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ON INP-MINIMAL ω -CATEGORICAL GROUPS AND RINGS

JAN DOBROWOLSKI AND FRANK O. WAGNER

ABSTRACT. An ω -categorical inp-minimal group is virtually finite-by-abelian; an ω -categorical inp-minimal ring is virtually finite-by-null; an ω -categorical NTP₂ ring is virtually nilpotent.

1. INTRODUCTION

The NIP property has recently been one of the most intensely studied subjects in model theory (see [14]). Dp-rank is a cardinal-valued rank well defined (i.e. not assuming value ∞) on the class of NIP theories, thus providing a hierarchy inside of this class. The simplest in the sense of this hierarchy of NIP structures are *dp-minimal structures*, that is, the structures of dp-rank one. Still, many of the most natural examples of NIP structures are dp-minimal, e.g. the field of real numbers (and its (weakly) o-minimal expansions), the valued fields of p -adic numbers for any prime p , non-trivially valued algebraically closed fields, Peano arithmetic $(\mathbb{Z}, 0, +, \cdot)$, as well as any strongly minimal structure. An analogous rank for NTP₂ structures (a generalization of NIP, including all simple structures) is inp-rank. In fact, both ranks coincide as long as the dp-rank is well-defined, i.e. under the NIP hypothesis (see Definition 3.2 for the definition of inp-rank). Thus, inp-minimal structures form a class containing that of dp-minimal structures; an example of an inp-minimal structure which is not dp-minimal is the random graph.

There is a long history of study of ω -categorical groups. In the general case, the main result is the theorem of Wilson on characteristically simple ω -categorical groups (Fact 5.1). A complete classification of ω -categorical groups and rings seems to be out of reach at the moment, yet the situation is clear under some model-theoretic assumptions: ω -categorical superstable groups are virtually abelian, and ω -categorical superstable rings are virtually null [3]; ω -categorical supersimple groups are virtually finite-by-abelian [6], and extraspecial p -groups yield an example showing that the finite normal subgroup cannot be avoided. Many variants of these now-classical results have been proven. In [10], it was shown that ω -categorical NIP rings are virtually nilpotent (we generalize this to NTP₂ setting in Theorem 6.2), and it was conjectured that ω -categorical groups are virtually nilpotent as well. The latter conjecture remains open, yet a special case of it was obtained in [8]: dp-minimal ω -categorical groups are virtually nilpotent. It was asked in [8, Problem 2.18] whether they are actually virtually abelian. The main result of this paper is Theorem 5.6, which in particular solves this problem in affirmative (see Corollary 5.7), and also solves [8, Problem 2.18]. To get the result, we prove that inp-minimal bilinear forms are virtually almost trivial, and, as a by-product, we obtain analogous results for rings (Theorem 5.8, Corollary 5.9). Throughout the paper, we do not refer to

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the definition of inp-rank — we only use the fact saying that in an inp-minimal group, definable subgroups are comparable up to finite index (Fact 3.3).

The paper is organized as follows: In Section 2, we recall the basic concepts and facts about measures and quasi-endomorphisms. In Section 3, we study properties of quasi-endomorphisms under assumptions of ω -categoricity and inp-minimality. In Section 4, we prove Theorem 4.8 about bilinear forms (using results of Section 3). In Section 5, we apply Theorem 4.8 to obtain the main results about groups and rings. In Section 6, we state some questions and we prove that ω -categorical rings with NTP_2 are virtually nilpotent.

2. PRELIMINARIES

2.1. Measure. Let \mathfrak{M} be a structure such that algebraic closure is locally finite (i.e. $\text{acl}(A)$ is finite for all finite A , in every sort S). We can then define a measure μ as follows: Let I be the collection of algebraically closed subsets of \mathfrak{M}^{eq} , and for $A \in I$ put $I_A = \{B \in I : A \subseteq B\}$. If \mathcal{U} is any ultrafilter on I extending the filter generated by the set $\{I_A : A \in \mathfrak{A}\}$ and X is a subset of \mathfrak{M}^{eq} , put

$$\mu(X) = \prod_{A \in I} |X \cap A| / \mathcal{U}.$$

Then in every sort μ is a finitely additive measure on the boolean algebra of sets of sort S and taking values in the non-standard natural numbers $\prod_I \mathbb{N} / \mathcal{U}$. Note that μ is in general *not* automorphism-invariant.

Lemma 2.1 (Multiplicativity). *If $f : X \rightarrow Y$ is a definable function whose fibres all have finite size n , then*

$$\mu(X) = n \mu(Y).$$

In particular, μ is invariant under definable bijections. More generally, if $F \subseteq X \times Y$ is a definable (k, ℓ) -correspondence, meaning that $|\{x \in X : (x, y) \in F\}| = k$ for all $y \in Y$ and $|\{y \in Y : (x, y) \in F\}| = \ell$ for all $x \in X$, then $\ell \mu(X) = k \mu(Y)$.

Proof. We choose $A \in I$ such that F, X, Y are defined over A . If $B \in I_A$, by algebraic closedness $|\{x \in X \cap B : (x, y) \in F\}| = k$ for all $y \in Y \cap B$ and $|\{y \in Y \cap B : (x, y) \in F\}| = \ell$ for all $x \in X \cap B$. The result follows, as $I_A \in \mathcal{U}$. \square

If X is a non-empty definable set in \mathfrak{M}^{eq} , we can relativize μ to X by putting

$$\mu_X(Y) = \text{st} \left(\frac{\mu(Y)}{\mu(X)} \right) = \text{st} \left(\prod_{A \in I} \frac{|Y \cap A|}{|X \cap A|} / \mathcal{U} \right).$$

(The first quotient is calculated in the non-standard rationals $\prod_I \mathbb{Q} / \mathcal{U}$.) Then μ_X is a finitely additive probability measure on X . We call a partial type π extending X *measure-generic in X* if $\mu_X(Y) > 0$ for all $Y \in \pi$. Note that by finite additivity every measure-generic partial type in X can be extended to a measure-generic complete type in X . We shall call a tuple y *measure-generic in X over A* if $\text{tp}(y/A)$ is such.

Lemma 2.2. *If G is an A -definable group in \mathfrak{M} and H an A -definable subgroup, the following are equivalent:*

- (1) H has finite index in G .
- (2) $\mu_G(H) > 0$.
- (3) There is a complete type p over A extending $x \in H$ which is measure-generic in G over A .

In this case, $|G : H| = 1/\mu_G(H) = \mu(G)/\mu(H)$.

Proof. If $|G : H| = n$, then $\mu(G) = n\mu(H)$, so $\mu_G(H) = 1/n > 0$. Conversely, if $|G : H| > n$ for all n , then $\mu_G(H) < 1/n$ for all n , so $\mu_G(H) = 0$.

If $\mu_G(H) > 0$, then H is measure-generic in G and can be completed to a type p over A measure-generic in G . Conversely, if $p \in S(A)$ extends H and is measure-generic in G over A , then clearly $\mu_G(H) > 0$. \square

2.2. Multiplicative relations and quasi-endomorphisms. Here, we slightly modify the construction of definable quasi-endomorphisms ring from [3] in order to be able to apply it to non-connected definable groups.

Definition 2.3. Let G and H be groups. A *multiplicative relation* between G and H is a subgroup $R \leq G \times G$. We call $\pi_1(R)$, the projection to the first coordinate, the *domain* $\text{dom } R$ and $\pi_2(R)$ the image $\text{im } R$ of R ; the subgroup $R1 = \{g \in G : (g, 1) \in R\}$ is the *kernel* $\ker R$, and $1R = \{h \in H : (1, h) \in R\}$ is the *cokernel* $\text{coker } R$. If $\text{dom } R$ has finite index in G and $\text{coker } R$ is finite, a multiplicative relation R is a *quasi-homomorphism* from G to H . A quasi-homomorphism R induces a homomorphism $\text{dom } R \rightarrow \text{im } R/\text{coker } R$. If $G = H$ we call R a *quasi-endomorphism*. Particular multiplicative relations are $\text{id}_g = \{(g, g) : g \in G\}$ and $1_G = G \times \{1\}$.

Definition 2.4. • If $R \leq G \times H$ is a multiplicative relation, $g \in G$ and $K \leq G$, put $R(g) = \{h \in H : (g, h) \in R\}$ and $R[K] = \bigcup_{g \in K} R(g)$.

- If $R, R' \leq G \times H$ are multiplicative relations, put

$$R \cdot R' = \{(a, bb') \in G \times H : (a, b) \in R, (a, b') \in R'\}.$$

If $\text{im } R$ and $\text{im } R'$ commute, this is again a multiplicative relation. If moreover R and R' are quasi-homomorphisms from G to H , so is $R \cdot R'$.

- We call $R, R' \leq G \times H$ *equivalent*, denoted $R \equiv R'$, if there is a subgroup G_1 of finite index in G and a finite group $F \leq H$ normalized by $R(G_1)$ and $R'(G_1)$, such that

$$(R \cap (G_1 \times H)) \cdot (\{1_G\} \times F) = (R' \cap (G_1 \times H)) \cdot (\{1_G\} \times F).$$

This is clearly an equivalence relation.

- If $R \leq G \times H$ and $R' \leq H \times K$ are multiplicative relations, put

$$R' \circ R = \{(a, c) \in G \times K : \exists b [(a, b) \in R \text{ and } (b, c) \in R']\},$$

a multiplicative relation between G and K . If R and R' are quasi-homomorphisms, so is $R' \circ R$. We denote the n -fold composition of R with itself by R^{on} .

- For a multiplicative relation $R \leq G \times H$ put

$$R^{-1} = \{(h, g) \in H \times G : (g, h) \in R\},$$

a multiplicative relation between H and G .

Remark 2.5. Note that

$$R^{-1} \circ R = \text{id}_{\text{dom } R} \cdot (\{1_G\} \times \ker R) \quad \text{and} \quad R \circ R^{-1} = \text{id}_{\text{im } R} \cdot (\{1_H\} \times \text{coker } R).$$

If $\text{im } R$ has finite index in H and $\ker R$ is finite, then R^{-1} is a quasi-homomorphism from H to G . If moreover R is a quasi-homomorphism, then $R \circ R^{-1} \equiv \text{id}_H$ and $R^{-1} \circ R \equiv \text{id}_G$.

If the groups are abelian and noted additively, we write $R + R'$ instead of $R \cdot R'$, and call them *additive* relations.

Lemma 2.6. *Let G be an abelian group. The sum, difference and product of definable quasi-endomorphisms of G is again a definable quasi-endomorphism. The set of definable quasi-endomorphisms of G modulo equivalence forms an associative ring.*

Proof. As in [3], taking subgroups of finite index where needed. \square

3. QUASI-HOMOMORPHISMS OF ω -CATEGORICAL GROUPS AND INP-MINIMALITY

Recall that a complete first order theory in a countable language is said to be ω -categorical if it has only one countable model up to isomorphisms, and a structure M is ω -categorical if $Th(M)$ is. By the Ryll-Nardzewski Theorem, this is equivalent to the following statement: for every $n < \omega$ there are only finitely many complete n -types over \emptyset . Hence, for any finite set A in an ω -categorical structure M there are only finitely many definable sets over A , and ω -categorical algebraic structures are uniformly locally finite.

Lemma 3.1. *Let G and H be definable groups in an ω -categorical structure, and $f, g \leq G \times H$ definable multiplicative relations such that $\ker f$ and $\operatorname{coker} g$ are finite, and $\operatorname{im} g$ has finite index in H . Then $\ker g$ and $\operatorname{coker} f$ are finite, and $\operatorname{im} f$ has finite index in H .*

Proof. Let A be a finite set over which all the above objects are definable.

Claim. *Suppose that $H_1 < H_2 \leq H$ are such that H_1 has infinite index in H_2 . Then $f[g^{-1}[H_1]]$ has infinite index in $f[g^{-1}[H_2]]$.*

Proof. As $\operatorname{im} g$ has finite index in H , the index of $H_1 \cap \operatorname{im} g$ in $H_2 \cap \operatorname{im} g$ is infinite. As $\operatorname{coker} g$ is finite, $g^{-1}[H_1]$ has infinite index in $g^{-1}[H_2]$. Hence $f[g^{-1}[H_1]]$ is a subgroup of infinite index in $f[g^{-1}[H_2]]$, since $\ker f$ is finite. \square

Suppose for a contradiction that $\ker(g)$ or $\operatorname{coker} f$ is infinite. Put $K_0 = \{e\} \leq H$ and define inductively $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 is infinite; by the claim K_n is a subgroup of infinite index in K_{n+1} for all $n < \omega$, contradicting ω -categoricity.

Now suppose that $\operatorname{im} f$ has infinite index in H . Put $K_0 = H$ and define as before $K_{n+1} = f[g^{-1}[K_n]]$. Then K_1 has infinite index in K_0 ; by the claim K_{n+1} is a subgroup of infinite index in K_n for all $n < \omega$, again contradicting ω -categoricity. \square

Although we will not refer to the definition of inp -rank in this paper, we write it here for completeness.

Definition 3.2. (1) An *inp-pattern* of depth κ in the partial type $\pi(\bar{x})$ is a sequence $\langle \varphi_i(\bar{x}; \bar{y}_i) : i < \kappa \rangle$ of formulas and an array $\{\bar{a}_{ij} : i < \kappa, j < \omega\}$ of parameters (from some model of T) such that:

- (a) For each $i < \kappa$, there is some $k_i < \omega$ such that $\{\varphi_i(\bar{x}; \bar{a}_{i,j}) : j < \omega\}$ is k_i -inconsistent; and
- (b) For each $\eta : \kappa \rightarrow \omega$, the partial type

$$\pi(\bar{x}) \cup \{\varphi_i(\bar{x}; \bar{a}_{i,\eta(i)}) : i < \kappa\}$$

is consistent.

- (2) The *inp-rank* (or *burden*) of a partial type $\pi(\bar{x})$ is the maximal κ such that there is an inp -pattern of depth κ in $\pi(\bar{x})$, if such a maximum exists. In case there are inp -patterns of depth λ in $\pi(\bar{x})$ for every cardinal $\lambda < \kappa$ but no inp -pattern of depth κ , we say that the inp -rank of $\pi(\bar{x})$ is κ_- .
- (3) The inp -rank of T is the inp -rank of $x = x$, and T is *inp-minimal* if its inp -rank is 1.

For the next lemma we introduce some notation for subgroups H and K of a group G . We say that H is *almost contained* in K , denoted $H \lesssim K$, if $H \cap K$ has finite index in H . If $H \lesssim K$ and $K \lesssim H$, the two groups are *commensurable*, denoted $H \simeq K$.

We shall only use the following consequence of inp-minimality, which was essentially observed in the proof of [13, Proposition 3.1]:

Fact 3.3. *If G is an inp-minimal group and $H, K \leq G$ are definable, then $H \lesssim K$ or $K \lesssim H$.*

In other words, \lesssim is a total pre-order on the set of definable subgroups of an inp-minimal group.

Lemma 3.4. *Let G and H be inp-minimal groups definable in an ω -categorical structure, and $f, g \leq G \times H$ definable quasi-homomorphisms. Then:*

- (1) $\ker f \lesssim \ker g$ if and only if $\operatorname{im} g \lesssim \operatorname{im} f$.
- (2) If K is a definable subgroup of G and $\ker f \lesssim \ker g$ then $g[K] \lesssim f[K]$.

Proof. (1) Replacing G by a subgroup of finite index, we may suppose $\operatorname{dom} f = \operatorname{dom} g = G$. Suppose $\ker f \lesssim \ker g$. Put $H_1 = \operatorname{im} f$ and

$$G_1 = g^{-1}[\operatorname{im} f \cap \operatorname{im} g](\ker f \cap \ker g)/(\ker f \cap \ker g).$$

and let $f_1, g_1 \leq G/(\ker f \cap \ker g) \times H_1$ be the multiplicative relations induced by f and g . Then $\ker f_1$ is finite since $\ker f \lesssim \ker g$.

Suppose that $\operatorname{im} f \lesssim \operatorname{im} g$. Then $g_1[G_1] = \operatorname{im} f \cap \operatorname{im} g$ has finite index in $H_1 = \operatorname{im} f$. Thus $f_1[G_1]$ has finite index in H_1 by Lemma 3.1 applied to $f_1|_{G_1}, g_1|_{G_1} \leq G_1 \times H_1$. As $\ker f_1$ is finite, G_1 has finite index in $G/(\ker f \cap \ker g)$, and $g_1[G_1] = \operatorname{im} f \cap \operatorname{im} g$ has finite index in $g_1[G/(\ker f \cap \ker g)] = \operatorname{im} g$. Thus $\operatorname{im} g \lesssim \operatorname{im} f$ as well. On the other hand, if $\operatorname{im} f \not\lesssim \operatorname{im} g$, then $\operatorname{im} g \lesssim \operatorname{im} f$ by inp-minimality. This shows that $\ker f \lesssim \ker g$ implies $\operatorname{im} g \lesssim \operatorname{im} f$.

For the converse, suppose $\operatorname{im} f \lesssim \operatorname{im} g$. If $\ker f \lesssim \ker g$, then we can apply Lemma 3.1 as above and obtain that $\ker g_1|_{G_1}$ is finite, i.e. $\ker f \cap \ker g$ has finite index in $\ker g$, whence $\ker g \lesssim \ker f$. Otherwise, $\ker g \lesssim \ker f$ by inp-minimality, and we are done in either case.

- (2) If $\ker f \lesssim \ker g$, then $\ker f|_K \lesssim \ker g|_K$. Apply part (1) to $f|_K, g|_K \leq K \times H$. \square

Lemma 3.5. *Let G and H be inp-minimal groups definable in an ω -categorical theory. Suppose $f, g \leq G \times H$ are definable quasi-homomorphisms such that $\ker(f) \not\lesssim \ker(g)$. Define inductively $D_0 = G$ and $D_{n+1} = g^{-1}[f[D_n]]$. Then there is some n such that $D_n \simeq \ker(g)$.*

Proof. The D_n form a descending chain of subgroups, all definable over the same finite set of parameters. By ω -categoricity there is some n such that $D_n = D_{n+1}$. Thus

$$g[D_n] = g[D_{n+1}] = g[g^{-1}[f[D_n]]] \lesssim f[D_n],$$

whence $\ker f|_{D_n} \lesssim \ker g|_{D_n}$ by Lemma 3.1. Clearly $\ker g \leq D_n$, whence $D_n \cap \ker f \lesssim \ker g$. Since $\ker f \lesssim D_n$ or $D_n \lesssim \ker f$ by inp-minimality, we obtain $\ker f \lesssim \ker g$ or $D_n \lesssim \ker g$. As the first option is excluded by hypothesis, we must have $D_n \simeq \ker g$. \square

Recall Schlichting's Theorem [12, 4].

Fact 3.6. *Let \mathfrak{H} be a family of uniformly commensurable subgroups of a group G , i.e. the index $|H : H \cap H^*|$ is finite and bounded independently of $H, H^* \in \mathfrak{H}$. Then there*

is a subgroup N of G commensurable with any $H \in \mathfrak{H}$, which is invariant under any automorphism of G stabilizing \mathfrak{H} setwise. Moreover, N is a finite extension of a finite intersection of groups in \mathfrak{H} ; if the latter are definable, so is N .

Lemma 3.7. *Let G be an inp-minimal torsion group, and $H \leq G$ a definable subgroup. Then H is uniformly commensurable with its G -conjugates, and there is a normal definable subgroup $N \leq G$ commensurable with H .*

Proof. For any $g \in G$ we have $H \lesssim H^g$ or $H^g \lesssim H$ by inp-minimality. In the first case we obtain $H^{g^n} \lesssim H^{g^{n+1}}$ for all $n < \omega$, whence $H \lesssim H^g \lesssim H^{g^n}$. Taking $n = o(g)$ the order of g , we get $H \simeq H^g$. Or, $\bigcap_{i < n} H^{g^i}$ has finite index in H and is g -invariant. It follows that $|H : H \cap H^g| = |H^g : H \cap H^g|$. If this index were not bounded, by compactness we could find g^* in some elementary extension with $H \not\lesssim H^{g^*}$ and $H^{g^*} \not\lesssim H$, contradicting inp-minimality. So H is uniformly commensurable with all its G -conjugates. The rest follows from Schlichting's Theorem. \square

4. BILINEAR FORMS

We will now introduce some notation related to bilinear forms. Suppose G and K are abelian groups, and $\lambda : G \times G \rightarrow K$ is a bilinear form. For $g \in G$ the *left annihilator* of g is the subgroup $\text{ann}_L(g) = \{g' \in G : \lambda(g', g) = 0\}$; the *right annihilator* is the subgroup $\text{ann}_R(g) = \{g' \in G : \lambda(g, g') = 0\}$.

If $\lambda : G \times G \rightarrow K$ is a bilinear form and $g \in G$, then λ induces a homomorphism $\lambda_g : G \rightarrow K$ given by $x \mapsto \lambda(g, x)$. Clearly $\ker \lambda_g = \text{ann}_R(g)$ and $\text{im} \lambda_g = \lambda(g, G)$. For $g' \in G$, we also consider the multiplicative relation $\lambda_{g, g'} = \lambda_{g'}^{-1} \circ \lambda_g \leq G \times G$ given by $\{(h, h') \in G \times G : \lambda(g, h) = \lambda(g', h')\}$. Clearly $\ker \lambda_{g, g'} = \text{ann}_R(g)$ and $\text{coker} \lambda_{g, g'} = \text{ann}_R(g')$. If $\text{ann}_R(g') \lesssim \text{ann}_R(g)$ and $\lambda(g, G) \lesssim \lambda(g', G)$ (which are equivalent conditions in the inp-minimal case by Lemma 3.4 applied to λ_g and $\lambda_{g'}$), then $\lambda_{g, g'}$ induces a quasi-endomorphism of $G/\text{ann}_R(g')$.

Definition 4.1. A bilinear form λ is *almost trivial* if there is a finite subgroup of K containing $\text{im} \lambda$. It is *virtually almost trivial* if there is a subgroup G_1 of finite index in G such that the restriction of λ to G_1 is almost trivial.

Proposition 4.2. *Let G and K be abelian groups and $\lambda : G \times G \rightarrow K$ a bilinear form. Then λ is almost trivial iff $\text{ann}_L(g)$ and $\text{ann}_R(g)$ have uniformly finite index in G for all $g \in G$.*

Proof. Clearly left implies right. So suppose $\text{ann}_L(g)$ and $\text{ann}_R(g)$ have uniformly finite index in G for all $g \in G$. Note that this implies that $\langle \text{im} \lambda \rangle$ has finite exponent. So it is enough to show that $\text{im} \lambda$ is finite. Note that $\lambda(g, G)$ is uniformly finite for all $g \in G$.

Consider $g \in G$ with $\lambda(g, G)$ maximal, and choose $g_0, \dots, g_n \in G$ with $\lambda(g, G) = \{\lambda(g, g_i) : i \leq n\}$. Then for $g' \in g + \bigcap_{i \leq n} \text{ann}_L(g_i)$ we have $\lambda(g', g_i) = \lambda(g, g_i)$, whence $\lambda(g', G) \supseteq \lambda(g, G)$, and $\lambda(g', G) = \lambda(g, G)$ by maximality. Note that $\bigcap_{i \leq n} \text{ann}_L(g_i)$ is a subgroup of boundedly finite index in G . It follows that there can only be finitely many maximal sets of the form $\lambda(g, G)$ for $g \in G$, and $\text{im} \lambda$ is finite. \square

Lemma 4.3. *In an ω -categorical theory let G and K be inp-minimal definable abelian groups, and $\lambda : G \times G \rightarrow K$ a definable bilinear map. If $(x_i : i \in \mathbb{Q})$ is 3-indiscernible in G and $\text{ann}_R(x_1) \lesssim \text{ann}_R(x_0)$, then $\lambda_{x_0, x_1}[G] \lesssim \text{ann}_R(x_i)$ for any $i < 1$.*

Proof. Note that the hypotheses and Lemma 3.4 imply $\lambda_{x_i}[H] \lesssim \lambda_{x_j}[H]$ for all definable subgroups $H \leq G$ and $i < j$, whence in particular $\text{im}\lambda_{x_i, x_k} \lesssim \text{im}\lambda_{x_j, x_k}$ for $i < j < k$. Moreover, $\lambda_{x_i}^{-1}$ and $\lambda_{x_j}^{-1}$ both induce quasi-homomorphisms from $\text{im}\lambda_{x_i}$ to $G/\text{ann}_R(x_i)$. The kernel of the former is finite, so $\lambda_{x_j}^{-1}[K'] \lesssim \lambda_{x_i}^{-1}[K']$ for all definable $K' \lesssim \text{im}\lambda_{x_i}$ by Lemma 3.4. Taking $K' = \lambda_{x_0}[H]$ for $H \leq G$ definable, we obtain $\lambda_{x_0, x_j}[H] \lesssim \lambda_{x_0, x_i}[H]$. Finally, $\lambda_g \circ \lambda_g^{-1} = \text{id}_{\text{im}\lambda_g}$ by Remark 2.5.

For $0 < i < j$ we have $\text{im}\lambda_{x_0} \lesssim \text{im}\lambda_{x_i}$, whence

$$\begin{aligned} \lambda_{x_0, x_j} &= \lambda_{x_j}^{-1} \circ \lambda_{x_0} = \lambda_{x_j}^{-1} \circ \text{id}_{\text{im}\lambda_{x_0}} \circ \lambda_{x_0} \\ &\equiv \lambda_{x_j}^{-1} \circ \text{id}_{\text{im}\lambda_{x_i}} \circ \lambda_{x_0} = \lambda_{x_j}^{-1} \circ \lambda_{x_i} \circ \lambda_{x_i}^{-1} \circ \lambda_{x_0} = \lambda_{x_i, x_j} \circ \lambda_{x_0, x_i}. \end{aligned}$$

Claim. *If $\text{im}\lambda_{x_0, x_1} \lesssim \text{im}\lambda_{x_1, x_2}$ then $\text{im}\lambda_{x_0, x_j} \lesssim \text{im}\lambda_{x_i, x_j}^{\circ n}$ for all $0 < i < j$ and $1 \leq n < \omega$.*

Proof. We proceed by induction on n . For $n = 1$ the statement follows from indiscernibility. Assume it holds for some n . Choose $0 < k < i$. Then

$$\begin{aligned} \text{im}\lambda_{x_0, x_j} &\simeq \text{im}(\lambda_{x_k, x_j} \circ \lambda_{x_0, x_k}) = \lambda_{x_k, x_j}[\text{im}\lambda_{x_0, x_k}] \lesssim \lambda_{x_k, x_j}[\text{im}\lambda_{x_k, x_j}] \\ &\lesssim \lambda_{x_k, x_j}[\text{im}\lambda_{x_i, x_j}^{\circ n}] \lesssim \lambda_{x_i, x_j}[\text{im}\lambda_{x_i, x_j}^{\circ n}] = \text{im}(\lambda_{x_i, x_j} \circ \lambda_{x_i, x_j}^{\circ n}) = \text{im}\lambda_{x_i, x_j}^{\circ(n+1)}. \quad \square \end{aligned}$$

Claim. *If $\text{im}\lambda_{x_1, x_2} \lesssim \text{im}\lambda_{x_0, x_1}$ then $\text{im}\lambda_{x_0, x_j} \lesssim \text{im}\lambda_{x_0, x_i}^{\circ n}$ for all $0 < i < j$ and $1 \leq n < \omega$.*

Proof. The case $n = 1$ is immediate, so assume the statement holds for some n . Take $i < k < j$, and consider λ_{x_i, x_j} and λ_{x_0, x_i} as quasi-homomorphisms from G to $G/\text{ann}_R(x_i)$. Then $\lambda_{x_i, x_j}[H] \lesssim \lambda_{x_0, x_i}[H]$ for all definable subgroups $H \leq G$ by Lemma 3.4. Thus

$$\begin{aligned} \text{im}\lambda_{x_0, x_j} &\simeq \text{im}(\lambda_{x_k, x_j} \circ \lambda_{x_0, x_k}) = \lambda_{x_k, x_j}[\text{im}\lambda_{x_0, x_k}] \lesssim \lambda_{x_k, x_j}[\text{im}\lambda_{x_0, x_i}^{\circ n}] \\ &\lesssim \lambda_{x_0, x_k}[\text{im}\lambda_{x_0, x_i}^{\circ n}] \lesssim \lambda_{x_0, x_i}[\text{im}\lambda_{x_0, x_i}^{\circ n}] \lesssim \text{im}(\lambda_{x_0, x_i} \circ \lambda_{x_0, x_i}^{\circ n}) = \text{im}\lambda_{x_0, x_i}^{\circ(n+1)}. \quad \square \end{aligned}$$

By Lemma 3.5 there is n such that $\text{im}\lambda_{x_i, x_j}^{\circ n} \lesssim \ker \lambda_{x_j}$ for any $i < j$. Since $\ker \lambda_{x_j} \lesssim \ker \lambda_{x_i}$ for $i < j$, the above claims yields $\text{im}\lambda_{x_0, x_1} \lesssim \ker \lambda_{x_i}$ for all $i < 1$. \square

Definition 4.4. For $g \in G$ let $\text{ann}_L(g)$ be the *left* annihilator of g , and for $H \leq G$ definable put

$$\begin{aligned} \widetilde{\text{ann}}_R(H) &= \{g \in G : H \lesssim \text{ann}_L(g)\} = \{g \in G : \lambda(H, g) \text{ is finite}\} \quad \text{and} \\ \widetilde{\text{ann}}_L(H) &= \{g \in G : H \lesssim \text{ann}_R(g)\} = \{g \in G : \lambda(g, H) \text{ is finite}\}, \end{aligned}$$

the *almost* (right or left) annihilator of H .

The almost annihilators are subgroups of G ; they are given as a countable increasing union of definable sets (over the same parameters as G and H). Thus they are definable in an ω -categorical theory. Note that $\widetilde{\text{ann}}_L(G)$ and $\widetilde{\text{ann}}_R(G)$ are ideals.

The next proposition is an adaptation of [7, Theorem 2.10] to bilinear forms.

Proposition 4.5. *Let H_1 and H_2 be definable subgroups of G . Then $H_1 \lesssim \widetilde{\text{ann}}_L(H_2)$ if and only if $H_2 \lesssim \widetilde{\text{ann}}_R(H_1)$.*

Proof. We may assume that G , H_1 and H_2 are defined over \emptyset . Suppose that $H_2 \not\lesssim \widetilde{\text{ann}}_R(H_1)$. Consider a sequence $(g_i : i < \omega)$ in H_2 representing different cosets of $\widetilde{\text{ann}}_R(H_1)$. So $g_i - g_j \notin \widetilde{\text{ann}}_R(H_1)$ for $i \neq j$, and the index $|H_1 : H_1 \cap \text{ann}_L(g_i - g_j)|$ is infinite. By Neumann's Lemma no finite union of cosets of the various $H_1 \cap \text{ann}_L(g_i - g_j)$ can cover H_1 . By compactness there is an infinite sequence $(h_k : k < \omega)$ such that $\lambda(h_k - h_\ell, g_i - g_j) \neq 0$ for all $i \neq j$ and $k \neq \ell$. It follows that $|H_2 : H_2 \cap \text{ann}_R(h_k - h_\ell)|$ is infinite and $h_k - h_\ell \notin \widetilde{\text{ann}}_L(H_2)$ for all $k \neq \ell$, whence $H_1 \not\lesssim \widetilde{\text{ann}}_L(H_2)$.

The other direction follows by symmetry. \square

Proposition 4.6. *Let G and K be inp-minimal abelian groups definable in some ω -categorical theory, and let $\lambda : G \times G \rightarrow K$ be a definable quasi-symmetric bilinear map. Let $(x_i)_{i \in \omega}$ be a 10-indiscernible measure generic sequence (over \emptyset). If $\text{ann}_L(x_i) \lesssim \text{ann}_R(x_i)$, then $\text{ann}_R(x_0) \simeq \text{ann}_R(x_1)$.*

Proof. Put $z_i = x_{2i} - x_{2i+1}$. So $(z_i : i < \omega)$ is 6-indiscernible.

Claim. $\text{ann}_R(z_1) \lesssim \text{ann}_R(z_0)$ and $\text{ann}_L(z_1) \lesssim \text{ann}_L(z_0)$.

Proof. Suppose $\text{ann}_R(z_0) \lesssim \text{ann}_R(z_1)$. Then $z_1 \in \widetilde{\text{ann}}_L(\text{ann}_R(z_0))$; as z_1 is measure-generic over z_0 we have $G \lesssim \widetilde{\text{ann}}_L(\text{ann}_R(z_0))$. By Proposition 4.5, $\text{ann}_R(z_0) \lesssim \widetilde{\text{ann}}_R(G)$.

By ω -categoricity there is a bound n on the index of $\text{ann}_L(g)$ in G for $g \in \widetilde{\text{ann}}_R(G)$. Choose $g \in \widetilde{\text{ann}}_R(G)$ measure-generic over x_0, \dots, x_n . Then $x_i - x_j \in \text{ann}_L(g)$ for some $0 \leq i < j \leq n$, whence $g \in \text{ann}_R(x_i - x_j)$. By measure-genericity, $\widetilde{\text{ann}}_R(G) \lesssim \text{ann}_R(x_i - x_j)$; by indiscernibility we obtain $\widetilde{\text{ann}}_R(G) \lesssim \text{ann}_R(x_0 - x_1) = \text{ann}_R(z_0)$.

It follows that $\text{ann}_R(z_i) \simeq \widetilde{\text{ann}}_R(G)$ for all $i \in \omega$, and $\text{ann}_R(z_1) \lesssim \text{ann}_R(z_0)$. The result follows by inp-minimality and symmetry (note that the assumption $\text{ann}_L(x_i) \lesssim \text{ann}_R(x_i)$ has not been used so far). \square

Now let $(y_i)_{i \in \mathbb{Q}}$ be an indiscernible sequence based on $(z_i)_{i \in \omega}$.

Claim. *Whenever $H \leq G$ is definable over a 3-element subset A of $(y_j : j \neq i)$ and $y \in H$, then $y \in H_A^0$.*

Proof. It is enough to show that the same is true about the sequence $(z_i)_{i \in \omega}$. A group H definable over a 3-element subset A of $(z_j : j \neq i)$ is definable over a 6-element subset B of the set $\{x_j : j \neq 2i, 2i+1\}$ with $A \subseteq \text{dcl}(B)$. We can extend $B \cup \{x_{2i}, x_{2i+1}\}$ to a 10-indiscernible sequence $(x'_j)_{j \in \mathbb{Q}}$ with $x_j = x'_j$ for $x_j \in B \cup \{x_{2i}, x_{2i+1}\}$. Since $z_i \in H$, by the pigeonhole principle and 10-indiscernibility, $x'_j - x'_k \in H$ for all $2i \leq j < k \leq 2i+1$. By Ramsey's Theorem, we can assume that for all $2i < j < k < 2i+1$ the element $x_j - x_k$ is in the same coset of H_A^0 , say $g + H_A^0$. If m is the order of g , then for $2i < j_0 < \dots < j_m < 2i+1$ we get that

$$x_{j_0} - x_{j_m} = (x_{j_0} - x_{j_1}) + \dots + (x_{j_{m-1}} - x_{j_m}) \in mg + H_A^0 = H_A^0.$$

By indiscernibility again, $z_i = x_{2i} - x_{2i+1} \in H_A^0$. \square

Claim. *If $\text{ann}_R(g) \not\lesssim \text{ann}_R(h)$, then $\text{ann}_R(g \pm h) \simeq \text{ann}_R(h)$.*

Proof. By inp-minimality we have $\text{ann}_R(h) \lesssim \text{ann}_R(g)$, whence

$$\text{ann}_R(h) \lesssim \text{ann}_R(h) \cap \text{ann}_R(g) \leq \text{ann}_R(g \pm h).$$

If $\text{ann}_R(g \pm h) \not\lesssim \text{ann}_R(h)$, then by inp-minimality

$$\text{ann}_R(h) \geq \text{ann}_R(g \pm h) \cap \text{ann}_R(g) \not\lesssim \text{ann}_R(h),$$

a contradiction. The result follows. \square

Note that if $\text{ann}_R(x_1) \not\lesssim \text{ann}_R(x_0)$, then $\text{ann}_R(z_i) \simeq \text{ann}_R(x_{2i})$ for all i , contradicting the first claim. For a contradiction, suppose that $\text{ann}_R(x_0) \not\lesssim \text{ann}_R(x_1)$. Then $\text{ann}_R(z_i) \simeq \text{ann}_R(x_{2i+1})$ for all i , and

$$(*) \quad \text{ann}_R(y_i) \not\lesssim \text{ann}_R(y_j) \quad \text{for all } i < j.$$

Fix $i < j < k < 1$. By Lemma 4.3, we get:

$$\text{im} \lambda_{y_i} \lesssim \lambda_{y_1}[\text{ann}_R(y_j)] \quad \text{and} \quad \text{im} \lambda_{y_i} \lesssim \lambda_{y_k}[\text{ann}_R(y_j)].$$

Moreover, $\lambda_{y_k}[\text{ann}_R(y_j)] \lesssim \lambda_{y_1}[\text{ann}_R(y_j)]$ by Lemma 3.4. Now

$$\lambda(y_i, y_1) \in (\text{im } \lambda_{y_i})_{y_i, y_j, y_k}^0 \leq \lambda_{y_k}[\text{ann}_R(y_j)].$$

Hence, putting $\lambda'_g(x) = \lambda(x, g)$, we obtain

$$y_i \in \lambda_{y_1}^{\prime -1}[\lambda_{y_k}[\text{ann}_R(y_j)]_{y_j, y_k, y_1}^0] \leq \lambda_{y_1}^{\prime -1}[\lambda_{y_1}[\text{ann}_R(y_j)]].$$

Thus, by Lemma 3.4 we get that

$$y_i \in (\text{ann}_R(y_j) + \text{ann}_L(y_1))_{y_j, y_1}^0 \leq (\text{ann}_R(y_j) + \text{ann}_R(y_1))_{y_j, y_1}^0 \leq \text{ann}_R(y_j).$$

Thus, $\lambda(y_i, y_j) = 0$. As $(y_i)_{i \in \mathbb{Q}}$ is based on $(z_i)_{i < \omega}$, we also have that $\lambda(z_0, z_1) = 0$. As $(z_i)_{i < \omega}$ is a measure generic sequence, we get that $\text{ann}_R(z_0)$ has finite index in G , so in particular $\text{ann}_R(z_i) \simeq G$ and hence $\text{ann}_R(y_i) \simeq G$ for all i , contradicting (*). \square

Lascar equivalence is, by definition, the finest bounded \emptyset -invariant relation on the monster model. Equivalently, it is the transitive closure of the relation Θ , where $\Theta(a, b)$ holds iff a and b are the first two elements of an indiscernible sequence. We say that a and b are at Lascar distance 1 if $\Theta(a, b)$ holds. For more details on Lascar equivalence see for example [16].

Corollary 4.7. *Let G and K be inp-minimal abelian groups definable in some ω -categorical theory, and let $\lambda : G \times G \rightarrow K$ be a definable bilinear map. If $x_0, x_1 \in G$ are Lascar-equivalent measure generics and $\text{ann}_L(x_0) \lesssim \text{ann}_R(x_0)$, then $\text{ann}_R(x_0) \simeq \text{ann}_R(x_1)$.*

Proof. We may assume that x_0 and x_1 are at Lascar distance 1, i.e. there is an indiscernible sequence $(x_i)_{i < \omega}$. Let $(y_i)_{i < \omega}$ be a 10-indiscernible measure generic sequence with $\text{tp}(y_0) = \text{tp}(x_0)$. By moving x_0 and x_1 we may assume that $x_0 = y_0$. As $(y_i)_{i < \omega}$ is measure generic and a formula of positive measure cannot divide over any set, we get that $\text{tp}(y_i : i > 0/x_0)$ does not divide over \emptyset . As $(x_i : i \in \omega)$ is indiscernible, there is a sequence $(y'_i)_{i > 0}$ such that

$$(x_0, y'_i : i > 0) \equiv (x_1, y'_i : i > 0) \equiv (y_i : i \in \omega).$$

Then $\text{ann}_R(y_0) \simeq \text{ann}_R(y_1)$ by Proposition 4.6, so $\text{ann}_R(x_0) \simeq \text{ann}_R(y'_1) \simeq \text{ann}_R(x_1)$. \square

Theorem 4.8. *Let G and K be inp-minimal abelian groups definable in some ω -categorical theory, and let $\lambda : G \times G \rightarrow K$ be a definable bilinear map. Then:*

- (1) $\widetilde{\text{ann}}_L(G)$ has finite index in G .
- (2) λ is virtually almost trivial.
- (3) If G is connected, then λ is trivial.

Proof. Assume $I = \widetilde{\text{ann}}_L(G)$ has finite index in G . Then $I \lesssim \text{ann}_R(g)$ for $g \in I$. Let $(x_i : i < \omega)$ be a measure-generic 4-indiscernible sequence, and put $z_i = x_{2i} - x_{2i+1}$. Then $z_i \in I$, so $z_0 \in I_{z_1}^0 \leq \text{ann}_R(z_1)$ (by the same argument as in the second claim from the proof of Proposition 4.6) and $\lambda(z_1, z_0) = 0$. But then $z_1 \in \text{ann}_L(z_0)$ is measure-generic over z_0 , and $\text{ann}_L(z_0)$ has finite index in G . It follows that $z_0 \in \widetilde{\text{ann}}_R(G)$, so $\widetilde{\text{ann}}_R(G)$ has finite index in G as well. Hence $J = \widetilde{\text{ann}}_L(G) \cap \widetilde{\text{ann}}_R(G)$ has finite index in G . By ω -categoricity and Lemma 4.2 we see that λ is almost trivial on J .

If moreover G is connected, then $G = \text{ann}_L(G) = \widetilde{\text{ann}}_R(G)$, whence $G = \text{ann}_L(g) = \text{ann}_R(g)$ for all $g \in G$. Thus it is enough to show (1).

Assume for a contradiction that neither $\widetilde{\text{ann}}_L(G)$ nor $\widetilde{\text{ann}}_R(G)$ has finite index in G . (We have just seen that if one does, so does the other.) In particular, λ is not virtually almost trivial. As G is locally finite, it is amenable, so in particular it admits a G -invariant

Keisler measure μ . By ω -categoricity, the Lascar equivalence relation is definable and there are only finitely many Lascar types. By naming representatives of all Lascar strong types, we can assume that all types over \emptyset are Lascar strong types.

By Corollary 4.7 and inp-minimality, $\text{ann}_L(x) + \text{ann}_R(x) \simeq \text{ann}_L(\phi(x)) + \text{ann}_R(\phi(x))$ for any automorphism ϕ and any measure-generic x . We choose a measure-generic Lascar strong type p such that $\text{ann}_L(x) + \text{ann}_R(x)$ is maximal possible up to commensurability for $x \models p$. We may assume $\text{ann}_L(x) \lesssim \text{ann}_R(x)$, so $\text{im}\lambda_x$ is minimal possible up to commensurability, among all $\text{im}\lambda_y$ and $\text{im}\lambda'_y$ for measure-generic y , by Lemma 3.4.

By Schlichting's Theorem there is an \emptyset -definable subgroup N of G commensurable with $\text{ann}_R(x)$ for all $x \models p$. Then for $x, y \models p$ the relation $\lambda_{x,y}$ induces a quasi-endomorphism $\bar{\lambda}_{x,y}$ on $\bar{G} = G/N$ with finite kernel. Let R be the ring of definable quasi-endomorphisms of \bar{G} . Note that \bar{G} is infinite, as $\text{ann}_R(x)$ has infinite index in G by assumption.

By inp-minimality G is not commensurable with a sum of two definable subgroups of G not commensurable with G . If H has infinite index in G and $r \in R$ such that rH has finite index in G , then $r^{-1}rH$ has finite index in G and for $g \in r^{-1}rH$ there is $h \in H$ with $rg = rh$. Thus $g \in H + \ker r$. It follows that

$$I = \{r \in R : G \not\lesssim \text{im } r\}$$

is a two-sided ideal of R . By Lemma 3.1, any definable quasi-endomorphism of \bar{G} with finite kernel must be almost surjective, and any definable almost surjective quasi-endomorphism must have finite kernel; in either case it is invertible. Hence R/I is a division ring, which is locally finite by ω -categoricity, whence a locally finite field by Wedderburn's Theorem.

Let $(x_i : i < \omega)$ be a measure-generic 2-indiscernible sequence of realizations of p . Then $\bar{\lambda}_{x_i, x_j} \in R \setminus I$ for $i \neq j$, and has a fixed finite order modulo I . By Ramsey's Theorem we can assume that $\bar{\lambda}_{x_i, x_j} + I$ does not depend on $i < j$. But $\bar{\lambda}_{x_j, x_k} \bar{\lambda}_{x_i, x_j} = \bar{\lambda}_{x_i, x_k}$ for $i < j < k$, so $\bar{\lambda}_{x_i, x_j} \in \text{id}_{\bar{G}} + I$.

Let $\bar{H} = \text{im}(\bar{\lambda}_{x_0, x_1} - \text{id}_{\bar{G}})$, a definable subgroup of infinite index in \bar{G} ; let H be its preimage in G . Then for all $g \in G$ there is $h \in H$ with $(g, g+h) \in \lambda_{x_0, x_1}$. Hence $\lambda_{x_1}(g+h) = \lambda_{x_0}(g)$, that is

$$\lambda(x_0, g) = \lambda(x_1, g+h) = \lambda(x_1, g) + \lambda(x_1, h),$$

whence

$$\lambda(x_0 - x_1, g) = \lambda(x_1, h).$$

But this means that $\text{im}\lambda_{x_0 - x_1} \leq \lambda_{x_1}[H]$.

Claim. $\text{im}\lambda_{x_1} \not\lesssim \lambda_{x_1}[H]$.

Proof. Otherwise $\text{id}_{\bar{G}} \equiv \bar{\lambda}_{x_1, x_1}$, so $H \simeq \bar{\lambda}_{x_1, x_1}[H] = G$, a contradiction. \square

Now $x_0 - x_1$ is again measure-generic, and $\text{im}\lambda_{x_1} \not\lesssim \text{im}\lambda_{x_0 - x_1}$. Hence $\text{ann}_R(x_0 - x_1) \not\lesssim \text{ann}_R(x_1)$ by Lemma 3.4, contradicting the choice of p . \square

5. MAIN RESULTS

5.1. ω -categorical inp-minimal Groups and Rings. Recall that each countable, ω -categorical group has a finite series of characteristic (i.e. invariant under the automorphism group) subgroups in which all successive quotients are characteristically simple groups (i.e. they do not have non-trivial, proper characteristic subgroups). On the other hand, Wilson [15] proved (see also [1] for an exposition of the proof):

Fact 5.1. *For each infinite, countable, ω -categorical, characteristically simple group H , one of the following holds.*

- (i) *For some prime number p , H is an elementary abelian p -group (i. e. an abelian group, in which every nontrivial element has order p).*
- (ii) *$H \cong B(F)$ or $H \cong B^-(F)$ for some non-abelian, finite, simple group F , where $B(F)$ is the group of all continuous functions from the Cantor space C to F , and $B^-(F)$ is the subgroup of $B(F)$ consisting of the functions f such that $f(x_0) = e$ for a fixed element $x_0 \in C$.*
- (iii) *H is a perfect p -group (perfect means that H equals its commutator subgroup).*

It remains a difficult open question whether there exist infinite, ω -categorical, perfect p -groups.

The following is Theorem 3.1 from [11]:

Fact 5.2. *There is a finite bound of the size of conjugacy classes in a group G if and only if the derived subgroup G' is finite.*

Remark 5.3. If a group G is virtually finite-by-abelian, then there is a definable finite-by-abelian subgroup $G_0 \leq G$ of finite index; if a ring R is virtually finite-by-null, there is a definable subring R_0 which is finite-by-null.

Proof. Let G be virtually finite-by-abelian, and consider a finite-by-abelian subgroup $H_0 \leq G$ of finite index; we may assume that H_0 is normal in G . Put $F := H'_0 \leq G$, and $G_h := \{g \in G : [g, h] \in F\}$ for any $h \in G$. As F is finite and normal, each G_h is a definable subgroup of G . Also, $G_h \geq H_0$ for any $h \in H_0$, so $H := \bigcap_{h \in H_0} G_h$ is an intersection of finitely many G_h , hence it is definable. Clearly $[H, H_0] \leq F$. Similarly, as $G_0 := H \cap \bigcap_{g \in H} G_g$ contains H_0 , it is definable, and $G'_0 \leq F$.

If R is virtually finite-by-null, let S_0 be a finite-by-null subring of finite (additive) index, and I a finite ideal of S_0 containing $S_0 \cdot S_0$. Then $S := \bigcap_{s \in S_0} \{r \in R : rs \in I\}$ contains S_0 and must be a definable subgroup of finite index, with $S \cdot S_0 \subseteq I$. Now $R_0 := S \cap \bigcap_{s \in S} \{r \in R : sr \in I\}$ contains S_0 and is again a definable subgroup of finite index. Since $R_0 \cdot R_0 \subseteq I \leq R_0$, this is the required subring. \square

We will use the following variant of Proposition 2.5 from [8]. As in our context we cannot use connected components, we have to modify the proof slightly.

Lemma 5.4. *Let \mathcal{C} be a class of countable, ω -categorical NTP_2 (pure) groups, closed under taking definable subgroups and quotients by definable normal subgroups. Suppose that every infinite, characteristically simple group in \mathcal{C} is solvable. Then every group in \mathcal{C} is nilpotent-by-finite.*

Proof. Let $G \in \mathcal{C}$. Let $\{e\} = G_0 \leq G_1 \leq \dots \leq G_n = G$ be a chain of characteristic subgroups of G of maximal length. We will show the assertion by induction n . Let i be maximal such that G_i is finite. Then $C_G(G_i)$ is a characteristic subgroup of G of finite index, so we can replace G by $C_G(G_i)/G_i$ without increasing n . We can thus assume that G_1 is infinite. Now, as G_1 is characteristically simple, it is solvable by the assumption. By the inductive hypothesis, G/G_1 is virtually nilpotent, so there is a normal definable subgroup N of G of finite index such that N/G_1 is nilpotent, so N is solvable. Since N is NTP_2 , it does not interpret the atomless boolean algebra, so by [2, Theorem 1.2] it is virtually nilpotent, and so is G . \square

Proposition 5.5. *An ω -categorical inp-minimal group is nilpotent-by-finite.*

Proof. We will additionally assume that the group G is characteristically simple, and we will show that G is solvable (even nilpotent), which is sufficient by Lemma 5.4.

Claim. *There is a finite tuple \bar{a} of elements of G such that $\tilde{Z}(C_G(\bar{a}))$ is infinite.*

Proof. If there is $\bar{a} \subseteq G$ such that $C_G(\bar{a}) \lesssim C_G(a')$ for every $a' \in C_G(\bar{a})$, then $C_G(\bar{a}) = \tilde{Z}(C_G(\bar{a}))$, so \bar{a} satisfies the conclusion (note that, by characteristic simplicity and Fact 5.1, centralizers of finite tuples are infinite).

Supposing otherwise, we can find a sequence $(a_i : i < \omega)$ such that $a_{i+1} \in C_G(a_0, \dots, a_i)$ and $C_G(a_0, \dots, a_i) \not\lesssim C_G(a_{i+1})$ for all $i < \omega$. In particular, all a_i are distinct and commute pairwise. Moreover $C_G(a_j) \lesssim C_G(a_i)$, so $a_i \in \tilde{Z}(C_G(\bar{a}_j))$ for $i \leq j$. By ω -categoricity, the groups $\tilde{Z}(C_G(a_i))$ are uniformly definable, so, by compactness, there is $a \in G$ such that $\tilde{Z}(C_G(a))$ is infinite. \square

Let $H = \tilde{Z}(C_G(\bar{a}))$ be as in the claim. Then H is commensurable with a normal group N by Lemma 3.7. So N is virtually abelian, and its Fitting subgroup $F(N)$ is nontrivial. As $F(N)$ is characteristic in N , it is normal in G , so $F(G)$ is non-trivial. As $F(G)$ is characteristic, characteristic simplicity of G implies that $G = F(G)$, so G is nilpotent. \square

Theorem 5.6. *An ω -categorical inp-minimal group is virtually finite-by-abelian.*

Proof. Let G be a counter-example; we may assume it is nilpotent of minimal class possible. Then $Z(G)$ is infinite, and $G/Z(G)$ is virtually finite-by-abelian. By Remark 5.3 there is a definable subgroup G_0 of finite index and a finite normal subgroup $F/Z(G)$ of $G_0/Z(G)$ such that G_0/F is abelian. Clearly we may replace G by $C_{G_0}(F/Z(G))$, a definable subgroup of finite index. Then $G' \leq F$ and $F \lesssim Z(G)$. By Neumann's Lemma F' is finite; we may assume it is trivial. Replacing G by a definable subgroup of finite index, we may assume that the index $|G' : G' \cap Z(G)|$ is minimal possible.

Consider $g \in G$. As $G'/Z(G)$ is central in $G/Z(G)$, the map $x \mapsto [g, x]Z(G)$ is a definable homomorphism from G to $G'/Z(G)$; its kernel H must have finite index. Then $x \mapsto [g, x]$ is a definable homomorphism from H to $Z(G)$ with abelian image; its kernel must hence contain $H'Z(G)$. As $H'Z(G) = G'Z(G)$ by minimality, we see that $G' \leq C_G(g)$. This holds for all $g \in G$, so $G' \leq Z(G)$.

Now commutation is a definable bilinear form from $G/Z(G)$ to $Z(G)$. By Theorem 4.8 it is virtually almost trivial. But this means that G is virtually finite-by-abelian, contradicting our assumption. \square

Corollary 5.7. *An ω -categorical dp-minimal group is virtually abelian.*

Proof. Let G be an ω -categorical dp-minimal group. As G is in particular NIP, by a result of Shelah, the absolute connected component G^{00} (i.e., the smallest type-definable subgroup of G of bounded index) exists. By ω -categoricity, G^{00} is definable and hence of finite index in G , so we may assume that G is connected. Then G is finite-by-abelian by Remark 5.3. Thus, the centralizer of any element in G has finite index in G , hence, by connectedness, is equal to G . This means that G is abelian. \square

Theorem 5.8. *An ω -categorical inp-minimal ring is virtually finite-by-null.*

Proof. This is immediate from Theorem 4.8, as multiplication is a definable bilinear map. \square

As for groups, we get a corollary:

Corollary 5.9. *An ω -categorical dp-minimal ring R is virtually null.*

Proof. Again, we may assume that R is connected (in the sense of the additive group). Then R is finite-by-null by Remark 5.3. Hence, the left annihilator of any element in R has finite index in R , and must be equal to R by connectedness. This shows that R is null. \square

6. QUESTIONS AND CONCLUDING REMARKS

One can ask various questions about generalizations of the above results to more general contexts, such as finite *inp*-rank, strong or NTP_2 theories.

Question 6.1. Are ω -categorical groups of finite *inp*-rank virtually finite-by-abelian?

A positive answer would seem to be a major improvement of Theorem 5.6, as the class of groups of finite *inp*-rank seems much richer than that of *inp*-minimal groups (containing for example all groups interpretable in \mathbb{R} , \mathbb{C} and \mathbb{Q}_p). Note that by [9, Theorem 2.1], a positive answer would also imply that every ω -categorical ring of finite dp-rank is virtually null.

By modifying the proof of Theorem 2.1 from [10] we prove its generalization from the NIP to the NTP_2 context:

Theorem 6.2. *Every ω -categorical NTP_2 ring is virtually nilpotent.*

Proof. As in [10], it is enough to show that a semisimple ω -categorical NTP_2 ring R is finite, and we can assume that R is a subring of $\prod_{i \in I} R_i$, where each R_i is finite, and $|\{R_i : i \in I\}| < \omega$. Let π_i be the projection onto the i -th coordinate. For $i_0, \dots, i_n \in I$ and $r_0 \in R_{i_0}, \dots, r_n \in R_{i_n}$, we define

$$R_{i_0, \dots, i_n}^{r_0, \dots, r_n} = \left\{ r \in R : \bigwedge_{j=0}^n \pi_{i_j}(r) = r_j \right\}.$$

Suppose for a contradiction that R is infinite. Again as in [10], we get the following claim:

Claim. *For any $N \in \omega$ there are pairwise distinct $i(0), \dots, i(N-1) \in I$ and non-nilpotent elements $r_i \in R_{i_i}$ for $i < N$ such that the sets*

$$R_{i_0, \dots, i_{N-1}}^{r_0, 0, \dots, 0}, R_{i_0, \dots, i_n}^{0, r_1, \dots, 0}, \dots, R_{i_0, \dots, i_n}^{0, 0, \dots, r_{N-1}}$$

are all non-empty.

Notice that, by ω -categoricity, the principal two-sided ideals RxR for $x \in R$ are uniformly definable. Hence, by [5, Theorem 2.4] and compactness, we obtain in particular that in order to contradict NTP_2 it is enough to find for any $n, m < \omega$ elements b_0, \dots, b_{n-1} such that

$$(*) \quad \left| \bigcap_{j \in n \setminus \{j_0\}} Rb_jR : \bigcap_{j \in n} Rb_jR \right| \geq m$$

for any $j_0 < n$ (where $n = \{0, 1, \dots, n-1\}$). So fix any $n, m < \omega$, and for $N = nm$ choose i_j and r_j as in the claim. Let $(i_{j,k})_{j < n, k < m}$ be another enumeration of $(i_j)_{j < N}$, and let $(r_{j,k})_{j < n, k < m}$ be the corresponding enumeration of $(r_j)_{j < N}$ and $(\pi_{j,k})_{j < n, k < m}$ the corresponding enumeration of $(\pi_j)_{j < N}$. For any $j_0 < n, k_0 < m$ let $s_{j_0, k_0} \in R$ be such that $\pi_{j,k}(s_{j_0, k_0}) = 0$ for $(j, k) \neq (j_0, k_0)$ and $\pi_{j_0, k_0}(s_{j_0, k_0}) = r_{j_0, k_0}$. Put $b_j = \sum_{j' \neq j, k < m} s_{j', k}$ for all $j < n$.

Claim. $|\bigcap_{j \in n \setminus \{j_0\}} Rb_j R : \bigcap_{j \in n} Rb_j R| \geq m$ for any $j_0 < n$.

Proof. Fix any $j_0 < n$ and put $b = b_0 b_1 \dots b_{j_0-1} b_{j_0+1} b_{j_0+2} \dots b_{n-1}$. Notice that for any $r \in \bigcap_{j \in n} Rb_j R$ and $k < m$ we have that $\pi_{j_0, k}(r) = 0$. On the other hand, for distinct $k_1, k_2 < m$ we have that

$$\pi_{j_0, k_1}(s_{j_0, k_1} b - s_{j_0, k_2} b) = \pi_{j_0, k_1}(s_{j_0, k_1} b) = \pi_{j_0, k_1}(s_{j_0, k_1}) \pi_{j_0, k_1}(b) = r_{j_0, k_1} r_{j_0, k_1}^{n-1} = r_{j_0, k_1}^n \neq 0.$$

Hence the elements

$$s_{j_0, 0} b, s_{j_0, 1} b, \dots, s_{j_0, m-1} b \in \bigcap_{j \in n \setminus \{j_0\}} Rb_j R$$

are in pairwise distinct cosets of $\bigcap_{j \in n} Rb_j R$. □

By the claim and (*) we obtain a contradiction. □

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