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Non-autonomous maximal regularity in weighted space

Mahdi Achache * Tebbani Hossni †

June 24, 2018

Abstract

We consider the problem of maximal regularity for non-autonomous Cauchy problems

$$u'(t) + A(t)u(t) = f(t) \quad (t \in [0, \tau]), \quad u(0) = u_0.$$

The time dependent operators $A(t)$ are associated with sesquilinear forms on a Hilbert space H . We prove the maximal regularity in the weighted space $L^2(0, \tau, t^\beta dt; H)$, with $\beta \in]-1, 1[$ and we prove also other regularity properties for the solution of the previous problem. Our result is motivated by boundary value problems.

keywords: Maximal regularity, non-autonomous evolution equations, weighted space.

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1 Introduction

The aim of this article is to study autonomous and non-autonomous evolution equation governed by forms.

Let $(H, (\cdot, \cdot), \|\cdot\|)$ be a Hilbert space over \mathbb{R} or \mathbb{C} . We consider another Hilbert space V which is densely and continuously embedded into H . We denote by V' the (anti-) dual space of V , so that

$$V \hookrightarrow_d H \hookrightarrow_d V'.$$

i.e., V is a dense subspace of H such that for some constant $C_H > 0$,

$$\|u\|_H \leq C_H \|u\|_V \quad (u \in V).$$

We denote by \langle, \rangle the duality $V' - V$ and note that $\langle \psi, v \rangle = (\psi, v)$ if $\psi, v \in H$. We consider a family of sesquilinear forms

$$\mathbf{a} : [0, \tau] \times V \times V \rightarrow \mathbb{C}$$

such that

- [H1]: $D(\mathbf{a}(t)) = V$ (constant form domain),
- [H2]: $|\mathbf{a}(t, u, v)| \leq M \|u\|_V \|v\|_V$ (uniform boundedness),
- [H3]: $\operatorname{Re} \mathbf{a}(t, u, u) + \nu \|u\|_H^2 \geq \delta \|u\|_V^2$ ($\forall u \in V$) for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We denote by $A(t), \mathcal{A}(t)$ the usual associated operators with $a(t)$ (as operators on H and V').

In 1961 J. L. Lions proved that the non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0 \end{cases} \quad (\text{P})$$

has L^2 -maximal regularity in V' :

Theorem 1.1 (Lions' theorem). *Given $f \in L^2(0, \tau; V')$ and $u_0 \in H$, there is a unique solution $u \in MR(V, V') := H^1(0, \tau; V') \cap L^2(0, \tau; V)$ of the Cauchy problem (P).*

Note that $MR(V, V') \hookrightarrow C([0, \tau]; H)$ so that the initial condition makes sense. In Theorem (1.1) only measurability of $t \rightarrow \mathbf{a}(t, \cdot, \cdot)$ with respect to the time variable is required to have a solution $u \in MR(V, V')$. However, considering boundary valued problems one is interested in strong solutions, i.e., solutions $u \in H^1(0, \tau; H)$ and not only in $H^1(0, \tau; V')$ (note that $H \hookrightarrow V'$ by the natural embedding).

In the recent decades, the maximal regularity approach has become very useful in application to parabolic partial differential equations. The maximal regularity in H (autonomous or non-autonomous cases) is so important for several reasons. First of all, if Robin boundary conditions are considered, only the operator $A(t)$ realizes these boundary conditions.

Problem 1.2. *Let $f \in L^2(0, \tau; H)$. Under which conditions on the forms $\mathfrak{a}(\cdot)$ the solution $u \in MR(V, V')$ of (P) satisfies $u \in H^1(0, \tau; H)$.*

Lions asked this question on maximal regularity for several conditions on the form and on the initial value. He also gave partial positive answers in [20][XVIII Chapter 3, p. 513]. More recently, this problem has been studied with a lot of progress. See the recent papers [2] or [5] for more details and references. The main reason for studying this problem is the importance for non-linear problems. They are mainly solved by applying the Banach or the Schauder fixed point theorem.

The main focus of this work is the presence of the temporal weights. The choice of the weighted spaces has a big advantages. One of them is to reduce the necessary regularity for initial conditions of evolution equations. Time-weights can be used also to exploit parabolic regularization which is typical for quasilinear parabolic problems.

In this paper we are mainly interested by proving the maximal regularity in the non-autonomous case, i.e. we prove the existence and uniqueness of solution to the Problem (P). We shall allow considerably less restrictive assumptions on f and the initial data u_0 . Here f belongs to the weighted Hilbert space $L^2(0, \tau, t^\beta dt; H)$, with $\beta \in [0, 1[$ and the initial data u_0 takes its values in a certain interpolation space $(H, D(A(0)))_{\frac{1-\beta}{2}, 2}$ between H and $D(A(0))$. The maximal regularity for the autonomous case in weighted spaces was the subject of treatment of many authors, see for instance [3].

In the non-autonomous case (see the Section 5) we prove that if $f \in L^2(0, \tau, t^\beta dt; H)$ and $u_0 \in (H, D(A(0)))_{\frac{1-\beta}{2}, 2}$ for arbitrary $\beta \geq 0$ with the assumption that the operator $\mathcal{A}(\cdot) \in W^{\frac{1}{2}, 2}(0, \tau; \mathcal{L}(V, V')) \cap C^\varepsilon([0, \tau], \mathcal{L}(V, V'))$ for some $\varepsilon > 0$, then the non-autonomous Problem (5.2) has a unique solution u such that $\dot{u}, \mathcal{A}(\cdot)u \in L^2(0, \tau, t^\beta dt; H)$. Throughout all of this paper we assume that the Kato square root property (see (3.1)) is satisfied.

In order to prove our results we appeal to classical tools from harmonic analysis such as square function estimate or functional calculus and from functional analysis such as interpolation theory or operator theory.

This work is structured as follows. In section (2) we present basic definitions and properties used throughout this paper, in particular those of weighted spaces. In Section (3) we prove some preparatory results, while in the Section (4) we prove the maximal regularity to the considered au-

tonomous Cauchy problem in the weighted space $L^2(0, \tau, t^\beta dt; H)$ and other regularity properties for the solution.

We illustrate our abstract results by two applications in the final section. One of them concerns the heat equation with Robin boundary conditions on a bounded Lipschitz domain Ω .

Notation.

- We denote by C , C' or $c\dots$ all inessential positive constants. Their values may change from line to line.
- On some cases we will use the notation $a \lesssim b$ to signify that there exists an inessential positive constant C such that $a \leq Cb$.

2 Properties of the weighted spaces

In this section we briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights.

Let $(X, \|\cdot\|_X)$ be a Banach space over \mathbb{R} or \mathbb{C} . For $-1 < \beta < 1$ we set $L_\beta^2(0, \tau; X) = L^2(0, \tau, t^\beta dt; X)$, endowed with norm

$$\|u\|_{L_\beta^2(0, \tau; X)}^2 = \int_0^\tau \|u(t)\|_X^2 t^\beta dt.$$

It's very seen that $L_\beta^2(0, \tau; X) \hookrightarrow L_{loc}^1(0, \tau; X)$. Indeed, for all $u \in L_\beta^2(0, \tau; X)$, we have

$$\int_0^\tau \|u(t)\|_X dt \leq \left(\int_0^\tau t^{-\beta} dt\right)^{\frac{1}{2}} \|u\|_{L_\beta^2(0, \tau; X)}.$$

It clearly holds $L^2(0, \tau; X) \hookrightarrow L_\beta^2(0, \tau; X)$ for $\beta > 0$ and $L_\beta^2(0, \tau; X) \hookrightarrow L^2(0, \tau; X)$ for $\beta < 0$.

We define the corresponding weighted Sobolev spaces

$$W_\beta^{1,2}(0, \tau; X) := \{u \in W^{1,1}(0, \tau; X) \text{ s.t } u, \dot{u} \in L_\beta^2(0, \tau; X)\},$$

$$W_{\beta,0}^{1,2}(0, \tau; X) := \{u \in W_\beta^{1,2}(0, \tau; X), \text{ s.t } u(0) = 0\},$$

which are Banach spaces endowed with norms, respectively

$$\|u\|_{W_\beta^{1,2}(0, \tau; X)}^2 = \|u\|_{L_\beta^2(0, \tau; X)}^2 + \|\dot{u}\|_{L_\beta^2(0, \tau; X)}^2,$$

$$\|u\|_{W_{\beta,0}^{1,2}(0, \tau; X)}^2 = \|\dot{u}\|_{L_\beta^2(0, \tau; X)}^2.$$

We set also

$$L_\beta^\infty(0, \tau; X) := \{u \in L^1(0, \tau; X) \text{ s.t } s \rightarrow s^{\frac{\beta}{2}} u(s) \in L^\infty(0, \tau; X)\},$$

endowed with norm $\|u\|_{L_\beta^\infty(0,\tau;X)} = \|\cdot^{\frac{\beta}{2}}u(\cdot)\|_{L^\infty(0,\tau;X)}$. We define also the fractional weighted Sobolev space $W_\beta^{s,2}(0,\tau;X)$, where

$$W_\beta^{s,2}(0,\tau;X) = (L_\beta^2(0,\tau;X); W_\beta^{1,2}(0,\tau;X))_{s,2},$$

endowed with norm

$$\|u\|_{W_\beta^{s,2}(0,\tau;X)}^2 = \|u\|_{L_\beta^2(0,\tau;X)}^2 + \int_0^\tau \int_0^t \frac{\|u(t) - u(s)\|_X^2}{|t-s|^{1+2s}} s^\beta ds dt,$$

with $s \in (0, 1)$.

Here $(\cdot; \cdot)_{s,2}$ is the real interpolation space. For more details see [24](2.6).

Lemma 2.1 (Weighted Hardy inequality). *For all $f \in L_\beta^2(0,\tau;X)$, we have the following inequality*

$$\int_0^\tau \left(\frac{1}{t} \int_0^t \|f(s)\|_X ds\right)^2 t^\beta dt \lesssim \|f\|_{L_\beta^2(0,\tau;X)}.$$

The Lemma 2.1 is proved in ([29], Lemma 6).

Proposition 2.2. *We have the following properties*

- 1- (a) For $p > 2$ and $\beta > \frac{2}{p} - 1$, we have $L^p(0,\tau;X) \hookrightarrow L_\beta^2(0,\tau;X)$,
(b) For $p < 2$ and $\beta < \frac{2}{p} - 1$, we obtain $L_\beta^2(0,\tau;X) \hookrightarrow L^p(0,\tau;X)$.
- 2- For all $u \in L_\beta^2(0,\tau;X)$, we have $t \rightarrow v(t) = \frac{1}{t} \int_0^t u(s) ds \in L_\beta^2(0,\tau;X)$.
- 3- We define the operator $\Phi : L_\beta^2(0,\tau;X) \rightarrow L^2(0,\tau;X)$, such that $(\Phi f)(t) = t^{\frac{\beta}{2}} f(t)$ for $f \in L_\beta^2(0,\tau;X)$ and $t \in [0,\tau]$. Then Φ is an isometric isomorphism. We note also that $\Phi \in \mathcal{L}(L^2(0,\tau;X), L_{-\beta}^2(0,\tau;X))$ and $\Phi \in \mathcal{L}(W_{\beta,0}^{1,2}(0,\tau;X), W_0^{1,2}(0,\tau;X))$.
- 4- We have $W_{\beta,0}^{1,2}(0,\tau;X) \hookrightarrow L_{\beta-2}^2(0,\tau;X)$.
- 5- $L_{-\beta}^2(0,\tau;X)$ is the dual space of $L_\beta^2(0,\tau;X)$ by the duality defined in $L^2(0,\tau;H)$.
- 6- If $u \in W_\beta^{1,2}(0,\tau;X)$, we obtain that u has a continuous extension on X and we have
$$W_\beta^{1,2}(0,\tau;X) \hookrightarrow C([0,\tau];X).$$
- 7- $C_c^\infty((0,\tau);X)$ is dense in $L_\beta^2(0,\tau;X)$ and $C^\infty([0,\tau];X)$ is dense in $W_\beta^{s,2}(0,\tau;X)$ for all $s \in [0, 1]$.

Proof. 1- (a) Let $p > 2$ and $\beta > \frac{2}{p} - 1$, we set $p' = \frac{p}{2} > 1$, $\frac{1}{p'} + \frac{1}{q} = 1$ and this implies that $q = \frac{p}{p-2}$. By using the Hölder's inequality and the condition above we get

$$\begin{aligned} \|u\|_{L^2_\beta(0,\tau;X)}^2 &= \int_0^\tau \|u(t)\|_X^2 t^\beta dt \\ &\leq \left(\int_0^\tau \|u(t)\|_X^p dt \right)^{\frac{2}{p}} \left(\int_0^\tau t^{\frac{\beta p}{p-2}} dt \right)^{\frac{p-2}{p}} \\ &= \left(\frac{1}{\frac{\beta p}{p-2} + 1} \tau^{\frac{\beta p}{p-2} + 1} \right)^{\frac{p-2}{p}} \|u\|_{L^p(0,\tau;X)}^2. \end{aligned}$$

(b) Similarly, by the above applied to the case $p' = \frac{2}{p} > 1$, we have

$$\begin{aligned} \|u\|_{L^p(0,\tau;X)}^p &= \int_0^\tau \|u(t)\|_X^p t^{-\frac{\beta p}{2}} t^{\frac{\beta p}{2}} dt \\ &\leq \left(\int_0^\tau \|u(t)\|_X^2 t^\beta dt \right)^{\frac{p}{2}} \left(\int_0^\tau t^{\frac{\beta p}{p-2}} dt \right)^{\frac{2-p}{2}} \\ &= C \|u\|_{L^2_\beta(0,\tau;X)}^p. \end{aligned}$$

2- Using the previous Hardy inequality we have

$$\|v\|_{L^2_\beta(0,\tau;X)}^2 = \int_0^\tau \left\| \frac{1}{t} \int_0^t u(s) ds \right\|_X^2 t^\beta dt \lesssim \|u\|_{L^2_\beta(0,\tau;X)}^2.$$

Now, since $u \in L^2_\beta(0, \tau; X)$, we get the result.

3- We see that $\|\Phi f\|_{L^2(0,\tau;X)} = \|f\|_{L^2_\beta(0,\tau;X)}$ and we have $\Phi^{-1} : L^2(0, \tau; X) \rightarrow L^2_\beta(0, \tau; X)$ such that $(\Phi^{-1}g)(t) = t^{-\frac{\beta}{2}}g(t)$ for all $g \in L^2(0, \tau; X)$.

4- Let $u \in W_\beta^{1,2}(0, \tau; X)$ such that $u(0) = 0$. We write $u(t) = \int_0^t \dot{u}(l) dl$. Then

$$\|u(t)\|_X^2 t^{\beta-2} = \left\| \int_0^t \dot{u}(l) dl \right\|_X^2 t^{\beta-2}.$$

This implies that

$$\begin{aligned} \|u\|_{L^2_{\beta-2}(0,\tau;X)}^2 &= \int_0^\tau \|u(t)\|_X^2 t^{\beta-2} dt \\ &= \int_0^\tau \frac{1}{t^2} \left\| \int_0^t \dot{u}(s) ds \right\|_X^2 t^\beta dt \\ &\leq \int_0^\tau \left(\frac{1}{t} \int_0^t \|\dot{u}(s)\|_X ds \right)^2 t^\beta dt \\ &\lesssim \|\dot{u}\|_{L^2_\beta(0,\tau;X)}^2 \leq \|u\|_{W_\beta^{1,2}(0,\tau;X)}^2, \end{aligned}$$

where here we used the Hardy inequality.

- 5- Use simple functions in $L^2_{-\beta}(0, \tau; X)$ which norm simple functions in $L^2_{\beta}(0, \tau; X)$ and the Cauchy-Schwartz inequality (the proof is similar to the non weighted case see ([14], p.98)).
- 6- Let $u \in W^{1,2}_{\beta}(0, \tau; X)$ and for $(t, s) \in [0, \tau]^2$. We obtain

$$\begin{aligned} \|u(t) - u(s)\|_X &= \left\| \int_s^t \dot{u}(l) dl \right\|_X \\ &\leq \left(\int_s^t l^{-\beta} dl \right)^{\frac{1}{2}} \|\dot{u}\|_{L^2_{\beta}(0, \tau; X)} \\ &= \frac{1}{\sqrt{1-\beta}} (t^{-\beta+1} - s^{-\beta+1})^{\frac{1}{2}} \|\dot{u}\|_{L^2_{\beta}(0, \tau; X)}. \end{aligned}$$

By letting $s \rightarrow t$ we get $u(s) \rightarrow u(t)$ in X . Therefore u has a continuous extension on X . Thus we can always identify a function in $W^{1,2}_{\beta}(0, \tau; X)$ by its continuous representative.

- 7- First we note that $C^{\infty}_c((0, \tau); X)$ is dense $L^2(0, \tau; X)$. Then for all $f \in L^2_{\beta}(0, \tau; X)$ and for any given $\varepsilon > 0$ there thus exists a function $\psi \in C^{\infty}_c((0, \tau); X)$ such that

$$\|(\Phi f) - \psi\|_{L^2(0, \tau; X)}^2 \leq \varepsilon.$$

It follows that

$$\begin{aligned} \|f - (\Phi^{-1}\psi)\|_{L^2_{\beta}(0, \tau; X)}^2 &\leq \|\Phi\|_{\mathcal{L}(L^2_{\beta}(0, \tau; X); L^2(0, \tau; X))} \|(\Phi f) - \psi\|_{L^2(0, \tau; X)}^2 \\ &\leq \varepsilon. \end{aligned}$$

Therefore $C^{\infty}_c((0, \tau); X)$ is dense in $L^2_{\beta}(0, \tau; X)$.

As in ([30], Theorem 2.9.1) for the scalar-valued case, one sees that the space of all function f in $C^{\infty}([0, \tau]; X)$ such that $f(0) = 0$, is dense in $W^{1,2}_{\beta,0}(0, \tau; X)$. Then for all $g \in W^{1,2}_{\beta,0}(0, \tau; X)$ and $\varepsilon > 0$ there exists $\phi \in C^{\infty}([0, \tau]; X)$ with $\phi(0) = 0$ such that

$$\|\phi - \Phi g\|_{W^{1,2}(0, \tau; X)}^2 \leq \varepsilon.$$

Then $\|\Phi^{-1}\phi - g\|_{W^{1,2}_{\beta}(0, \tau; X)}^2 \leq \varepsilon$. This shows that the space of all function f in $C^{\infty}([0, \tau]; X)$ such that $f(0) = 0$, is dense in $W^{1,2}_{\beta,0}(0, \tau; X)$. Let $f \in W^{1,2}_{\beta,0}(0, \tau; X)$ and $\phi \in C^{\infty}([0, \tau]; X)$ such that $\phi(0) = f(0)$. Then $f - \phi \in W^{1,2}_{\beta,0}(0, \tau; X)$ and there is $\xi \in C^{\infty}([0, \tau]; X)$ with $\xi(0) = 0$, such that $\|f - \xi - \phi\|_{W^{1,2}_{\beta}(0, \tau; X)}^2 \leq \varepsilon$. Since $\xi + \phi \in C^{\infty}([0, \tau]; X)$,

then $C^\infty([0, \tau]; X)$ is dense in $W_\beta^{1,2}(0, \tau; X)$.
 Since $C^\infty([0, \tau]; X)$ is dense in $W_\beta^{1,2}(0, \tau; X)$ and

$$W_\beta^{s,2}(0, \tau; X) = (L_\beta^2(0, \tau; X); W_\beta^{1,2}(0, \tau; X))_{s,2},$$

we obtain that $C^\infty([0, \tau]; X)$ is also dense in $W_\beta^{s,2}(0, \tau; X)$ by ([30], p.39). □

3 Preliminaries

In this section we prove several estimates which will play an important role in the proof of our results.

From now we assume without loss of generality that the forms are coercive, that is $[H3]$ holds with $\nu = 0$. The reason is that by replacing $A(t)$ by $A(t) + \nu$, the solution v of (P) is $v(t) = e^{-\nu t}u(t)$ and it is clear that $u \in W_\beta^{1,2}(0, \tau; H)$ if and only if $v \in W_\beta^{1,2}(0, \tau; H)$.

We denote by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |\arg(z)| < \theta\}$ with vertex 0.

The following lemma is proved in [17] (Proposition 2.1)

Lemma 3.1. *For any $t \in [0, \tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - \arctan(\frac{M}{\delta})$ on H and V' respectively. In addition, there exist constants C and C_θ , independent of t , such that*

- 1- $\|e^{-zA(t)}\|_{\mathcal{L}(H)} \leq 1$ and $\|e^{-z\mathcal{A}(t)}\|_{\mathcal{L}(V')} \leq C$ for all $z \in S_\gamma$.
- 2- $\|A(t)e^{-sA(t)}\|_{\mathcal{L}(H)} \leq \frac{C}{s}$ and $\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}\|_{\mathcal{L}(V')} \leq \frac{C}{s}$ for all $s \in \mathbb{R}$.
- 3- $\|e^{-sA(t)}\|_{\mathcal{L}(H,V)} \leq \frac{C}{\sqrt{s}}$.
- 4- $\|(z - A(t))^{-1}\|_{\mathcal{L}(H,V)} \leq \frac{C}{\sqrt{|z|}}$ and $\|(z - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(V',H)} \leq \frac{C}{\sqrt{|z|}}$ for all $z \notin S_\theta$ with fixed $\theta > \gamma$.

The following lemma is proved in [22](Corollary 4.312)

Lemma 3.2. *Let H_1, H_2 be two Hilbert spaces, with $H_2 \subset H_1$, H_2 dense in H_1 . Then for every $\theta \in (0, 1)$,*

$$[H_1, H_2]_\theta = (H_1, H_2)_{\theta,2},$$

with $\|u\|_{[H_1, H_2]_\theta} = C\|u\|_{(H_1, H_2)_{\theta,2}}$, where C is a positive constant independent of H_1 and H_2 .

As consequence from the previous Lemma and ([22], Theorem 4.2.6) we have that for all $\gamma \in (0, 1), t \in [0, \tau]$

$$(H, D(A(t)))_{\gamma,2} = [H, D(A(t))]_{\gamma} = D(A(t)^{\gamma}).$$

Lemma 3.3. *For all $x \in (H, D(A(t)))_{\frac{1}{2},2}$, we get*

$$\int_0^{\infty} \|A(t)e^{-sA(t)}x\|^2 ds \leq C\|x\|_{(H,D(A(t)))_{\frac{1}{2},2}}^2,$$

where $C > 0$ is independent of t .

Proof. Note that $\|e^{-sA(t)}\|_{\mathcal{L}(H)} \leq 1$ and $\|sA(t)e^{-sA(t)}\|_{\mathcal{L}(H)} \leq M_1$, where M_1 is independent of t . Let $x \in (H, D(A(t)))_{\frac{1}{2},2}$. We write $x = a + b$, where $a \in H$ and $b \in D(A(t))$ to obtain

$$\begin{aligned} s^{\frac{1}{2}}\|A(t)e^{-sA(t)}x\| &\leq M_1s^{-\frac{1}{2}}\|a\| + s^{\frac{1}{2}}\|b\|_{D(A(t))} \\ &\leq \max\{M_1, 1\}s^{-\frac{1}{2}}\{\|a\| + s\|b\|_{D(A(t))}\} \\ &\leq \max\{M_1, 1\}s^{-\frac{1}{2}}K(s, x; H, D(A(t))). \end{aligned}$$

So $\|A(t)e^{-sA(t)}x\| \leq \max\{M_1, 1\}s^{-1}K(s, x; H, D(A(t)))$, where

$$K(s, x; H, D(A(t))) = \inf_{x=a+b; a \in H, b \in D(A(t))} \left(\|a\| + s\|b\|_{D(A(t))} \right).$$

Since $\|x\|_{(H,D(A(t)))_{\frac{1}{2},2}}^2 = \int_0^{\infty} |K(s, x; H, D(A(t)))|^2 \frac{ds}{s^2}$ (See [22], Definition 1.1.1), then

$$\int_0^{\infty} \|A(t)e^{-sA(t)}x\|^2 ds \leq \max\{M_1, 1\}\|x\|_{(H,D(A(t)))_{\frac{1}{2},2}}^2.$$

Then we get the desired result. \square

In the next lemma we prove the quadratic estimate, this lemma was proved in [2] with the assumption (3.1), here we prove this estimate without this assumption.

Lemma 3.4. *Let $x \in H$ and $t \in [0, \tau]$. We have the following estimate*

$$\int_0^{\tau} \|A(t)^{\frac{1}{2}}e^{-sA(t)}x\|^2 ds \leq c\|x\|^2,$$

where $c > 0$ is independent of t .

Proof. Note that by ([19], (A1) p. 269)

$$A(t)^{-\beta} = \frac{1}{\pi} \int_0^\infty \mu^{-\beta} (\mu + A(t))^{-1} d\mu.$$

Then by Lemma 3.1 one has $\|A(t)^{-\frac{1}{2}}\|_{\mathcal{L}(H)} \leq C'$, with $C' > 0$ independent of t .

Let $x \in H$ and $t \in [0, \tau]$. We get by the previous lemma

$$\begin{aligned} \int_0^1 \|A(t)^{\frac{1}{2}} e^{-sA(t)} x\|^2 ds &= \int_0^1 \|A(t) e^{-sA(t)} A(t)^{-\frac{1}{2}} x\|^2 ds \\ &\leq \|A(t)^{-\frac{1}{2}} x\|_{(H; D(A(t))^{\frac{1}{2}, 2})}^2 \\ &= \|x\|_H^2 + \|A(t)^{-\frac{1}{2}} x\|^2 \\ &\leq (C'^2 + 1) \|x\|^2. \end{aligned}$$

□

In the next of this paper we suppose that $D(A(t)^{\frac{1}{2}}) = V$, for all $t \in [0, \tau]$ and there exist $c_1, c^1 > 0$ such that for all $v \in V$

$$c_1 \|v\|_V \leq \|A(t)^{\frac{1}{2}} v\| \leq c^1 \|v\|_V, \quad (3.1)$$

this also holds for the adjoint-operator and we have

$$c_1 \|v\|_V \leq \|A^*(t)^{\frac{1}{2}} v\| \leq c^1 \|v\|_V.$$

Note that this assumption is always true for symmetric forms and we get $c_1 = \sqrt{\delta}$ and $c^1 = \sqrt{M}$.

Lemma 3.5. *For all $t \in [0, \tau]$ we have $D(\mathcal{A}(t)^{\frac{1}{2}}) = H$ and $D([\mathcal{A}(t)^*]^{\frac{1}{2}}) = V$.*

Proof. We write

$$\mathcal{A}(t)^{\frac{1}{2}} u = \mathcal{A}(t) A(t)^{-\frac{1}{2}} u.$$

Therefore

$$\frac{\alpha}{c^1} \|u\| \leq \|\mathcal{A}(t)^{\frac{1}{2}} u\|_{V'} \leq \frac{M}{c_1} \|u\|.$$

So that $\mathcal{A}(t)^{\frac{1}{2}} \in \mathcal{L}(H, V')$. By duality we have $A(t)^{* \frac{1}{2}} \in \mathcal{L}(V, H)$. □

Every $f \in L^2(0, t; H)$, defines an operator by putting

$$(R(t)f) = \int_0^t e^{-(t-s)A(t)} f(s) ds.$$

The next lemma shows that $R(t)$ is bounded in $\mathcal{L}(L^2(0, t; H), V)$.

Lemma 3.6. *We have that for all $t \in [0, \tau]$, $R(t) \in \mathcal{L}(L^2(0, t; H), V)$.*

The Lemma 3.6 is proved in ([2], Lemma 4.1).

We define the space

$$L^2_\beta(0, \tau; D(A(\cdot))) = \{u \in L^2_\beta(0, \tau; H) \text{ s.t. } A(\cdot)u \in L^2_\beta(0, \tau; H)\}$$

endowed with norm

$$\|u\|_{L^2_\beta([0, \tau], D(A(\cdot)))} = \|u\|_{L^2_\beta(0, \tau, H)} + \|A(\cdot)u\|_{L^2_\beta(0, \tau, H)}.$$

Lemma 3.7. *We assume that $\mathcal{A}(\cdot) \in C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))$, $\varepsilon > 0$. Then for all $\lambda \in (0, \infty)$, we have $(\lambda + A(\cdot))^{-1} \in C^\varepsilon([0, \tau]; \mathcal{L}(H))$ and $\|(\lambda + A(\cdot))^{-1}\|_{C^\varepsilon([0, \tau]; \mathcal{L}(H))} \leq \frac{C}{\lambda}$.*

Proof. Let $\lambda \in (0, \infty)$, $t, s \in [0, \tau]$. We get

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(s))(\lambda + A(s))^{-1}.$$

Therefore by the Lemma (3.1) we have

$$\begin{aligned} & \|(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\|_{\mathcal{L}(H)} \\ & \leq \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(V', H)} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V', V)} \|(\lambda + A(s))^{-1}\|_{\mathcal{L}(H, V)} \\ & \leq C \frac{|t - s|^\varepsilon}{|\lambda|}. \end{aligned}$$

□

Lemma 3.8. *We suppose that $\mathcal{A}(\cdot) \in C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))$, then $L^2_\beta(0, \tau; D(A(\cdot)))$ is dense in $L^2_\beta(0, \tau; H)$.*

Proof. Let $f \in L^2_\beta(0, \tau; H)$, and set $f_n(t) = n(n + A(t))^{-1}f(t)$ for $n \in \mathbb{N}$. Since $t \rightarrow (n + A(t))^{-1} \in C^\varepsilon([0, \tau]; \mathcal{L}(H))$, then for all $n \in \mathbb{N}$ the function $f_n : [0, \tau] \rightarrow H$ is measurable and satisfies $f_n(t) \in D(A(t))$ almost everywhere as well as $\|A(t)f_n(t)\| \leq Cn\|f(t)\|$. Moreover

$$\|f_n(t) - f(t)\| = \|(n(n + A(t))^{-1} - I)f(t)\|.$$

Hence, the convergence $f_n \rightarrow f$ in $L^2_\beta(0, \tau; H)$ holds by the dominated convergence theorem. □

Proposition 3.9. *Assume that $\mathcal{A}(\cdot) \in C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))$, for some $\varepsilon > 0$. Then for all $f \in L^2_\beta(0, \tau; H)$, with $\beta < 1$ the operator L defined by*

$$(Lf)(t) = A(t) \int_0^t e^{-(t-s)A(s)} f(s) ds$$

is bounded in $L^2_\beta(0, \tau; H)$.

Proof. Let $f \in L^2_\beta(0, \tau; D(A(\cdot)))$. We split the integral into two parts to get

$$\begin{aligned} (Lf)(t) &= A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s) ds + A(t) \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s) ds \\ &:= I_1(t) + I_2(t). \end{aligned}$$

We begin by estimating the first integral

$$\begin{aligned} \|I_1(t)\| &= \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s) ds\|_H \lesssim \int_0^{\frac{t}{2}} \frac{1}{t-s} \|f(s)\| ds \\ &\lesssim \frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\| ds. \end{aligned}$$

Therefore by Hardy inequality (See Lemma 2.1) we have

$$\begin{aligned} &\int_0^\tau \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s) ds\|_H^2 t^\beta dt \\ &\lesssim \int_0^\tau \left(\frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\| ds \right)^2 t^\beta dt \\ &\lesssim \|f\|_{L^2_\beta(0, \tau; H)}^2. \end{aligned}$$

As before we estimate the second integral, so for all $x \in H$, we obtain

$$\begin{aligned} |(I_2(t), x)| &= \left| \int_{\frac{t}{2}}^t \left(A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s), A(t)^{\frac{1}{2}*} e^{-\frac{1}{2}(t-s)A(t)*} x \right) ds \right| \\ &\leq \left(\int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2 ds \right)^{\frac{1}{2}} \left(\int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}*} e^{-\frac{1}{2}(t-s)A(t)*} x\|^2 ds \right)^{\frac{1}{2}} \\ &\lesssim^{(i)} \left(\int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2 ds \right)^{\frac{1}{2}} \|x\|. \end{aligned}$$

In (i) we have used the quadratic estimate (3.4).

Taking the supremum over all $x \in H$, we obtain the following estimate

$$\begin{aligned} \int_0^\tau t^\beta \|I_2(t)\| dt &= \int_0^\tau t^\beta \|A(t) \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s) ds\|^2 dt \\ &\lesssim \int_0^\tau t^\beta \int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2 ds dt \\ &\lesssim \int_0^\tau \int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} \left(s^{\frac{\beta}{2}} f(s) \right)\|^2 ds dt. \end{aligned}$$

Let g be the function defined by $g = (\Phi f)$. Using the Fubini theorem and

the inequality $(x + y)^2 \leq 2x^2 + 2y^2$, we obtain

$$\begin{aligned}
& \int_0^\tau \int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} [s^{\frac{\beta}{2}} f(s)]\|^2 ds dt \\
& \leq 2 \int_0^\tau \int_{\frac{t}{2}}^t \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} g(s)\|^2 ds dt \\
& + 2 \int_0^\tau \int_{\frac{t}{2}}^t \|(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}) g(s)\|^2 ds dt \\
& \leq 2 \int_0^\tau \int_s^{2s} \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} g(s)\|^2 dt ds \\
& + 2 \int_0^\tau \int_{\frac{t}{2}}^t \|(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}) g(s)\|^2 ds dt \\
& \lesssim \|g\|_{L^2(0,\tau;H)}^2 + \int_0^\tau \int_{\frac{t}{2}}^t \|(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}) g(s)\|^2 ds dt.
\end{aligned}$$

The functional calculus for a sectorial operators gives

$$\begin{aligned}
& A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} \\
& = \int_\Gamma \lambda^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)\lambda} (\lambda - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda - \mathcal{A}(s))^{-1} d\lambda.
\end{aligned}$$

Hence

$$\begin{aligned}
& \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(H)} \\
& \leq \int_\Gamma |\lambda|^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)\operatorname{Re} \lambda} \|(\lambda - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(V',H)} \|(\mathcal{A}(t) - \mathcal{A}(s))\|_{\mathcal{L}(V,V')} \|(\lambda - \mathcal{A}(s))^{-1}\|_{\mathcal{L}(H,V)} |d\lambda|.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(H)} \\
& \leq \int_0^\infty |\lambda|^{-\frac{1}{2}} e^{-\frac{1}{2}(t-s)\cos(\gamma)|\lambda|} d|\lambda| \|(\mathcal{A}(t) - \mathcal{A}(s))\|_{\mathcal{L}(V,V')}.
\end{aligned}$$

Then

$$\|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(H)} \lesssim \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')}}{(t-s)^{\frac{1}{2}}}.$$

Therefore

$$\begin{aligned}
& \int_0^\tau \int_{\frac{t}{2}}^t \|(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}) g(s)\|^2 ds dt \\
& \lesssim \int_0^\tau \int_{\frac{t}{2}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')}^2}{t-s} \|g(s)\|^2 ds dt \\
& \lesssim \sup_{s \in [0,\tau]} \int_s^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')}^2}{t-s} dt \|g\|_{L^2(0,\tau;H)}^2 \\
& \lesssim \tau^{2\epsilon} \|\mathcal{A}\|_{C^\epsilon([0,\tau];\mathcal{L}(V,V'))}^2 \|f\|_{L^2_\beta(0,\tau;H)}^2.
\end{aligned}$$

This completes the proof of the Proposition 3.9. \square

Proposition 3.10. For $\beta \geq 1$ we have that the operator L is not bounded in $L^2_\beta(0, \tau; H)$ in general.

Proof. Let $u \in H$ and $g \in L^2_{-\beta}(0, \tau; H)$. It's very seen that

$$(L^*g)(t) = \int_t^\tau A(s)^* e^{-(s-t)A(s)^*} g(s) ds$$

and $L \in \mathcal{L}(L^2_\beta(0, \tau; H))$ if and only if $L^* \in \mathcal{L}(L^2_{-\beta}(0, \tau; H))$.

If $A(s)^* = A(0)^*$ for all $s \in [0, \tau]$, then $(L^*g)(t) = \int_t^\tau A(0)^* e^{-(s-t)A(0)^*} g(s) ds$. Assume now that $\tau > 1$ and take $g(s) = \mathbb{1}_{[1, \tau]}(s)u$, so

$$(L^*g)(t) = e^{-(\tau-t)A(0)^*} u - e^{-(1-t)A(0)^*} u,$$

which converges to $e^{-\tau A(0)^*} u - e^{-A(0)^*} u$ as $t \rightarrow 0$. We claim that

$$e^{-\tau A(0)^*} u - e^{-A(0)^*} u \neq 0,$$

then

$$\begin{aligned} \|L^*g\|_{L^2_{-\beta}(0, \tau; H)}^2 &\geq \|L^*g\|_{L^2_{-\beta}(0, 1; H)}^2 \\ &= \int_0^1 \|e^{-(\tau-t)A(0)^*} u - e^{-(1-t)A(0)^*} u\|_H^2 \frac{dt}{t^\beta} = \infty. \end{aligned}$$

Now, suppose that $e^{-\tau A(0)^*} u - e^{-A(0)^*} u = 0$, then we have

$$e^{-(2\tau-1)A(0)^*} u = e^{-A(0)^*} u.$$

So for all $n \in \mathbb{N}$ and by using induction we get

$$e^{-(n(\tau-1)+1)A(0)^*} u - e^{-A(0)^*} u = 0.$$

Since $\|A(0)^* e^{-(n(\tau-1)+1)A(0)^*} A(0)^{* -1} u\|_H \lesssim \frac{1}{(n(\tau-1)+1)} \|A(0)^{* -1} u\|$, by letting $n \rightarrow \infty$ it follows that $e^{-A(0)^*} u = 0$. Hence, for all $t \geq 1$, we get $e^{-tA(0)^*} u = 0$. We deduce that $u = 0$ by an application of the isolated point theorem and the analyticity of the semigroup. \square

Lemma 3.11. For all $f \in L^2_\beta(0, \tau; H)$, $t \in [0, \tau]$ and $\beta < 1$, we have $(L_1 f)(t) \in V$, where

$$(L_1 f)(t) = t^{\frac{\beta}{2}} \int_0^t e^{-(t-s)A(t)} f(s) ds.$$

Proof. We write

$$(L_1 f)(t) = t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s) ds + t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s) ds.$$

A straightforward computation gives

$$\begin{aligned} \|t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s) ds\|_V &\lesssim t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \|e^{-(t-s)A(t)}\|_{\mathcal{L}(H,V)} \|f(s)\| ds \\ &\lesssim t^{\frac{\beta}{2}} \left(\int_0^{\frac{t}{2}} s^{-\beta-1} ds \right)^{\frac{1}{2}} \|f\|_{L^2_\beta(0,\tau;H)} \\ &\lesssim \|f\|_{L^2_\beta(0,\tau;H)}. \end{aligned}$$

Using the Lemma (3.4), to deduce

$$\begin{aligned} \|t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s) ds\|_V &\lesssim \left\| \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} \left(s^{\frac{\beta}{2}} f(s) \right) ds \right\|_V \\ &\lesssim \|f\|_{L^2_\beta(0,\tau;H)}. \end{aligned}$$

Then we get the result. \square

Lemma 3.12. For all $u_0 \in (H; D(A(0)))_{\frac{1-\beta}{2},2}$ and $\beta \in [0, 1)$, we have

$$\int_0^\tau \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0\|_H^2 dt \simeq \|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2},2}}^2.$$

Proof. Note that $(H; D(A(0)))_{\frac{1-\beta}{2},2} = D(A(0))^{\frac{1-\beta}{2}}$. If $\beta \in [0, 1)$, by using the quadratic estimate we obtain

$$\begin{aligned} &\int_0^\tau \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0\|_H^2 dt \\ &= \int_0^\tau \|t^{\frac{\beta}{2}} A(0)^{\frac{1+\beta}{2}} e^{-tA(0)} A(0)^{\frac{1-\beta}{2}} u_0\|_H^2 dt \\ &\lesssim \int_0^\tau \|A(0)^{\frac{1}{2}} e^{-\frac{t}{2}A(0)} A(0)^{\frac{1-\beta}{2}} u_0\|_H^2 dt \\ &\lesssim \|A(0)^{\frac{1-\beta}{2}} u_0\|^2 = \|u_0\|_{[H; D(A(0))]_{\frac{1-\beta}{2}}}^2 \\ &\lesssim \|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2},2}}^2. \end{aligned}$$

Conversely, we know that (See [22], Definition 1.1.1)

$$\|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2},2}}^2 = \int_0^1 t^{\beta-2} \|K(t, u_0)\|_H^2 dt,$$

where

$$K(t, u_0) = \inf_{u_0=a+b; a \in H, b \in D(A(0))} \left(\|a\|_H + t \|b\|_{D(A(0))} \right).$$

This allows us to write for $t \in [0, \tau]$

$$\begin{aligned} u_0 &= \left(u_0 - e^{-tA(0)} u_0 \right) + e^{-tA(0)} u_0 \\ &= - \int_0^t A(0) e^{-lA(0)} u_0 dl + e^{-tA(0)} u_0. \end{aligned}$$

Since $e^{-tA(0)}u_0 \in D(A(0))$ a.e $t \in [0, \tau]$ and $(u_0 - e^{-tA(0)}u_0) \in H$, it follows that

$$\|K(t, u_0)\|_H \leq \int_0^t \|A(0)e^{-lA(0)}u_0\| dl + t\|A(0)e^{-tA(0)}u_0\|.$$

Roughly speaking, by Hardy inequality (2.1), we have

$$\|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}}^2 \lesssim \int_0^\tau \|t^{\frac{\beta}{2}}A(0)e^{-tA(0)}u_0\|_H^2 dt.$$

This completes the proof of the Lemma. \square

Remark 3.13. As consequence of the previous lemma the orbit $t \rightarrow e^{-tA(0)}u_0$ belongs to the space $W_\beta^{1,2}(0, \tau; H) \cap L_\beta^2(0, \tau; D(A(0)))$ if and only if $u_0 \in (H; D(A(0)))_{\frac{1-\beta}{2}, 2}$.

We define the space

$$W_\beta(D(A(\cdot)), H) = \{u \in W^{1,1}(0, \tau; H), \text{ s.t } A(\cdot)u \in L_\beta^2(0, \tau; H), \dot{u} \in L_\beta^2(0, \tau; H)\},$$

endowed with norm

$$\|u\|_{W_\beta(D(A(\cdot)), H)} = \|A(\cdot)u\|_{L_\beta^2(0, \tau; H)} + \|\dot{u}\|_{L_\beta^2(0, \tau; H)}.$$

It is easy to see that $W_\beta(D(A(\cdot)), H) \hookrightarrow W_\beta^{1,2}(0, \tau; H)$.

Lemma 3.14. For all $\gamma \leq \frac{1}{2}$, we have $(H, D(A(0)))_{\gamma, 2} = [H, V]_{2\gamma}$ and for $\gamma > \frac{1}{2}$ we get $(H, D(A(0)))_{\gamma, 2} \hookrightarrow V$.

Proof. As a consequence of the interpolation method (See [22], Remark 1.3.6) we have for $\gamma \leq \frac{1}{2}$,

$$(H, D(A(0)))_{\gamma, 2} = (H, D(A(0)^{\frac{1}{2}}))_{2\gamma, 2} = (H, V)_{2\gamma, 2}.$$

Since H and V are Hilbert spaces we get by the Lemma 3.2

$$(H, D(A(0)))_{\gamma, 2} = (H, V)_{2\gamma, 2} = [H, V]_{2\gamma}.$$

Let $v \in D(A(0))$ and $\gamma > \frac{1}{2}$. We obtain

$$\begin{aligned} \delta\|v\|_V^2 &\leq \operatorname{Re} (A(0)v, v) \\ &\lesssim \|A(0)^\gamma v\|_H \|[A(0)^*]^{1-\gamma} v\|_H \\ &\lesssim \|A(0)^\gamma v\|_H \|v\|_{[H, V]_{2(1-\gamma)}} \\ &\lesssim \|A(0)^\gamma v\|_H \|v\|_V. \end{aligned}$$

Therefore we have that for all $\gamma > \frac{1}{2}$ and $v \in D(A(0))$

$$\|v\|_V \lesssim \|v\|_{D(A(0)^\gamma)}.$$

Finally, by the density of $D(A(0))$ in $D(A(0)^\gamma)$ we get the desired result. \square

4 Maximal regularity for autonomous problem

In this section we are interesting in the regularity of the following problem

$$\begin{cases} \dot{u}(t) + \mathcal{A}(0)u(t) = f(t) \\ u(0) = u_0. \end{cases} \quad (4.1)$$

Theorem 4.1. *For all $f \in L^2_\beta(0, \tau, H)$ and $u_0 \in (H; D(A(0)))_{\frac{1-\beta}{2}, 2}$ if $\beta \geq 0$ and $u_0 = 0$ if $\beta < 0$, there is a unique $u \in W_\beta(D(A(0)), H) \cap L^\infty_\beta(0, \tau; V)$ be the solution of the problem (4.1). We have also the following embedding*

$$W_\beta(D(A(0)), H) \hookrightarrow C([0, \tau]; (H; D(A(0)))_{\frac{1-\beta}{2}, 2}).$$

For $\beta \in [0, 1[$ we have

$$W_\beta(D(A(0)), H) \hookrightarrow W^{\frac{1}{2}, 2}_\beta(0, \tau; V).$$

Proof. Since $A(0)$ is an analytic semigroup in H , it is very knowing that by the variation of constants formula the solution of the Problem (4.1) is given by

$$u(t) = e^{-tA(0)}u_0 + \int_0^t e^{-(t-s)A(0)}f(s) ds.$$

Thus, it follows

$$\begin{aligned} A(0)u(t) &= A(0)e^{-tA(0)}u_0 + A(0) \int_0^t e^{-(t-s)A(0)}f(s) ds \\ &:= (Fu_0)(t) + (Lf)(t). \end{aligned}$$

Then by the Lemmas 3.12 and 3.11 and the Proposition 3.9, we obtain

$$\begin{aligned} \|A(0)u\|_{L^2_\beta(0, \tau; H)} &\leq \|Fu_0\|_{L^2_\beta(0, \tau; H)} + \|Lf\|_{L^2_\beta(0, \tau; H)} \\ &\leq C \left(\|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} + \|f\|_{L^2_\beta(0, \tau; H)} \right). \end{aligned}$$

Since $\dot{u} = f - A(0)u \in L^2_\beta(0, \tau; H)$, we get finally

$$\|u\|_{W_\beta(D(A(0)), H)} \leq C' \left(\|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} + \|f\|_{L^2_\beta(0, \tau; H)} \right). \quad (4.2)$$

Using Proposition 5.1 and (4.2) we obtain for all $t \in [0, \tau]$

$$\begin{aligned} \|u(t)\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} &\lesssim \|u\|_{W_\beta(D(A(0)), H) \cap L^\infty_\beta(0, \tau; V)} \\ &\lesssim \|u_0\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} + \|f\|_{L^2_\beta(0, \tau; H)}. \end{aligned} \quad (4.3)$$

For $0 \leq s \leq l \leq t \leq \tau$ we set $v(l) = e^{-(t-l)A(0)}u(l)$. Then

$$u(t) - u(s) = \left(e^{-(t-s)A(0)} - I\right)u(s) + \int_s^t e^{-(t-l)A(0)}f(l) dl. \quad (4.4)$$

Observe that $e^{-(t-s)A(0)}$ is strongly continuous on $(H; D(A(0)))_{\frac{1-\beta}{2}, 2}$. In particular, this ensures that $\|(e^{-(t-s)A(0)} - I)u(s)\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} \rightarrow 0$ as $t \rightarrow s$.

The Estimate (4.3) for the case $u_0 = 0$ gives that

$$\left\| \int_s^t e^{-(t-l)A(0)}f(l) dl \right\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}} \lesssim \|f\|_{L_\beta^2(s, t; H)}.$$

It follows that $u(t)$ is right continuous on $(H; D(A(0)))_{\frac{1-\beta}{2}, 2}$.

Now, we set $v(l) = e^{-(l-s)A(0)}u(r)$, for $0 \leq s \leq l \leq t$.

Then

$$u(s) - u(t) = \left(e^{-(t-s)A(0)} - I\right)u(t) + \int_s^t e^{-(l-s)A(0)}\left(f(l) + 2A(0)u(l)\right) dl.$$

The same argument with the right continuous gives that u is left continuous in $(H; D(A(0)))_{\frac{1-\beta}{2}, 2}$. Thus $u \in C([0, \tau]; (H; D(A(0)))_{\frac{1-\beta}{2}, 2})$.

In the next we prove that $W_\beta(D(A(0)), H) \hookrightarrow W_\beta^{\frac{1}{2}, 2}(0, \tau; V)$.

Let $\beta \in [0, 1[$ and $u \in C^\infty([0, \tau]; D(A(0)))$. We recall that

$$\|u\|_{W_\beta^{\frac{1}{2}, 2}(0, \tau; V)}^2 = \|u\|_{L_\beta^2(0, \tau; V)}^2 + \int_0^\tau \int_0^t \frac{\|u(t) - u(s)\|_V^2}{|t - s|^2} s^\beta ds dt.$$

By (4.4) it holds that for all $0 \leq s \leq t \leq \tau$

$$\begin{aligned} u(t) - u(s) &= \left(e^{-(t-s)A(0)}u(s) - u(s)\right) + \int_s^t e^{-(t-l)A(0)}f(l) dl \\ &:= L_1(t, s) + L_2(t, s), \end{aligned}$$

where $f(l) = A(0)u(l) + \dot{u}(l)$.

So that

$$\begin{aligned} \|u\|_{W_\beta^{\frac{1}{2}, 2}(0, \tau; V)}^2 &\leq \|u\|_{L_\beta^2(0, \tau; V)}^2 + 2 \int_0^\tau \int_0^t \frac{\|L_1(t, s)\|_V^2}{|t - s|^2} s^\beta ds dt \\ &\quad + 2 \int_0^\tau \int_0^t \frac{\|L_2(t, s)\|_V^2}{|t - s|^2} s^\beta ds dt. \end{aligned}$$

We write

$$L_1(t, s) = e^{-(t-s)A(0)}u(s) - u(s) = \int_0^{t-s} e^{-lA(0)}A(0)u(s) dl.$$

Therefore by the Hardy inequality of the Lemma 2.1 and the quadratic estimate we have

$$\begin{aligned}
& \int_0^\tau \int_0^t \frac{\|L_1(t, s)\|_V^2}{|t-s|^2} s^\beta ds dt \\
& \leq \int_0^\tau \int_s^\tau \left(\frac{\int_0^{t-s} \|e^{-lA(0)} A(0) u(s)\|_V dl}{|t-s|} \right)^2 dt s^\beta ds \\
& \leq C \int_0^\tau \int_s^\tau \|e^{-tA(0)} A(0) u(s)\|_V^2 dt s^\beta ds \\
& \leq C' \int_0^\tau \|A(0) u(s)\|_H^2 s^\beta ds \\
& \leq C' \|A(0) u\|_{L_\beta^2(0, \tau; H)}^2.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& \int_0^\tau \int_0^t \frac{\|L_2(t, s)\|_V^2}{|t-s|^2} s^\beta ds dt \\
& \leq \int_0^\tau \int_0^t \left(\frac{\int_s^t \|e^{(t-l)A(0)} (\Phi f)(l)\|_V dl}{|t-s|} \right)^2 ds dt \\
& \leq C \int_0^\tau \int_0^t \|e^{(t-s)A(0)} (\Phi f)(s)\|_V^2 ds dt \\
& = C \int_0^\tau \int_s^\tau \|e^{(t-s)A(0)} (\Phi f)(s)\|_V^2 dt ds \\
& \leq C \|\Phi f\|_{L^2(0, \tau; H)}^2 = C \|f\|_{L_\beta^2(0, \tau; H)}^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u\|_{W_\beta^{\frac{1}{2}, 2}(0, \tau; V)} & \lesssim \|A(0)u\|_{L_\beta^2(0, \tau; H)} + \|f\|_{L_\beta^2(0, \tau; H)} \\
& \lesssim \|u\|_{W_\beta(D(A(0)), H)}.
\end{aligned}$$

We note that $C^\infty([0, \tau]; D(A(0)))$ is dense in $W_\beta(D(A(0)), H)$. This shows that

$$W_\beta(D(A(0)), H) \hookrightarrow W_\beta^{\frac{1}{2}, 2}(0, \tau; V).$$

□

Remark 4.2. *The following embeddings hold*

$$1) W_\beta(D(A(0)), H) \hookrightarrow C([0, \tau]; [H, V]_{2(1-\beta)}), \text{ for } \beta \geq 0.$$

$$2) W_\beta(D(A(0)), H) \hookrightarrow C([0, \tau]; V), \text{ for } \beta \leq 0.$$

Theorem 4.3. *For all $f \in W_{\beta, 0}^{1, 2}(0, \tau, H)$, there exists a unique*

$$u \in C^1([0, \tau]; (H; D(A(0)))_{\frac{1-\beta}{2}, 2}) \cap C([0, \tau]; D(A(0))),$$

which satisfies the following equation

$$\begin{cases} \dot{u}(t) + \mathcal{A}(0)u(t) = f(t) \\ u(0) = 0. \end{cases} \quad (4.5)$$

In addition,

$$\|u\|_{C^1([0,\tau];(H;D(A(0)))_{\frac{1-\beta}{2},2}) \cap C([0,\tau];D(A(0)))} \leq C\|f\|_{W_\beta^{1,2}(0,\tau;H)}.$$

Assume now that $\tau = +\infty$ and f is a periodic function with period p . Then u satisfies

$$u(t+p) = e^{-tA}u(p) + u(t), \quad t \in [0, \infty),$$

and u is periodic with the same period p if and only if $u(p) = 0$.

Proof. According to the Theorem 4.1, there exists a unique solution u of the Problem (4.5) and for all $f \in L_\beta^2(0, \tau; H)$ the solution is given by the formula

$$u(t) = \int_0^t e^{-(t-s)\mathcal{A}(0)} f(s) ds. \quad (4.6)$$

We have also $u \in W_\beta(D(A(0)), H)$ and

$$\|u\|_{W_\beta(D(A(0)), H)} \leq C\|f\|_{L_\beta^2(0,\tau;H)}. \quad (4.7)$$

For $t \in [0, \tau]$ and $f \in W_{\beta,0}^{1,2}(0, \tau, H)$, we have by integration by parts

$$\begin{aligned} \mathcal{A}(0)u(t) &= \mathcal{A}(0) \int_0^t e^{-(t-s)\mathcal{A}(0)} f(s) ds \\ &= f(t) - \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s) ds \\ &= \dot{u}(t) + \mathcal{A}(0)u(t) - \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s) ds. \end{aligned}$$

Hence

$$\dot{u}(t) = \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s) ds = (L\dot{f})(t).$$

Now, by the Theorem 4.1 we get that $u \in C^1([0, \tau]; (H; D(A(0)))_{\frac{1-\beta}{2}, 2})$.

Since for $t \in [0, \tau]$, $\mathcal{A}(0)u(t) = f(t) - \dot{u}(t)$ we have $\mathcal{A}(0)u \in C([0, \tau]; H)$.

As a consequence, we obtain the final estimate

$$\|u\|_{C^1([0,\tau];(H;D(A(0)))_{\frac{1-\beta}{2},2}) \cap C([0,\tau];D(A(0)))} \leq C\|f\|_{W_\beta^{1,2}(0,\tau;H)}.$$

Now we take $\tau = +\infty$ and we assume that f is a periodic function with period p , i.e., $f(t+p) = f(t)$ for all $t \in [0, +\infty)$. It is clear that if u is

periodic with period p , then $u(p) = u(0) = 0$.
 By the formula (4.6), we get that for $t \in [0, \infty)$

$$u(t+p) = \int_0^{t+p} e^{-(t+p-s)A(0)} f(s) ds.$$

Then

$$\begin{aligned} u(t+p) &= \int_0^p e^{-(t+p-s)A(0)} f(s) ds + \int_p^{p+t} e^{-(t+p-s)A(0)} f(s) ds \\ &\stackrel{(i)}{=} e^{-tA} \int_0^p e^{-(p-s)A(0)} f(s) ds + \int_0^t e^{-(t-l)A(0)} f(l+p) dl \\ &\stackrel{(ii)}{=} e^{-tA} u(p) + u(t). \end{aligned}$$

In (i) we made a change of variable and the periodicity of the function f has been used in (ii).

Then u is periodic with period p if and only if $e^{-tA}u(p) = 0$ for all $t \in [0, \infty)$. Therefore by the analyticity of the semigroup we get that $u(p) = 0$ is a necessary condition for u to be periodic. \square

5 Maximal regularity for non-autonomous problem

In this section we focus with the maximal regularity for the non-autonomous problem, i.e. we prove the existence and the uniqueness of the solution for the Problem (P) in the weighted space $W_\beta^{1,2}(0, \tau; H)$.

Proposition 5.1. 1. Assume that

$$\int_0^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(V, V')}}{t} dt < \infty.$$

Then for all $s \in [0, \tau]$

$$\begin{aligned} TR_s : W_\beta(D(A(\cdot), H) \cap L^\infty(0, \tau; V)) &\longrightarrow (H; D(A(s)))_{\frac{1-\beta}{2}, 2} \\ u &\longrightarrow u(s) \end{aligned}$$

is a bounded operator.

2. For $u_0 \in (H; D(A(0)))_{\frac{1-\beta}{2}, 2}$, we obtain that

$$t \rightarrow (Fu_0)(t) = t^{\frac{\beta}{2}} A(t) e^{-tA(t)} u_0 \in L^2(0, \tau; H).$$

Proof. 1. First we consider the case $s = 0$. We have

$$\begin{aligned}
\|u(0)\|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}}^2 &= \int_0^1 \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u(0)\|_H^2 dt + \|u(0)\|_H^2 \\
&\leq 2 \int_0^1 \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} (u(0) - u(t))\|_H^2 dt + \|u(0)\|_H^2 \\
&\quad + 2 \int_0^1 \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u(t)\|_H^2 dt \\
&\lesssim \int_0^1 t^\beta \left(\frac{1}{t} \int_0^t \|\dot{u}(l)\|_H ds \right)^2 dl \\
&\quad + \int_0^\tau t^\beta \|A(t) u(t)\|_H^2 dt \\
&\quad + \int_0^\tau \|t^{\frac{\beta}{2}} [A(0) e^{-tA(0)} - A(t) e^{-tA(t)}] u(t)\|_H^2 dt + \|u(0)\|_H^2 \\
&\lesssim \|\dot{u}\|_{L_\beta^2(0, \tau; H)}^2 + \|A(\cdot) u\|_{L_\beta^2(0, \tau; H)}^2 \\
&\quad + \int_0^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(V, V')}}{t} dt \|u\|_{L_\beta^\infty(0, \tau; V)} + \|u(0)\|_H^2 \\
&\lesssim \|u\|_{W_\beta(D(A(\cdot), H))}^2 + \|u\|_{L_\beta^\infty(0, \tau; V)}^2 + \|u(0)\|_H^2,
\end{aligned}$$

where we have used the estimate

$$\|A(0) e^{-tA(0)} - A(t) e^{-tA(t)}\|_{\mathcal{L}(V, H)} \lesssim \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(V, V')}}{t^{\frac{1}{2}}}.$$

Now, we prove the result for all $s \in]0, \tau[$. Indeed, let $l \in]0, \tau[$ and we set

$$v(t) = \begin{cases} u(t+s), & t \in [0, \tau-s]. \\ u(\frac{\tau}{s}(\tau-t)), & t \in [\tau-s, \tau]. \end{cases}$$

Similarly

$$B(t) = \begin{cases} A(t+s), & t \in [0, \tau-s]. \\ A(\frac{\tau}{s}(\tau-t)), & t \in [\tau-s, \tau]. \end{cases}$$

Since $v(t) \in W_\beta(D(B(\cdot), H))$, therefore

$$v(0) = u(s) \in (H; D(B(0)))_{\frac{1-\beta}{2}, 2} = (H; D(A(s)))_{\frac{1-\beta}{2}, 2}.$$

For the case $s = \tau$, we take $v(t) = u(\tau - t)$ and $B(t) = A(\tau - t)$.

2. Note that

$$\begin{aligned}
(Fu_0)(t) &= t^{\frac{\beta}{2}} A(t) e^{-tA(t)} u_0 \\
&= t^{\frac{\beta}{2}} (A(t) e^{-tA(t)} u_0 - A(0) e^{-tA(0)} u_0) \\
&\quad + t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0.
\end{aligned}$$

For $\beta > 0$ we have by interpolation

$$\|(\lambda - A(0))^{-1}\|_{\mathcal{L}((H; D(A(0)))_{\frac{1-\beta}{2}, 2}, V)} \lesssim \frac{1}{|\lambda|^{1-\frac{\beta}{2}}}.$$

Therefore

$$\begin{aligned} & \| (Fu_0)(t) \|_H \\ & \lesssim \frac{\| \mathcal{A}(0) - \mathcal{A}(t) \|_{\mathcal{L}(V, V')}}{t} \| u_0 \|_{(H; D(A(0)))_{\frac{1-\beta}{2}}} \\ & + \| t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0 \|_H. \end{aligned}$$

Hence,

$$\begin{aligned} & \| (Fu_0) \|_{L^2(0, \tau; H)}^2 \\ & \lesssim \int_0^\tau \frac{\| \mathcal{A}(0) - \mathcal{A}(t) \|_{\mathcal{L}(V, V')}^2}{t} dt \| u_0 \|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}}^2 \\ & + \int_0^\tau \| t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0 \|_H^2 dt \\ & \lesssim \| u_0 \|_{(H; D(A(0)))_{\frac{1-\beta}{2}, 2}}^2. \end{aligned}$$

This shows 2. □

In the next of this paper we consider only the case $\beta \in [0, 1[$.

Proposition 5.2. *Suppose that $\mathcal{A} \in C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))$. Then for all $f \in L_\beta^2(0, \tau; H)$, $u_0 \in (H; D(A(0)))_{\frac{1-\beta}{2}, 2}$, and for τ is small enough, there is a unique u such that $u \in L_\beta^\infty(0, \tau; V)$, where u is the solution of the non-autonomous Cauchy problem*

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0. \end{cases} \quad (5.1)$$

Proof. Let $f \in L_\beta^2(0, \tau; H)$. We set $v(s) = e^{-(t-s)A(t)}u(s)$. Since $u(t) = e^{-tA(t)}u_0 + \int_0^t \dot{v}(s) ds$, therefore

$$\begin{aligned} u(t) &= e^{-tA(t)}u_0 + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) ds + \int_0^t e^{-(t-s)A(t)}f(s) ds \\ &:= (Mu_0)(t) + (M_1u)(t) + (L_1f)(t). \end{aligned}$$

Moreover, for $\beta > 0$ and $u_0 \in (H, D(A(0)))_{\frac{1-\beta}{2}, 2}$ we have by interpolation argument

$$\|e^{-tA(t)}u_0\|_V \lesssim t^{-\frac{\beta}{2}}\|u_0\|_{(H,D(A(0)))_{\frac{1-\beta}{2},2}}.$$

In view of the Lemma (3.11) $\cdot^{\frac{\beta}{2}}M, \cdot^{\frac{\beta}{2}}L_1$ are bounded in V .

In what follows, we prove that $t^{\frac{\beta}{2}}(M_1u)(t) \in L^\infty(0, \tau; V)$, for all $t^{\frac{\beta}{2}}u \in L^\infty(0, \tau; V)$. We write

$$\begin{aligned} (M_1u)(t) &= \int_0^{\frac{t}{2}} e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &\quad + \int_{\frac{t}{2}}^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &:= (M_{11}u)(t) + (M_{12}u)(t). \end{aligned}$$

By taking $x \in V'$ we obtain the following estimate

$$\begin{aligned} &|((M_{12}u)(t), x)_{V' \times V}| \\ &= \left| \int_{\frac{t}{2}}^t (e^{-\frac{(t-s)}{2}A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s), [A(t)^*]^{\frac{1}{2}}e^{-\frac{(t-s)}{2}A(t)^*}[A(t)^*]^{-\frac{1}{2}}x)_H ds \right| \\ &\leq \left(\int_{\frac{t}{2}}^t \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(V',H)}^2 \|\mathcal{A}(t) - \mathcal{A}(s)u(s)\|_{V'}^2 ds \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\frac{t}{2}}^t \|[A(t)^*]^{\frac{1}{2}}e^{-\frac{(t-s)}{2}A(t)^*}[A(t)^*]^{-\frac{1}{2}}x\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Now, we estimate M_{11} by the following

$$\begin{aligned} &t^{\frac{\beta}{2}}\|(M_{11}v)(t)\|_V \\ &\lesssim t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(V',V)} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')} s^{-\frac{\beta}{2}} ds \|s^{\frac{\beta}{2}}u(\cdot)\|_{L^\infty(0, \frac{t}{2}; V)} \\ &\lesssim t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \frac{s^{-\frac{\beta}{2}}}{(t-s)^{1-\varepsilon}} ds \sup_{s \in [0, \frac{t}{2}]} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')}}{(t-s)^\varepsilon} \|s^{\frac{\beta}{2}}u(\cdot)\|_{L^\infty(0, \frac{t}{2}; V)}. \end{aligned}$$

Note that

$$t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \frac{s^{-\frac{\beta}{2}}}{(t-s)^{1-\varepsilon}} ds = t^\varepsilon \int_0^{\frac{1}{2}} \frac{l^{-\frac{\beta}{2}}}{(1-l)^{1-\varepsilon}} dl.$$

Therefore

$$\begin{aligned} &t^{\frac{\beta}{2}}\|(M_1v)(t)\|_V \\ &\lesssim t^\varepsilon \|\mathcal{A}\|_{C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))} \|s^{\frac{\beta}{2}}u(\cdot)\|_{L^\infty(0, \frac{t}{2}; V)} \\ &\quad + \left(\int_{\frac{t}{2}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V, V')}^2}{t-s} ds \right)^{\frac{1}{2}} \|u\|_{L^\infty_\beta(\frac{t}{2}, t; V)} \\ &\lesssim t^\varepsilon \|\mathcal{A}(\cdot)\|_{C^\varepsilon([0, \tau]; \mathcal{L}(V, V'))} \|u\|_{L^\infty_\beta(0, t; V)}. \end{aligned}$$

By taking τ small enough we may arrange that

$$M_1 \in \mathcal{L}(L_\beta^\infty(0, \tau; V)),$$

with norm $\|M_1\|_{\mathcal{L}(L_\beta^\infty(0, \tau; V))} < 1$, therefore $(I - M_1)$ is invertible in $L_\beta^\infty(0, \tau; V)$. Hence

$$u = (I - M_1)^{-1}(Mu_0 + L_1f) \in L_\beta^\infty(0, \tau; V).$$

This finishes the proof. \square

Our principal result in the non-autonomous case is the following

Theorem 5.3. *Suppose that $\mathcal{A} \in W^{\frac{1}{2}, 2}(0, \tau; \mathcal{L}(V, V')) \cap C^\varepsilon([0, \tau], \mathcal{L}(V, V'))$, with $\varepsilon > 0$, then for all $f \in L_\beta^2(0, \tau; H)$ and $u_0 \in (H; D(A(0))_{\frac{1-\beta}{2}})$, there is a unique $u \in W_\beta(D(A(\cdot), H))$ be the solution of the Cauchy problem*

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0. \end{cases} \quad (5.2)$$

Proof. Let τ be small enough and $f \in L_\beta^2(0, \tau; H)$, $u_0 \in (H; D(A(0))_{\frac{1-\beta}{2}, 2})$. Then by the Proposition 5.2 we have $u \in L_\beta^\infty(0, \tau; V)$, while u is the solution of the Cauchy problem (5.2).

Let $0 \leq s \leq t \leq \tau$ and we set $v(s) = e^{-(t-s)A(t)}u(s)$. Since $v(t) = v(0) + \int_0^t \dot{v}(s) ds$, therefore

$$\begin{aligned} A(t)u(t) &= A(t)e^{-tA(t)}u_0 + A(t) \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &\quad + A(t) \int_0^t e^{-(t-s)A(t)} f(s) ds \\ &:= (Fu_0)(t) + (Su)(t) + (Lf)(t). \end{aligned}$$

The only thing to check is that $Su \in L_\beta^2(0, \tau; H)$. In fact, take $g \in L^2(0, \tau; H)$, then the following relation holds

$$\begin{aligned} &|(\cdot^{\frac{\beta}{2}} Su, g)_{L^2(0, \tau; H)}| \\ &= \left| \int_0^\tau t^{\frac{\beta}{2}} \int_0^t \langle (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t) \rangle_{V' \times V} ds dt \right| \\ &= \left| \int_0^\tau t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \langle (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t) \rangle_{V' \times V} ds dt \right| \\ &\quad + \left| \int_0^\tau t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t \langle (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t) \rangle_{V' \times V} ds dt \right| \\ &:= I_1 + I_2. \end{aligned}$$

For I_2 we have

$$\begin{aligned}
I_2 &\lesssim \int_0^\tau t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')} \|e^{-\frac{(t-s)}{2}A(t)^*}\|_{\mathcal{L}(H,V)} \\
&\quad \times \|[A(t)^*]^{\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*}\|_{\mathcal{L}(H)} \|A(t)^{\frac{1}{2}*} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|_H s^{-\frac{\beta}{2}} ds dt \|\cdot\|^{\frac{\beta}{2}} u\|_{L^\infty(0,\tau;V)} \\
&\lesssim \int_0^\tau \int_{\frac{t}{2}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V,V')}}{t-s} \|A(t)^{\frac{1}{2}*} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|_H ds dt \|\cdot\|^{\frac{\beta}{2}} u\|_{L^\infty(0,\tau;V)} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(V,V'))} \left(\int_0^\tau \int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}*} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|_H^2 ds dt \right)^{\frac{1}{2}} \|u\|_{L^\infty(0,\tau;V)} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(V,V'))} \|g\|_{L^2(0,\tau,H)} \|u\|_{L^\infty(0,\tau;V)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_1 &\lesssim \int_0^\tau t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \frac{s^{-\frac{\beta}{2}}}{(t-s)^{\frac{3}{2}-\varepsilon}} \|g(t)\|_H ds dt \\
&\quad \times \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(V,V'))} \|\cdot\|^{\frac{\beta}{2}} u\|_{L^\infty(0,\tau;V)} \\
&\lesssim \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(V,V'))} \|g\|_{L^2(0,\tau,H)} \|u\|_{L^\infty(0,\tau;V)}.
\end{aligned}$$

Finally, we get the final estimate

$$\begin{aligned}
\|A(\cdot)u\|_{L_\beta^2(0,\tau;H)} &\lesssim \|Fu_0\|_{L_\beta^2(0,\tau;H)} + \|Su\|_{L_\beta^2(0,\tau;H)} + \|Lf\|_{L_\beta^2(0,\tau;H)} \\
&\lesssim \|u_0\|_{(H;D(A(0)))_{\frac{1-\beta}{2},2}} + \|u\|_{L_\beta^\infty(0,\tau;V)} + \|f\|_{L_\beta^2(0,\tau;H)} \\
&\lesssim \|u_0\|_{(H;D(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L_\beta^2(0,\tau;H)}.
\end{aligned}$$

Therefore $A(\cdot)u \in L_\beta^2(0,\tau;H)$. Since $\dot{u} = f - Au$, then $\dot{u} \in L_\beta^2(0,\tau;H)$, and so $u \in W_\beta(D(A(\cdot)), H)$. We note that by the Proposition 5.1 we have $u(t) \in (H; D(A(t)))_{\frac{1-\beta}{2},2}$ for all $t \in [0, \tau]$.

For arbitrary τ we split the interval $[0, \tau]$ into union of small intervals and similarly we use the same procedure as before to each subinterval. Finally we stick the solutions and we get the desired result. \square

Remark 5.4. *By the same assumption of the Theorem 5.3 and as in the proof of the Theorem 4.1 we may prove that*

$$W_\beta(D(A(\cdot)), H) \hookrightarrow W_\beta^{\frac{1}{2},2}(0,\tau;V).$$

Proposition 5.5. *For all $g \in L^2(0,\tau;H)$ and $0 \leq \beta < 1$ there exists a unique $v \in W_0(D(A(\cdot)), H)$ be the solution of the singular equation*

$$\begin{cases} \dot{v}(t) + \mathcal{A}(t)v(t) - \frac{\beta}{2} \frac{v(t)}{t} = g(t) \\ v(0) = 0. \end{cases} \quad (5.3)$$

Proof. We set $f(t) = (\Phi g)(t) = t^{\frac{\beta}{2}}g(t)$ with $t \in [0, \tau]$, so that $f \in L^2_{\beta}(0, \tau; H)$. Let $u \in W_{\beta}(D(A(\cdot), H))$ be the solution of the Problem

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = 0. \end{cases} \quad (5.4)$$

Now, we set $v = (\Phi^{-1}u)$. Then $v \in W_0(D(A(\cdot), H))$ and v is the unique solution of the Problem (5.3). \square

6 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality.

6.1 Elliptic operators in the divergence form

Let Ω be a bounded Lipschitz domain of \mathbb{R}^n . We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$ and we define the sesquilinear forms

$$\mathfrak{a}(t, u, v) := \int_{\Omega} C(t, x) \nabla u \overline{\nabla v} dx$$

where here $u, v \in V$ and $C : [0, \tau] \times \Omega \rightarrow \mathbb{C}^{n \times n}$ is a bounded and measurable function for which there exists $\alpha, M > 0$ such that

$$\alpha |\xi|^2 \leq \operatorname{Re} (C(t, x) \xi \cdot \bar{\xi}) \text{ and } |C(t, x) \xi \cdot \nu| \leq M |\xi| |\nu|$$

for all $t \in [0, \tau]$ and a.e $x \in \Omega$, and all $\xi, \nu \in \mathbb{C}^n$. We define the gradient operator $\nabla : V \rightarrow H$ and $\nabla^* : H \rightarrow V'$. The non-autonomous form $\mathfrak{a}(t)$ induces the operators

$$\mathcal{A}(t) := -\nabla^* C(t, x) \nabla \in \mathcal{L}(V, V').$$

The form $\mathfrak{a}(t)$ is $H^1(\Omega)$ -bounded and coercive. The part of $\mathcal{A}(t)$ in H is the operator

$$A(t) := -\operatorname{div} C(t, x) \nabla$$

under Neumann boundary conditions. We note that

$$\|\mathcal{A}(t)\|_{\mathcal{L}(V, V')} \simeq \|C(t, \cdot)\|_{L^\infty(\Omega; \mathbb{C}^{n \times n})} = M.$$

Next, we suppose that $C \in W^{\frac{1}{2}, 2}(0, \tau; L^\infty(\Omega; \mathbb{C}^{n \times n})) \cap C^\varepsilon([0, \tau]; L^\infty(\Omega; \mathbb{C}^{n \times n}))$, with $\varepsilon > 0$. which is equivalent to

$$\int_0^\tau \int_0^\tau \sup_{x \in \Omega} \frac{\|C(t, x) - C(s, x)\|^2}{|t - s|^2} ds dt < \infty,$$

$$\frac{\|C(t, x) - C(s, x)\|}{|t - s|^\varepsilon} < \infty$$

a.e for $x \in \Omega$.

We note that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(V, V')} \lesssim \|C(t, \cdot) - C(s, \cdot)\|_{L^\infty(\Omega; \mathbb{C}^{n \times n})}.$$

Therefore we get

$$\mathcal{A} \in W^{\frac{1}{2}, 2}(0, \tau; \mathcal{L}(V, V')) \cap C^e([0, \tau]; \mathcal{L}(V, V')).$$

Remark 6.1. $D(A(t)^{\frac{1}{2}}) = V = H^1(\Omega)$ for all $t \in [0, \tau]$ and

$$c_1 \|u\|_{H^1(\Omega)} \leq \|u\|_{D(A(t)^{\frac{1}{2}})} \leq c^1 \|u\|_{H^1(\Omega)}$$

where c_1, c^1 are two positive constants independent of t (see [7], Theorem 1).

In the next theorem we assume that $\beta \in [0, 1[$.

Proposition 6.2. For all $f \in L^2_\beta(0, \tau; L^2(\Omega))$, $u_0 \in H^{1-\beta}(\Omega)$ there is a unique $u \in W_\beta(D(A(\cdot), L^2(\Omega)))$, be the solution of the following problem

$$\begin{cases} \dot{u}(t) - \operatorname{div} C(t, x) \nabla u(t) = f(t) \\ \frac{\partial u(t, \sigma)}{\partial n} = 0 \quad (\sigma \in \partial\Omega) \\ u(0) = u_0. \end{cases} \quad (6.1)$$

The proposition follows by Theorems 5.3.

6.2 Robin boundary conditions

Let Ω be a bounded domain of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0, \tau] \times \partial\Omega \rightarrow [0, \infty)$ be bounded function and $H := L^2(\Omega)$. We define the form

$$\mathfrak{a}(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta(\cdot) \operatorname{Tr}(u) \operatorname{Tr}(v) \, d\sigma,$$

for all $u, v \in V := H^1(\Omega)$. The form \mathfrak{a} is $H^1(\Omega)$ -bounded, symmetric and quasi-coercive. The first statement follows readily from the continuity of the trace operator and the boundedness of β . The second one is a consequence of the inequality

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leq \delta \|u\|_{H^1(\Omega)}^2 + C_\delta \|u\|_{L^2(\Omega)}^2$$

which is valid for all $\delta > 0$ (C_δ is a constant depending on δ). Note that this is a consequence of compactness of the trace as an operator from $H^1(\Omega)$

into $L^2(\partial\Omega, d\sigma)$.

Formally, the associated operator A is (minus) the Laplacian with the time dependent Robin boundary condition

$$\frac{\partial u}{\partial n} + \beta(\cdot)u = 0 \text{ on } \partial\Omega.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative in the weak sense.

By using Theorems 4.1 and 4.3, we get the following result

Proposition 6.3. *Let $\beta \in]-1, 1[$ and $f \in L^2_\beta(0, \tau; L^2(\Omega))$. There exists a unique $u \in W_\beta(D(A), L^2(\Omega)) \cap C([0, \tau], (L^2(\Omega); D(A))_{\frac{1-\beta}{2}, 2})$ be the solution of the problem*

$$\begin{cases} \dot{u}(t) - \Delta u(t) = f(t) \\ \frac{\partial u}{\partial n} + \beta(\cdot)u = 0 \text{ on } \partial\Omega \\ u(0) = 0. \end{cases} \quad (6.2)$$

If we assume moreover that $f \in W_{\beta,0}^{1,2}(0, \tau; L^2(\Omega))$, then the solution u is in $C^1([0, \tau]; (L^2(\Omega); D(A))_{\frac{1-\beta}{2}, 2}) \cap C([0, \tau]; D(A))$.

Remark 6.4. *We note that for $\beta \in [0, 1[$ we have*

$$(L^2(\Omega); D(A))_{\frac{1-\beta}{2}, 2} = [L^2(\Omega); D(A)]_{\frac{1-\beta}{2}} = [L^2(\Omega); H^1(\Omega)]_{1-\beta} = H^{1-\beta}(\Omega).$$

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