{\text{on}}$ for all $0 < i < j$ and $1 \leq n < \omega$.

Proof of Claim. We proceed by induction on n . For $n = 1$ this is clear, as $\text{im} \lambda_{y_0, y_j} \lesssim \text{im} \lambda_{y_i, y_j}$ by Claim 4. Assume it holds for some n . Choose $0 < k < \ell < i$. Then

$$\begin{aligned} \text{im} \lambda_{y_0, y_j} &\sim \text{im}(\lambda_{y_i, y_j} \circ \lambda_{y_0, y_i}) = \lambda_{y_i, y_j}[\text{im} \lambda_{y_0, y_i}] \lesssim \lambda_{y_i, y_j}[\text{im} \lambda_{y_0, y_k}] \\ &\lesssim \lambda_{y_i, y_j}[\text{im} \lambda_{y_\ell, y_j}] \lesssim \lambda_{y_i, y_j}[\text{im} \lambda_{y_i, y_j}^{\text{on}}] = \text{im}(\lambda_{y_i, y_j} \circ \lambda_{y_i, y_j}^{\text{on}}) = \text{im} \lambda_{y_i, y_j}^{\circ(n+1)} \end{aligned}$$

(the first inequality follows by the second part of Claim 4, the second inequality follows by the assumption of the claim, and the last one by the inductive assumption). \square

Claim 12. If $\text{im} \lambda_{y_2, y_3} \lesssim \text{im} \lambda_{y_0, y_1}$ then $\text{im} \lambda_{y_0, y_k} \lesssim \text{im} \lambda_{y_i, y_j}^{\text{on}}$ for all $0 < i < j < k$ and $1 \leq n < \omega$.

Proof of Claim. The case $n = 1$ follows from Claims 4 and 5, so assume the statement holds for some n . Choose $0 < i < j < \ell < m < k$. Let $\bar{\lambda}_{y, y'}$ be the quasi-homomorphism from H to $H/\text{ann}_H(y_j)$ induced by $\lambda_{y, y'}$. By the assumption and indiscernibility we have $\text{im} \bar{\lambda}_{y_m, y_k} \lesssim \text{im} \bar{\lambda}_{y_i, y_j}$. Hence, Claim 10 and Lemma 4.4 imply $\ker \bar{\lambda}_{y_i, y_j} \lesssim \ker \bar{\lambda}_{y_m, y_k}$, so the same holds for the restrictions to $B := \text{im} \lambda_{y_0, y_\ell}$. But now by Lemma 4.4 and Claim 9 we have $\lambda_{y_m, y_k}[B] \lesssim \lambda_{y_i, y_j}[B]$. Then

$$\begin{aligned} \text{im} \lambda_{y_0, y_k} &\sim \text{im}(\lambda_{y_m, y_k} \circ \lambda_{y_0, y_m}) = \lambda_{y_m, y_k}[\text{im} \lambda_{y_0, y_\ell}] \lesssim \lambda_{y_i, y_j}[\text{im} \lambda_{y_0, y_\ell}] \\ &\lesssim \lambda_{y_i, y_j}[\text{im} \lambda_{y_i, y_j}^{\text{on}}] = \text{im}(\lambda_{y_i, y_j}^{\text{on}} \circ \lambda_{y_i, y_j}) = \text{im} \lambda_{y_i, y_j}^{\circ(n+1)}. \quad \square \end{aligned}$$

Claim 13. $\text{im} \lambda_{y_i, y_k} \lesssim \text{ann}_H(y_j)$ for all $i, j < k$.

Proof of Claim. By Claim 8, the quasi-endomorphism $\bar{\lambda}_{y_i, y_j}$ of $H/\text{ann}_H(y_j)$ induced by λ_{y_i, y_j} is not invertible, so it must be nilpotent by Claim 6. The assertion now follows from Claims 9, 11 and 12. \square

Of course, all of the previous claims also hold with the roles of G and H exchanged.

Claim 14. For any $i \neq j$ we have $\text{im} \lambda_{y_i} \lesssim \text{im} \lambda'_{y_j}$ or $\text{im} \lambda'_{y_j} \lesssim \text{im} \lambda_{y_i}$.

Proof of Claim. Suppose not. Put $A = \{g \in G : \text{im}\lambda_g \lesssim \text{im}\lambda_{y_i}\}$. Then $y_k \in A_{y_i, y'_j}^0$ for $j \neq k < i$ by Claims 1 and 4. Consider the induced bilinear quasi-form

$$\bar{\lambda} : A \times H \rightarrow \text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \text{im}\lambda'_{y'_j}).$$

As $\text{bdn}(\text{im}\lambda_{y_i}/(\text{im}\lambda_{y_i} \cap \text{im}\lambda'_{y'_j})) < \text{bdn}(K)$, the quasi-form $\bar{\lambda}$ must be virtually almost trivial. But then $\text{im}\lambda_{y_k} \lesssim \text{im}\lambda'_{y'_j}$ for all $k < i$, a contradiction, as there is $k < i$ with $y_k \equiv_{y'_j} y_i$. \square

By Claim 14 and symmetry we may assume that $\text{im}\lambda'_{y'_i} \lesssim \text{im}\lambda_{y_j}$ for all $i < j$. Fix i and k , and choose $i < j < \ell$ and $k < \ell < 1$ with $k \notin \{i, j\}$. By Claim 13 we get:

$$\text{im}\lambda_{y_j} \lesssim \lambda_{y_\ell}[\text{ann}_H(y_k)].$$

Moreover, $\lambda_{y_\ell}[\text{ann}_H(y_k)] \lesssim \lambda_{y_1}[\text{ann}_H(y_k)]$ by Claim 4. Then

$$\lambda(y_1, y'_i) \in (\text{im}\lambda'_{y'_i})_{y'_i, y_j, y_k, y_\ell}^0 \leq (\text{im}\lambda_{y_j})_{y'_i, y_j, y_k, y_\ell}^0 \leq \lambda_{y_\ell}[\text{ann}_H(y_k)].$$

Hence,

$$y'_i \in \lambda_{y_1}^{-1}[\lambda_{y_\ell}[\text{ann}_H(y_k)]_{y_k, y_\ell, y_1}^0] \leq \lambda_{y_1}^{-1}[\lambda_{y_1}[\text{ann}_H(y_k)]].$$

Thus,

$$y'_i \in (\text{ann}_H(y_k) + \text{ann}_H(y_1))_{y_k, y_1}^0 \leq \text{ann}_H(y_k),$$

and $\lambda(y_k, y'_i) = 0$. As $(y_i, y'_i)_{i < \omega}$ is a subgroup-generic sequence, $\text{ann}_G(y'_0)$ has finite index in G and $\text{ann}_H(y_0)$ has finite index in H . Since (y_0, y'_0) is subgroup-generic, $\widetilde{\text{ann}}_H(G)$ and $\widetilde{\text{ann}}_G(H)$ have finite index in H and G respectively. So λ is virtually almost trivial by Corollary 5.7.

Finally, if G and H are connected, then λ is almost trivial. But then for every $g \in G$ and $h \in H$ the annihilators $\text{ann}_H(g)$ and $\text{ann}_G(h)$ have finite index in H and in G , and must be equal to H and G , respectively, by connectivity. Thus λ is trivial. \square

7. ON GROUPS AND RINGS

Recall that each countable, ω -categorical group has a finite series of characteristic (i.e. invariant under the automorphism group) subgroups in which all successive quotients are characteristically simple groups (i.e. they do not have non-trivial, proper characteristic subgroups). On the other hand, Wilson [21] proved (see also [2] for an exposition of the proof):

Fact 7.1. *For each infinite, countable, ω -categorical, characteristically simple group H , one of the following holds.*

- (i) *For some prime number p , H is an elementary abelian p -group (i.e. an abelian group, in which every nontrivial element has order p).*
- (ii) *$H \cong B(F)$ or $H \cong B^-(F)$ for some non-abelian, finite, simple group F , where $B(F)$ is the group of all continuous functions from the Cantor space C to F , and $B^-(F)$ is the subgroup of $B(F)$ consisting of the functions f such that $f(x_0) = e$ for a fixed element $x_0 \in C$.*
- (iii) *H is a perfect p -group (perfect means that H equals its commutator subgroup).*

It remains a difficult open question whether there exist infinite, ω -categorical, perfect p -groups.

Remark 7.2. The groups $B(F)$ and $B^-(F)$ above have TP_2 (in particular, they do not have finite burden).

Proof. Let $f \in F$ be a non-central element, and let D_i , $i < \omega$ be pairwise disjoint clopen sets in C . Let $g_i \in B(F)$ be given by $g_i[A_i] = \{f\}$ and $g_i[C \setminus A_i] = \{e\}$ for each i . Then the centralizers of the g_i do not satisfy the conclusion of [7, Theorem 2.4], hence $B(F)$ has TP_2 . The argument for $B^-(F)$ is analogous. \square

The following is Theorem 3.1 from [17]:

Fact 7.3. *There is a finite bound of the size of conjugacy classes in a group G if and only if the derived subgroup G' is finite.*

This implies in particular that if the almost centre $\tilde{Z}(G)$ of a group G is definable, then it is finite-by-abelian.

Remark 7.4. If a group G is virtually finite-by-abelian, then there is a characteristic definable finite-by-abelian subgroup $G_0 \leq G$ of finite index; if a ring R is virtually finite-by-null, there is a definable subring R_0 which is finite-by-null.

Proof. Let G be virtually finite-by-abelian. Then $\tilde{Z}(G)$ is characteristic and definable of finite index (this does not even need ω -categoricity), and finite-by-abelian.

If R is virtually finite-by-null, let S_0 be a finite-by-null subring of finite (additive) index, and I a finite ideal of S_0 containing $S_0 \cdot S_0$. Then $S := \bigcap_{s \in S_0} \{r \in R : rs \in I\}$ contains S_0 and must be a definable subgroup of finite index, with $S \cdot S_0 \subseteq I$. Now $R_0 := S \cap \bigcap_{s \in S} \{r \in R : sr \in I\}$ contains S_0 and is again a definable subgroup of finite index. Since $R_0 \cdot R_0 \subseteq I \leq R_0$, this is a required subring. \square

We will use the following variant of Proposition 2.5 from [14]. As in our context we cannot use connected components, we have to modify the proof slightly.

Lemma 7.5. *Let \mathcal{C} be a class of countable, ω -categorical NTP_2 (pure) groups, closed under taking definable subgroups and quotients by definable normal subgroups. Suppose that every infinite, characteristically simple group in \mathcal{C} is solvable. Then every group in \mathcal{C} is nilpotent-by-finite.*

Proof. Let $G \in \mathcal{C}$. Let $\{e\} = G_0 \leq G_1 \leq \dots \leq G_n = G$ be a chain of characteristic subgroups of G of maximal length. We will show the assertion by induction n . Let i be maximal such that G_i is finite. Then $C_G(G_i)$ is a characteristic subgroup of G of finite index, so we can replace G by $C_G(G_i)/G_i$ without increasing n . We can thus assume that G_1 is infinite. Now, as G_1 is characteristically simple, it is solvable by the assumption. By the inductive hypothesis, G/G_1 is virtually nilpotent, so there is a normal definable subgroup N of G of finite index such that N/G_1 is nilpotent, so N is solvable. Since N is NTP_2 , it does not interpret the atomless boolean algebra, so by [3, Theorem 1.2] it is virtually nilpotent, and so is G . \square

Proposition 7.6. *A nilpotent ω -categorical group of finite burden is virtually finite-by-abelian.*

Proof. Let G be a counter-example; we may assume it is nilpotent of minimal class possible. Then $Z(G)$ is infinite, and $G/Z(G)$ is virtually finite-by-abelian. By Remark 7.4 there is a definable subgroup G_0 of finite index and a finite normal subgroup $F/Z(G)$ of $G_0/Z(G)$ such that G_0/F is abelian. Clearly we may replace G by $C_{G_0}(F/Z(G))$, a definable subgroup of finite index. Then $G' \leq F$ so $G'/Z(G)$ is central in $G/Z(G)$ (*), and $F \lesssim Z(G)$. Thus F' is finite by Fact 7.3; we may assume it is trivial. Replacing G

by a definable subgroup of finite index, we may assume that the index $|G' : G' \cap Z(G)|$ is not greater than $|G'_0 : G'_0 \cap Z(G)|$ for any definable $G_0 \leq G$ of finite index (\dagger).

Consider $g \in G$. By $(*)$, the map $x \mapsto [g, x]Z(G)$ is a definable homomorphism from G to $G'/Z(G)$; its kernel H must have finite index. Then $x \mapsto [g, x]$ is a definable homomorphism from H to $Z(G)$ with abelian image; its kernel must hence contain $H'Z(G)$. As $H'Z(G) = G'Z(G)$ by (\dagger) , we see that $G' \leq C_G(g)$. This holds for all $g \in G$, so $G' \leq Z(G)$.

Now commutation is a definable bilinear form from $G/Z(G)$ to $Z(G)$. By Theorem 6.2 it is virtually almost trivial. But this means that G is virtually finite-by-abelian, contradicting our assumption. \square

Proposition 7.7. *An ω -categorical group of finite burden is nilpotent-by-finite.*

Proof. If the proposition does not hold, there is a non-soluble ω -categorical characteristically simple group G by Lemma 7.5; it must be a perfect p -group for some prime p by Fact 7.1 and Remark 7.2. We choose such a G of minimal possible burden k . Note that $\tilde{Z}(G)$ is trivial, as it is characteristic and finite-by-abelian (so soluble, as it is a p -group). Hence there are no finite normal subgroups, and all non-trivial conjugacy classes are infinite.

Claim 1. *The soluble radical $R(G)$ of G is trivial.*

Proof of Claim. Suppose $R(G)$ is non-trivial. Then there is some non-trivial $a \in R(G)$ such that a^G generates an infinite (definable) abelian normal subgroup A_a of G . Let $\mathfrak{A} = \{A_a : A_a \text{ abelian}\}$, a definable invariant collection of definable abelian normal subgroups.

Any finite product S of groups in \mathfrak{A} is nilpotent, whence virtually finite-by-abelian by Proposition 7.6. But then $\tilde{Z}(S)'$ is a finite characteristic subgroup of S , whence normal in G , and thus trivial. So S is virtually abelian. It follows that for $A \in \mathfrak{A}$ the almost centraliser $\tilde{C}_G(A)$ almost contains A' for all $A' \in \mathfrak{A}$. Hence

$$A \lesssim \tilde{C}_A(\tilde{C}_G(A)) \leq \tilde{C}_A(A')$$

(the first inequality follows from Fact 2.4). But $[\tilde{C}_A(A'), \tilde{C}_{A'}(A)]$ is normal; applying Proposition 5.6 to the bilinear form $(x, y) \mapsto [x, y]$ from $\tilde{C}_A(A') \times \tilde{C}_{A'}(A)$ to $A \cap A'$, we see that it is finite, whence trivial. Hence $\mathfrak{A}' = \{\tilde{C}_A(\tilde{C}_G(A)) : A \in \mathfrak{A}\}$ is an invariant family of pairwise commuting abelian groups, and generates a characteristic abelian subgroup, which must be the whole of G . This contradiction finishes the proof of the claim. \square

Suppose every centraliser of a non-trivial element is soluble. Then by compactness there is a bound on the derived length of any proper centraliser. As every finite subset of G is contained in a centralizer of a nontrivial element, this would imply that G is soluble, a contradiction. Hence there is $n \in G \setminus \{1\}$ such that $H := C_G(n)$ is non-soluble. Put $N := \langle n^G \rangle$, an infinite normal subgroup, which is definable by ω -categoricity.

Since $\tilde{C}_G(N) \cap N$ is normal and finite-by-abelian (by Fact 7.3 and ω -categoricity), whence soluble, it must be trivial.

Claim 2. $\tilde{C}_G(N) = \{1\}$.

Proof. Suppose $\tilde{C}_G(N)$ is nontrivial, whence infinite. The map $\tilde{C}_G(N) \times N \rightarrow G$ given by multiplication is a definable injection, so $\text{bdn}(G) \geq \text{bdn}(N) + \text{bdn}(\tilde{C}_G(N))$ by Remark 2.2. As $\tilde{C}_G(N)$ is infinite, $\text{bdn}(\tilde{C}_G(N)) \geq 1$, and $\text{bdn}(N) < \text{bdn}(G) = k$. By inductive hypothesis N is nilpotent-by-finite, whence solvable, a contradiction. \square

Claim 3. *Any definable normal subgroup M of G with $M \lesssim H$ is trivial.*

Proof. If $M \lesssim H = C_G(n)$, then $n \in \tilde{C}_G(M)$; by normality of M we get $n^G \subseteq \tilde{C}_G(M)$, and hence $N \leq \tilde{C}_G(M)$. Then $M \lesssim \tilde{C}_G(N) = \{1\}$ by Fact 2.4. \square

Consider a nontrivial definable normal subgroup M of G . Since $M \cap H$ is normalized by H , we have a definable injection

$$M/(M \cap H) \times H/(M \cap H) \rightarrow G/(M \cap H)$$

given by multiplication. As $M/M \cap H$ is infinite by the claim, we conclude that

$$\text{bdn}(H/M \cap H) < \text{bdn}(G/(M \cap H)) \leq \text{bdn}(G) = k.$$

By inductive hypothesis, $H/(M \cap H)$ is nilpotent-by-finite, whence soluble. If M runs through the family \mathcal{M} of 1-generated normal subgroups, the family $\{H/(M \cap H) : M \in \mathcal{M}\}$ is uniformly definable by ω -categoricity, and by compactness there is $d < \omega$ such that $H/(M \cap H)$ has derived length at most d for all $M \in \mathcal{M}$. But this means that $H^{(d)}$ is contained in M for all $M \in \mathcal{M}$, and thus is contained in all nontrivial normal subgroups.

Since H is not soluble, $H^{(d)}$ generates a non-abelian minimal normal subgroup L . But then L is finite by [2, Theorem D], a contradiction. This completes the proof. \square

Theorem 7.8. *An ω -categorical group of finite burden is virtually finite-by-abelian.*

Proof. This follows immediately from Propositions 7.6 and 7.7. \square

Corollary 7.9. *An ω -categorical NIP group of finite burden is virtually abelian.*

Proof. Let G be an ω -categorical NIP group of finite burden. By a result of Shelah, the absolute connected component G^{00} (i.e. the smallest type-definable subgroup of G of bounded index) exists (see [13, Theorem 6.1] for a proof). By ω -categoricity, G^{00} is definable and hence of finite index in G , so we may assume that G is connected. Then G is finite-by-abelian by Remark 7.4. Thus, the centralizer of any element in G has finite index in G , hence, by connectedness, is equal to G . This means that G is abelian. \square

Theorem 7.10. *An ω -categorical ring of finite burden is virtually finite-by-null.*

Proof. This is immediate from Theorem 6.2, as multiplication is a definable bilinear map. \square

As for groups, we get a corollary:

Corollary 7.11. *An ω -categorical NIP ring of finite burden is virtually null.*

Proof. Let R be such a ring. We may again assume that R is connected (in the sense of the additive group). Then R is finite-by-null by Remark 7.4. Hence, the left annihilator of any element in R has finite index in R , and must be equal to R by connectedness. This shows that R is null. \square

8. QUESTIONS AND CONCLUDING REMARKS

One can ask various questions about generalizations of the above results to more general contexts, such as strong or NTP_2 theories. For example, one can ask:

Question 8.1. Are ω -categorical strong groups

- (1) virtually nilpotent-by-finite?
- (2) virtually abelian-by-finite?

An analogue of Question 8.1(1) for rings has positive answer by Theorem 8.3 below. As to the stronger version, we do not know:

Question 8.2. Are ω -categorical strong rings null-by-finite?

The proof below is a modification of the proof of Theorem 2.1 from [16], generalizing that result from the NIP to the NTP₂ context.

Theorem 8.3. *Every ω -categorical NTP₂ ring is nilpotent-by-finite.*

Proof. As in [16], it is enough to show that a semisimple ω -categorical NTP₂ ring R is finite, and we can assume that R is a subring of $\prod_{i \in I} R_i$, where each R_i is finite, and $|\{R_i : i \in I\}| < \omega$. Let π_i be the projection onto the i -th coordinate. For $i_0, \dots, i_n \in I$ and $r_0 \in R_{i_0}, \dots, r_n \in R_{i_n}$, we define

$$R_{i_0, \dots, i_n}^{r_0, \dots, r_n} = \left\{ r \in R : \bigwedge_{j=0}^n \pi_{i_j}(r) = r_j \right\}.$$

Suppose for a contradiction that R is infinite. Again as in [16], we get the following claim:

Claim 1. *For any $N \in \omega$ there are pairwise distinct $i(0), \dots, i(N-1) \in I$ and non-nilpotent elements $r_i \in R_i$ for $i < N$ such that the sets*

$$R_{i_0, \dots, i_{N-1}}^{r_0, 0, \dots, 0}, R_{i_0, \dots, i_n}^{0, r_1, \dots, 0}, \dots, R_{i_0, \dots, i_n}^{0, 0, \dots, r_{N-1}}$$

are all non-empty.

Notice that, by ω -categoricity, the principal two-sided ideals RxR for $x \in R$ are uniformly definable. Hence, by [7, Theorem 2.4] and compactness, we obtain in particular that in order to contradict NTP₂ it is enough to find for any $n, m < \omega$ elements b_0, \dots, b_{n-1} such that

$$(*) \quad \left| \bigcap_{j \in n \setminus \{j_0\}} Rb_jR : \bigcap_{j \in n} Rb_jR \right| \geq m$$

for any $j_0 < n$ (where $n = \{0, 1, \dots, n-1\}$). So fix any $n, m < \omega$, and for $N = nm$ choose i_j and r_j as in the claim. Let $(i_{j,k})_{j < n, k < m}$ be another enumeration of $(i_j)_{j < N}$, and let $(r_{j,k})_{j < n, k < m}$ be the corresponding enumeration of $(r_j)_{j < N}$ and $(\pi_{j,k})_{j < n, k < m}$ the corresponding enumeration of $(\pi_j)_{j < N}$. For any $j_0 < n, k_0 < m$ let $s_{j_0, k_0} \in R$ be such that $\pi_{j,k}(s_{j_0, k_0}) = 0$ for $(j, k) \neq (j_0, k_0)$ and $\pi_{j_0, k_0}(s_{j_0, k_0}) = r_{j_0, k_0}$. Put $b_j = \sum_{j' \neq j, k < m} s_{j', k}$ for all $j < n$.

Claim 2. $\left| \bigcap_{j \in n \setminus \{j_0\}} Rb_jR : \bigcap_{j \in n} Rb_jR \right| \geq m$ for any $j_0 < n$.

Proof. Fix any $j_0 < n$ and put $b = b_0 b_1 \dots b_{j_0-1} b_{j_0+1} b_{j_0+2} \dots b_{n-1}$. Notice that for any $r \in \bigcap_{j \in n} Rb_jR$ and $k < m$ we have that $\pi_{j_0, k}(r) = 0$. On the other hand, for distinct $k_1, k_2 < m$ we have that

$$\pi_{j_0, k_1}(s_{j_0, k_1} b - s_{j_0, k_2} b) = \pi_{j_0, k_1}(s_{j_0, k_1} b) = \pi_{j_0, k_1}(s_{j_0, k_1}) \pi_{j_0, k_1}(b) = r_{j_0, k_1} r_{j_0, k_1}^{n-1} = r_{j_0, k_1}^n \neq 0.$$

Hence the elements

$$s_{j_0, 0} b, s_{j_0, 1} b, \dots, s_{j_0, m-1} b \in \bigcap_{j \in n \setminus \{j_0\}} Rb_jR$$

are in pairwise distinct cosets of $\bigcap_{j \in n} Rb_jR$. □

By the claim and (*) we obtain a contradiction. □

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