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# QUANTITATIVE HOMOGENIZATION OF DIFFERENTIAL FORMS

Paul Dario

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# QUANTITATIVE HOMOGENIZATION OF DIFFERENTIAL FORMS

PAUL DARIO

ABSTRACT. We develop a quantitative theory of stochastic homogenization in the more general framework of differential forms. Inspired by recent progress in the uniformly elliptic setting, the analysis relies on the study of certain subadditive quantities. We establish an algebraic rate of convergence from these quantities and deduce from this an algebraic error estimate for the homogenization of the Dirichlet problem. Most of the ideas needed in this article comes from two distinct theory, the theory of quantitative stochastic homogenization, and the generalization of the main results of functional analysis and of the regularity theory of second-order elliptic equations to the setting of differential forms.

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## 1. INTRODUCTION

The classical theory of stochastic homogenization focuses on the study of the second-order elliptic equation

$$(1.1) \quad \nabla \cdot (\mathbf{a}(x)\nabla u) = 0,$$

where  $\mathbf{a}$  is a random, rapidly oscillating, uniformly elliptic coefficient field with law  $\mathbb{P}$ . The basic qualitative result roughly states that, under appropriate assumptions on  $\mathbb{P}$ , a solution  $u_r$  of (1.1) in  $B(0, r)$ , the ball of center 0 and radius  $r$ , converges as  $r \rightarrow \infty$ ,  $\mathbb{P}$ -a.s, to a solution  $\bar{u}_r$  of the equation

$$(1.2) \quad \nabla \cdot (\bar{\mathbf{a}}\nabla \bar{u}_r) = 0,$$

where  $\bar{\mathbf{a}}$  is a constant, symmetric, definite-positive matrix, in the sense that

$$(1.3) \quad \frac{1}{r^d} \int_{B(0,r)} |u_r(x) - \bar{u}_r(x)|^2 dx \xrightarrow{r \rightarrow \infty} 0.$$

This second equation (1.2) is frequently called the homogenized equation. Obtaining quantitative information, for instance rates of convergence in (1.3), drew a lot of attention in the recent years, and there has been some notable progress, in particular by the works of Armstrong, Kuusi, Mourrat and Smart [5, 4, 2, 3] and the works of Gloria, Neukamm and Otto [13, 14, 15, 16]. Quantitative rates of convergence are also interesting in particular because they can provide information on the performance of numerical algorithms for the computation of the homogenized coefficients [26].

The purpose of this article is to develop a theory of quantitative stochastic homogenization for the more general equation

$$(1.4) \quad d(\mathbf{a}(x)du) = 0,$$

where  $u$  is an  $r$ -form,  $d$  is the exterior derivative and  $\mathbf{a}$  is a random, rapidly oscillating tensor which maps the space of  $r$ -forms into the space of  $(d-r)$ -forms, satisfying some suitable properties which will be described below. When  $r = 0$ ,  $u$  is a 0-form, that is to say a function, and the differential equation (1.4) reduces to (1.1) and we recover the classical theory of stochastic homogenization.

When  $r = 1$  and the underlying space is 4-dimensional, the system of equations in (1.4) has the same structure as *Maxwell's equations* (see e.g. [20, Section 1.2]), with the fundamental difference that here we assume  $\mathbf{a}(x)$  to be Riemannian, that is, elliptic in the sense of (2.16), while for Maxwell's equations the underlying geometric structure is Lorentzian. Replacing a Lorentzian geometry by a Riemannian one, a procedure sometimes referred to as "Wick's rotation", is very common in constructive quantum field theory, see e.g. [12, Section 6.1(ii)]. While the objects we study here are minimizers of the random Lagrangian in (2.22), we believe that the techniques developed in this paper will be equally informative for the study of the Gibbs measures associated with such Lagrangians.

The main result of this article, Theorem 2 below, is to prove a quantitative homogenization theorem for differential forms, i.e a quantitative version of (1.3) for differential forms. In our last main result, stated in Theorem 3 below, we prove that homogenization commutes with the natural duality structure of differential forms. This duality structure is behind certain exact formulas for the homogenized matrix which are known to hold in dimension  $d = 2$  (see for instance [18, Chapter 1]). We note that similar results were obtained independently by Serre [28] in the case of periodic coefficients.

Note that the system (1.4), under natural assumptions on the coefficient field  $\mathbf{a}$ , is elliptic but not uniformly elliptic (since the operator vanishes on every closed form). To our knowledge, the results in this paper are the first quantitative stochastic homogenization estimates for such degenerate elliptic systems. The proof of our main results are based on an adaptation of the theory of quantitative stochastic homogenization developed in [3].

## 2. NOTATIONS, ASSUMPTIONS AND STATEMENTS OF THE MAIN RESULTS

In this section, we introduce the main notation and assumptions needed in this paper as well as a statement of the main theorems, Theorems 1 and 2.

**2.1. General Notations and Definitions.** We begin by recalling some definitions and recording some properties about differential forms which will be useful in this article. We consider the space  $\mathbb{R}^d$  for some positive integer  $d$ , equipped with the standard  $|\cdot|$ . Denote by  $\mathbf{e}_1, \dots, \mathbf{e}_d$  the canonical basis of  $\mathbb{R}^d$ . A cube of  $\mathbb{R}^d$ , generally denoted by  $\square$ , is a set of the form

$$(2.1) \quad z + R(-1, 1)^d.$$

Given a cube  $\square := z + R(-1, 1)^d$ , we also denote by  $\text{size}(\square)$  the size of the edges of the cube, in this case  $\text{size} = R$ . A triadic cube of  $\mathbb{R}^d$  is a cube of the specific form

$$z + \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d, \quad m \in \mathbb{N}, z \in 3^m \mathbb{Z}^d.$$

We use the notation, for  $m \in \mathbb{N}$ ,

$$\square_m := \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d.$$

If  $U$  is a measurable subset of  $\mathbb{R}^d$ , we denote its Lebesgue measure by  $|U|$ . The normalized integral for a function  $u : U \rightarrow \mathbb{R}$  for a measurable subset  $U \subseteq \mathbb{R}^d$  is denoted by

$$\int_U u(x) dx := \frac{1}{|U|} \int_U u(x) dx.$$

Given two sets  $U, V \subseteq \mathbb{R}^d$ , we denote by  $\text{dist}(U, V) := \inf_{x \in U, y \in V} |x - y|$ .

For  $0 \leq r \leq d$ , we denote by  $\Lambda^r(\mathbb{R}^d)$  the space of  $r$ -linear forms. This is a vector space of dimension  $\binom{d}{r}$ , a canonical basis is given by

$$dx_{i_1} \wedge \dots \wedge dx_{i_r}, \quad 1 \leq i_1 < \dots < i_r \leq d.$$

We will denote by

$$dx_I := dx_{i_1} \wedge \dots \wedge dx_{i_r}, \quad \text{for } I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, d\}.$$

Given  $U$  an open subset of  $\mathbb{R}^d$ , a differential form is a map

$$u : \begin{cases} U \rightarrow \Lambda^r(\mathbb{R}^d), \\ x \rightarrow \sum_{|I|=r} u_I(x) dx_I. \end{cases}$$

Given  $\xi := \xi_1 e_1 + \dots + \xi_d e_d \in \mathbb{R}^d$ , we denote by  $d\xi := \xi_1 dx_1 + \dots + \xi_d dx_d \in \Lambda^1(\mathbb{R}^d)$ .

In practice, we need to assume some regularity on  $u$ , so we introduce the following spaces.

- The space of smooth differential forms on  $U$  up to the boundary, denoted by  $C^\infty \Lambda^r(U)$ , i.e,

$$C^\infty \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) dx_I : \forall I, u_I \in C^\infty(U) \right\}.$$

- The space of compactly supported smooth differential forms on  $U$ , denoted by  $C_c^\infty \Lambda^r(U)$ , i.e,

$$C_c^\infty \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) dx_I : \forall I, u_I \in C_c^\infty(U) \right\}.$$

With this definition in mind, we denote by  $\mathcal{D}_r(U)$  the space of  $r$ -currents, i.e, the space of formal sums

$$\sum_{|I|=r} u_I dx_I$$

where the  $u_I$  are distributions on  $\Omega$ . It is equivalently defined as the topological dual of  $C_c^\infty \Lambda^r(U)$ .

- For  $1 \leq p \leq \infty$  the set of  $L^p$  differential forms on  $U$ , denoted by  $L^p \Lambda^r(U)$  i.e,

$$L^p \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) dx_I : \forall I, u_I \in L^p(U) \right\}$$

equipped with the norm

$$\|u\|_{L^p \Lambda^r(U)} := \sum_{|I|=r} \|u_I\|_{L^p \Lambda^r(U)},$$

and, for  $1 \leq p < \infty$ , the normalized  $L^p$ -norm

$$\|u\|_{\underline{L}^p \Lambda^r(U)}^p := \sum_{|I|=r} \int_U |u_I(x)|^p dx = \frac{1}{|U|} \sum_{|I|=r} \int_U |u_I(x)|^p dx.$$

We also equip the space  $L^2 \Lambda^r(U)$  with the scalar product  $\langle u, v \rangle_U := \sum_{|I|=r} \langle u_I, v_I \rangle_{L^2(U)}$ .

- For  $s \in \mathbb{R}$ , the set of  $H^s$  differential forms on  $U$ , denoted by  $H^s \Lambda^r(U)$ , i.e.,

$$H^s \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) dx_I : \forall I, u_I \in H^s(U) \right\}$$

equipped with the scalar product  $\langle u, v \rangle_{H^s \Lambda^r(U)} := \sum_{|I|=r} \langle u_I, v_I \rangle_{H^s(U)}$ .

If  $U \subseteq \mathbb{R}^d$  and  $u : U \rightarrow \Lambda^r(\mathbb{R}^d)$ , we denote the  $i$ th-partial derivative of  $u$  by  $\partial_i u$ , it is understood in the sense of currents according to the formula

$$\partial_i u = \sum_{|I|=r} \partial_i u_I dx_I,$$

where  $\partial_i u_I$  is understood in the sense of distribution. The gradient of  $u$ , denoted by  $\nabla u := (\partial_1 u, \dots, \partial_d u)$ , is a vector-valued differential form. Higher derivatives, which are also vector-valued forms, are denoted by, for  $l \geq 1$ ,

$$\nabla^l u := (\partial_{i_1} \dots \partial_{i_l} u)_{i_1, \dots, i_l \in \{1, \dots, d\}}.$$

Given an  $m$ -form  $\alpha$  and an  $r$ -form  $\omega$ , we consider the exterior product  $\alpha \wedge \omega$  which is an  $(m+r)$ -form and satisfies the following property

$$\alpha \wedge \omega = (-1)^{mr} \omega \wedge \alpha.$$

If  $m+r > d$ , we set  $\omega \wedge \alpha = 0$ .

We then define the exterior derivative which maps  $C^\infty \Lambda^r(U)$  to  $C^\infty \Lambda^{r+1}(U)$  according to the formula,

$$du = \sum_{|I|=r} \sum_{k \notin I} \frac{\partial u_I}{\partial x_k} dx_k \wedge dx_I,$$

and can then be extended to currents. In particular, if  $u$  is a differential form of degree  $d$ , then  $du = 0$ . This operator satisfies the following properties

$$(2.2) \quad d \circ d = 0 \text{ and } d(u \wedge v) = (du) \wedge v + (-1)^m u \wedge (dv).$$

Given a form  $u := \sum_{|I|=r} u_I dx_I \in C^\infty \Lambda^r(U)$ , an open set  $V \subseteq \mathbb{R}^d$  and a smooth map  $\Phi : V \rightarrow U$ , we define the pullback  $u$  by  $\Phi$  to be the smooth form

$$\Phi^* u := \begin{cases} V \rightarrow \Lambda^r(\mathbb{R}^d), \\ x \rightarrow \sum_{I=\{i_1, \dots, i_r\}} u_I(\Phi(x)) d(\Phi(x)_{x_{i_1}}) \wedge \dots \wedge d(\Phi(x)_{x_{i_r}}). \end{cases}$$

where  $d\Phi(x)$  denotes the differential of  $\Phi$  evaluated at  $x$ . The pullback satisfies the following properties, given an  $m$ -form  $u$  and an  $r$ -form  $v$ ,

$$(2.3) \quad \Phi^* du = d\Phi^* u \text{ and } \Phi^*(u \wedge v) = \Phi^* u \wedge \Phi^* v.$$

Given another open set  $W \subseteq \mathbb{R}^d$  and another smooth map  $\Psi : W \rightarrow V$ , we have the composition rule

$$\Psi^*(\Phi^* u) = (\Phi \circ \Psi)^* u.$$

Moreover, if we assume that  $\Phi$  is a smooth diffeomorphism from  $V$  to  $U$  such that  $\Phi, \Phi^{-1}$  and all their derivatives are bounded then, for  $s \in \mathbb{R}$ ,  $\Phi^*$  maps  $H^s \Lambda^r(U)$  into  $H^s \Lambda^r(V)$  and we have the estimate

$$\|\Phi^* u\|_{H^s \Lambda^r(V)} \leq C \|u\|_{H^s \Lambda^r(U)},$$

for some  $C := C(d, s, \Phi) < \infty$ .

We can also define a scalar product on  $\Lambda^r(\mathbb{R}^d)$  such that  $(dx_I)_{|I|=r}$  is an orthonormal basis, i.e.,

$$(2.4) \quad \left( \sum_{|I|=r} \alpha_I dx_I, \sum_{|I|=r} \beta_I dx_I \right) = \sum_{|I|=r} \alpha_I \beta_I.$$

We will use the notation, for  $\alpha \in \Lambda^r(\mathbb{R}^d)$

$$|\alpha| = \sqrt{(\alpha, \alpha)}.$$

We denote by  $B_1\Lambda^r(\mathbb{R}^d)$  the unit ball of  $\Lambda^r(\mathbb{R}^d)$ , i.e,

$$B_1\Lambda^r(\mathbb{R}^d) := \{\alpha \in \Lambda^r(\mathbb{R}^d) : |\alpha| \leq 1\}.$$

Moreover for each  $r$ , notice that

$$\dim \Lambda^r(\mathbb{R}^d) = \dim \Lambda^{(d-r)}(\mathbb{R}^d) = \binom{d}{r}.$$

There is a canonical bijection between these spaces, the Hodge star operator, denoted by  $\star$ , which sends  $\Lambda^r(\mathbb{R}^d)$  to  $\Lambda^{(d-r)}(\mathbb{R}^d)$  and satisfies the property, for each  $\alpha, \beta \in \Lambda^r(\mathbb{R}^d)$

$$\alpha \wedge (\star\beta) = (\alpha, \beta) dx_1 \wedge \cdots \wedge dx_d.$$

It is defined on the canonical basis by

$$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_r}) := dx_{i_{r+1}} \wedge \cdots \wedge dx_{i_d}$$

where  $(i_1, \dots, i_d)$  is an even permutation of  $\{1, \dots, d\}$ . An important property of this operator is the following, for each  $\alpha \in \Lambda^r(\mathbb{R}^d)$ ,

$$(2.5) \quad \star \star \alpha = (-1)^{r(d-r)} \alpha.$$

We then define the integral of a  $d$ -form over a domain  $U$ . Let  $u = u_{\{1, \dots, d\}} dx_1 \wedge \cdots \wedge dx_d$  be a  $d$ -form over  $U$ . If  $u_{\{1, \dots, d\}} \in L^1(U)$ , we say that  $u$  is integrable and define

$$(2.6) \quad \int_U u := \int_U u_{\{1, \dots, d\}}(x) dx.$$

In particular, the scalar product on  $L^2\Lambda^r(U)$  can be rewritten, for each  $\alpha, \beta \in L^2\Lambda^r(U)$ ,

$$\langle u, v \rangle_U = \int_U u \wedge (\star v).$$

Additionally, if  $\Phi$  is a smooth diffeomorphism mapping  $V$  to  $U$  positively oriented, i.e if  $\det d\Phi > 0$ , then the change of variables formula reads, for each integrable  $d$ -form  $u$ ,

$$(2.7) \quad \int_V \Phi^* u = \int_U u.$$

We then want to define the normal and tangential components of a form  $u$  on the boundary of a smooth bounded domain  $U$ . To achieve this, consider  $U \subseteq \mathbb{R}^d$  a smooth bounded domain of  $\mathbb{R}^d$ , denote by  $\nu$  the outward normal of  $\partial U$  and fix  $u \in C^\infty\Lambda^r(\mathbb{R}^d)$  a smooth  $r$ -form. For each  $x \in \partial U$ , we define  $\mathbf{n}u(x) \in \Lambda^r(\mathbb{R}^d)$ , the normal component of  $u(x)$ , to be the orthogonal projection of  $u(x)$  with respect to the scalar product  $(\cdot, \cdot)$  defined in (2.4) on the kernel of the mapping

$$(2.8) \quad d\nu(x) \wedge \cdot : \begin{cases} \Lambda^r(\mathbb{R}^d) \rightarrow \Lambda^{r+1}(\mathbb{R}^d), \\ v \rightarrow d\nu(x) \wedge v. \end{cases}$$

The tangential component of  $u(x)$ , denoted by  $\mathbf{t}u(x)$ , is given by the formula

$$(2.9) \quad \mathbf{t}u(x) = u(x) - \mathbf{n}u(x).$$

Let now  $u \in C^\infty\Lambda^{d-1}(U)$ , using the previous notation there exists a smooth function  $v : \partial U \rightarrow \mathbb{R}$  such that, for each  $x \in \partial U$ ,

$$\mathbf{t}u(x) = v(x) de_1^x \wedge \cdots \wedge de_{d-1}^x,$$

where  $e_1^x, \dots, e_{d-1}^x \in \mathbb{R}^d$  are such that  $(e_1^x, \dots, e_{d-1}^x, \nu(x))$  is an orthonormal basis positively oriented of  $\mathbb{R}^d$ . With this notation, we define the integral of  $u$  on  $\partial U$  by the formula

$$(2.10) \quad \int_{\partial U} u = \int_{\partial U} v(x) d\mathcal{H}^{d-1}(x),$$

where  $\mathcal{H}^{d-1}$  is the Hausdorff measure of dimension  $(d-1)$  on  $\mathbb{R}^d$ .

The two definition of integrals (2.6) and (2.10) are linked together by the Stokes' formula: for each smooth bounded domain  $U \subseteq \mathbb{R}^d$  and each  $u \in C^\infty \Lambda^{d-1}(U)$ ,

$$(2.11) \quad \int_{\partial U} u = \int_U du.$$

We can now define  $\delta$ , the formal adjoint of  $d$  with respect to the scalar product  $\langle \cdot, \cdot \rangle_{L^2 \Lambda^r(U)}$ , i.e., the operator which satisfies for each  $(u, v) \in C_c^\infty \Lambda^{r-1}(U) \times C_c^\infty \Lambda^r(U)$ ,

$$\langle du, v \rangle_{L^2 \Lambda^r(U)} = \langle u, \delta v \rangle_{L^2 \Lambda^{r-1}(U)}.$$

This operator can be explicitly computed using the second equality in (2.2), the equality (2.5), and the Stokes' formula (2.11). Indeed we have

$$\begin{aligned} 0 &= \int_{\partial U} u \wedge (\star v) = \int_U du \wedge (\star v) + (-1)^{r-1} \int_U u \wedge d(\star v) \\ &= \int_U du \wedge \star v + (-1)^{r-1+(r-1)(d-r+1)} \int_U u \wedge \star(\star d \star v). \end{aligned}$$

Consequently,

$$(2.12) \quad \delta = (-1)^{(r-1)d+1} \star d \star.$$

We now define the set of  $L^2$  forms  $u$  such that  $du$  is also  $L^2$ . This will play a crucial role in this article. Note that this space is different from the Sobolev space  $H^1 \Lambda^r(U)$  introduced earlier.

**Definition 2.1.** For each open subset  $U \subseteq \mathbb{R}^d$ , and each  $0 \leq r < d$ , we define the space  $H_{\text{d}}^1 \Lambda^r(U)$  to be the set of forms in  $L^2 \Lambda^r(U)$  such that  $du \in L^2 \Lambda^{r+1}(U)$ , i.e.,

$$H_{\text{d}}^1 \Lambda^r(U) := \left\{ u \in L^2 \Lambda^r(U) : \exists f \in L^2 \Lambda^{r+1}(U), \forall v \in C_c^\infty \Lambda^{d-r-1}(U), \int_U (u \wedge \delta v + (-1)^r f \wedge v) = 0 \right\}.$$

If  $u \in H_{\text{d}}^1 \Lambda^r(U)$ , we denote by  $du$  the unique form in  $L^2 \Lambda^{r+1}(U)$  which satisfies, for every  $v \in C_c^\infty \Lambda^{d-r-1}(U)$ ,

$$(2.13) \quad \int_U (u \wedge dv + (-1)^r du \wedge v) = 0.$$

This space is a Hilbert space equipped with the norm

$$\|u\|_{H_{\text{d}}^1 \Lambda^r(U)} = \langle u, u \rangle_U + \langle du, du \rangle_U.$$

In the case  $r = d$ , we have  $du = 0$  for each  $u \in L^2 \Lambda^d(U)$  and  $H_{\text{d}}^1 \Lambda^d(U) = L^2 \Lambda^d(U)$ . We also denote by  $H_{\text{d},0}^1 \Lambda^r(U)$  the closure of  $C_c^\infty \Lambda^r(U)$  in  $H_{\text{d}}^1 \Lambda^r(U)$ , i.e.,

$$H_{\text{d},0}^1 \Lambda^r(U) := \overline{C_c^\infty \Lambda^r(U)}^{H_{\text{d}}^1 \Lambda^r(U)}.$$

Symmetrically, for each  $0 < r \leq d$ , we define  $H_{\delta}^1 \Lambda^r(U)$  to be the set of forms in  $L^2 \Lambda^r(U)$  such that  $\delta u \in L^2 \Lambda^{r-1}(U)$ , i.e.,

$$H_{\delta}^1 \Lambda^r(U) := \left\{ u \in L^2 \Lambda^r(U) : \exists f \in L^2 \Lambda^{r-1}(U), \forall v \in C_c^\infty \Lambda^{d-r+1}(U), \int_U (u \wedge dv + (-1)^{d-r} f \wedge v) = 0 \right\}.$$

and in that case, we denote by  $\delta u = f$ . In the case  $r = 0$ , we have  $\delta u = 0$  for each  $u \in L^2(U)$  and  $H_{\delta}^1 \Lambda^0(U) = L^2(U)$ . We also denote by  $H_{\delta,0}^1 \Lambda^r(U)$  the closure of  $C_c^\infty \Lambda^r(U)$  in  $H_{\delta}^1 \Lambda^r(U)$ , i.e.,

$$H_{\delta,0}^1 \Lambda^r(U) := \overline{C_c^\infty \Lambda^r(U)}^{H_{\delta}^1 \Lambda^r(U)}.$$

We then introduce the subspaces of closed (resp. co-closed) forms of  $H_{\text{d}}^1 \Lambda^r(U)$  (resp.  $H_{\delta}^1 \Lambda^r(U)$ ).

**Definition 2.2.** For each open  $U \subseteq \mathbb{R}^d$  and each  $0 \leq r \leq d$ , we say that a form  $u \in H_d^1 \Lambda^r(U)$  is closed (resp. co-closed) if and only if  $du = 0$  (resp.  $\delta u = 0$ ). We denote by  $C_d^r(U)$  the subset of closed  $r$  forms, i.e.,

$$C_d^r(U) := \{u \in H_d^1 \Lambda^r(U) : du = 0\}.$$

We also define

$$C_{d,0}^r(U) := C_d^r(U) \cap H_{d,0}^1 \Lambda^r(U).$$

Symmetrically, we denote by  $C_\delta^r(U)$  the subset of co-closed  $r$  forms, i.e.,

$$C_\delta^r(U) := \{u \in H_d^1 \Lambda^r(U) : \delta u = 0\}.$$

We also define

$$C_{\delta,0}^r(U) := C_\delta^r(U) \cap H_{\delta,0}^1 \Lambda^r(U).$$

**2.2. Notation related to the probability space.** For a random variable  $X$ , an exponent  $s \in (0, +\infty)$  and a constant  $C \in (0, \infty)$ , we write

$$X \leq \mathcal{O}_s(C)$$

to mean that

$$\mathbb{E} \left[ \exp \left( \left( \frac{X_+}{C} \right)^s \right) \right] \leq 2,$$

where  $X_+ := \max(X, 0)$ . The notation is clearly homogeneous:

$$X \leq \mathcal{O}_s(C) \iff \frac{X}{C} \leq \mathcal{O}_s(1).$$

More generally, for  $\theta_0, \theta_1, \dots, \theta_n \in \mathbb{R}^+$  and  $C_1, \dots, C_n \in \mathbb{R}_*^+$ , we write

$$X \leq \theta_0 + \theta_1 \mathcal{O}_s(C_1) + \dots + \theta_n \mathcal{O}_s(C_n)$$

to mean that there exist nonnegative random variables  $X_1, \dots, X_n$  satisfying  $X_i \leq \mathcal{O}_s(C_n)$  such that

$$X \leq \theta_0 + \theta_1 X_1 + \dots + \theta_n X_n.$$

We now record an important property about this notation, the proof of which can be found in [3, Lemma A.4].

**Proposition 2.3.** *For each  $s \in (0, \infty)$ , there exists a constant  $C_s < \infty$  such that the following holds. Let  $\mu$  be a measure over an arbitrary measurable space  $E$ , let  $\theta : E \rightarrow (0, \infty)$  be a measurable function and  $(X(x))_{x \in E}$  be a jointly measurable family of nonnegative random variables such that, for every  $x \in E$ ,  $X(x) \leq \mathcal{O}_s(C(x))$ . We have*

$$(2.14) \quad \int_E X(x) \mu(dx) \leq \mathcal{O}_s \left( C_s \int_E C(x) \mu(dx) \right).$$

We then record a corollary which will be useful in Section 5.

**Corollary 2.4.** (i) *Given positive random variables  $X_1, \dots, X_n$  such that, for each  $i \in \{1, \dots, n\}$ ,  $X_i \leq \mathcal{O}_s(C_i)$ , then*

$$\sum_{i=1}^n X_i \leq \mathcal{O}_s \left( C_s \sum_{i=1}^n C_i \right),$$

where  $C_s$  is the constant in Proposition 2.3.

(ii) *Given a real number  $r > 1$  and  $X_1, \dots, X_n$  such that for each  $i \in \{1, \dots, n\}$ ,  $X_i \leq \mathcal{O}_s(C)$ , then*

$$\sum_{i=1}^n r^i X_i \leq \mathcal{O}_s \left( C_s C \frac{r^{n+1}}{r-1} \right),$$

where  $C_s$  is the constant in Proposition 2.3.



**2.3. Notation and assumptions related to homogenization.** Given  $\lambda \in (0, 1]$  and  $1 \leq r \leq d$ , we consider the space of measurable functions from  $\mathbb{R}^d$  to  $\mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$  satisfying the symmetry assumption, for each  $x \in \mathbb{R}^d$ ,

$$(2.15) \quad p \wedge \mathbf{a}(x)q = q \wedge \mathbf{a}(x)p, \quad \forall p, q \in \Lambda^r(\mathbb{R}^d),$$

and the ellipticity assumption, for each  $x \in \mathbb{R}^d$ ,

$$(2.16) \quad \lambda|p|^2 \leq \star(p \wedge \mathbf{a}(x)p) \leq \frac{1}{\lambda}|p|^2, \quad \forall p \in \Lambda^r(\mathbb{R}^d).$$

We denote by  $\Omega_r$  the collection of all such measurable functions,

$$(2.17) \quad \Omega_r := \left\{ \mathbf{a}(\cdot) : \mathbf{a} : \mathbb{R}^d \rightarrow \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d)) \text{ is Lebesgue measurable} \right. \\ \left. \text{and satisfies (2.15) and (2.16)} \right\}.$$

We endow  $\Omega_r$  with the translation group  $(\tau_y)_{y \in \mathbb{R}^d}$ , acting on  $\Omega_r$  via

$$(\tau_y \mathbf{a})(x) := \mathbf{a}(x + y)$$

and with the family  $\{\mathcal{F}_r(U)\}$  of  $\sigma$ -algebras on  $\Omega_r$ , with  $\mathcal{F}_r(U)$  defined for each Borel subset  $U \subseteq \mathbb{R}^d$  by

$$\mathcal{F}_r(U) := \left\{ \sigma\text{-algebra on } \Omega_r \text{ generated by the family of maps} \right. \\ \left. \mathbf{a} \rightarrow \int_U p \wedge \mathbf{a}(x)q\phi(x), p, q \in \Lambda^r(\mathbb{R}^d), \phi \in C_c^\infty(U) \right\}.$$

The largest of these  $\sigma$ -algebras is  $\mathcal{F}_r(\mathbb{R}^d)$ , simply denoted by  $\mathcal{F}_r$ . The translation group may be naturally extended to  $\mathcal{F}_r$  itself by defining, for  $A \in \mathcal{F}_r$ ,

$$(2.18) \quad \tau_y A := \{\tau_y \mathbf{a} : \mathbf{a} \in A\}.$$

We then endow the measurable space  $(\Omega_r, \mathcal{F}_r)$  with a probability measure  $\mathbb{P}_r$  satisfying the two following conditions:

- $\mathbb{P}_r$  is invariant under  $\mathbb{Z}^d$ -translations: for every  $z \in \mathbb{Z}^d$ ,  $A \in \mathcal{F}_r$ ,

$$(2.19) \quad \mathbb{P}[\tau_z A] = \mathbb{P}[A].$$

- $\mathbb{P}_r$  has a unit range dependence: for every pair of Borel subsets  $U, V \subseteq \mathbb{R}^d$  with  $\text{dist}(U, V) \geq 1$ ,

$$(2.20) \quad \mathcal{F}_r(U) \text{ and } \mathcal{F}_r(V) \text{ are independent.}$$

The expectation of an  $\mathcal{F}_r$ -measurable random variable  $X$  with respect to  $\mathbb{P}_r$  is denoted by  $\mathbb{E}_r[X]$  or simply  $\mathbb{E}[X]$  when there is no confusion about the value of  $r$ .

**Definition 2.5.** Given an integer  $1 \leq r \leq d$ , an environment  $\mathbf{a} \in \Omega_r$  and an open subset  $U \subseteq \mathbb{R}^d$ , we say that  $u \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U)$  is a solution of the equation

$$\mathbf{d}(\mathbf{a}u) = 0,$$

if for every smooth compactly supported form  $v \in C_c^\infty \Lambda^r(U)$ ,

$$\int_U \mathbf{d}u \wedge \mathbf{a}v = 0.$$

We denote by  $\mathcal{A}_r^{\mathbf{a}}(U)$  the set of solutions, i.e.,

$$(2.21) \quad \mathcal{A}_r^{\mathbf{a}}(U) := \left\{ u \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U) : \forall v \in C_c^\infty \Lambda^r(U), \int_U \mathbf{d}u \wedge \mathbf{a}v = 0 \right\}.$$

When there is no confusion, we omit the subscripts  $r$  and  $\mathbf{a}$  and only write  $\mathcal{A}(U)$ .

## 2.4. Statement of the main results.

**Definition 2.6.** For every convex bounded domain  $U \subseteq \mathbb{R}^d$ , we define, for  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{(d-r)}(\mathbb{R}^d)$ ,

$$(2.22) \quad J(U, p, q) := \sup_{v \in \mathcal{A}(U)} \int_U \left( -\frac{1}{2} dv \wedge \mathbf{a}dv - p \wedge \mathbf{a}dv + dv \wedge q \right).$$

The quantity  $J$  is nonnegative and satisfies a subadditivity property with respect to the domain  $U$ : see [3, Chapter 2] or Proposition 5.1 below. In particular the mapping

$$n \mapsto \mathbb{E}[J(\square_n, p, q)]$$

is decreasing and nonnegative, thus it converges as  $n \rightarrow \infty$ . The idea is then to show that there exists a linear mapping  $\bar{\mathbf{a}} \in \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$  such that for each  $r$ -form  $p$ ,  $J(\square_n, p, \bar{\mathbf{a}}p)$  tends to 0 and to quantify this statement. Precisely, we prove the following result.

**Theorem 1** (Quantitative homogenization). *Given  $1 \leq r \leq d$ , there exist an exponent  $\alpha(d, \Lambda) > 0$ , a constant  $C(d, \Lambda) < \infty$  and a unique linear mapping  $\bar{\mathbf{a}} \in \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$ , which is symmetric and satisfies the ellipticity condition (2.16), such that for every  $n \in \mathbb{N}$ ,*

$$(2.23) \quad \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} J(\square_n, p, \bar{\mathbf{a}}p) \leq \mathcal{O}_1(C3^{-n\alpha}).$$

This is the subject of Section 5. In Section 6, we study the solvability of the equation  $\mathbf{d}\mathbf{a}du = 0$  on a smooth bounded domain  $U$ . The first main proposition is the following, which establishes the well-posedness of the Dirichlet problem for differential forms.

**Proposition 2.7.** *Let  $U$  be a bounded smooth domain of  $\mathbb{R}^d$  and  $1 \leq r \leq d$ . Let  $f \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U)$ , then for any measurable map  $\mathbf{a} : \mathbb{R}^d \rightarrow \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$  satisfying (2.16) and (2.15), there exists a unique solution in  $f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U) \cap (C_{\mathbf{d},0}^{r-1}(U))^{\perp}$  of the equation*

$$(2.24) \quad \begin{cases} \mathbf{d}(\mathbf{a}du) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U, \end{cases}$$

in the sense that, for each  $v \in H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$ ,

$$\int_U du \wedge \mathbf{a}dv = 0.$$

Moreover if we enlarge the space of admissible solutions to the space  $f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$ , we loose the uniqueness property, but if  $v, w \in f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$  are two solutions of (2.24), then

$$v - w \in C_{\mathbf{d},0}^{r-1}.$$

Before stating the homogenization theorem, there are two things to note about this proposition. First the suitable notion to replace the trace of a function when the degree of the form is not 0 is the tangential part of the form. This is the only information which is available when one has access to the form  $u$  and its differential derivative  $du$ . It will become clear in the next section when Propositions 3.2 and 3.3 are stated. Also note that for functions, or 0-forms, the notion of trace and tangential trace are the same.

Second, note that if  $v \in C_{\mathbf{d},0}^{r-1}(U)$  and  $u$  is a solution of (2.24), then  $u + v$  is also a solution of (2.24). This problem does not appear when one works with functions (or 0-forms) because in that case  $C_{\mathbf{d},0}^0(U) = \{0\}$ . This explains why we need to be careful when solving (2.24).

We then deduce from the previous proposition and Theorem 1 the homogenization theorem.

**Theorem 2** (Homogenization Theorem). *Let  $U$  be a bounded smooth domain of  $\mathbb{R}^d$  and  $1 \leq r \leq d$ , fix  $\varepsilon \in (0, 1]$  and  $f \in H^2\Lambda^{r-1}(U)$ . Let  $u^\varepsilon, u \in f + H_{d,0}^1\Lambda^{r-1}(U) \cap (C_{d,0}^r(U))^\perp$  respectively denote the solutions of the Dirichlet problems*

$$\begin{cases} d\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)du^\varepsilon\right) = 0 & \text{in } U \\ \mathbf{t}u^\varepsilon = \mathbf{t}f & \text{on } \partial U. \end{cases} \quad \text{and} \quad \begin{cases} d(\bar{\mathbf{a}}du) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U. \end{cases}$$

Then there exist an exponent  $\alpha := \alpha(d, \lambda, U) > 0$  and a constant  $C := C(d, \lambda, U) < \infty$  such that

$$\|u^\varepsilon - u\|_{L^2\Lambda^r(U)} + \|du^\varepsilon - du\|_{H^{-1}\Lambda^r(U)} \leq \mathcal{O}_1\left(C\|df\|_{H^1\Lambda^r(U)}\varepsilon^\alpha\right).$$

The previous theorem is often stated, when one is dealing with functions (or 0-forms) in the case that  $U$  is a bounded Lipschitz domain and with a boundary condition  $f \in W^{1,2+\delta}(U)$  for some  $\delta > 0$ : see for instance [3, Theorem 2.16]. This is convenient since this assumption ensures that the energy of the solution does not concentrate in a region of small Lebesgue measure near  $\partial U$ . Indeed, the global Meyers estimate gives some extra regularity on the function  $u$ ,

$$\left(\int_U |\nabla u|^{2+\varepsilon}(x)dx\right)^{\frac{1}{2+\varepsilon}} \leq C\left(\int_U |\nabla f|^{2+\varepsilon}(x)dx\right)^{\frac{1}{2+\varepsilon}},$$

for some tiny  $\varepsilon > 0$ . On the other hand, this assumption is natural in view of the interior Meyers estimate, which ensures that the restriction of any solution to the heterogeneous equation to a smaller domain will possess such regularity.

Unfortunately, we were not able to prove a global Meyers-type estimate for the solutions of

$$\begin{cases} d(\bar{\mathbf{a}}du) = 0 & \text{in } U, \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U. \end{cases}$$

To bypass this difficulty, we made the extra assumptions  $U$  smooth and  $df \in H^1\Lambda^r(U)$ , this implies, by Proposition A.4 that  $du \in H^1\Lambda^r(U)$  with the estimate

$$\|du\|_{H^1\Lambda^r(U)} \leq C\|df\|_{H^1\Lambda^r(U)}.$$

Then, via the Sobolev embedding Theorem, we obtain that  $du$  belongs to some  $L^p$ , for some  $p := p(d) > 2$ . This allows to control the  $L^2$  norm of  $du$  in a boundary layer of small volume, as it used to be done with the Meyers' estimate.

The last section is devoted to the study of the following dual problem. If  $\mathbf{a} \in \Omega_r$ , then for each  $x \in \mathbb{R}^d$ ,  $\mathbf{a}(x)$  is invertible and  $\mathbf{a}^{-1} \in \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d))$  satisfies the symmetry assumption (2.15) and the following ellipticity condition

$$\frac{1}{\lambda}|p|^2 \leq \mathbf{a}(x)^{-1}p \wedge p \leq \lambda|p|^2, \quad \forall p \in \Lambda^{(d-r)}(\mathbb{R}^d).$$

We can thus define, for each  $(p, q) \in \Lambda^{(d-r)}(\mathbb{R}^d) \times \Lambda^r(\mathbb{R}^d)$  and each  $m \in \mathbb{N}$ , the random variable

$$J_{\text{inv}}(\square_m, p, q) := \sup_{u \in \mathcal{A}^{\text{inv}}(\square_m)} \int_{\square_m} \left( -\frac{1}{2}\mathbf{a}^{-1}du \wedge du - \mathbf{a}^{-1}du \wedge p + q \wedge du \right),$$

where  $\mathcal{A}^{\text{inv}}(\square_m)$  is the set of solution under the environment  $\mathbf{a}^{-1}$ , i.e.,

$$\mathcal{A}^{\text{inv}}(\square_m) := \left\{ u \in H_d^1\Lambda^{(d-r-1)}(\square_m) : \forall v \in C_c^\infty\Lambda^{r-1}(\square_m), \int_{\square_m} du \wedge \mathbf{a}^{-1}dv = 0 \right\}.$$

In Section 7, we prove that there exist a constant  $C(d, \lambda) < \infty$ , an exponent  $\alpha(d, \lambda) > 0$  a linear operator  $\overline{\text{inv}}\mathbf{a} \in \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d))$  such that, for each  $m \in \mathbb{N}$ ,

$$\sup_{p \in B_1\Lambda^{(d-r)}(\mathbb{R}^d)} J_{\text{inv}}(\square_m, p, \overline{\text{inv}}\mathbf{a}p) \leq \mathcal{O}_1(C3^{-m\alpha}).$$

We also prove that  $\overline{\text{inv}}\mathbf{a}$  is linked to  $\bar{\mathbf{a}}$  according to the following theorem.

**Theorem 3** (Duality). *The homogenized linear maps  $\bar{\mathbf{a}}$  and  $\overline{\text{inv } \mathbf{a}}$  satisfy*

$$\overline{\text{inv } \mathbf{a}} = (\bar{\mathbf{a}})^{-1}.$$

**Outline of the paper.** The rest of this article is organized as follows. In Section 3, we state without proof some important properties of differential forms, in particular we give a trace theorem for differential forms, study the solvability of the equation  $df = u$  and state the Hodge-Morrey decomposition theorem. In Section 4, we generalize some inequalities known for functions to the setting of differential forms, in particular the Caccioppoli inequality and the multiscale Poincaré inequality. In Section 5, we combine all the ingredients established in the previous sections and prove the first main theorem of this article, Theorem 1. In Section 6, we use the results from Section 5 and the regularity estimates (pointwise interior estimate and boundary  $H^2$ -regularity) proved in the Appendix A, to show the second main theorem of this article, Theorem 2. In Section 7, we study a duality structure between  $r$ -forms and  $(d-r)$ -forms and we deduce from that some results about the homogenized matrix in the case  $d = 2$  and  $r = 1$ . Finally Appendix A is devoted to the proof of some regularity estimates (more specifically pointwise interior estimate and  $H^2$  boundary estimate) for the solution of the elliptic degenerate system  $d\bar{\mathbf{a}}du = 0$ , where  $\bar{\mathbf{a}}$  is a linear mapping sending  $r$ -forms to  $(d-r)$ -forms satisfying some suitable properties of symmetry and ellipticity, more formally explained in Section 2.

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### 3. SOME RESULTS PERTAINING TO FORMS

In this section, we record some properties related to the spaces  $H_{\mathbf{d}}^1\Lambda^r(U)$ ,  $H_{\delta}^1\Lambda^r(U)$  and  $C_{\mathbf{d}}^r(U)$ . Most of these results and their proofs can be found in [23] and [24].

Given  $U \subseteq \mathbb{R}^d$  Lipschitz and bounded, we define the Sobolev space  $H^{1/2}(\partial U)$  as the set of functions of  $L^2(\partial U)$  which satisfy

$$[g]_{H^{1/2}(\partial U)} := \left( \int_{\partial U} \int_{\partial U} \frac{|g(x) - g(y)|^2}{|x - y|^{d+1}} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{\frac{1}{2}} < \infty.$$

It is a Hilbert space equipped with the norm

$$\|g\|_{H^{1/2}(\partial U)} := \|g\|_{L^2(\partial U)} + [g]_{H^{1/2}(\partial U)}.$$

Define  $H^{-1/2}(\partial U)$  to be the dual of  $H^{1/2}(\partial U)$ , i.e.,

$$H^{-1/2}(\partial U) := \left( H^{1/2}(\partial U) \right)^*.$$

We can then extend this definition to differential forms by defining, for each  $0 \leq r \leq d$ ,

$$H^{1/2}\Lambda^r(\partial U) := \left\{ u \in L^2\Lambda^r(\partial U) \text{ s.t. } u = \sum_{|I|=r} u_I dx_I \text{ and } \forall I, [u_I]_{H^{1/2}(\partial U)} < \infty \right\}.$$

This is also a Hilbert space, equipped with the norm,

$$\|u\|_{H^{1/2}\Lambda^r(\partial U)} := \|u\|_{L^2\Lambda^r(\partial U)} + \sum_{|I|=r} [u_I]_{H^{1/2}(\partial U)}.$$

We can also define  $H^{-1/2}\Lambda^r(\partial U)$  by duality, according to the formula,

$$H^{-1/2}\Lambda^r(\partial U) := \left( H^{1/2}\Lambda^{d-r}(\partial U) \right)^*.$$

We then recall the classical Sobolev Trace Theorem for Lipschitz domains, it is a special case of [19, Chapter VII, Theorem 1] (see also [21]). The second half of this result is a consequence of the solvability of the Dirichlet problem for the Poisson equation in Lipschitz domains, which was proved in [17] or [8, Theorem 10.1].

**Proposition 3.1** (Sobolev Trace Theorem). *Let  $U$  be a bounded Lipschitz domain. The linear operator  $C^\infty(\overline{U}) \rightarrow \text{Lip}(\partial U)$  that restricts a smooth function on  $\overline{U}$  to  $\partial U$  has an extension to a bounded linear mapping  $H^1(U) \rightarrow H^{1/2}(\partial U)$ . That is, there exists a linear operator*

$$\text{Tr} : H^1(U) \rightarrow H^{1/2}(\partial U),$$

and a constant  $C(d, U) < \infty$  such that for each  $u \in H^1(U)$ ,

$$\|\text{Tr} u\|_{H^{1/2}(\partial U)} \leq C \|u\|_{H^1(U)}$$

and for each  $u \in C^\infty(\overline{U})$ ,

$$\text{Tr} u = u \text{ on } \partial U.$$

Moreover this map has a bounded right-inverse

$$E : H^{1/2}(\partial U) \rightarrow H^1(U).$$

In particular, the map  $\text{Tr}$  is surjective.

The trace can then be extended to differential forms by setting, for  $u = \sum_{|I|=r} u_I dx_I \in H^1 \Lambda^r(U)$ ,

$$\text{Tr} u = \sum_{|I|=r} \text{Tr} u_I dx_I \in H^{1/2} \Lambda^r(\partial U).$$

In the case when  $u$  does not belong to the space  $H^1 \Lambda^r(U)$  but only belongs to the larger space  $H_d^1 \Lambda^r(U)$ , one still has a Sobolev trace theorem, but one can only get information about the tangential component of the trace of  $u$ . The following proposition is a specific case of [24, Proposition 4.1 and Proposition 4.3].

**Proposition 3.2** ([24], Proposition 4.1 and Proposition 4.3). *For each  $u \in H_d^1 \Lambda^{r-1}(U)$ , the map*

$$\langle \mathbf{t}u, \cdot \rangle : \begin{cases} H^{1/2} \Lambda^{(d-r)}(\partial U) \rightarrow \mathbb{R}, \\ \psi \rightarrow \int_U (du \wedge \Psi + (-1)^r u \wedge d\Psi), \end{cases}$$

where  $\Psi \in H^1 \Lambda^{d-r}(U)$  is chosen such that  $\text{Tr} \Psi = \psi$ , is well-defined, linear and bounded. The tangential trace

$$\mathbf{t} : \begin{cases} H_d^1 \Lambda^r(U) \rightarrow H^{-1/2} \Lambda^r(\partial U), \\ u \rightarrow \langle \mathbf{t}u, \cdot \rangle. \end{cases}$$

is linear and continuous. Moreover this notation is consistent with the tangential component introduced in (2.9). Similarly, we can define the normal trace for  $H_\delta^1 \Lambda^{r+1}(U)$  according to the formula, for each  $v \in H_\delta^1 \Lambda^r(U)$

$$\langle \mathbf{n}v, \cdot \rangle : \begin{cases} H^{1/2} \Lambda^{(d-r)}(\partial U) \rightarrow \mathbb{R}, \\ \psi \rightarrow \int_U (\delta v \wedge \Psi + (-1)^{d-r} v \wedge \delta \Psi), \end{cases}$$

where  $\Psi \in H^1 \Lambda^{d-r}(U)$  is chosen such that  $\text{Tr} \Psi = \psi$ . The linear operator  $v \rightarrow \mathbf{n}v$  sends  $H_\delta^1 \Lambda^r(U)$  to  $H^{-1/2} \Lambda^r(\partial U)$ , is continuous and the notation is consistent with the normal component introduced in (2.8).

The following property shows that, when  $U$  is Lipschitz, the space  $H_{d,0}^1 \Lambda^r(U)$  (resp.  $H_{\delta,0}^1 \Lambda^r(U)$ ) is also the space of differential forms in  $H_d^1 \Lambda^r(U)$  (resp.  $H_\delta^1 \Lambda^r(U)$ ) with tangential (resp. normal) trace equal to 0. A proof for these results can be found in [23, Lemma 2.13].

**Proposition 3.3** ([23], Lemma 2.13). *Let  $U$  be an open bounded Lipschitz subset of  $\mathbb{R}^d$ . For each  $0 \leq r \leq d$ , the following results hold:*

- The space of smooth differential forms  $C^\infty \Lambda^r(U)$  is dense in  $H_d^1 \Lambda^r(U)$  (resp.  $H_\delta^1 \Lambda^r(U)$ ).

- The space  $C_c^\infty \Lambda^r(U)$  of smooth and compactly supported differential forms is dense in  $\{u \in H_d^1 \Lambda^r(U) : \mathbf{t}u = 0\}$  and in  $\{u \in H_\delta^1 \Lambda^r(U) : \mathbf{n}u = 0\}$ . In particular, we have

$$H_{d,0}^1 \Lambda^r(U) = \{u \in H_d^1 \Lambda^r(U) : \mathbf{t}u = 0\} \quad \text{and} \quad H_{\delta,0}^1 \Lambda^r(U) = \{u \in H_\delta^1 \Lambda^r(U) : \mathbf{n}u = 0\}.$$

An interesting corollary of this proposition is that the space of solutions  $\mathcal{A}(U)$ , defined by (2.21), can be equivalently defined by the formula

$$(3.1) \quad \mathcal{A}(U) := \left\{ u \in H_d^1 \Lambda^r(U) : \forall v \in H_{d,0}^1 \Lambda^{d-r}(U), \int_U du \wedge \mathbf{a}dv = 0 \right\}.$$

We then record one important result concerning the solvability of the equation  $du = f$  on bounded star-shaped domains.

**Proposition 3.4** ([23], Theorem 1.5 and Theorem 4.1). *Let  $U \subseteq \mathbb{R}^d$  be a bounded star-shaped domain. The following statements hold.*

- For  $1 \leq r \leq d$  (resp.  $0 \leq r \leq d-1$ ), given  $f \in L^2 \Lambda^r(U)$ , the problem

$$(3.2) \quad \begin{cases} du = f \text{ in } U, \\ u \in H_d^1 \Lambda^{r-1}(U), \end{cases} \quad \text{resp.} \quad \begin{cases} \delta u = f \text{ in } U, \\ u \in H_\delta^1 \Lambda^{r+1}(U), \end{cases}$$

has a solution if and only if  $f$  satisfies  $df = 0$  (resp.  $\delta f = 0$ ). In this case, there exists a constant  $C(d, U) < \infty$  and a solution  $u$  of (3.2) which belongs to  $H^1 \Lambda^{r-1}(U)$  (resp.  $u \in H^1 \Lambda^{r+1}(U)$ ) and satisfies

$$\|u\|_{H^1 \Lambda^{r-1}(U)} \leq C \|f\|_{L^2 \Lambda^r(U)} \quad \text{resp.} \quad \|u\|_{H^1 \Lambda^{r+1}(U)} \leq C \|f\|_{L^2 \Lambda^r(U)}.$$

- For  $1 \leq r \leq d-1$ , given  $f \in L^2 \Lambda^r(U)$ , the problem

$$(3.3) \quad \begin{cases} du = f, \\ u \in H_{d,0}^1 \Lambda^{r-1}(U), \end{cases} \quad \text{resp.} \quad \begin{cases} \delta u = f \text{ in } U, \\ u \in H_{\delta,0}^1 \Lambda^{r+1}(U), \end{cases}$$

has a solution if and only if  $f$  satisfies

$$\begin{cases} df = 0, \\ \mathbf{t}f = 0. \end{cases} \quad \text{resp.} \quad \begin{cases} \delta f = 0, \\ \mathbf{n}f = 0. \end{cases}$$

In this case, there exists a constant  $C(d, U) < \infty$  and a solution  $u$  of (3.3) which belongs to  $H^1 \Lambda^{r-1}(U)$  (resp.  $u \in H^1 \Lambda^{r+1}(U)$ ) and satisfies

$$(3.4) \quad \|u\|_{H^1 \Lambda^{r-1}(U)} \leq C \|f\|_{L^2 \Lambda^r(U)} \quad \text{resp.} \quad \|u\|_{H^1 \Lambda^{r+1}(U)} \leq C \|f\|_{L^2 \Lambda^r(U)}.$$

- For  $r = d$  (resp.  $r = 0$ ), given  $f \in L^2 \Lambda^r(U)$ , the problem

$$\begin{cases} du = f, \\ u \in H_{d,0}^1 \Lambda^{d-1}(U), \end{cases} \quad \text{resp.} \quad \begin{cases} \delta u = f \text{ in } U, \\ u \in H_{\delta,0}^1 \Lambda^1(U), \end{cases}$$

has a solution if and only if  $f$  satisfies

$$\int_U f = 0 \quad \text{resp.} \quad \int_U \star f = 0.$$

Moreover there exists a solution  $u \in H^1 \Lambda^{d-1}(U)$  (resp.  $u \in H^1 \Lambda^1(U)$ ) which satisfies (3.4).

The next important result of this section is the Hodge-Morrey Decomposition Theorem, but before stating this result, we need to introduce the subspaces of exact, co-exact and harmonic forms.

**Definition 3.5.** For each open  $U \subseteq \mathbb{R}^d$  and each  $1 \leq r \leq d$ , we say that a form  $u \in H_d^1 \Lambda^r(U)$  is exact if and only if there exists  $\alpha \in H_{d,0}^1 \Lambda^{r-1}(U)$  such that  $d\alpha = u$ . We denote by  $\mathcal{E}^r(U)$  the subset of exact  $r$  forms with null tangential trace, i.e.,

$$\mathcal{E}^r(U) := \{u \in H_d^1 \Lambda^r(U) : \exists \alpha \in H_{d,0}^1 \Lambda^{r-1}(U) \text{ such that } d\alpha = u\} \subseteq C_d^r(U),$$

the subset of co-exact  $r$  forms with null normal trace  $\mathcal{C}^r(U)$ , i.e.,

$$\mathcal{C}^r(U) := \{v \in H_{\mathbb{d}}^1 \Lambda^r(U) : \exists \beta \in H_{\delta,0}^1 \Lambda^{r+1}(U) \text{ such that } \delta\beta = v\} \subseteq C_{\delta}^r(U),$$

and the subset of  $r$  harmonic forms, i.e.,

$$\mathcal{H}^r(U) := \{w \in L^2 \Lambda^r(U) : dw = 0 \text{ and } \delta w = 0\}.$$

We now state the Hodge decomposition Theorem. This theorem is stated for two kinds of bounded domains, the convex domains in which case the situation is simple and the result can be deduced from Proposition 3.4, and the smooth domains. In the latter case the proof is more complicated and we refer to [27, Theorem 2.4.2] for the demonstration.

**Proposition 3.6** (Hodge-Morrey Decomposition, Theorem 2.4.2 of [27]). *Let  $U \subseteq \mathbb{R}^d$  be an open, bounded domain. We assume that this domain is either convex or smooth, then for each  $0 \leq r \leq d$ ,*

- (i) *the spaces  $\mathcal{E}^r(U)$ ,  $\mathcal{C}^r(U)$  and  $\mathcal{H}^r(U)$  are closed in the  $L^2 \Lambda^r(U)$  topology.*
- (ii) *the following orthogonal decomposition holds*

$$L^2 \Lambda^r(U) = \mathcal{E}^r(U) \dot{\oplus} \mathcal{C}^r(U) \dot{\oplus} \mathcal{H}^r(U).$$

#### 4. MULTISCALE POINCARÉ AND CACCIOPPOLI INEQUALITIES

The goal of this section is to prove some functional inequalities which will be important in the proof of Theorem 1 in Section 6. To do so, we first deduce from the results of the previous section the Poincaré inequality for differential forms on convex or smooth bounded domains of  $\mathbb{R}^d$ , Proposition 4.2 and Proposition 4.1. We then state, without proof, the Gaffney-Friedrichs inequality for convex or smooth bounded domains of  $\mathbb{R}^d$ . We deduce from these propositions the multiscale Poincaré inequality, Proposition 4.6. We finally conclude this section by stating and proving the Caccioppoli inequality for differential forms.

**Proposition 4.1** (Poincaré). *Let  $U$  be a bounded domain of  $\mathbb{R}^d$ . We assume that  $U$  is either smooth or convex. There exists a constant  $C := C(U) < \infty$ , such that for all  $0 \leq r \leq d$ , for all  $v \in H_{\mathbb{d},0}^1 \Lambda^r(U)$ ,*

$$(4.1) \quad \inf_{\alpha \in C_{\mathbb{d},0}^r(U)} \|v - \alpha\|_{L^2 \Lambda^r(U)} \leq C \|dv\|_{L^2 \Lambda^{r+1}(U)}.$$

Moreover, the constant  $C$  has the following scaling property, for each  $\lambda > 0$ ,

$$C(U) = \lambda C(\lambda^{-1}U).$$

**Proposition 4.2** (Poincaré-Wirtinger). *Let  $U$  be a bounded domain of  $\mathbb{R}^d$ . We assume that  $U$  is either smooth or convex. There exists a constant  $C := C(U) < \infty$ , such that for all  $v \in H_{\mathbb{d}}^1 \Lambda^r(U)$ ,*

$$(4.2) \quad \inf_{\alpha \in C_{\mathbb{d}}^r(U)} \|v - \alpha\|_{L^2 \Lambda^r(U)} \leq C \|dv\|_{L^2 \Lambda^{r+1}(U)}.$$

Moreover, the constant  $C$  has the following scaling property, for each  $\lambda > 0$ ,

$$C(U) = \lambda C(\lambda^{-1}U).$$

*Proof of Propositions 4.1 and 4.2.* First notice that both estimates are easy when  $r = d$  since in that case  $C_{\mathbb{d}}^r(U) = H_{\mathbb{d}}^1 \Lambda^d(U)$ . From now on, we assume  $0 \leq r \leq d - 1$ . In the case  $U$  convex, both inequalities (4.1) and (4.2) are a consequence of Proposition 3.4. We thus assume that  $U$  is smooth. The proof can be split into two steps.

- In Step 1, we prove that the space

$$\{u \in L^2 \Lambda^{r+1}(U) : \exists \alpha \in H_{\mathbb{d}}^1 \Lambda^r(U) \text{ such that } u = d\alpha\}$$

is closed in the  $L^2 \Lambda^{r+1}$  topology.

- In Step 2, we deduce, from Step 1 and Proposition 3.6, the estimates (4.1) and (4.2).

*Step 1.* The argument relies on a decomposition of the space  $\mathcal{H}^{r+1}(U)$  of harmonic forms, called the Friedrichs decomposition. By [27, Theorem 2.4.8], we have the following orthogonal decomposition,

$$\mathcal{H}^{r+1}(U) = (\mathcal{H}^{r+1}(U) \cap H_{\delta,0}^1 \Lambda^{r+1}(U)) \overset{\perp}{\oplus} \{u \in \mathcal{H}^{r+1}(U) \mid \exists \alpha \in H_{\text{d}}^1 \Lambda^r(U) \text{ such that } u = \text{d}\alpha\}.$$

Combining this result with Proposition 3.6 shows that

$$\begin{aligned} \{u \in L^2 \Lambda^{r+1}(U) : \exists \alpha \in H_{\text{d}}^1 \Lambda^r(U) \text{ such that } u = \text{d}\alpha\} \\ = \mathcal{E}^r(U) \overset{\perp}{\oplus} \{u \in \mathcal{H}^{r+1}(U) \mid \exists \alpha \in H_{\text{d}}^1 \Lambda^r(U) \text{ such that } u = \text{d}\alpha\} \end{aligned}$$

is closed for the  $L^2 \Lambda^{r+1}$  topology.

*Step 2.* We first prove (4.1). By Proposition 3.6, we know that the space  $\mathcal{E}^r$  is closed in  $L^2 \Lambda^{r+1}(U)$ . This yields that the range of the linear operator

$$\text{d} : \begin{cases} H_{\text{d},0}^1 \Lambda^r(U) \rightarrow L^2 \Lambda^{r+1}(U), \\ u \rightarrow \text{d}u. \end{cases}$$

is closed. Thus, by [6, Corollary 2.7], there exists a constant  $C(d, U) < \infty$  such that for each  $v \in H_{\text{d},0}^1 \Lambda^r(U)$ ,

$$\inf_{\alpha \in \ker \text{d}} \|v - \alpha\|_{L^2 \Lambda^r(U)} \leq C \|\text{d}v\|.$$

But we have  $\ker \text{d} = C_{\text{d}}^r(U) \cap H_{\text{d},0}^1 \Lambda^r(U)$ . This completes the proof of (4.1).

The proof of (4.2) is similar, the only difference is that we use Step 1, instead of Proposition 3.6, to obtain that

$$\{u \in L^2 \Lambda^{r+1}(U) : \exists \alpha \in H_{\text{d}}^1 \Lambda^r(U) \text{ such that } u = \text{d}\alpha\}$$

is closed in the  $L^2 \Lambda^{r+1}$  topology.

The scaling of the constant comes from the change of variable  $x \rightarrow \lambda x$ . □

We now state the Gaffney-Friedrichs inequality. The idea behind this inequality is to measure the global smoothness of a form  $u$  satisfying

$$(4.3) \quad \text{d}u \in L^2 \Lambda^{r+1}(U), \quad \delta u \in L^2 \Lambda^{r-1}(U) \text{ and } \mathbf{t}u = 0 \text{ on } \partial U.$$

According to a result from Gaffney [10] and Friedrich [9], provided that  $U$  is smooth, the former assumption (4.3) implies that  $u$  is  $H^1 \Lambda^r(U)$  with the estimate

$$(4.4) \quad \|u\|_{H^1 \Lambda^r(U)} \leq C \left( \|\text{d}u\|_{L^2 \Lambda^{r+1}(U)} + \|\delta u\|_{L^2 \Lambda^{r-1}(U)} + \|u\|_{L^2 \Lambda^r(U)} \right),$$

for some  $C := C(d, U) < \infty$ . Conversely, one clearly has

$$\left( \|\text{d}u\|_{L^2 \Lambda^{r+1}(U)} + \|\delta u\|_{L^2 \Lambda^{r-1}(U)} \right) \leq C \|\nabla u\|_{L^2 \Lambda^r(U)}.$$

Thus one can wonder whether the former inequality (4.4) can be refined into

$$(4.5) \quad \|\nabla u\|_{L^2 \Lambda^r(U)} \leq C \left( \|\text{d}u\|_{L^2 \Lambda^{r+1}(U)} + \|\delta u\|_{L^2 \Lambda^{r-1}(U)} \right).$$

This inequality is false in general, indeed the set of harmonic forms with Dirichlet boundary condition

$$\mathcal{H}_D^r := \{u \in L^2 \Lambda^r(U) : \text{d}u = 0, \delta u = 0 \text{ and } \mathbf{t}u = 0 \text{ on } \partial U\}$$

is known to be finite dimensional and of dimension  $\beta^{d-r}(U)$ , the Betti number of the set  $U$ , cf [27, Theorem 2.2.2]. In particular, as soon as  $\dim \mathcal{H}_D^r > 0$ , the inequality (4.5) cannot hold. Nevertheless it is the only obstruction and we have the following result, which is a consequence of [27, Proposition 2.2.3].



**Proposition 4.3** (Gaffney-Friedrich inequality for smooth domains). *Let  $U$  be a bounded smooth domain of  $\mathbb{R}^d$ , then there exists a constant  $C := C(d, U) < \infty$  such that if  $\omega \in L^2\Lambda^r(U)$  satisfies  $d\omega \in L^2\Lambda^{r+1}(U)$ ,  $\delta\omega \in L^2\Lambda^{r-1}(U)$ ,  $\mathbf{t}\omega = 0$  on  $\partial U$  and  $\omega \in (\mathcal{H}_D^r)^\perp$ , then  $\omega \in H^1\Lambda^r(U)$  and*

$$\|\nabla\omega\|_{L^2\Lambda^r(U)} \leq C \left( \|d\omega\|_{L^2\Lambda^{r+1}(U)} + \|\delta\omega\|_{L^2\Lambda^{r-1}(U)} \right).$$

One can also expect the inequality (4.5) to be true on convex domains, which are not necessarily smooth but satisfy  $\beta^r(U) = 0$  for each  $0 \leq r \leq d$ . This result is stated in the following proposition and can be found in [25, Theorem 5.5].

**Proposition 4.4** (Gaffney-Friedrichs inequality for convex domains). *Let  $U$  be a convex bounded domain of  $\mathbb{R}^d$ . Then there exists a constant  $C := C(d, U) < \infty$  such that if  $\omega \in L^2\Lambda^r(U)$  satisfies  $d\omega \in L^2\Lambda^{r+1}(U)$ ,  $\delta\omega \in L^2\Lambda^{r-1}(U)$  and either  $\mathbf{t}\omega = 0$  or  $\mathbf{n}\omega = 0$  on  $\partial U$ , then  $\omega \in H^1\Lambda^r(U)$  and*

$$\|\nabla\omega\|_{H^1\Lambda^r(U)} \leq C \left( \|d\omega\|_{L^2\Lambda^{r+1}(U)} + \|\delta\omega\|_{L^2\Lambda^{r-1}(U)} \right).$$

These inequalities are a key ingredient in the proofs of Theorem 1 and Theorem 2. Another important ingredient needed in the proof of Theorem 1 is the so called multiscale Poincaré inequality stated below (Proposition 4.6). This inequality is valid for cubes and the statement and the proofs of Theorems 1 and 2 only require to apply the following results to cubes of  $\mathbb{R}^d$ . Thus, from now on and until the end of Section 4, we will only be dealing with cubes of  $\mathbb{R}^d$ , denoted by  $\square$ , instead of convex bounded domains. Recall that a cube of  $\mathbb{R}^d$  is a set of the form

$$z + R(-1, 1)^d \text{ with } z \in \mathbb{R}^d, R \in \mathbb{R}_+$$

and a triadic cube, denoted by  $\square_m$ , for  $m \in \mathbb{N}$ , is defined according to the formula

$$\square_m := \left( -\frac{3^m}{2}, \frac{3^m}{2} \right)^d.$$

We then define the mean value of a form on a cube according to the following proposition.

**Definition 4.5.** Given  $\square$  a cube of  $\mathbb{R}^d$  and  $0 \leq r \leq d$  and a form  $\alpha = \sum_{|I|=r} \alpha_I dx_I \in L^2\Lambda^r(\square)$ . We denote by

$$(\alpha)_\square := \sum_{|I|=r} \left( \int_\square \alpha_I(x) dx \right) dx_I \in \Lambda^r(\mathbb{R}^d).$$

The multiscale Poincaré inequality then reads:

**Proposition 4.6** (Multiscale Poincaré). *Fix  $m \in \mathbb{N}$  and, for each  $0 \leq r \leq d$ , each  $n \in \mathbb{N}$ ,  $n \leq m$ , define  $\mathcal{Z}_{m,n} = 3^n \mathbb{Z}^d \cap \square_m$ . There exists a constant  $C(d) < \infty$  such that, for every  $u \in C_d^r(U)^\perp$ ,*

$$\|u\|_{\underline{L}^2(\square_m)} \leq C \|du\|_{\underline{L}^2(\square_m)} + C \sum_{n=0}^{m-1} 3^n \left( |\mathcal{Z}_{m,n}|^{-1} \sum_{z \in \mathcal{Z}_{m,n}} |(du)_{z+\square_n}|^2 \right)^{\frac{1}{2}}.$$

To prove this estimate, we first need to introduce the following  $H^{-1}$  norm for cubes.

**Definition 4.7.** For each cube  $\square$  of  $\mathbb{R}^d$  and each  $\omega \in L^2\Lambda^r(\square)$ , we define the following  $H^{-1}$  norm

$$\|\omega\|_{\underline{H}^{-1}\Lambda^r(\square)} := \sup \left\{ \frac{1}{|\square|} \langle \omega, \alpha \rangle_\square : \alpha \in H^1\Lambda^r(\square), \text{size}(\square)^{-1} |(\alpha)_\square| + \|\nabla\alpha\|_{\underline{L}^2\Lambda^r(\square)} \leq 1 \right\}.$$

By the Poincaré-Wirtinger inequality, there exists a constant  $C(d) < \infty$  such that,

$$\|\omega\|_{\underline{H}^{-1}\Lambda^r(\square)} \leq C \text{size}(\square) \|\omega\|_{\underline{L}^2\Lambda^r(\square)}.$$

The Multiscale Poincaré inequality is a consequence of this improved version of the Poincaré-Wirtinger inequality. The particular case  $r = 0$  of this statement can be found in [3, Lemma 1.9].

**Proposition 4.8.** *There exists a constant  $C := C(d) < \infty$  such that for every cube  $\square \in \mathbb{R}^d$ , every  $0 \leq r \leq d$  and every  $u \in H^1_{\square} \Lambda^r(\square)$ ,*

$$\inf_{\alpha \in C^r_{\square}(\square)} \|u - \alpha\|_{\underline{L}^2 \Lambda^r(\square)} \leq C \|du\|_{\underline{H}^{-1} \Lambda^{r+1}(\square)}.$$

Before starting the proof, we need to state and prove the following lemma.

**Lemma 4.9.** *There exists  $C := C(d) < \infty$  such that for each cube  $\square \in \mathbb{R}^d$ , each  $0 \leq r \leq (d-1)$  and each  $u \in C^r_{\square}(\square)^{\perp}$  there exists a unique  $w \in H^1_{\square} \Lambda^r(\square) \cap C^r_{\square}(\square)^{\perp}$  solution of the Neumann problem*

$$(4.6) \quad \begin{cases} \delta dw = u \text{ in } \square, \\ \mathbf{n}dw = 0 \text{ on } \partial\square, \end{cases}$$

in the sense that, for each  $v \in H^1_{\square} \Lambda^r(U)$ ,

$$\langle dw, dv \rangle_{\square} = \langle u, v \rangle_{\square}.$$

Moreover,  $dw \in H^1 \Lambda^{r+1}(\square)$  and

$$(4.7) \quad \|\nabla dw\|_{L^2 \Lambda^{r+1}(\square)} \leq C \|u\|_{L^2 \Lambda^r(\square)}.$$

*Proof.* The proof can be split in two steps, first we need to prove that there exists a function  $w$  in  $H^1_{\square} \Lambda^r(\square)$  solution of the Neumann problem (4.6) and then that the function  $w$  satisfies  $dw \in H^1 \Lambda^{r+1}(\square)$  with the regularity estimate (4.7).

*Step 1.* To solve (4.6), denote for  $v \in H^1_{\square} \Lambda^r(\square)$ , by

$$\mathcal{J}(v) := \langle dv, dv \rangle_{\square} - \langle u, v \rangle_{\square}$$

and look at the variational problem

$$\inf_{v \in H^1_{\square} \Lambda^r(\square) \cap C^r_{\square}(\square)^{\perp}} \mathcal{J}(v).$$

By the standard minimization techniques of the calculus of variations and the Poincaré-Wirtinger inequality (Proposition 4.2), it is straightforward to prove that there exists a unique minimizer  $w$  of this problem. By the first variation,  $w$  solves (4.6).

*Step 2.* The main ingredient of this step is the Gaffney-Friedrichs inequality (Proposition 4.4) applied with  $U = \square$  and  $\omega = dw$ . This form satisfies  $w \in L^2 \Lambda^{r+1}(\square)$ ,  $dw = ddu = 0 \in L^2 \Lambda^{r+2}(\square)$ ,  $\delta w = u \in L^2 \Lambda^r(\square)$  and  $\mathbf{n}\omega = 0$ . Thus, by the Gaffney-Friedrichs inequality,  $\omega \in H^1 \Lambda^{r+1}(\square)$ , and for some  $C := C(\square) < \infty$ ,

$$\|\nabla \omega\|_{L^2 \Lambda^{r+1}(\square)} \leq C \|u\|_{L^2 \Lambda^r(\square)}.$$

By translation and scaling invariance, we obtain that there exists a constant  $C := C(d) < \infty$  such that

$$\|\nabla \omega\|_{L^2 \Lambda^{r+1}(\square)} \leq C \|u\|_{L^2 \Lambda^r(\square)}.$$

This is exactly (4.7). □

We know apply Lemma 4.9 to prove Proposition 4.8.

*Proof of Proposition 4.8.* First notice that it is enough to prove the result when  $u \in H^1_{\square} \Lambda^r(\square) \cap (C^r_{\square}(\square))^{\perp}$ . Using the function  $w \in H^1_{\square} \Lambda^r(\square)$  solution of the Neumann problem (4.6) in the cube  $\square$ , we have

$$\begin{aligned} \|u\|_{\underline{L}^2 \Lambda^r(\square)}^2 &= \frac{1}{|\square|} \langle u, u \rangle_{\square} \\ &= \frac{1}{|\square|} \langle du, dw \rangle_{\square} \\ &\leq \|du\|_{\underline{H}^{-1} \Lambda^{r+1}(\square)} \left( \text{size}(\square)^{-1} |(dw)_{\square}| + \|\nabla dw\|_{\underline{L}^2 \Lambda^{r+1}(\square)} \right). \end{aligned}$$

By Lemma 4.9,

$$\|\nabla dw\|_{\underline{L}^2\Lambda^{r+1}(\square)} \leq C\|u\|_{\underline{L}^2\Lambda^r(\square)}.$$

To complete the proof, there remains to estimate  $|(dw)_\square|$ , to do so denote by

$$p = \sum_{i_1 < \dots < i_p} p_{i_1, \dots, i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p} := \frac{(dw)_\square}{|(dw)_\square|}$$

and

$$(4.8) \quad l_p : \begin{cases} \mathbb{R}^d \rightarrow \Lambda^p(\mathbb{R}^d), \\ x \rightarrow \sum_{i_1 < \dots < i_p} p_{i_1, \dots, i_p} x_{i_1} dx_{i_2} \wedge \dots \wedge dx_{i_p}, \end{cases}$$

such that  $dl_p = p$ . Testing the equation (4.6) with  $\alpha = l_p$ , we obtain

$$\begin{aligned} |(dw)_\square| &= \frac{1}{|\square|} |\langle p, dw \rangle_\square| = \frac{1}{|\square|} |\langle dl_p, dw \rangle_\square| \\ &= \frac{1}{|\square|} |\langle l_p, u \rangle_\square| \\ &\leq C \text{size}(\square) \|u\|_{\underline{L}^2\Lambda^r(\square)}. \end{aligned}$$

Combining the previous results completes the proof of the proposition.  $\square$

We then apply the Multiscale Poincaré inequality stated below. A proof of this inequality can be found in [3, Proposition 1.8].

**Proposition 4.10** (Multiscale Poincaré, Proposition 1.8 of [3]). *Fix  $m \in \mathbb{N}$  and, for each  $n \in \mathbb{N}$ , such that  $n \leq m$ , define  $\mathcal{Z}_{m,n} = 3^n \mathbb{Z}^d \cap \square_m$ . There exists a constant  $C(d) < \infty$  such that, for every  $f \in L^2(\square_m)$ ,*

$$\|f\|_{\underline{H}^{-1}(\square_m)} \leq C\|f\|_{\underline{L}^2(\square_m)} + C \sum_{n=0}^{m-1} 3^n \left( |\mathcal{Z}_{m,n}|^{-1} \sum_{z \in \mathcal{Z}_{m,n}} |(f)_{z+\square_n}|^2 \right)^{\frac{1}{2}}.$$

*Proof of Proposition 4.6.* The result is then a consequence of Proposition 4.8 and Proposition 4.10 applied with  $f = du$ .  $\square$

We complete this section by proving a version of the Caccioppoli inequality for differential forms. Recall the definitions of the space  $\Omega_r$  in (2.17) and, given an environment  $\mathbf{a} \in \Omega_r$ , the definition of the space of solutions  $\mathcal{A}(U)$  in (2.21).

**Proposition 4.11** (Caccioppoli inequality). *There exist a constant  $C := C(d, \lambda) < \infty$  such that, for every  $1 \leq r \leq d$ , every open subsets  $V, U \subseteq \mathbb{R}^d$  satisfying  $\bar{V} \subseteq U$ , and every  $u \in \mathcal{A}(U)$ ,*

$$\|du\|_{L^2\Lambda^{r+1}(V)} \leq \frac{C}{\text{dist}(V, \partial U)} \|u\|_{L^2\Lambda^r(U \setminus V)}.$$

*Proof.* Let  $\eta \in C_c^\infty(U)$  be such that

$$\mathbf{1}_V \leq \eta \leq 1, \quad |\nabla \eta| \leq \frac{C}{\text{dist}(V, \partial U)}.$$

The  $r$ -form  $\eta u$  belongs to  $H_{d,0}^1 \Lambda^r(U)$  from this we deduce that

$$\int_U du \wedge \mathbf{ad}(\eta u) = 0,$$

which gives

$$\begin{aligned} 0 &= \int_U du \wedge \mathbf{a}d(\eta^2 u) \\ &= \int_U (du \wedge \mathbf{a}d\eta^2 \wedge u + du \wedge \mathbf{a}\eta^2 du) \\ &= \int_U (du \wedge \mathbf{a}2\eta d\eta \wedge u + du \wedge \mathbf{a}\eta^2 du). \end{aligned}$$

Thus, since by the ellipticity assumption (2.16) and the symmetry assumption (2.15), for each  $x \in \mathbb{R}^d$ , the bilinear form  $(p, p') \rightarrow p \wedge \mathbf{a}(x)p'$  is a scalar product on  $\Lambda^r(\mathbb{R}^d)$ . In particular we can apply the Cauchy-Schwarz inequality in the following computation.

$$\begin{aligned} \int_U du \wedge \mathbf{a}\eta^2 du &= \int_U du \wedge \mathbf{a}(2\eta d\eta \wedge u) \\ &\leq 2 \left( \int_U du \wedge \mathbf{a}(\eta^2 du) \right)^{\frac{1}{2}} \left( \int_U d\eta \wedge u \wedge (\mathbf{a}d\eta \wedge u) \right)^{\frac{1}{2}}. \end{aligned}$$

Using the ellipticity condition (2.16), we obtain

$$\begin{aligned} \|du\|_{L^2\Lambda^{r+1}(V)} &\leq C \|d\eta \wedge u\|_{L^2\Lambda^r(U \setminus V)} \\ &\leq \frac{C}{\text{dist}(V, \partial U)} \|u\|_{L^2\Lambda^r(U \setminus V)}. \end{aligned}$$

The proof is complete.  $\square$

## 5. QUANTITATIVE HOMOGENIZATION

The goal of this section is to study the quantity  $J$  defined, according to Definition 2.6, by the formula, for  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$

$$J(U, p, q) := \sup_{v \in \mathcal{A}(U)} \int_U \left( -\frac{1}{2} dv \wedge \mathbf{a}dv - p \wedge \mathbf{a}dv + dv \wedge q \right).$$

Thanks to the Poincaré-Wirtinger inequality, Proposition 4.2, one can prove that there exists a unique maximizer in  $\mathcal{A}(U) \cap C_d^{r-1}(U)^\perp$ , denoted by  $v(\cdot, U, p, q)$ . The proof is very similar to Step 1 of the proof of Lemma 4.9 and the details are omitted.

We first record some useful properties about  $J$ , Proposition 5.1. We then establish a series of Lemmas, Lemmas 5.2 to 5.8, before proving the main result of this section, namely Theorem 1. We eventually deduce from Theorem 1 a corollary pertaining to the maximizer  $v(\cdot, U, p, q)$ , Proposition 5.10.

**Proposition 5.1** (Basic properties of  $J$ ). *Fix a bounded Lipschitz domain  $U \subseteq \mathbb{R}^d$ . For each  $1 \leq r \leq d$ , the quantity  $J(U, p, q)$  and its maximizer  $v(\cdot, U, p, q)$  satisfy the following properties:*

- (1) Decomposition of the maximizer  $v(\cdot, U, p, q)$ . *The map*

$$(5.1) \quad \begin{cases} \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d) \rightarrow \mathcal{A}(U) \cap C_d^{r-1}(U)^\perp, \\ (p, q) \rightarrow v(\cdot, U, p, q), \end{cases}$$

*is linear. Moreover,  $v(\cdot, U, p, 0)$  is, up to a closed form, equal to a solution of the Dirichlet problem*

$$(5.2) \quad \begin{cases} d(\mathbf{a}du) = 0 \in U, \\ \mathbf{t}u = \mathbf{t}l_{-p} \text{ on } \partial U, \end{cases}$$

*where  $l_p$  is defined by (4.8). The precise interpretation of (5.2) is:*

$$u \text{ solves (5.2)} \Leftrightarrow u \in l_{-p} + H_{d,0}^1 \Lambda^{(r-1)}(U) \text{ and } \forall w \in H_{d,0}^1 \Lambda^{(r-1)}(U), \int_U du \wedge \mathbf{a}dw = 0.$$

Similarly  $v(\cdot, U, 0, q)$  is a solution of the Neumann problem

$$(5.3) \quad \begin{cases} d(\mathbf{a}du) = 0 \text{ in } U, \\ \mathbf{t}(\mathbf{a}du) = \mathbf{t}q \text{ on } \partial U. \end{cases}$$

the precise interpretation of (5.3) is:

$$u \text{ solves (5.3)} \Leftrightarrow u \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U) \text{ and } \forall w \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U), \int_U du \wedge \mathbf{a}dw - dw \wedge q = 0.$$

- (2) Decomposition of  $J(U, p, q)$ . For each  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ , the quantity  $J(U, p, q)$  can be decomposed

$$(5.4) \quad J(U, p, q) = \nu(U, p) + \nu^*(U, q) - \star(p \wedge q),$$

where  $p \rightarrow \nu(U, p)$  and  $q \rightarrow \nu^*(U, q)$  are quadratic forms given by the formulas

$$(5.5) \quad \nu(U, p) = \inf_{u \in \ell_{-p} + H_{\mathbf{d}, 0}^1 \Lambda^{r-1}(U)} \int_U du \wedge \mathbf{a}du$$

and

$$(5.6) \quad \nu^*(U, q) = \sup_{u \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U)} \int_U \left( -\frac{1}{2} du \wedge \mathbf{a}du + du \wedge q \right).$$

As a remark note that there is a star before  $q \wedge p$  in (5.4) because  $q \wedge p$  is a  $d$ -form and all the other terms are real numbers.

- (3) Upper and lower bound on  $\nu(U, p)$  and  $\nu^*(U, q)$ . There exists a constant  $C(d, \lambda) < \infty$  such that for every  $p \in \Lambda^r(\mathbb{R}^d)$ ,  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.7) \quad \frac{1}{C} |p|^2 \leq \nu(U, p) \leq C |p|^2$$

and

$$(5.8) \quad \frac{1}{C} |q|^2 \leq \nu^*(U, q) \leq C |q|^2.$$

This implies, according to (5.4), for some  $C := C(d, \lambda) < \infty$ ,

$$(5.9) \quad J(U, p, q) \leq C(|p|^2 + |q|^2)$$

and

$$(5.10) \quad \|dv(\cdot, U, p, q)\|_{\underline{L}^2 \Lambda^r(U)} \leq C(|p|^2 + |q|^2).$$

- (4) Uniform convexity and  $C^{1,1}$  in  $p$  and  $q$  separately. There exists  $C(d, \lambda) < \infty$  such that for every  $p_1, p_2 \in \Lambda^r(\mathbb{R}^d)$  and  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.11) \quad \frac{1}{C} |p_1 - p_2|^2 \leq \frac{1}{2} J(U, p_1, q) + \frac{1}{2} J(U, p_2, q) - J\left(U, \frac{p_1 + p_2}{2}, q\right) \leq C |p_1 - p_2|^2.$$

For every  $q_1, q_2 \in \Lambda^{d-r}(\mathbb{R}^d)$  and  $p \in \Lambda^r(\mathbb{R}^d)$ ,

$$(5.12) \quad \frac{1}{C} |q_1 - q_2|^2 \leq \frac{1}{2} J(U, p, q_1) + \frac{1}{2} J(U, p, q_2) - J\left(U, p, \frac{q_1 + q_2}{2}\right) \leq C |q_1 - q_2|^2.$$

- (5) Subadditivity. Let  $U_1, \dots, U_n \subseteq U$  be bounded Lipschitz domains that form a partition of  $U$ , in the sense that

$$(5.13) \quad U_i \cap U_j = \emptyset \quad \text{if } i \neq j \quad \text{and} \quad \left| U \setminus \bigcup_{i=1}^N U_i \right| = 0,$$

then, for every  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.14) \quad J(U, p, q) \leq \sum_{i=1}^N J(U_i, p, q).$$

- (6) First variation for  $J$ . For each  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ , the function  $v(\cdot, U, p, q)$  is characterized as the unique element of  $\mathcal{A}(U) \cap C_d^{r-1}(U)^\perp$  which satisfies, for each  $u \in \mathcal{A}(U)$ ,

$$(5.15) \quad \int_U dv \wedge \mathbf{a}du = \int_U (-p \wedge \mathbf{a}du + du \wedge q)$$

- (7) Quadratic response For every  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$  and  $w \in \mathcal{A}(U)$ ,

$$(5.16) \quad \frac{1}{C} \|dw - dv(\cdot, U, p, q)\|_{\underline{L}^2 \Lambda^r(U)}^2 \\ \leq J(U, p, q) - \int_U \left( -\frac{1}{2} dw \wedge \mathbf{a}dw - p \wedge \mathbf{a}dw + dw \wedge q \right) \\ \leq C \|dw - dv(\cdot, U, p, q)\|_{\underline{L}^2 \Lambda^r(U)}^2.$$

- (8) Control of the difference of the optimizers by the subadditivity. Let  $U_1, \dots, U_n \subseteq U$  be bounded Lipschitz domains that form a partition of  $U$ , in the sense of (5.13). Then for each  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.17) \quad \sum_{i=1}^n \frac{|U_i|}{|U|} \|dv(\cdot, U, p, q) - dv(\cdot, U_i, p, q)\|_{\underline{L}^2 \Lambda^r(U_i)}^2 \leq C \sum_{i=1}^n \frac{|U_i|}{|U|} (J(U_i, p, q) - J(U, p, q)).$$

*Proof.* These properties are easy to check and their proofs are almost the same of those of [3, Lemma 2.2], so we omit the details.  $\square$

We now turn to the proof of a series of lemmas, which will be then used in the proof of Theorem 1. In the following lemma, we denote by  $\mathcal{Z}_{m,n} := 3^n \mathbb{Z}^d \cap \square_m$ . It is a finite set of cardinality  $3^{d(m-n)}$ .

**Lemma 5.2.** Fix  $m, n \in \mathbb{N}$  with  $n < m$ ,  $(p, q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$  and  $\{q'_z\}_{z \in \mathcal{Z}_{m,n}} \in \Lambda^{d-r}(\mathbb{R}^d)$ . Then

$$(5.18) \quad \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \left| \int_{z+\square_n} (dv - dv_z) \wedge q'_z \right| \leq \left( \sum_{z \in \mathcal{Z}_{n,m}} |q'_z|^2 \right)^{\frac{1}{2}} \left( \sum_{z \in \mathcal{Z}_{n,m}} J(z + \square_n, p, q) - J(\square_m, p, q) \right)^{\frac{1}{2}}.$$

*Proof.* We shorten the notation by setting, for each  $z \in \mathcal{Z}_{m,n}$ ,

$$v := v(\cdot, \square_m, p, q), \quad v_z := v(\cdot, z + \square_n, p, q), \quad v'_z := v(\cdot, z + \square_n, 0, q'_z).$$

Notice that for each  $x \in \mathbb{R}^d$ , the quadratic map  $(p, p') \rightarrow p \wedge \mathbf{a}(x)p'$  is a scalar product. In particular we can apply the Cauchy-Schwarz inequality, this reads

$$p \wedge \mathbf{a}(x)p' \leq (p \wedge \mathbf{a}(x)p)^{\frac{1}{2}} (p' \wedge \mathbf{a}(x)p')^{\frac{1}{2}} \leq \Lambda |p||p'|,$$

by the ellipticity assumption (2.16). Keeping this in mind, we compute, using Hölder's inequality,

$$\begin{aligned} & \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \left| \int_{z+\square_n} q'_z \wedge (dv - dv_z) \right| \\ &= \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \left| \int_{z+\square_n} dv'_z \wedge \mathbf{a}(dv - dv_z) \right| \\ &\leq \frac{C}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \|dv'_z\|_{L^2(z+\square_n)} \|dv - dv_z\|_{L^2(z+\square_n)} \\ &\leq C \left( \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \|dv'_z\|_{L^2(z+\square_n)}^2 \right)^{\frac{1}{2}} \left( \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \|dv - dv_z\|_{L^2(z+\square_n)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

by (5.10) and (5.17),

$$\leq C \left( \sum_{z \in \mathcal{Z}_{n,m}} |q'_z|^2 \right)^{\frac{1}{2}} \left( \sum_{z \in \mathcal{Z}_{n,m}} J(z + \square_n, p, q) - J(\square_m, p, q) \right)^{\frac{1}{2}}. \quad \square$$

**Lemma 5.3.** *Let  $m, n \in \mathbb{N}$  with  $0 \leq n \leq m - 2$ . Then there exists  $C(d, \lambda) < \infty$  such that, for every  $(p, q) \in B_1 \Lambda^r(\mathbb{R}^d) \times B_1 \Lambda^{d-r}(\mathbb{R}^d)$ ,*

$$(5.19) \quad \text{var} \left[ (dv(\cdot, \square_m, p, q))_{\square_m} \right] \leq C 3^{-d(m-n)} \text{var} \left[ (dv(\cdot, \square_n, p, q))_{\square_n} \right] + C \mathbb{E} [J(\square_n, p, q) - J(\square_m, p, q)].$$

*Proof.* We first fix  $n \in \mathbb{N}$  with  $n \leq m - 2$ ,  $q' \in B_1 \Lambda^{d-r}(\mathbb{R}^d)$  and apply Lemma 5.2 with  $q'_z := q'$  to get

$$(5.20) \quad \frac{1}{|\square_m|} \left| \int_{\square_m} dv(\cdot, \square_m, p, q) \wedge q' - \sum_{z \in \mathcal{Z}_{n,m}} \int_{z + \square_n} dv(\cdot, z + \square_n, p, q) \wedge q' \right| \leq C \left( \sum_{z \in \mathcal{Z}_{n,m}} J(z + \square_n, p, q) - J(\square_m, p, q) \right)^{\frac{1}{2}}.$$

From this we obtain

$$\text{var} \left[ \int_{\square_m} dv(\cdot, \square_m, p, q) \wedge q' \right] \leq 2 \text{var} \left[ \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \int_{z + \square_n} dv(\cdot, z + \square_n, p, q) \wedge q' \right] + 2C \mathbb{E} \left[ \left( \sum_{z \in \mathcal{Z}_{n,m}} J(z + \square_n, p, q) - J(\square_m, p, q) \right)^2 \right].$$

We take an enumeration  $\{z_{i,j} : 1 \leq i \leq 3^d, 1 \leq j \leq 3^{d(m-n-1)}\}$  of  $\mathcal{Z}_{m,n}$  such that for each  $1 \leq i \leq 3^d$  and each  $1 \leq j, j' \leq 3^{d(m-n-1)}$ ,

$$|z_{i,j} - z_{i,j'}| \geq 2 \cdot 3^n.$$

This gives in particular

$$\text{dist}(z_{i,j} + \square_n, z_{i,j'} + \square_n) \geq 3^n,$$

and thus, according to the independence assumption (2.20),

$$\mathcal{F}_r(z_{i,j} + \square_n) \text{ and } \mathcal{F}_r(z_{i,j'} + \square_n) \text{ are independent.}$$

We can thus estimate the first term on the right side of (5.18), using the previous display and the stationarity (2.19) to get

$$\begin{aligned}
 & \text{var} \left[ \frac{1}{|\square_m|} \sum_{z \in \mathcal{Z}_{n,m}} \int_{z+\square_n} dv(\cdot, z + \square_n, p, q) \wedge q' \right] \\
 &= 3^{-2dm} \text{var} \left[ \sum_{i=1}^{3^d} \sum_{j=1}^{3^{d(m-n-1)}} \int_{z_{i,j} + \square_n} dv(\cdot, z_{i,j} + \square_n, p, q) \wedge q' \right] \\
 &\leq 3^{-2dm+d} \sum_{i=1}^{3^d} \text{var} \left[ \sum_{j=1}^{3^{d(m-n-1)}} \int_{z_{i,j} + \square_n} dv(\cdot, z_{i,j} + \square_n, p, q) \wedge q' \right] \\
 &\leq 3^{-2dm+d} \sum_{i=1}^{3^d} \sum_{j=1}^{3^{d(m-n-1)}} \text{var} \left[ \int_{z_{i,j} + \square_n} dv(\cdot, z_{i,j} + \square_n, p, q) \wedge q' \right] \\
 &\leq 3^{d(-m+1-n)} \text{var} \left[ \int_{\square_n} dv(\cdot, \square_n, p, q) \wedge q' \right] \\
 &\leq C 3^{-d(m-n)} \text{var} \left[ \int_{\square_n} dv(\cdot, \square_n, p, q) \wedge q' \right].
 \end{aligned}$$

Combining the previous display with (5.20) and taking the supremum over  $q' \in B_1 \Lambda^{d-r}(\mathbb{R}^d)$  completes the proof of the lemma.  $\square$

We then define:

**Definition 5.4.** For  $n \in \mathbb{N}$ , we define by

$$\begin{aligned}
 \tau_n &:= \sup_{(p,q) \in B_1 \Lambda^r(\mathbb{R}^d) \times B_1 \Lambda^{d-r}(\mathbb{R}^d)} \mathbb{E} [J(\square_n, p, q) - J(\square_{n+1}, p, q)] \\
 &= \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E} [\nu(\square_n, p) - \nu(\square_{n+1}, p)] + \sup_{q \in B_1 \Lambda^{d-r}(\mathbb{R}^d)} \mathbb{E} [\nu^*(\square_n, q) - \nu^*(\square_{n+1}, q)]
 \end{aligned}$$

With this definition, one can prove:

**Lemma 5.5.** For each  $n \in \mathbb{N}$ , there exists a constant  $C(d, \lambda) < \infty$  and an exponent  $\beta := \beta(d, \lambda) > 0$  such that for every  $(p, q) \in B_1 \Lambda^r(\mathbb{R}^d) \times B_1 \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.21) \quad \text{var} \left[ (dv(\cdot, \square_m, p, q))_{\square_m} \right] \leq C \sum_{n=0}^m 3^{\beta(n-m)} \tau_n + C 3^{-\beta m}.$$

*Proof.* Denote by  $C := C(d, \lambda) < \infty$  the constant of Lemma 5.3 and select  $l := l(d, \lambda) \in \mathbb{N}$  such that

$$\frac{1}{9} < C 3^{-dl} \leq \frac{1}{3}.$$

The inequality (5.19) applied with  $n = m - l$  yields

$$\text{var} \left[ (dv(\cdot, \square_m, p, q))_{\square_{m-l}} \right] \leq \frac{1}{3} \text{var} \left[ (dv(\cdot, \square_m, p, q))_{\square_{m-l}} \right] + C \sum_{k=n-l}^n \tau_k.$$

Iterating this estimate and using the bound on the  $L^2$  norm of  $dv$  (5.10) gives, for some  $C := C(d, \lambda) < \infty$ ,

$$\text{var} \left[ (dv(\cdot, \square_m, p, q))_{\square_m} \right] \leq C 3^{-\frac{n}{l}} + C \sum_{k=0}^n 3^{\frac{n-k}{l}} \tau_k.$$

This completes the proof of the lemma with  $\beta = \frac{1}{l}$ .  $\square$



**Definition 5.6.** Consider  $U$  a bounded open subset of  $\mathbb{R}^d$ . Since  $q \rightarrow J(U, 0, q)$  is quadratic and bounded from above and below according to (5.8), there exists a linear mapping, denoted by  $\bar{\mathbf{a}}_U$ , from  $\Lambda^r(\mathbb{R}^d)$  to  $\Lambda^{d-r}(\mathbb{R}^d)$ , satisfying the symmetry assumption (2.15) and such that, for every  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.22) \quad \mathbb{E}[J(U, 0, q)] = \frac{1}{2} \star (\bar{\mathbf{a}}_U^{-1} q \wedge q).$$

We also write  $\bar{\mathbf{a}}_n = \bar{\mathbf{a}}_{\square_n}$  for short.

There are two properties to notice about this quantity. First since  $J$  satisfies the subadditivity property (5.14), and by the stationarity assumption (2.19), the sequence  $(\mathbb{E}[J(\square_n, 0, q)])_{n \in \mathbb{N}}$  is decreasing. From this we deduce that it converges for each  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ . From this, we deduce that there exists a linear symmetric map  $\bar{\mathbf{a}} \in \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d))$  such that, for each  $q \in \Lambda^{d-r}(\mathbb{R}^d)$

$$\mathbb{E}[J(\square_n, 0, q)] \xrightarrow{n \rightarrow \infty} \frac{1}{2} \star (\bar{\mathbf{a}}^{-1} q \wedge q)$$

which also implies

$$\bar{\mathbf{a}}_n \rightarrow \bar{\mathbf{a}} \text{ in } \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d)).$$

Moreover, by (5.8), one can check that there exists a constant  $C(d, \lambda) < \infty$  such that, for each  $p \in \Lambda^r(\mathbb{R}^d)$  and each  $n \in \mathbb{N}$ ,

$$\frac{1}{C} |p|^2 \leq p \wedge \bar{\mathbf{a}}_n p \leq C |p|^2.$$

Sending  $n \rightarrow \infty$  shows that the same estimate is true for  $\bar{\mathbf{a}}$ .

Second, one has the formula, for  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$(5.23) \quad \bar{\mathbf{a}}_n^{-1} q = \mathbb{E}[(dv(\cdot, \square_n, 0, q))_{\square_n}].$$

To prove this formula, one has, according to the first variation (5.15), we have, for each  $q \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$J(\square_n, 0, q) = \int_{\square_n} dv(\cdot, \square_n, 0, q) \wedge q$$

Taking the expectation proves

$$\bar{\mathbf{a}}_n^{-1} q \wedge q = \mathbb{E}[(dv(\cdot, \square_n, 0, q))_{\square_n}] \wedge q.$$

To prove (5.23) it is thus sufficient to prove that  $q \rightarrow \mathbb{E}[(dv(\cdot, \square_n, 0, q))_{\square_n}]$  satisfies the following symmetry property, for each  $q, q' \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$\mathbb{E}[(dv(\cdot, \square_n, 0, q))_{\square_n}] \wedge q' = \mathbb{E}[(dv(\cdot, \square_n, 0, q'))_{\square_n}] \wedge q.$$

It is a consequence of the following computation

$$\begin{aligned} \mathbb{E}[(dv(\cdot, \square_n, 0, q))_{\square_n}] \wedge q' &= \mathbb{E}\left[\int_{\square_n} dv(\cdot, \square_n, 0, q) \wedge q'\right] \\ &= \mathbb{E}\left[\int_{\square_n} dv(\cdot, \square_n, 0, q) \wedge \mathbf{a} dv(\cdot, \square_n, 0, q')\right] \\ &= \mathbb{E}\left[\int_{\square_n} dv(\cdot, \square_n, 0, q') \wedge \mathbf{a} dv(\cdot, \square_n, 0, q)\right] \\ &= \mathbb{E}\left[\int_{\square_n} dv(\cdot, \square_n, 0, q') \wedge q\right] \\ &= \mathbb{E}\left[(dv(\cdot, \square_n, 0, q'))_{\square_n}\right] \wedge q. \end{aligned}$$

We then note that, for every  $q \in B_1\Lambda^{d-r}(\mathbb{R}^d)$ ,  $m, n \in \mathbb{N}$  such that  $n < m$ , we have

$$\begin{aligned}
 (5.24) \quad & |\bar{\mathbf{a}}_m^{-1}q - \bar{\mathbf{a}}_n^{-1}q|^2 \\
 &= \left| \mathbb{E} \left[ \left( dv(\cdot, \square_m, 0, q) \right)_{\square_m} - 3^{d(n-m)} \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} \left( dv(\cdot, z + \square_n, 0, q) \right)_{z + \square_n} \right] \right|^2 \\
 &\leq \mathbb{E} \left[ 3^{d(n-m)} \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} \|dv(\cdot, \square_m, 0, q) - dv(\cdot, z + \square_n, 0, q)\|_{\underline{L}^2 \Lambda^r(U)}^2 \right] \\
 &\leq C \mathbb{E} [J(\square_n, 0, q) - J(\square_m, 0, q)] \\
 &\leq C \sum_{k=n}^{m-1} \tau_k.
 \end{aligned}$$

For  $p \in \Lambda^r(\mathbb{R}^d)$  and  $m \in \mathbb{N}$ , we denote by  $l_p^m$  the unique element of  $C_d^{r-1}(\square_m)^\perp$  such that  $dl_p^m = p$ . It is the projection of the function  $l_p$  defined in (4.8) on  $C_d^{r-1}(\square_m)^\perp$ .

**Lemma 5.7.** *There exists  $C := C(d, \lambda) < \infty$  such that, for every  $m \in \mathbb{N}$ ,  $(p, q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$ ,*

$$\mathbb{E} \left[ \left\| v(\cdot, \square_{m+1}, p, q) - l_{\bar{\mathbf{a}}_m^{-1}q-p}^{m+1} \right\|_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})}^2 \right] \leq C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

*Proof.* Fix  $(p, q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$  and denote by  $\mathcal{Z}_{m,n} := 3^n \mathbb{Z}^d \cap \square_{m+1}$ . We split the proof into two steps.

*Step 1.* Since, by definition, both  $v(\cdot, \square_{m+1}, p, q)$  and  $l_{\bar{\mathbf{a}}_m^{-1}q-p}^{m+1}$  are in  $C_d^{r-1}(\square_{m+1})^\perp$ , the difference belongs to  $C_d^{r-1}(\square_{m+1})^\perp$ , thus we can apply the Multiscale Poincaré inequality (4.6),

$$\begin{aligned}
 (5.25) \quad & \left\| v(\cdot, \square_{m+1}, p, q) - l_{\bar{\mathbf{a}}_m^{-1}q-p}^{m+1} \right\|_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})}^2 \\
 &\leq C \|dv(\cdot, \square_{m+1}, p, q) - \bar{\mathbf{a}}_m^{-1}q + p\|_{\underline{L}^2 \Lambda^r(\square_{m+1})}^2 \\
 &+ C \left( \sum_{n=0}^m 3^n \left( |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \left| (dv(x, \square_{m+1}, p, q) dx - \bar{\mathbf{a}}_m^{-1}q + p)_{z + \square_n} \right|^2 \right)^{\frac{1}{2}} \right)^2.
 \end{aligned}$$

We first bound the first term on the right-hand side

$$\|dv(\cdot, \square_{m+1}, p, q) - \bar{\mathbf{a}}_m^{-1}q + p\|_{\underline{L}^2 \Lambda^r(\square_{m+1})}^2 \leq 2 \left| -\bar{\mathbf{a}}_m^{-1}q + p \right|^2 + 2 \|dv(\cdot, \square_{m+1}, p, q)\|_{\underline{L}^2 \Lambda^r(\square_{m+1})}^2 \leq C.$$

*Step 2.* We prove the estimate, for every  $0 \leq n \leq m$ ,

$$\mathbb{E} \left[ |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \left| (dv(\cdot, \square_{m+1}, p, q) - \bar{\mathbf{a}}_m^{-1}q + p)_{y + \square_n} \right|^2 \right] \leq C \left( 3^{-n} + \sum_{k=0}^n 3^{k-n} \tau_k + \sum_{k=n}^m \tau_k \right).$$

By (5.17), we have, for every  $(p, q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$ ,

$$\begin{aligned}
 |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \|dv(\cdot, \square_{m+1}, p, q) - dv(\cdot, z + \square_n, p, q)\|_{\underline{L}^2 \Lambda^r(y + \square_n)}^2 \\
 \leq C |\mathcal{Z}_{m,n}|^{-1} \sum_{z \in \mathcal{Z}_{m,n}} (J(z + \square_n, p, q) - J(\square_m, p, q)).
 \end{aligned}$$

Taking expectations and using the stationarity yields

$$|\mathcal{Z}_{m,n}|^{-1} \mathbb{E} \left[ \sum_{y \in \mathcal{Z}_{m,n}} \|dv(\cdot, \square_{m+1}, p, q) - dv(\cdot, z + \square_n, p, q)\|_{\underline{L}^2 \Lambda^r(y + \square_n)}^2 \right] \\ \leq C \mathbb{E} [J(\square_n, p, q) - J(\square_m, p, q)] \leq C \sum_{k=n}^m \tau_k.$$

The triangle inequality, the previous display and Lemma 5.5 then yield,

$$|\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \mathbb{E} \left[ \left| (dv(\cdot, \square_{m+1}, p, q) dx - \bar{\mathbf{a}}_n^{-1} q + p)_{y + \square_n} \right|^2 \right] \\ \leq 3 |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \mathbb{E} \left[ \left| (dv(\cdot, \square_{m+1}, p, q) - dv(x, y + \square_n, p, q))_{y + \square_n} \right|^2 \right] \\ + 3 |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \mathbb{E} \left[ \left| (dv(\cdot, y + \square_n, p, q) dx - \bar{\mathbf{a}}_n^{-1} q + p)_{y + \square_n} \right|^2 \right] \\ + 3 |\bar{\mathbf{a}}_m^{-1} q - \bar{\mathbf{a}}_n^{-1} q|^2 \\ \leq C \sum_{k=n}^m \tau_k + C \sum_{k=0}^n 3^{\beta(k-n)} + C 3^{-\beta n}.$$

Combining this estimate and inequality (5.25) shows

$$(5.26) \quad \left\| v(\cdot, \square_{m+1}, p, q) - l_{\bar{\mathbf{a}}_m^{-1} q - p}^{m+1} \right\|_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})}^2 \leq C \left( 1 + \left( \sum_{n=0}^m 3^n X_n^{\frac{1}{2}} \right)^2 \right)$$

where the random variable

$$X_n := |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \left| (dv(x, \square_{m+1}, p, q) dx - \bar{\mathbf{a}}_m^{-1} q + p)_{y + \square_n} \right|^2$$

satisfies

$$\mathbb{E} [X_n] \leq C \sum_{k=n}^m \tau_k + C \sum_{k=0}^n 3^{\beta(k-n)} \tau_k + C 3^{-\beta n}.$$

By the Cauchy-Schwarz inequality,

$$\left( \sum_{n=0}^m 3^n X_n^{\frac{1}{2}} \right)^2 \leq \left( \sum_{n=0}^m 3^n \right) \left( \sum_{n=0}^m 3^n X_n \right) \leq C 3^m \sum_{n=0}^m 3^n X_n.$$

Taking the expectation thus yields

$$\mathbb{E} \left[ \left( \sum_{n=0}^m 3^n X_n^{\frac{1}{2}} \right)^2 \right] \leq C 3^m \left( \sum_{n=0}^m \sum_{k=n}^m 3^n \tau_k + C \sum_{n=0}^m \sum_{k=0}^n 3^{(1-\beta)n} 3^{\beta k} \tau_k + C \sum_{n=0}^m 3^{(1-\beta)n} \right).$$

We then compute the term on the right-hand side

$$\sum_{n=0}^m \sum_{k=n}^m 3^n \tau_k = \sum_{k=0}^m \sum_{n=0}^k 3^n \tau_k \leq C \sum_{k=0}^m 3^k \tau_k$$

and

$$\sum_{n=0}^m \sum_{k=0}^n 3^{(1-\beta)n} 3^{\beta k} \tau_k \leq \sum_{k=0}^m \sum_{n=k}^m 3^{(1-\beta)n} 3^{\beta k} \tau_k \leq C 3^{(1-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

Combining the three previous displays shows

$$(5.27) \quad \mathbb{E} \left[ \left( \sum_{n=0}^m 3^n X_n^{\frac{1}{2}} \right)^2 \right] \leq C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k + C 3^m \sum_{k=0}^m 3^k \tau_k.$$

Moreover, since  $0 < \beta \leq 1$ , we notice that for each  $k, m \in \mathbb{N}$  with  $k \leq m$ ,  $3^{(k-m)} \leq 3^{\beta(k-m)}$ . In particular the third term on the right-hand side of (5.27) is smaller than the second term on the right-hand side. Consequently, estimate (5.27) can be simplified to obtain

$$\mathbb{E} \left[ \left( \sum_{n=0}^m 3^n X_n^{\frac{1}{2}} \right)^2 \right] \leq C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

Thus estimate (5.26) becomes

$$\mathbb{E} \left[ \left\| v(\cdot, \square_{m+1}, p, q) - v(\cdot, \square_{m+1}, p, q) \right\|_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})}^2 \right] \leq C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

□

Now that we have some control on the flatness of the maximizers of  $J(\square_m, p, q)$ , we can estimate  $J(\square_m, p, \bar{\mathbf{a}}_m p)$  thanks to the Caccioppoli inequality.

**Lemma 5.8.** *There exists a constant  $C(d, \lambda) < \infty$  such that, for every  $m \in \mathbb{N}$  and  $p \in B_1 \Lambda^r(\mathbb{R}^d)$ ,*

$$\mathbb{E} [J(\square_m, p, \bar{\mathbf{a}}_m p)] \leq C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

*Proof.* Fix  $p \in B_1 \Lambda^r(\mathbb{R}^d)$ , by Lemma 5.7,

$$\mathbb{E} \left[ \left\| v(\cdot, \square_{m+1}, p, \bar{\mathbf{a}}_m p) \right\|_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})}^2 \right] \leq C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

Applying the Caccioppoli inequality, Proposition 4.11, one obtains

$$(5.28) \quad \mathbb{E} \left[ \left\| dv(\cdot, \square_{m+1}, p, \bar{\mathbf{a}}_m p) \right\|_{\underline{L}^2 \Lambda^r(\square_m)}^2 \right] \leq C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

By (5.17), we have

$$3^{-d} \sum_{y \in 3^m \mathbb{Z}^d \cap \square_{m+1}} \mathbb{E} \left[ \left\| dv(\cdot, \square_{m+1}, p, \bar{\mathbf{a}}_m p) - dv(\cdot, y + \square_m, p, \bar{\mathbf{a}}_m p) \right\|_{\underline{L}^2 \Lambda^r(y + \square_m)}^2 \right] \leq C \tau_m.$$

In particular, this yields

$$\mathbb{E} \left[ \left\| dv(\cdot, \square_{m+1}, p, \bar{\mathbf{a}}_m p) - dv(\cdot, \square_m, p, \bar{\mathbf{a}}_m p) \right\|_{\underline{L}^2 \Lambda^r(\square_m)}^2 \right] \leq C \tau_m.$$

Combining the previous display with (5.28) gives

$$\begin{aligned} \mathbb{E} \left[ \left\| dv(\cdot, \square_m, p, \bar{\mathbf{a}}_m p) \right\|_{\underline{L}^2 \Lambda^r(\square_m)}^2 \right] &\leq C \tau_m + C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^m 3^{\beta k} \tau_k \\ &\leq C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^m 3^{\beta k} \tau_k. \end{aligned}$$

By (5.16) with  $w = 0$ , we deduce

$$\mathbb{E} [J(\square_m, p, \bar{\mathbf{a}}_m p)] \leq C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^m 3^{\beta k} \tau_k.$$

The proof of the Lemma is complete. □

We are now able to prove Theorem 1.

*Proof of Theorem 1.* First note that since, for each  $m \in \mathbb{N}$ , the mapping  $p \rightarrow \mathbb{E}[J(\square_m, p, \bar{\mathbf{a}}_m p)]$  is a positive definite quadratic form, we have

$$\frac{1}{d} \sum_{i=1}^d \mathbb{E}[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i)] \leq \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}[J(\square_m, p, \bar{\mathbf{a}}_m p)] \leq \sum_{i=1}^d \mathbb{E}[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i)].$$

Thus if we denote by

$$D_m = \sum_{i=1}^d \mathbb{E}[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i)],$$

we get from the previous remark that the estimate (2.23) is equivalent to

$$(5.29) \quad D_m \leq C 3^{-\alpha m}.$$

The reason we consider this particular quantity is because of the bound, for some  $c := c(d, \lambda) > 0$ ,

$$(5.30) \quad D_m - D_{m+1} \geq c \tau_m.$$

Moreover notice that using the definition of  $\bar{\mathbf{a}}_{m+1}$  (5.22) and the decomposition of  $J$  (5.4), for each  $p \in \Lambda^r(\mathbb{R}^d)$ , the quadratic form

$$q \rightarrow \mathbb{E}[J(\square_{m+1}, p, q)] = \mathbb{E}[\nu(\square_{m+1}, p)] + \star \left( \frac{1}{2} \bar{\mathbf{a}}_{m+1}^{-1} q \wedge q - p \wedge q \right)$$

attains its minimum at  $q = \bar{\mathbf{a}}_{m+1} p$ . Consequently

$$D_{m+1} = \sum_{i=1}^d \mathbb{E}[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_{m+1} e_i)] \leq \sum_{i=1}^d \mathbb{E}[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_m e_i)].$$

Thus we can compute

$$\begin{aligned} D_m - D_{m+1} &= \sum_{i=1}^d (\mathbb{E}[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i)] - \mathbb{E}[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_{m+1} e_i)]) \\ &\geq \sum_{i=1}^d (\mathbb{E}[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i)] - \mathbb{E}[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_m e_i)]) \\ &\geq \sum_{i=1}^d (\mathbb{E}[\nu(\square_m, e_i)] - \mathbb{E}[\nu(\square_{m+1}, e_i)]) \\ &\quad + \sum_{i=1}^d (\mathbb{E}[\nu^*(\square_m, \bar{\mathbf{a}}_m e_i)] - \mathbb{E}[\nu^*(\square_{m+1}, \bar{\mathbf{a}}_m e_i)]) \\ &\geq c \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}[\nu^*(\square_m, e_i)] - \mathbb{E}[\nu^*(\square_{m+1}, e_i)] \\ &\quad + c \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}[\nu^*(\square_m, e_i)] - \mathbb{E}[\nu^*(\square_{m+1}, e_i)] \\ &\geq c \tau_m. \end{aligned}$$

The main ingredient in the proof of Theorem 1 is to define the alternative quantity

$$\tilde{D}_m := 3^{-\frac{\beta m}{2}} \sum_{n=0}^m 3^{\frac{\beta n}{2}} D_n,$$

where  $\beta := \beta(d, \lambda)$  is the exponent which appears in Lemmas 5.7 and 5.8, and to use Lemma 5.8 to prove the estimate

$$(5.31) \quad \tilde{D}_m \leq C 3^{-\alpha m}.$$

for some  $\alpha := \alpha(d, \lambda) > 0$ . The estimate (5.29) follows since, for each  $m \in \mathbb{N}$

$$D_m \leq \tilde{D}_m.$$

The proof of (5.31) can be split in 5 steps

*Step 1.* We show that there exist  $\theta(d, \lambda) \in (0, 1)$  and  $C(d, \lambda) < \infty$  such that, for every  $m \in \mathbb{N}$ ,

$$(5.32) \quad \tilde{D}_{m+1} \leq \theta \tilde{D}_m + C3^{-\frac{\beta m}{2}}.$$

By (5.30) and  $D_0 \leq C$ , we have

$$\tilde{D}_m - \tilde{D}_{m+1} = 3^{-\frac{\beta m}{2}} \sum_{n=0}^m 3^{\frac{\beta n}{2}} (D_n - D_{n+1}) - C3^{-\frac{\beta m}{2}} \geq 3^{-\frac{\beta m}{2}} \sum_{n=0}^m 3^{\frac{\beta n}{2}} \tau_n - C3^{-\frac{\beta m}{2}}.$$

In particular, the previous estimate gives

$$\tilde{D}_{m+1} \leq \tilde{D}_m + C3^{-\frac{\beta m}{2}}$$

From this and Lemma 5.8, we compute

$$\begin{aligned} \tilde{D}_{m+1} &\leq \tilde{D}_m + D_0 3^{-\frac{\beta m}{2}} = 3^{-\frac{\beta m}{2}} \sum_{n=0}^m 3^{\frac{\beta n}{2}} D_n + D_0 3^{-\frac{\beta m}{2}} \\ &\leq C3^{-\frac{\beta m}{2}} \sum_{n=0}^m 3^{\frac{\beta n}{2}} \left( 3^{-\beta n} + 3^{-\beta n} \sum_{k=0}^n 3^{\beta k} \tau_k \right) + C3^{-\frac{\beta m}{2}} \\ &\leq C3^{-\frac{\beta m}{2}} \sum_{n=0}^m \sum_{k=0}^n 3^{-\frac{\beta n}{2}} 3^{\beta k} \tau_k + C3^{-\frac{\beta m}{2}} \\ &\leq C3^{-\frac{\beta m}{2}} \sum_{k=0}^m \sum_{n=k}^m 3^{-\frac{\beta n}{2}} 3^{\beta k} \tau_k + C3^{-\frac{\beta m}{2}} \\ &\leq C3^{-\frac{\beta m}{2}} \sum_{k=0}^m 3^{\frac{\beta k}{2}} \tau_k + C3^{-\frac{\beta m}{2}}. \end{aligned}$$

Combining the two previous displays gives

$$\tilde{D}_{m+1} \leq C(\tilde{D}_m - \tilde{D}_{m+1}) + C3^{-\frac{\beta m}{2}}.$$

A rearrangement of this estimates yields (5.32).

*Step 2.* Iterating (5.32) gives

$$\tilde{D}_m \leq \theta^m D_0 + C \sum_{k=0}^m \theta^k 3^{-\frac{\beta(m-k)}{2}}.$$

Without loss of generality, we can assume  $\theta > 3^{-\frac{\beta}{2}}$  (since we can make  $\theta$  closer to 1 if necessary). With this assumption, the second term on the right-hand side can be estimated,

$$\sum_{k=0}^m \theta^k 3^{-\frac{\beta(m-k)}{2}} \leq C\theta^m.$$

Combining this with the fact that  $\tilde{D}_0 = D_0 \leq C$ , we obtain

$$\tilde{D}_m \leq C\theta^m,$$

which can be rewritten, with  $\alpha = -\frac{\ln \theta}{\ln 3} > 0$ ,

$$\tilde{D}_m \leq C3^{-\alpha m}.$$

*Step 3.* We need to get the same estimate as (5.29) but with  $\bar{\mathbf{a}}$  instead of  $\bar{\mathbf{a}}_m$ . First notice that by (5.29) and (5.30),

$$c\tau_m \leq D_m - D_{m+1} \leq D_m \leq C3^{-\alpha m}.$$

Thus by (5.24), for every  $q \in B_1 \Lambda^{d-r}(\mathbb{R}^d)$ , every  $m \in \mathbb{N}$ ,

$$|\bar{\mathbf{a}}^{-1}q - \bar{\mathbf{a}}_m^{-1}q|^2 = \lim_{l \rightarrow \infty} |\bar{\mathbf{a}}_l^{-1}q - \bar{\mathbf{a}}_m^{-1}q|^2 \leq \sum_{k=m}^{\infty} \tau_k \leq C \sum_{k=m}^{\infty} 3^{-\alpha k} \leq C3^{-\alpha m}.$$

Using the ellipticity assumption (2.16), we deduce, for each  $p \in B_1 \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$|\bar{\mathbf{a}}p - \bar{\mathbf{a}}_m p|^2 \leq C3^{-\alpha m}.$$

Using that  $J$  is a quadratic form according to (5.9), one obtains that there exists a constant  $C(d, \lambda) < \infty$  such that, for each  $m \in \mathbb{N}$ , each  $p, p' \in \Lambda^r(\mathbb{R}^d)$  and each  $q, q' \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$|J(\square_m, p, q) - J(\square_m, p', q')| \leq C(|p - p'| + |q - q'|)(|p| + |p'| + |q| + |q'|).$$

Consequently, for each  $p \in B_1 \Lambda^r(\mathbb{R}^d)$  and each  $m \in \mathbb{N}$

$$\begin{aligned} |J(\square_m, p, \bar{\mathbf{a}}p) - J(\square_m, p, \bar{\mathbf{a}}_m p)| &\leq C|\bar{\mathbf{a}}p - \bar{\mathbf{a}}_m p|(1 + |\bar{\mathbf{a}}p| + |\bar{\mathbf{a}}_m p|) \\ &\leq C3^{-\frac{\alpha}{2}m}. \end{aligned}$$

Redefining  $\alpha = \frac{\alpha}{2}$  completes the proof of the quantitative homogenization estimate (2.23).

*Step 4.* We need to show that the mapping  $\bar{\mathbf{a}}$  is unique. Given two maps  $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d))$  such that the estimate (2.23) is satisfied, we have, by (5.12), for each  $m \in \mathbb{N}$ , and each  $p \in B_1 \Lambda^r(\mathbb{R}^d)$ ,

$$\begin{aligned} \frac{1}{C}|\bar{\mathbf{a}}p - \bar{\mathbf{a}}'p|^2 &\leq \mathbb{E} \left[ \frac{1}{2}J(\square_m, p, \bar{\mathbf{a}}p) + \frac{1}{2}J(\square_m, p, \bar{\mathbf{a}}'p) \right] \\ &\leq C3^{-\alpha m}. \end{aligned}$$

Sending  $m \rightarrow \infty$  gives, for each  $p \in B_1 \Lambda^r(\mathbb{R}^d)$ ,  $\bar{\mathbf{a}}p = \bar{\mathbf{a}}'p$ . Consequently  $\bar{\mathbf{a}} = \bar{\mathbf{a}}'$  and the proof of the first part of Theorem 1 is complete.

*Step 5.* We can now complete the proof of Theorem 1 by upgrading the stochastic integrability. This is a consequence of the following Lemma, the proof of which can be found in [3, Lemma 2.14].

**Lemma 5.9.** *Suppose that  $U \rightarrow \rho(U)$  is a (random) map from the set of bounded Lipschitz domains to  $[0, +\infty)$  and satisfies, for a fixed  $K \geq 0$ :*

$$\begin{aligned} \rho(U) &\text{ is } \mathcal{F}(U) \text{ - measurable} \\ \rho(U) &\leq K. \end{aligned}$$

and, whenever  $U$  is the disjoint union of  $U_1, \dots, U_k$  up to a set of zero Lebesgue measure, we have

$$\rho(U) \leq \sum_{i=1}^k \frac{|U_i|}{|U|} \rho(U_i)$$

Then there exists a universal constant  $C < \infty$  such that, for every  $m, n \in \mathbb{N}$ ,

$$\rho(\square_{n+m+1}) \leq 2\mathbb{E}[\rho(\square_n)] + \mathcal{O}_1(CK3^{-md}).$$

Applying this result to

$$\rho(U) := \sup_{p \in B_1} J(U, p, \bar{\mathbf{a}}p),$$

gives, for each  $m, n \in \mathbb{N}$ ,

$$\rho(\square_{n+m+1}) \leq 2\mathbb{E}[\rho(\square_n)] + \mathcal{O}_1(C3^{-md}) \leq C3^{-n\alpha} + \mathcal{O}_1(C3^{-md}).$$

Taking  $n = m$  yields, for every  $n \in \mathbb{N}$ ,

$$\rho(\square_{2n+1}) \leq C3^{-n\alpha} + \mathcal{O}_1(C3^{-nd}).$$

By redefining  $\alpha := \min(\frac{\alpha}{2}, \frac{d}{2})$ , we obtain, for each  $n \in \mathbb{N}$ ,

$$\rho(\square_n) \leq C3^{-n\alpha} + \mathcal{O}_1(C3^{-n\alpha}) \leq \mathcal{O}_1(C3^{-n\alpha}).$$

The proof of Theorem 1 is now complete.  $\square$

Before turning to the proof of Theorem 2 in the next section, we state and prove the following proposition, which is a consequence of Theorem 1 and gives some information about the flatness of the minimizers.

**Proposition 5.10.** *There exists  $\alpha := \alpha(d, \lambda) > 0$  and  $C := C(d, \lambda) < \infty$  such that for each  $1 \leq r \leq d$ , each  $(p, q) \in B_1 \Lambda^r(\mathbb{R}^d) \times B_1 \Lambda^{d-r}(\mathbb{R}^d)$  and each  $m \in \mathbb{N}$ ,*

$$(5.33) \quad 3^{-m} \left\| \mathrm{d}v(\cdot, \square_m, p, q) - (\bar{\mathbf{a}}^{-1} q - p) \right\|_{\underline{H}^{-1} \Lambda^r(\square)} + 3^{-m} \left\| \mathrm{ad}v(\cdot, \square_m, p, q) - (q - \bar{\mathbf{a}} p) \right\|_{\underline{H}^{-1} \Lambda^{d-r}(\square)} \leq \mathcal{O}_1(C 3^{-m\alpha}).$$

*Proof.* The proof is split into 2 steps.

- *Step 1.* We prove that, for each  $q \in B_1 \Lambda^{d-r}(\mathbb{R}^d)$  and every  $m, n \in \mathbb{N}$  such that  $m \geq n$

$$(5.34) \quad 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (\mathrm{d}v(\cdot, \square_m, 0, q) - \bar{\mathbf{a}}^{-1} q)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{-\alpha n}).$$

Similarly, for each  $p \in B_1 \Lambda^r(\mathbb{R}^d)$  and every  $m, n \in \mathbb{N}$  such that  $m \geq n$

$$(5.35) \quad 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (\mathrm{ad}v(\cdot, \square_m, p, 0) - \bar{\mathbf{a}} p)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{-\alpha n}).$$

- *Step 2.* We deduce from the previous step and the multiscale Poincaré inequality, Proposition 4.10, the estimate (5.33).

*Step 1.* We first deal with the case  $m = n$ , in this specific case, the estimate (5.34) reads

$$\left| (\mathrm{d}v(\cdot, \square_n, 0, q) - \bar{\mathbf{a}}^{-1} q)_{\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{-\alpha n}).$$

To argue this, note that, by the first variation for  $J$ ,

$$J(\square_n, 0, q) = \frac{1}{2} (\mathrm{d}v(\cdot, \square_n, 0, q))_{\square_n} \wedge q.$$

Moreover, the map  $q \rightarrow (\mathrm{d}v(\cdot, \square_n, 0, q))_{\square_n}$  is bounded by (5.10) and symmetric since, for each  $q, q' \in \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$\begin{aligned} (\mathrm{d}v(\cdot, \square_n, 0, q'))_{\square_n} \wedge q &= \int_{\square_n} \mathrm{d}v(\cdot, \square_n, 0, q') \wedge \mathrm{d}v(\cdot, \square_n, 0, q) \\ &= \int_{\square_n} \mathrm{d}v(\cdot, \square_n, 0, q) \wedge \mathrm{ad}v(\cdot, \square_n, 0, q') \\ &= (\mathrm{d}v(\cdot, \square_n, 0, q))_{\square_n} \wedge q'. \end{aligned}$$

A combination of the two previous ideas and Theorem 1 gives

$$(5.36) \quad \sup_{q \in B_1 \Lambda^{d-r}} \left| (\mathrm{d}v(\cdot, \square_n, 0, q) - \bar{\mathbf{a}}^{-1} q)_{\square_n} \right|^2 \leq C \sup_{q \in B_1 \Lambda^{d-r}} \left| J(\square_n, 0, q) - \bar{\mathbf{a}}^{-1} q \wedge q \right| \leq \mathcal{O}_1(C 3^{-n\alpha}).$$



To obtain the general case  $m \geq n$  from the specific case  $m = n$ , we compute

$$\begin{aligned}
& 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, \square_m, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \\
& \leq C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, \square_m, 0, q) - dv(\cdot, y + \square_n, 0, q))_{y+\square_n} \right|^2 \\
& \quad + C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \\
& \leq C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \|dv(\cdot, \square_m, 0, q) - dv(\cdot, y + \square_n, 0, q)\|_{\underline{L}^2(y+\square_n)}^2 \\
& \quad + C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \\
& \leq C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} J(y + \square_n, 0, q) - J(\square_m, 0, q) \\
& \quad + C 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2
\end{aligned}$$

To deal with the first term on the right-hand side, we note that, for each  $y \in \mathcal{Z}_{m,n}$ ,

$$\begin{aligned}
J(y + \square_n, 0, q) - J(\square_m, 0, q) & \leq |J(y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q \wedge q| + |J(\square_m, 0, q) - \bar{\mathbf{a}}^{-1}q \wedge q| \\
& \leq \mathcal{O}_1(C 3^{-\alpha n}) + \mathcal{O}_1(C 3^{-\alpha m}) \\
& \leq \mathcal{O}_1(C 3^{-\alpha n}),
\end{aligned}$$

by the stationarity assumption (2.19). Using the inequality (2.14), we eventually obtain

$$3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} J(y + \square_n, 0, q) - J(\square_m, 0, q) \leq \mathcal{O}_1(C 3^{-n\alpha}).$$

To deal with the second term on the right-hand side, we have by the stationarity assumption (2.19) and (5.36), for each  $y \in \mathcal{Z}_{m,n}$ ,

$$\left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{-\alpha n}).$$

Using the inequality (2.14), we obtain

$$3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{-\alpha n}).$$

The proof of (5.34) is thus complete. Thus proof of (5.35) is similar, the details are left to the reader.

*Step 2.* From Step 1 and (ii) of Corollary 2.4, we have

$$\sum_{n=0}^{m-1} 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (dv(\cdot, y + \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{(1-\alpha)m})$$

and

$$\sum_{n=0}^{m-1} 3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| (\mathbf{a}dv(\cdot, y + \square_n, p, 0) - \bar{\mathbf{a}}p)_{y+\square_n} \right|^2 \leq \mathcal{O}_1(C 3^{(1-\alpha)m}).$$

By the multiscale Poincaré inequality, Proposition 4.6, the bound on the  $L^2$  norm of  $dv$ , estimate (5.10), and the previous estimates, one obtains for each  $(p, q) \in B_1 \Lambda^r(\mathbb{R}^d) \times B_1 \Lambda^{d-r}(\mathbb{R}^d)$ ,

$$\|dv(\cdot, \square_m, p, q) - (\bar{\mathbf{a}}^{-1}q - p)\|_{\underline{H}^{-1}\Lambda^r(\square)} + \|\mathbf{a}dv(\cdot, \square_m, p, q) - (q - \bar{\mathbf{a}}p)\|_{\underline{H}^{-1}\Lambda^{d-r}(\square)} \leq \mathcal{O}_1(C 3^{(1-\alpha)m}).$$

Dividing both sides of the previous inequality by  $3^m$  yields (5.33) and completes the proof of Proposition 5.10.  $\square$

## 6. HOMOGENIZATION OF THE DIRICHLET PROBLEM

The goal of this section is to study the Dirichlet problem for the equation  $\mathbf{d}\mathbf{a}u = 0$  and to establish Theorem 2. We first prove existence and uniqueness of solution for this equation.

**Proposition 2.7.** *Let  $U$  be a bounded smooth domain of  $\mathbb{R}^d$  and  $1 \leq r \leq d$ . Let  $f \in H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$ , then for any measurable map  $\mathbf{a} : \mathbb{R}^d \rightarrow \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$  satisfying (2.16) and (2.15), there exists a unique solution in  $f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U) \cap (C_{\mathbf{d},0}^{r-1}(U))^\perp$  of the equation*

$$(6.1) \quad \begin{cases} \mathbf{d}(\mathbf{a}u) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U, \end{cases}$$

in the sense that, for each  $v \in H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$ ,

$$\int_U \mathbf{d}u \wedge \mathbf{a}dv = 0.$$

The solution satisfies the estimate, for some  $C := C(d, \lambda, U) < \infty$ ,

$$\|u\|_{H_{\mathbf{d}}^1 \Lambda^{r-1}(U)} \leq C \|\mathbf{d}f\|_{L^2 \Lambda^r(U)}.$$

Moreover if we enlarge the space of admissible solutions to the space  $f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$ , we loose the uniqueness property, but if  $v, w \in f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U)$  are two solutions of (6.1), then

$$v - w \in C_{\mathbf{d},0}^{r-1}.$$

*Proof.* The existence and uniqueness of such a solution are obtained by minimizing the quantity

$$\mathcal{J}(v) := \langle \mathbf{d}f + \mathbf{d}v, \mathbf{d}f + \mathbf{d}v \rangle_U$$

on the space  $H_{\mathbf{d},0}^1 \Lambda^{r-1}(U) \cap (C_{\mathbf{d},0}^r)^\perp$  and requires to use the Poincaré inequality, Proposition 4.1. The techniques are standard, we thus omit the details.  $\square$

We now turn to the statement and the proof of the main theorem of this section, Theorem 2.

**Theorem 2** (Homogenization Theorem). *Let  $U$  be a bounded smooth domain of  $\mathbb{R}^d$  and  $1 \leq r \leq d$ . Fix  $\varepsilon \in (0, 1]$  and  $f \in H_{\mathbf{d}}^1 \Lambda^{r-1}(U)$  such that  $\mathbf{d}f \in H_{\mathbf{d}}^1 \Lambda^r(U)$ . Let  $u^\varepsilon, u \in f + H_{\mathbf{d},0}^1 \Lambda^{r-1}(U) \cap (C_{\mathbf{d},0}^r(U))^\perp$  respectively denote the solutions of the Dirichlet problems*

$$\begin{cases} \mathbf{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathbf{d}u^\varepsilon\right) = 0 & \text{in } U \\ \mathbf{t}u^\varepsilon = \mathbf{t}f & \text{on } \partial U. \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{d}(\bar{\mathbf{a}}\mathbf{d}u) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U. \end{cases}$$

Then there exist an exponent  $\alpha := \alpha(d, \lambda, U) > 0$  and a constant  $C := C(d, \lambda, U) < \infty$  such that

$$\|u^\varepsilon - u\|_{L^2 \Lambda^r(U)} + \|\mathbf{d}u^\varepsilon - \mathbf{d}u\|_{\underline{H}^{-1} \Lambda^r(U)} \leq \mathcal{O}_1\left(C \|\mathbf{d}f\|_{H^1 \Lambda^r(U)} \varepsilon^\alpha\right).$$

*Proof.* Without loss of generality, one can assume that  $|U| = 1$ . Fix  $l > 0$ , this parameter represents the thickness of a boundary layer we need to remove in the argument, it will be chosen at the end of the proof and will depend only on  $\varepsilon$ . For  $R > 0$ , denote by  $U_R := \{x \in U : \text{dist}(x, \partial U) > R\}$ . For  $I \subseteq \{1, \dots, d\}$  of cardinality  $r$  and  $m \in \mathbb{N}$ , we denote by  $\phi_{m,I}$  the unique solution in  $l_{\mathbf{d}x_I} + H_{\mathbf{d},0}^1 \Lambda^{r-1}(\square_m) \cap (C_{\mathbf{d},0}^r(\square_m))^\perp$  of

$$\begin{cases} \mathbf{d}(\mathbf{a}u) = 0 & \text{in } \square_m \\ \mathbf{t}u = \mathbf{t}l_{\mathbf{d}x_I} & \text{on } \partial \square_m, \end{cases}$$

where  $l_{dx_I}$  is defined in (4.8) satisfies  $dl_{dx_I} = dx_I$ . In particular, one has

$$d\phi_{m,I} = dv(\cdot, \square_m, p).$$

Let  $m$  be the smallest integer such that

$$U \subseteq \varepsilon \square_m,$$

and define the two scale expansion, with the convention  $du := \sum_{|I|=r} (du)_I dx_I$ ,

$$(6.2) \quad w_0^\varepsilon(x) := u(x) + \varepsilon \zeta_l(x) \sum_{|I|=r} (du)_I(x) \phi_{m,I} \left( \frac{x}{\varepsilon} \right),$$

where  $\zeta_l \in C_c^\infty(U)$  is a smooth cutoff function satisfying, for every  $k \in \mathbb{N}$ :

$$(6.3) \quad 0 \leq \zeta_l \leq 1, \quad \zeta_l = 1 \text{ in } U_{2l}, \quad \zeta_l = 0 \text{ in } U \setminus U_l, \quad |\nabla^k \zeta_l| \leq C(k, d, U) l^{-k}.$$

Note that  $w_0^\varepsilon \in f + H_{d,0}^1 \Lambda^{r-1}(U)$ . Since it is more convenient to work with an element of  $f + H_{d,0}^1 \Lambda^{r-1}(U) \cap (C_{d,0}^r(U))^\perp$  (to have the Poincaré inequality), we further define

$$w^\varepsilon := f + \text{Proj}_{(C_{d,0}^r(U))^\perp} (w_0^\varepsilon - f).$$

where  $\text{Proj}_{(C_{d,0}^r(U))^\perp}$  denotes the  $L^2$ -orthogonal projection on the space  $(C_{d,0}^r(U))^\perp$ . Note that  $w^\varepsilon \in f + H_{d,0}^1 \Lambda^{r-1}(U) \cap (C_{d,0}^r(U))^\perp$  by construction and that it satisfies

$$(6.4) \quad dw_0^\varepsilon = dw^\varepsilon \text{ in } U.$$

We then consider the map

$$d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) : \begin{cases} H_{d,0}^1 \Lambda^{r-1}(U) \rightarrow \mathbb{R}, \\ v \rightarrow \int_U dw^\varepsilon \wedge \mathbf{a} \left( \frac{x}{\varepsilon} \right) dv. \end{cases}$$

and denote by

$$(6.5) \quad \left\| d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) \right\|_{H_{d,0}^1 \Lambda^{r-1}(U)} := \sup \left\{ \int_U dw^\varepsilon \wedge \mathbf{a} \left( \frac{x}{\varepsilon} \right) dv : v \in H_{d,0}^1 \Lambda^{r-1}(U) \text{ s.t. } \|v\|_{H_d^1 \Lambda^r(U)} \leq 1 \right\}.$$

The idea of the proof is to compare  $w^\varepsilon$  to the function  $w^\varepsilon$ . The proof is split into 7 steps.

*Step 1.* In this step, we show that the norm  $H_d^1 \Lambda^{d-r+1}(U)$  defined in (6.5) is equivalent to the norm

$$\left\| d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) \right\|_{H^{-1} \Lambda^{d-r+1}(U)} := \sup \left\{ \int_U dw^\varepsilon \wedge \mathbf{a} \left( \frac{x}{\varepsilon} \right) dv : v \in H_0^1 \Lambda^{r-1}(U) \text{ s.t. } \|v\|_{H^1 \Lambda^r(U)} \leq 1 \right\}.$$

This result is a consequence of the following property, for some  $C := C(d, \lambda, U) < \infty$ ,

$$\forall v \in H_{d,0}^1 \Lambda^{r-1}(U), \exists w \in H_0^1 \Lambda^{r-1}(U) \text{ such that } dw = dv \text{ and } \|w\|_{H^1 \Lambda^r(U)} \leq C \|v\|_{H_d^1 \Lambda^r(U)}.$$

To prove this, we mimic the proof of [23, Theorem 1.1]. Let  $(O_j)_{1 \leq j \leq N}$  be a finite, open covering of  $\bar{U}$  such that  $O_j \cap U$  is a smooth star-shaped domain. Then let  $(\phi_j)_{1 \leq j \leq N}$  be a smooth partition of unity such that  $\text{supp } \phi_j \subseteq O_j$  for  $1 \leq j \leq N$ . Note that the form  $\phi_j v$  belongs to  $H_{d,0}^1 \Lambda^{r-1}(O_j \cap U)$ . Thus by Proposition 3.4, there exist a function  $w_j \in H_0^1 \Lambda^{r-1}(O_j \cap U)$  satisfying

$$\|w_j\|_{H^1 \Lambda^{r-1}(O_j \cap U)} \leq C \|\phi_j v\|_{H_d^1 \Lambda^{r-1}(O_j \cap U)}.$$

We then extend the forms  $\phi_j v$  and  $w_j$  by 0 to  $\mathbb{R}^d$ , so that  $\phi_j v \in H_{d,0}^1 \Lambda^{r-1}(\mathbb{R}^d)$  and  $w_j \in H^1 \Lambda^{r-1}(\mathbb{R}^d)$  satisfy

$$dw_j = d(\phi_j v) \text{ in } \mathbb{R}^d.$$

We then define

$$w := \sum_{j=1}^N w_j,$$

so that

$$w \in H_0^1 \Lambda^{r-1}(U) \text{ and } dw = \sum_{j=1}^N d(\phi_j v) = dv.$$

We also have the estimate

$$\|w\|_{H^1 \Lambda^{r-1}(U)} \leq C \|v\|_{H_d^1 \Lambda^{r-1}(U)}.$$

This completes the proof of Step 1.

*Step 2.* We show the  $H^{-1} \Lambda^r(U)$  estimate

$$(6.6) \quad \left\| d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) \right\|_{H^{-1} \Lambda^{r-1}(U)} \leq \begin{cases} C \|df\|_{H^1 \Lambda^r(U)} \left( l^{\frac{1}{d-2}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d \geq 3, \\ C \|df\|_{H^1 \Lambda^r(U)} \left( l^{\frac{1}{4}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d = 2. \end{cases}$$

We first compute the exterior derivative of  $w^\varepsilon$ , by (6.2) and (6.4),

$$\begin{aligned} dw^\varepsilon &= du + \zeta_l \sum_{|I|=r} (du)_I d\phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) + \varepsilon \sum_{|I|=r} d(\zeta_l (du)_I) \wedge \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \\ &= (1 - \zeta_l) du + \sum_{|I|=r} \zeta_l (du)_I \left( dx_I + d\phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right) + \varepsilon \sum_{|I|=r} d(\zeta_l (du)_I) \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right). \end{aligned}$$

From this we deduce, in the weak sense

$$\begin{aligned} d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) &= d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) \left( (1 - \zeta_l) du + \varepsilon \sum_{|I|=r} d(\zeta_l (du)_I) \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right) \right) \\ &\quad + \sum_{|I|=r} d(\zeta_l (du)_I) \wedge \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) \left( dx_I + d\phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right). \end{aligned}$$

On the other hand, since  $u$  satisfies  $d(\bar{\mathbf{a}} du) = 0$ , we see that

$$\sum_{|I|=r} d(\zeta_l (du)_I) \wedge \bar{\mathbf{a}} dx_I = d(\zeta_l \bar{\mathbf{a}} du) = -d((1 - \zeta_l) \bar{\mathbf{a}} du).$$

Consequently, in the weak sense

$$\begin{aligned} d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) dw^\varepsilon \right) &= d \left( \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) - \bar{\mathbf{a}} \right) (1 - \zeta_l) du \right) + \varepsilon \sum_{|I|=r} d \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) d(\zeta_l (du)_I) \wedge \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right) \\ &\quad + \sum_{|I|=r} d(\zeta_l (du)_I) \wedge \left( \mathbf{a} \left( \frac{\cdot}{\varepsilon} \right) \left( dx_I + d\phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right) - \bar{\mathbf{a}} dx_I \right). \end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| d \left( \mathbf{a} \left( \frac{\dot{\cdot}}{\varepsilon} \right) dw^\varepsilon \right) \right\|_{H^{-1}\Lambda^{r-1}(U)} \\
& \leq \sum_{|I|=r} \|d(\zeta_l(du)_I)\|_{W^{1,\infty}(U)} \left\| \mathbf{a} \left( \frac{\dot{\cdot}}{\varepsilon} \right) \left( dx_I + d\phi_{m,I} \left( \frac{\dot{\cdot}}{\varepsilon} \right) \right) - \bar{\mathbf{a}} dx_I \right\|_{H^{-1}\Lambda^r(U)} \\
& + \left\| \left( \mathbf{a} \left( \frac{\dot{\cdot}}{\varepsilon} \right) - \bar{\mathbf{a}} \right) (1 - \zeta_l) du \right\|_{L^2\Lambda^r(U)} \\
& + \varepsilon \sum_{|I|=r} \left\| \mathbf{a} \left( \frac{\dot{\cdot}}{\varepsilon} \right) d(\zeta_l(du)_I) \wedge \phi_{m,I} \left( \frac{\dot{\cdot}}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)} \\
& =: T_1 + T_2 + T_3
\end{aligned}$$

To bound the term on the right we appeal to the interior regularity estimate, Proposition A.3 and the assumption (6.3) on  $\zeta_l$ , we have

$$(6.7) \quad \|d(\zeta_l(du)_I)\|_{W^{1,\infty}(U)} \leq \frac{C}{l^{3+d/2}} \|df\|_{L^2\Lambda^r(U)},$$

hence by Proposition 5.10,

$$T_1 \leq \frac{C}{l^{3+d/2}} \|df\|_{H^1(U)} \mathcal{O}_1(\varepsilon^\alpha).$$

The bound for  $T_3$  is similar, by Proposition (5.10) and the Popincaré inequality, Proposition 4.1, we have

$$\varepsilon \left\| \phi_{m,I} \left( \frac{\dot{\cdot}}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)} \leq \mathcal{O}_1(C\varepsilon^\alpha).$$

So by (6.7), one has

$$T_3 \leq \frac{C}{l^{3+d/2}} \|df\|_{H^1(U)} \mathcal{O}_1(\varepsilon^\alpha).$$

To estimate the second term  $T_2$ , the idea is to apply the boundary regularity result proved in the appendix, Proposition A.4. Since  $df$  is assumed to be in  $H^1\Lambda^r(U)$  and  $U$  is assumed to be smooth, we have

$$\|du\|_{H^1\Lambda^r(U)} \leq C \|d\bar{\mathbf{a}}df\|_{L^2\Lambda^{d-r+1}(U)} \leq C \|df\|_{H^1\Lambda^r(U)}.$$

This implies, via the Sobolev imbedding Theorem, that  $du$  is in  $L^{\frac{2d}{d-2}}\Lambda^r(U)$  if  $d \geq 3$  and any  $L^p\Lambda^r(U)$  if  $d = 2$ , with the estimate

$$\begin{cases} \|du\|_{L^{\frac{2d}{d-2}}\Lambda^r(U)} \leq C \|df\|_{H^1\Lambda^r(U)} & \text{if } d \geq 3, \\ \|du\|_{L^p\Lambda^r(U)} \leq C_p \|df\|_{H^1\Lambda^r(U)} & \text{if } d = 2, \end{cases}$$

for some  $C := C(d, U) < \infty$  and  $C_p := C(p, U) < \infty$ . We now set  $p = 4$  (but any  $p > 2$  would work). Using this estimate and the fact that  $(1 - \zeta_l)$  is supported in  $U \setminus U_{2l}$ , gives, by Hölder inequality,

$$T_2 \leq C \|du\|_{L^2\Lambda^r(U \setminus U_{2r})} \leq \begin{cases} C |U \setminus U_{2l}|^{\frac{1}{d-2}} \|du\|_{L^{\frac{2d}{d-2}}\Lambda^r(U)} & \text{if } d \geq 3, \\ C |U \setminus U_{2l}|^{\frac{1}{4}} \|du\|_{L^4\Lambda^r(U)} & \text{if } d = 2, \end{cases}$$

Combining the few previous results completes the proof of (6.6).

*Step 3.* We deduce from the previous step the  $H^1$  estimate

$$(6.8) \quad \|u^\varepsilon - w^\varepsilon\|_{H^1\Lambda^{r-1}(U)} \leq \begin{cases} C \|df\|_{H^1\Lambda^r(U)} \left( l^{\frac{1}{d-2}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d \geq 3, \\ C \|df\|_{H^1\Lambda^r(U)} \left( l^{\frac{1}{4}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d = 2. \end{cases}$$

Indeed, testing (6.6) with  $u^\varepsilon - w^\varepsilon \in H_{d,0}^1(U)$ , and using Step 1, we obtain

$$\begin{aligned} \left| \int_U d(u^\varepsilon - w^\varepsilon)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) dw^\varepsilon(x) \right| &\leq \|u^\varepsilon - w^\varepsilon\|_{H_d^1 \Lambda^{r-1}(U)} \left\| d\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) dw^\varepsilon\right) \right\|_{H_d^{-1} \Lambda^{r-1}(U)} \\ &\leq \|u^\varepsilon - w^\varepsilon\|_{H_d^1 \Lambda^{r-1}(U)} \left\| d\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) dw^\varepsilon\right) \right\|_{H^{-1} \Lambda^{r-1}(U)}. \end{aligned}$$

Meanwhile, testing this equation for  $u^\varepsilon$ , we get

$$\int_U d(u^\varepsilon - w^\varepsilon)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) du^\varepsilon(x) = 0.$$

Combining the two previous displays with the Poincaré inequality yields,

$$\begin{aligned} \|du^\varepsilon - dw^\varepsilon\|_{L^2 \Lambda^r(U)}^2 &\leq C \int_U d(u^\varepsilon - w^\varepsilon)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) d(u^\varepsilon - w^\varepsilon)(x) \\ &\leq C \|u^\varepsilon - w^\varepsilon\|_{H_d^1 \Lambda^r(U)} \left\| d\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) dw^\varepsilon\right) \right\|_{H_d^{-1} \Lambda^{r-1}(U)} \\ &\leq C \|du^\varepsilon - dw^\varepsilon\|_{L^2 \Lambda^r(U)} \left\| d\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) dw^\varepsilon\right) \right\|_{H^{-1} \Lambda^{r-1}(U)}. \end{aligned}$$

Thus

$$\|du^\varepsilon - dw^\varepsilon\|_{L^2 \Lambda^r(U)} \leq C \left\| d\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) dw^\varepsilon \right\|_{H^{-1} \Lambda^{r-1}(U)}.$$

Using the estimate (6.6) and another application of the Poincaré inequality completes the proof of (6.8)

*Step 4.* Recall that at the beginning of the proof, we assumed  $|U| = 1$ . We extend Definition 4.7 to the set  $U$  by setting, for each  $\omega \in L^2 \Lambda^r(U)$ ,

$$\|\omega\|_{\underline{H}^{-1} \Lambda^r(U)} := \sup \left\{ \langle \omega, \alpha \rangle_U : \alpha \in H^1 \Lambda^r(U), |(\alpha)_U| + \|\nabla \alpha\|_{L^2 \Lambda^r(U)} \leq 1 \right\}.$$

Note that this norm is a bit stronger than the standard  $H^{-1}$  norm which only requires to have test function in  $H_0^1$ . In this step, we prove that for each  $w \in H_{d,0}^1 \Lambda^{r-1}(U) \cap (C_{d,0}^r)^{\perp}$ , we have

$$\|w\|_{L^2 \Lambda^{r-1}(U)} \leq \|dw\|_{\underline{H}^{-1} \Lambda^r(U)}$$

To do so, let  $v$  be the unique solution in  $H_{d,0}^1 \Lambda^{r-1}(U) \cap (C_{d,0}^r)^{\perp}$  of the problem

$$(6.9) \quad \begin{cases} \delta dv = w \text{ in } U \\ \mathbf{t}v = 0 \text{ on } \partial U. \end{cases}$$

The existence and uniqueness of such a solution are obtained by minimizing the quantity

$$\mathcal{J}(v) := \langle dv, dv \rangle_U - \langle w, v \rangle_U$$

on the space  $H_{d,0}^1 \Lambda^{r-1}(U) \cap (C_{d,0}^r)^{\perp}$  and requires to use the Poincaré inequality, Proposition 4.1. The details are left to the reader.

If  $v$  is a solution (6.9), note that  $ddv = 0 \in L^2 \Lambda^{r+1}(U)$ ,  $\delta dv = w \in L^2 \Lambda^{r-1}(U)$  and  $\mathbf{t}dw = 0$  (this last property is implied by the condition  $\mathbf{t}v = 0$ , see for instance [24]). As a consequence, we have by the Gaffney-Friedrich inequality, Proposition 4.3, that  $dv \in H^1 \Lambda^r(U)$ , with the estimate

$$\|dv\|_{H^1 \Lambda^r(U)} \leq C \left( \|w\|_{L^2 \Lambda^{r-1}(U)} + \|dv\|_{L^2 \Lambda^r(U)} \right).$$

Testing (6.9) with  $v$  and using the Poincaré inequality also shows

$$\|dv\|_{L^2 \Lambda^r(U)} \leq C \|w\|_{L^2 \Lambda^{r-1}(U)}.$$

Combining the two previous displays shows

$$(6.10) \quad \|dv\|_{H^1 \Lambda^r(U)} \leq C \|w\|_{L^2 \Lambda^{r-1}(U)}.$$

testing (6.9) with  $w$  then shows,

$$\langle dw, dv \rangle_U = \langle w, w \rangle_U = \|w\|_{L^2\Lambda^{r-1}(U)}.$$

On the other hand, by the definition of the  $\underline{H}^{-1}$  and (6.10), we have

$$\begin{aligned} \langle dw, dv \rangle_U &\leq \|dw\|_{\underline{H}^{-1}\Lambda^r(U)} \|dv\|_{H^1\Lambda^r(U)} \\ &\leq \|dw\|_{\underline{H}^{-1}\Lambda^r(U)} \|w\|_{L^2\Lambda^{r-1}(U)}. \end{aligned}$$

Combining the two previous displays completes the proof of Step 4.

*Step 5.* We prove that

$$(6.11) \quad \|dw^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} \leq \mathcal{O}_1 \left( \frac{C \|df\|_{H^1\Lambda^r(U)} \varepsilon^\alpha}{l^{2+d/2}} \right).$$

We have that

$$dw^\varepsilon - du = d \left( \varepsilon \zeta_l \sum_{|I|=r} (du)_I \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right)$$

and therefore, since  $w^\varepsilon - u \in H_{d,0}^1\Lambda^{r-1}(U)$ , we have

$$\|dw^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} \leq C \|du\|_{L^\infty(U_r)} \sum_{|I|=r} \varepsilon \left\| \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)}.$$

But with the same proof as in Step 4, with  $\varepsilon\Box_m$  instead of  $U$ , and Proposition 5.10, we have

$$\varepsilon \left\| \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)} \leq \varepsilon \left\| \phi_{m,I} \left( \frac{\cdot}{\varepsilon} \right) \right\|_{L^2\Lambda^r(\varepsilon\Box_m)} \leq \mathcal{O}_1(C\varepsilon^\alpha),$$

then, by Proposition A.3, we have

$$\|dw^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} \leq \|df\|_{L^2\Lambda^r(U)} \mathcal{O}_1 \left( \frac{C\varepsilon^\alpha}{l^{1+d/2}} \right).$$

This completes the proof of (6.11).

*Step 6.* The conclusion. By Steps 2 and 3, we can compute

$$\begin{aligned} \|du^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} &\leq \|du^\varepsilon - dw^\varepsilon\|_{\underline{H}^{-1}\Lambda^r(U)} + \|dw^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} \\ &\leq \|du^\varepsilon - dw^\varepsilon\|_{L^2\Lambda^r(U)} + \|dw^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)}. \end{aligned}$$

This yields

$$\|du^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)} \leq \begin{cases} C \|df\|_{H^1\Lambda^r(U)} \left( l^{\frac{1}{d-2}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d \geq 3, \\ C \|df\|_{H^1\Lambda^r(U)} \left( l^{\frac{1}{4}} + \mathcal{O}_1 \left( \frac{\varepsilon^\alpha}{l^{3+d/2}} \right) \right) & \text{if } d = 2. \end{cases}$$

Finally, the bound for  $\|u^\varepsilon - u\|_{L^2(U)}$  is obtained from the previous inequality and Step 4. Indeed, since  $u - u^\varepsilon \in H_{d,0}^1\Lambda^{r-1}(U) \cap (C_{d,0}^r)^\perp$ , we have

$$\|du^\varepsilon - du\|_{L^2\Lambda^r(U)} \leq C \|du^\varepsilon - du\|_{\underline{H}^{-1}\Lambda^r(U)}.$$

*Step 7.* The conclusion. The estimate obtained is valid for any  $0 < l \leq 1$ , in particular we can choose  $l$  to be a small power of  $\varepsilon$  such that  $\frac{\varepsilon^\alpha}{l^{3+d/2}}$  is still a small power of  $\varepsilon$ . This completes the proof of Theorem 2.  $\square$

## 7. DUALITY

The goal of this section is to study a duality property between the homogenization of  $r$ -forms and  $(d-r)$ -forms. We note that similar results were obtained independently by Serre [28] in the case of periodic coefficients. For each  $\mathbf{a} \in \Omega_r$  and each  $x \in \mathbb{R}^d$ , the operator  $\mathbf{a}(x) \in \mathcal{L}(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d))$  satisfies the ellipticity assumption (2.16), so it is invertible and one can define the inverse operator  $(\mathbf{a}(x))^{-1} \in \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d))$ , which satisfies the symmetry assumption (2.15) and the following ellipticity condition

$$(7.1) \quad \frac{1}{\lambda}|p|^2 \leq \mathbf{a}(x)^{-1}p \wedge p \leq \lambda|p|^2, \quad \forall p \in \Lambda^{(d-r)}(\mathbb{R}^d).$$

We denote by

$$\Omega'_{d-r} := \left\{ \mathbf{a}(\cdot) : \mathbf{a} : \mathbb{R}^d \rightarrow \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d)) \text{ is Lebesgue measurable} \right. \\ \left. \text{and satisfies (2.15) and (7.1)} \right\}.$$

We equip this set with a family of sigma algebras, for each  $U \subseteq \mathbb{R}^d$ ,

$$\mathcal{F}'_r(U) := \left\{ \sigma\text{-algebra on } \Omega'_r \text{ generated by the family of maps} \right. \\ \left. \mathbf{a} \rightarrow \int_U p \wedge \mathbf{a}(x)q\phi(x), p, q \in \Lambda^r(\mathbb{R}^d), \phi \in C_c^\infty(U) \right\}.$$

One also defines  $\text{inv}$  to be the mapping

$$\text{inv} : \begin{cases} \Omega_r \rightarrow \Omega'_{d-r}, \\ \mathbf{a} \rightarrow \mathbf{a}^{-1}. \end{cases}$$

We then define  $\text{inv}_*\mathbb{P}$  the probability measure defined on the measured space  $(\Omega'_{d-r}, \mathcal{F}'_{d-r})$  by, for each  $A \in \mathcal{F}'_{d-r}$ ,

$$\text{inv}_*\mathbb{P}_r(A) := \mathbb{P}_r(\text{inv}^{-1}A),$$

the probability space  $(\Omega'_{d-r}, \mathcal{F}'_{d-r}, \text{inv}_*\mathbb{P}_r)$  satisfies the stationarity assumption (2.19) and the independence assumption (2.20). The idea is then to define, for each  $(p, q) \in \Lambda^{(d-r)}(\mathbb{R}^d) \times \Lambda^r(\mathbb{R}^d)$  and each  $m \in \mathbb{N}$ ,

$$J_{\text{inv}}(\square_m, p, q) := \sup_{u \in \mathcal{A}^{\text{inv}}(\square_m)} \int_{\square_m} \left( -\frac{1}{2} \mathbf{a}^{-1} du \wedge du - \mathbf{a}^{-1} du \wedge p + q \wedge du \right),$$

where  $\mathcal{A}^{\text{inv}}(\square_m)$  is the set of solution under the environment  $\mathbf{a}^{-1}$ , i.e,

$$\mathcal{A}_{\text{inv}}(\square_m) := \left\{ u \in H^1_d \Lambda^{(d-r-1)}(\square_m) : \forall v \in C_c^\infty \Lambda^{r-1}(\square_m), \int_{\square_m} du \wedge \mathbf{a}^{-1} dv = 0 \right\},$$

and this quantity satisfies the conclusions of Proposition 5.1 and Theorem 1. In particular, there exist a constant  $C(d, \lambda) < \infty$ , an exponent  $\alpha(d, \lambda) > 0$  and a linear operator

$$\overline{\text{inv } \mathbf{a}} \in \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d))$$

such that, for each  $m \in \mathbb{N}$ ,

$$\sup_{p \in B_1 \Lambda^{(d-r)}(\mathbb{R}^d)} \mathbb{E} \left[ J_{\text{inv}}(\square_m, p, \overline{\text{inv } \mathbf{a}} p) \right] \leq C 3^{-m\alpha}.$$

The following theorem determines  $\overline{\text{inv } \mathbf{a}}$ .

**Theorem 3** (Duality). *The homogenized linear maps  $\overline{\mathbf{a}}$  and  $\overline{\text{inv } \mathbf{a}}$  satisfy*

$$\overline{\text{inv } \mathbf{a}} = (\overline{\mathbf{a}})^{-1}.$$



*Proof.* First we need to prove the following result, for each  $0 \leq r \leq d$  and each bounded  $m \in \mathbb{N}$ ,

$$\mathcal{A}_{\text{inv}}(\square_m) = \left\{ v \in H_d^1 \Lambda^{(d-r-1)}(\square_m) : dv = \mathbf{a}du \text{ with } u \in \mathcal{A}(\square_m) \right\}.$$

We split the proof into 2 steps

- We prove that each  $v \in H_d^1 \Lambda^{(d-r-1)}(\square_m)$  satisfying  $dv = \mathbf{a}du$  for some  $u \in \mathcal{A}(\square_m)$  belongs to  $\mathcal{A}_{\text{inv}}(\square_m)$ . Indeed, for each  $w \in C_c^\infty \Lambda^{r-1}(\square_m)$ , we have, by the symmetry assumption (2.15), and (2.13),

$$(7.2) \quad \int_{\square_m} dv \wedge \mathbf{a}^{-1}dw = \int_{\square_m} dw \wedge \mathbf{a}^{-1}dv = \int_{\square_m} dw \wedge du = 0.$$

- We prove that for each  $v \in \mathcal{A}_{\text{inv}}(\square_m)$ , there exists  $u \in \mathcal{A}(\square_m)$  such that  $dv = \mathbf{a}du$ . Indeed, if  $v \in \mathcal{A}_{\text{inv}}(\square_m)$ , then  $\mathbf{a}^{-1}dv$  belongs to  $L^2 \Lambda^r(\square_m)$  and satisfies

$$d(\mathbf{a}^{-1}dv) = 0 \text{ in } \square_m.$$

Consequently  $\mathbf{a}^{-1}dv \in H_d^1 \Lambda^r(\square_m)$ . We can apply Proposition 3.4, to prove that there exists  $u \in H_d^1 \Lambda^r(\square_m)$  such that

$$\mathbf{a}^{-1}dv = du \text{ in } \square_m.$$

There only remains to prove that  $u \in \mathcal{A}(\square_m)$ , it is a consequence of the following computation. For each  $w \in C_c^\infty \Lambda^{(d-r-1)}(\square_m)$ , we have, by the symmetry assumption (2.15) and (2.13),

$$\int_{\square_m} du \wedge \mathbf{a}dw = \int_{\square_m} dw \wedge \mathbf{a}du = \int_{\square_m} dw \wedge dv = 0.$$

Using (7.2), we have

$$\begin{aligned} J_{\text{inv}}(\square_m, p, q) &= \sup_{u \in \mathcal{A}_{\text{inv}}(\square_m)} \int_{\square_m} \left( -\frac{1}{2} \mathbf{a}^{-1}du \wedge du - \mathbf{a}^{-1}du \wedge p + q \wedge du \right) \\ &= \sup_{v \in \mathcal{A}(\square_m)} \int_{\square_m} \left( -\frac{1}{2} (\mathbf{a}dv) \wedge (\mathbf{a}dv) - \mathbf{a}^{-1}(\mathbf{a}dv) \wedge p + q \wedge (\mathbf{a}dv) \right) \\ &= \sup_{v \in \mathcal{A}(\square_m)} \int_{\square_m} \left( -\frac{1}{2} dv \wedge \mathbf{a}dv - dv \wedge p + q \wedge \mathbf{a}dv \right) \\ &= J(\square_m, -q, -p) \\ &= J(\square_m, q, p). \end{aligned}$$

Thus, by Theorem 1, for each  $m \in \mathbb{N}$ ,

$$\sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}[J_{\text{inv}}(\square_m, \bar{\mathbf{a}}p, p)] = \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}[J(\square_m, p, \bar{\mathbf{a}}p)] \leq C3^{-\alpha m}.$$

The previous inequality can be rewritten

$$\sup_{q \in B_1 \Lambda^{(d-r)}(\mathbb{R}^d)} \mathbb{E}[J_{\text{inv}}(\square_m, p, \bar{\mathbf{a}}^{-1}p)] \leq C3^{-\alpha m}.$$

Since the homogenized matrix is unique, we have  $\bar{\mathbf{a}}^{-1} = \overline{\text{inv } \mathbf{a}}$ . This gives the expected result.  $\square$

**Remark 7.1.** The previous result can be applied in the particular case  $d = 2, r = 1$  and the standard homogenization problem

$$\nabla \cdot (\mathbf{a} \nabla u) = 0$$

can be rewritten with the formalism of forms

$$d(\star \mathbf{a}du) = 0$$

(we identify the space  $\mathbb{R}^2$  with the space  $\Lambda^1(\mathbb{R}^2)$  canonically). Thus, we can compute the dual problem,

$$d(\mathbf{a}^{-1} \star du) = 0$$

which can be rewritten in the standard formalism,

$$(7.3) \quad \nabla^\perp \cdot (\mathbf{a}^{-1} \nabla^\perp u) = 0,$$

where we used the notation

$$\nabla^\perp f = \begin{pmatrix} -\partial_2 f \\ \partial_1 f \end{pmatrix}.$$

Denote by

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Performing the change of variable  $u(x) \rightarrow u(Px)$ , the equation (7.3) becomes

$$(7.4) \quad \nabla \cdot ((\mathbf{a}^{-1} \circ P) \nabla u) = 0,$$

where  $\mathbf{a}^{-1} \circ P$  is defined by, for each  $x \in \mathbb{R}^d$ ,

$$\mathbf{a}^{-1} \circ P(x) = \mathbf{a}^{-1}(Px).$$

With this in mind, one can compute the homogenized matrix  $\overline{\mathbf{a}^{-1} \circ P}$  of the problem (7.4). We obtain according Theorem 3

$$(7.5) \quad \overline{\mathbf{a}^{-1} \circ P} = \bar{\mathbf{a}}^{-1}.$$

In particular, indeed if we assume that the environment satisfies, for some positive constant  $k$  and for each  $x \in \mathbb{R}^d$ ,

$$\mathbf{a}(x)\mathbf{a}(Px) = kI_d,$$

then  $\mathbf{a}(x) = k\mathbf{a}^{-1}(Px)$  and (7.5) gives

$$\frac{\bar{\mathbf{a}}}{k} = \bar{\mathbf{a}}^{-1},$$

which implies

$$\bar{\mathbf{a}} = \sqrt{k}I_d.$$

This formula is known as the *Dykhne formula* which was originally proved in [7].

## APPENDIX A. REGULARITY ESTIMATES FOR DIFFERENTIAL FORMS

In this appendix, we record some properties about the regularity of the solutions of the constant coefficient equation  $d\bar{\mathbf{a}}u$ . The two main results are the pointwise interior estimate, Proposition A.3 and the  $H^2$  boundary estimate, Proposition A.4. Both these results are used in the proof of Theorem 2. Most of these proofs are an adaptation of the classical proofs of the regularity theory of uniformly elliptic equations (cf [11]).

We first state two propositions, Proposition A.1 and Proposition A.2. We then use these two ingredients to prove the pointwise interior estimate, Proposition A.3. We finally prove a global  $H^2$  regularity result for the solution of  $d\bar{\mathbf{a}}u = 0$ , Proposition A.4.

The following proposition is an interior version of the Gaffney-Friedrich inequality, Propositions 4.3 and 4.4. The result is weaker because it is only an interior estimate, but it does not require any regularity for the domain  $U$  nor any assumption on the value of the form on the boundary of the domain.

**Proposition A.1** (Interior Gaffney-Friedrich inequality). *There exists a constant  $C := C(d) < \infty$  such that, for every  $0 \leq r \leq d$ , every open bounded subset  $V, U \subseteq \mathbb{R}^d$  satisfying  $\bar{V} \subseteq U$ , and every  $u \in L^2\Lambda^r(U)$  such that  $du \in L^2\Lambda^{r+1}(U)$  and  $\delta u \in L^2\Lambda^{r-1}(U)$ , we have  $u \in H^1\Lambda^r(V)$  with the estimate,*

$$(A.1) \quad \|\nabla u\|_{L^2\Lambda^r(V)} \leq C \left( \|du\|_{L^2\Lambda^{r+1}(U)} + \|\delta u\|_{L^2\Lambda^{r-1}(U)} + \frac{1}{\text{dist}(V, \partial U)} \|u\|_{L^2\Lambda^r(U)} \right).$$

*Proof.* The proof relies on the following observation, given a form  $u = \sum_{|I|=r} u_I dx_I \in C^\infty \Lambda^r(U)$ , we have

$$(d\delta + \delta d)u = \sum_{|I|=r} \Delta u_I dx_I.$$

Select a function  $\eta \in C_c^\infty(U)$  such that

$$\mathbf{1}_V \leq \eta \leq 1, \quad |\nabla \eta| \leq \frac{C}{\text{dist}(V, \partial U)}.$$

We then compute

$$\begin{aligned} \|\nabla u\|_{L^2 \Lambda^r(V)}^2 &= \sum_I \int_V |\nabla u_I|^2(x) dx \\ &\leq \sum_I \int_U |\nabla u_I \eta|^2(x) dx \\ &= \sum_I \int_U (u_I \eta)(x) \Delta (u_I \eta)(x) dx \\ &= \langle u\eta, (\delta d + d\delta)u\eta \rangle_U \\ &= \langle d(u\eta), d(u\eta) \rangle_U + \langle \delta(u\eta), \delta(u\eta) \rangle_U \end{aligned}$$

By (2.2), we have

$$\begin{aligned} \langle d(u\eta), d(u\eta) \rangle_U &= \langle \eta du + d\eta \wedge u, \eta du + d\eta \wedge u \rangle_U \\ &\leq 2 \langle \eta du, \eta du \rangle_U + 2 \langle d\eta \wedge u, d\eta \wedge u \rangle_U \\ &\leq C \left( \|du\|_{L^2 \Lambda^{r+1}(U)}^2 + \frac{1}{\text{dist}(V, \partial U)^2} \|u\|_{L^2 \Lambda^r(U)}^2 \right). \end{aligned}$$

A similar computation yields

$$\langle \delta(u\eta), \delta(u\eta) \rangle_U \leq C \left( \|\delta u\|_{L^2 \Lambda^{r-1}(U)}^2 + \frac{1}{\text{dist}(V, \partial U)^2} \|u\|_{L^2 \Lambda^r(U)}^2 \right).$$

Combining the three previous displays completes the proof of (A.1).  $\square$

We then use the previous interior Gaffney-Friedrich inequality to prove the following interior  $H^2$  estimate. The proof of the following proposition is an adaptation of the standard interior  $H^2$  estimate for the solutions of uniformly elliptic equations, cf [11, Theorem 8.8].

**Proposition A.2** (Interior  $H^2$  regularity estimate). *For every open bounded subset  $U, V \subseteq \mathbb{R}^d$  such that  $\overline{V} \subseteq U$ , every  $1 \leq r \leq d$  and every  $u \in H_d^1 \Lambda^{(r-1)}(U)$  solution of the equation*

$$(A.2) \quad d(\overline{\mathbf{a}} du) = 0 \text{ in } U.$$

*we have  $du \in H^1 \Lambda^r(V)$  and it satisfies the interior estimate*

$$\|\nabla du\|_{L^2 \Lambda^r(V)} \leq C \left( \frac{1}{\text{dist}(V, \partial U)} \|du\|_{L^2 \Lambda^r(U)} + \frac{1}{\text{dist}(V, \partial U)^2} \|u\|_{L^2 \Lambda^{(r-1)}(U)} \right)$$

*for a constant  $C := C(d, \lambda) < \infty$ .*

*Proof.* The main idea of this proof is to mimic the proof of [11, Theorem 8.8] and combine it with the interior Gaffney-Friedrich inequality.

First note that without loss of generality, one can assume that  $u \in C_d^{r-1}(U)^\perp$ . Select an open space  $V \subseteq U$  such that  $\overline{V} \subseteq U$  and select two other open spaces  $W, W_1$  space such that  $V \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$  such that

$$(A.3) \quad \text{dist}(V, \partial W) = \text{dist}(W, \partial W_1) = \text{dist}(W_1, \partial U) = \frac{\text{dist}(V, \partial W)}{3}$$

and select a cutoff function  $\eta \in C_c^\infty(U)$  such that

$$(A.4) \quad \mathbb{1}_V \leq \eta \leq \mathbb{1}_W, \quad \|\nabla \eta\| \leq \frac{C}{\text{dist}(V, \partial U)}.$$

Let  $h > 0$  be small, choose  $k \in \{1, \dots, d\}$  and denote by

$$v := D_k^{-h}(\eta^2 D_k^h u),$$

where  $D_k^h$  is the difference quotient, defined by

$$D_k^h u(x) = \frac{u(x + h e_k) - u(x)}{h}.$$

If  $h$  is small enough then  $v \in H_{d,0}^1 \Lambda^{r-1}(U)$  can be used as a test function in (A.2). We obtain

$$\langle du, \bar{\mathbf{a}} dv \rangle_U = 0.$$

Thanks to (2.2) and the equality  $dD_k^h u = D_k^h du$ , we compute

$$dv = D_k^{-h}(d(\eta^2 D_k^h u)) = D_k^{-h}(2\eta d\eta \wedge D_k^h u) + D_k^{-h}(\eta^2 D_k^h du).$$

Combining the two previous displays yields

$$\langle du, \bar{\mathbf{a}} D_k^{-h}(\eta^2 D_k^h du) \rangle_U = - \langle du, \bar{\mathbf{a}} D_k^{-h}(2\eta d\eta \wedge D_k^h u) \rangle_U.$$

The discrete integration by parts gives

$$\langle D_k^h du, \bar{\mathbf{a}}(\eta^2 D_k^h du) \rangle_U = - \langle D_k^h du, \bar{\mathbf{a}}(2\eta d\eta \wedge D_k^h u) \rangle_U.$$

By the Cauchy-Schwarz inequality, we obtain

$$\langle D_k^h du, \bar{\mathbf{a}}(\eta^2 D_k^h du) \rangle_U \leq 2 \left( \langle D_k^h du, \bar{\mathbf{a}}(\eta^2 D_k^h du) \rangle_U \right)^{\frac{1}{2}} \left( \langle d\eta \wedge D_k^h u, \bar{\mathbf{a}} d\eta \wedge D_k^h u \rangle_U \right)^{\frac{1}{2}}.$$

and consequently, by (A.1) and the ellipticity assumption (2.16)

$$\langle D_k^h du, D_k^h du \rangle_V \leq \frac{C}{\text{dist}(V, \partial U)^2} \langle D_k^h u, D_k^h u \rangle_W.$$

Since we assumed  $u \in C_d^{r-1}(U)^\perp$ , we have  $\delta u = 0$  in  $U$  and in particular  $\delta u \in L^2 \Lambda^{r-2}(V)$ . From this we deduce that  $u$  satisfies the assumptions of Proposition A.1 and consequently  $u$  is in  $H^1 \Lambda^{r-1}(W)$  and satisfies the estimate

$$\begin{aligned} \|\nabla u\|_{L^2 \Lambda^r(W_1)} &\leq C \left( \|du\|_{L^2 \Lambda^r(U)} + \frac{1}{\text{dist}(W_1, \partial U)} \|u\|_{L^2 \Lambda^{(r-1)}(U)} \right) \\ &\leq C \left( \|du\|_{L^2 \Lambda^r(U)} + \frac{1}{\text{dist}(V, \partial U)} \|u\|_{L^2 \Lambda^{(r-1)}(U)} \right), \end{aligned}$$

where we used (A.4) in the second inequality. Moreover, according to [11, Lemma 7.23], we have the inequality

$$(A.5) \quad \|D_k^h u\|_{L^2(W)} \leq C \|\nabla u\|_{L^2 \Lambda^r(W_1)}$$

for  $h > 0$  small enough. Combining the three previous displays shows

$$\langle D_k^h du, D_k^h du \rangle_V \leq \frac{C}{\text{dist}(V, \partial U)^2} \left( \|du\|_{L^2 \Lambda^r(U)}^2 + \frac{1}{\text{dist}(W_1, \partial U)^2} \|u\|_{L^2 \Lambda^{(r-1)}(U)}^2 \right).$$

Since this inequality is true for every  $|h| > 0$  small enough, we have, according to [11, Lemma 7.24],  $du \in H^1 \Lambda^{r-1}(V)$  and

$$\|\nabla du\|_{L^2 \Lambda^r(V)}^2 \leq \frac{C}{\text{dist}(V, \partial U)^2} \left( \|du\|_{L^2 \Lambda^r(U)}^2 + \frac{1}{\text{dist}(V, \partial U)^2} \|u\|_{L^2 \Lambda^{(r-1)}(U)}^2 \right)$$

and the proof is complete.  $\square$

**Proposition A.3** (Elliptic regularity). *There exists a constant  $C := C(d, k, \lambda) < \infty$  such that for every open bounded subset  $U \subseteq \mathbb{R}^d$ , every  $0 \leq r \leq d$ , every  $k \in \mathbb{N}$ , every  $R > 0$ , and every solution of the equation*

$$d(\bar{\mathbf{a}}du) = 0 \text{ in } U,$$

the following pointwise estimate holds

$$(A.6) \quad \|\nabla^k du\|_{L^\infty \Lambda^r(U_R)} \leq \frac{C}{R^{k+d/2}} \|du\|_{L^2 \Lambda^r(U)},$$

where we denoted by  $U_R := \{x \in U : \text{dist}(x, \partial U) > R\}$ .

*Proof.* Select an integer  $k \in \mathbb{N}$ , a non-negative real number  $R > 0$ , and a point  $x \in U_R$ . It is sufficient to prove (A.6), to show the estimate

$$(A.7) \quad |\nabla^k du(x)| \leq \frac{C}{R^{k+d/2}} \|du\|_{L^2 \Lambda^r(B_R(x))},$$

for some constant  $C = C(d, k, \Lambda) < \infty$ . We split the proof into two steps

*Step 1.* We prove that there exists a constant  $C = C(d, \lambda) < \infty$ , such that for every  $l \in \mathbb{N}$ ,  $du \in H^l \Lambda^r(B_{R/2^l}(x))$  and

$$(A.8) \quad \|\nabla^l du\|_{L^2 \Lambda^r(B_{R/2^l}(x))} \leq \frac{C^l 2^{l^2/2}}{R^l} \|du\|_{L^2 \Lambda^r(B_R(x))}.$$

This inequality can be proved by induction on  $l$ . It is true for  $l = 0$ . We can use Proposition A.2 to go from  $l$  to  $l + 1$ . Assume that (A.8) holds with  $l$ . In that case we have  $\nabla^l du \in L^2 \Lambda^r(B_{R/2^l}(x))$ . It is easy to check

$$d(\nabla^l du) = 0.$$

Thus by Proposition 3.4, there exists a form  $v_l \in H_d^1 \Lambda^{r-1}(B_{R/2^l}(x))$  such that

$$v_l \in C_d^{r-1}(B_{R/2^l}(x))^\perp \text{ and } dv_l = \nabla^l du.$$

It is moreover a straightforward computation to check

$$d(\bar{\mathbf{a}}dv_l) = 0.$$

Consequently, we can apply Proposition A.2 to  $v_l$  with  $U = B_{R/2^l}(x)$  and  $V = B_{R/2^{l+1}}(x)$ . This gives  $\nabla^{l+1} du \in H^1 \Lambda^r(B_{R/2^{l+1}}(x))$ , and thus  $du \in H^{l+1} \Lambda^r(B_{R/2^{l+1}}(x))$  with the estimate

$$\|\nabla^{l+1} du\|_{L^2 \Lambda^r(B_{R/2^{l+1}}(x))} \leq \frac{C 2^{l+1}}{R} \left( \|\nabla^l du\|_{L^2 \Lambda^r(B_{R/2^l}(x))} + \frac{2^{l+1}}{R} \|v_l\|_{L^2 \Lambda^{(r-1)}(B_{R/2^l}(x))} \right).$$

By Proposition 4.1,  $v_l$  satisfies the Poincaré estimate

$$\|v_l\|_{L^2 \Lambda^{(r-1)}(B_{R/2^l}(x))} \leq C \frac{2^l}{R} \|\nabla^l v_l\|_{L^2 \Lambda^{(r-1)}(B_{R/2^l}(x))},$$

with  $C := C(d) < \infty$ . Combining the two previous displays yields

$$\|\nabla^{l+1} du\|_{L^2 \Lambda^r(B_{R/2^{l+1}}(x))} \leq \frac{C 2^{l+1}}{R} \|\nabla^l du\|_{L^2 \Lambda^r(B_{R/2^l}(x))}$$

for some  $C := C(d, \Lambda) < \infty$ . Applying the induction hypothesis completes the proof.

*Step 2.* From the first step, we get that for every  $l \in \mathbb{N}$ ,  $\nabla^{k+l} du \in L^2 \Lambda^r(B_{R/2^{k+l}}(x))$ . In particular, by the Sobolev injection, see for instance [1, Chapter 4], we have

$\nabla^k du \in L^\infty \Lambda^r(B_{R/2^{k+d/2+1}}(x))$  with the estimate

$$\|\nabla^k du\|_{L^\infty \Lambda^r(B_{R/2^{k+d/2+1}}(x))} \leq \frac{C}{R^{k+d/2}} \|du\|_{L^2 \Lambda^r(B_R(x))},$$

for some constant  $C = C(d, k, \lambda) < \infty$ . This completes the proof of (A.7).  $\square$

We then establish the following global  $H^2$  estimate for the solution of  $d\bar{\mathbf{a}}du = 0$ .

**Proposition A.4** (Global  $H^2$  regularity). *Let  $U \subseteq \mathbb{R}^d$  be a smooth bounded domain of  $\mathbb{R}^d$ . For  $0 \leq r \leq d$ , let  $f \in H^1_d \Lambda^{r-1}(U)$  satisfying  $df \in H^1 \Lambda^{r-1}(U)$ . Let  $u \in H^1_{d,0} \Lambda^{r-1}(U)$  be a solution of the equation*

$$(A.9) \quad \begin{cases} d(\bar{\mathbf{a}}du) = 0 \text{ in } U, \\ \mathbf{t}u = f \text{ on } \partial U, \end{cases}$$

then  $du \in H^1 \Lambda^r(U)$  and we have the estimate

$$(A.10) \quad \|du\|_{H^1 \Lambda^r(U)} \leq \|df\|_{H^1 \Lambda^r(U)}.$$

*Proof.* First note that two solutions of (A.9) differ by a form of  $C^r_{d,0}$ , this implies that two solutions of (A.9) have the same exterior derivative. Thus to prove (A.10), it is enough to prove it for a particular solution of (A.4).

The strategy of the proof is the following. We want to apply the result from the regularity theory of strongly elliptic operators to the differential form  $u$ , see (A.24) for a definition and [22] for a reference on the topic of strongly differential operators. Unfortunately the operator  $d\bar{\mathbf{a}}d$  is not strongly elliptic, thus the result cannot directly apply. The strategy is then to solve the problem  $d\bar{\mathbf{a}}d + (-1)^r \star d\delta u = 0$  with appropriate boundary conditions so that  $\star d\delta u = 0$  and  $u$  is in fact a solution of (A.9). Contrary to  $d\bar{\mathbf{a}}d$ , the operator  $d\bar{\mathbf{a}}d + (-1)^r \star d\delta$  is strongly elliptic and a regularity theory exists for these operators. Thanks to this, we are able to obtain  $H^2$  boundary regularity for the function  $u$ , this implies (A.10) by the previous remark.

The main ideas of the proof are standard and can be found in [27, Chapter 2] and [22, Chapter 4]. We recall the notation for the set of harmonic forms with Dirichlet boundary condition introduced in Proposition 4.4,

$$\mathcal{H}_D^{r-1}(U) := \mathcal{H}^{r-1}(U) \cap H^1_{d,0}(U) := \{u \in L^2 \Lambda^{r-1}(U) : du = 0, \delta u = 0 \text{ in } U \text{ and } \mathbf{t}u = 0 \text{ on } \partial U\}.$$

We split the proof into 5 steps

- In Step 1, we show that there exists a unique solution in  $u \in \mathcal{H}_D^{r-1}(U)^\perp$  to the system

$$(A.11) \quad \begin{cases} d\bar{\mathbf{a}}du + (-1)^r \star d\delta u = d\bar{\mathbf{a}}df, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases}$$

- In Step 2, we show that the form  $u$  defined in Step 1 satisfies  $d\delta u = 0$  and is actually a solution of (A.9).
- Steps 3, 4 and 5 are the technical steps, we show the  $H^2$  boundary regularity for the solution of the more general problem,

$$\begin{cases} d\bar{\mathbf{a}}du + (-1)^r \star d\delta u = g, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases}$$

for  $g \in L^2 \Lambda^{d-r+1}$ , using the theory of strongly elliptic operators developed in [22].

- In Step 6, we combine the results of Steps 1 to 4 with  $g = d\bar{\mathbf{a}}df$  to prove (A.10).

*Step 1.* First, we prove that there exists a unique solution  $u \in \mathcal{H}_D^{r-1}(U)^\perp$  of the system

$$\begin{cases} d\bar{\mathbf{a}}du + (-1)^r \star d\delta u = d\bar{\mathbf{a}}df, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases}$$

This equation can be rewritten variationally the following way, there exists  $u \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$  such that  $\mathbf{t}u = 0$  and for each  $v \in H^1\Lambda^{r-1}(U)$  satisfying  $\mathbf{t}v = 0$ ,

$$(A.12) \quad \int_U du \wedge \bar{\mathbf{a}}dv + \int_U \delta u \wedge \star \delta v = \int_U df \wedge \bar{\mathbf{a}}dv.$$

To solve this, we look at the associated energy: for  $v \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$  satisfying the boundary condition  $\mathbf{t}v = 0$ , we define

$$\mathcal{J}(v) := \int_U dv \wedge \bar{\mathbf{a}}dv + \int_U \delta v \wedge \star \delta v - \int_U df \wedge \bar{\mathbf{a}}dv.$$

Since  $\bar{\mathbf{a}}$  satisfies the ellipticity assumption (2.16), we have

$$\int_U dv \wedge \bar{\mathbf{a}}dv + \int_U \delta v \wedge \star \delta v \geq \lambda \|dv\|_{L^2\Lambda^r(U)} + \|\delta v\|_{L^2\Lambda^r(U)}.$$

Moreover, by the Gaffney-Friedrich inequality, Proposition 4.4, we have

$$\begin{aligned} \int_U dv \wedge \bar{\mathbf{a}}dv + \int_U \delta v \wedge \star \delta v &\geq \lambda \|dv\|_{L^2\Lambda^r(U)} + \|\delta v\|_{L^2\Lambda^r(U)} \\ &\geq \lambda \|\nabla v\|_{L^2\Lambda^{r-1}(U)} \end{aligned}$$

This implies that the functional  $\mathcal{J}$  is coercive on the space

$$\{u \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp : \mathbf{t}u = 0 \text{ on } \partial U\}.$$

Moreover, this functional is also uniformly convex. The standard techniques of the calculus of variations then show that there exists a unique minimizer of  $\mathcal{J}$  denoted by  $u$ . By the first variation, we have for each  $v \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$  satisfying the boundary condition  $\mathbf{t}v = 0$ ,

$$\int_U du \wedge \bar{\mathbf{a}}dv + \int_U \delta u \wedge \star \delta v = \int_U df \wedge \bar{\mathbf{a}}dv.$$

Also, for each  $v \in \mathcal{H}_D^{r-1}(U)$ , we have

$$\int_U du \wedge \bar{\mathbf{a}}dv + \int_U \delta u \wedge \star \delta v = \int_U df \wedge \bar{\mathbf{a}}dv = 0.$$

Thus for each  $v \in H^1\Lambda^{r-1}(U)$  satisfying  $\mathbf{t}v = 0$ , we have

$$\int_U du \wedge \bar{\mathbf{a}}dv + \int_U \delta u \wedge \star \delta v = \int_U df \wedge \bar{\mathbf{a}}dv$$

and the proof of Step 1 is complete. As a remark, note that since  $df \in H^1\Lambda^r(U)$ ,  $d\bar{\mathbf{a}}df \in L^2\Lambda^{d-r+1}(U)$ . Thus, if we denote by  $g := d\bar{\mathbf{a}}df \in L^2\Lambda^{d-r+1}(U)$ , we have

$$(A.13) \quad \int_U du \wedge \bar{\mathbf{a}}dv + \int_U \delta u \wedge \star \delta v = \int_U g \wedge v,$$

for each  $v \in H^1\Lambda^{r-1}(U)$  satisfying  $\mathbf{t}v = 0$ .

*Step 2.* We show that the solution  $u$  constructed in the previous step actually satisfies

$$\begin{cases} d\bar{\mathbf{a}}du = d\bar{\mathbf{a}}df \text{ in } U, \\ \mathbf{t}u = 0 \text{ on } \partial U. \end{cases}$$

To prove this, it is enough, by Proposition 3.3, to show that for each  $v \in H^1\Lambda^{r-1}(U)$  satisfying the boundary condition  $\mathbf{t}v = 0$ ,

$$\int_U du \wedge \bar{\mathbf{a}}dv = \int_U df \wedge \bar{\mathbf{a}}dv.$$

To prove this select some  $v \in H^1\Lambda^{r-1}(U)$  satisfying  $\mathbf{t}v = 0$ . Denote  $\alpha_v$  the form of  $C_{d,0}^{r-1}(U)$  such that

$$\alpha_v = \operatorname{argmin}_{\alpha \in C_{d,0}^{r-1}(U)} \|v - \alpha\|_{L^2\Lambda^{r-1}(U)},$$

and set  $w = v - \alpha_v$ . In particular, this form satisfies, for each  $\gamma \in C_c^\infty\Lambda^{r-2}(U)$ ,

$$\langle w, d\gamma \rangle_{L^2\Lambda^{r-1}(U)} = 0,$$

this implies  $\delta w = 0$ . Moreover it is clear that  $dw = dv$  and that  $\mathbf{t}w = 0$ . Thus, by the Gaffney-Friedrich inequality,  $w \in H^1\Lambda^{r-1}(U)$ . So  $w$  can be tested in (A.12), this gives

$$\int_U du \wedge \bar{\mathbf{a}}dw = \int_U df \wedge \bar{\mathbf{a}}dw$$

and since  $dw = dv$ , the previous equality can be rewritten

$$\int_U du \wedge \bar{\mathbf{a}}dv = \int_U df \wedge \bar{\mathbf{a}}dv,$$

which is the desired result. The proof of Step 2 is complete.

*Step 3.* From now on, we consider the problem

$$\begin{cases} d\bar{\mathbf{a}}du + (-1)^r \star d\delta u = g, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases}$$

and want to prove the boundary regularity estimate, assuming that  $u \in \mathcal{H}_D^{d-1}(U)^\perp$ ,

$$\|u\|_{H^2\Lambda^{r-1}(U)} \leq C \|g\|_{L^2\Lambda^{r-1}(U)},$$

for some  $C := C(d, \lambda, U) < \infty$ . This is the subject of Steps 3, 4 and 5.

The first part of this step is to reduce the problem to the half-ball denoted by  $B^+ := \{x \in B(0, 1) : x_n \geq 0\}$ . To do so, let  $x \in \partial U$ , since  $\partial U$  is assumed to be smooth there exists an open set  $V \subseteq \mathbb{R}^d$  such that  $x \in V$  and a smooth positively oriented diffeomorphism  $\Phi : V \rightarrow B(0, 1)$  such that

$$\Phi(B^+) = V \cap \bar{U} \text{ and } \Phi(0) = x.$$

Without loss of generality, one can further assume that for each  $y \in \{x \in B(0, 1) : x_n = 0\}$ ,

$$d\Phi(y)(-e_n) = \nu(y),$$

where  $\nu(y) \in \mathbb{R}^d$  denotes the outward unit normal to  $\partial U$  at  $y$ . This extra assumption ensures that for each  $v \in H_d^1\Lambda^{r-1}(U)$ , we have

$$\Phi^*\mathbf{t}v = \mathbf{t}\Phi^*v \text{ on } \{x \in B(0, 1) : x_n = 0\}.$$

In particular, if  $v \in H_d^1\Lambda^{r-1}(U)$  is such that  $\mathbf{t}v = 0$  on  $\partial U \cap V$ , then  $\mathbf{t}\Phi^*v = 0$  on  $\{x \in B(0, 1) : x_n = 0\}$ .

From the previous remark, the equality (A.13), the change of variable formula (2.7) and the properties of the pullback (2.3) one obtains, via a straightforward computation, for each  $v \in H^1\Lambda^{r-1}(B_+)$  such that  $\mathbf{t}v = 0$  on  $\{x \in B(0, 1) : x_n = 0\}$  (the tangential component of  $v$  vanishes on the flat part of  $\partial B_+$ ) and  $v = 0$  on  $\partial B_+ \setminus \{x \in B(0, 1) : x_n = 0\}$  ( $v$  vanishes on the curved part of  $\partial B_+$ ),

$$(A.14) \quad \int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi dv + \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi v = \int_{B^+} g_\Phi \wedge v,$$

where we used the notation.

$$\begin{cases} u_\Phi = \Phi^*u, \\ \bar{\mathbf{a}}_\Phi = \Phi^*\bar{\mathbf{a}}(\Phi^{-1})^*, \\ \star_\Phi = \Phi^*\star(\Phi^{-1})^*, \\ \delta_\Phi = \Phi^*\delta(\Phi^{-1})^*, \\ g_\Phi = \Phi^*g. \end{cases}$$

The goal is then to prove the following  $H^2$  regularity estimate

$$\|u_\Phi\|_{H^2\Lambda^{r-1}(B_{1/2}^+)} \leq C \left( \|g_\Phi\|_{L^2\Lambda^{d-r+1}(B^+)} + \|u_\Phi\|_{L^2\Lambda^{r-1}(B^+)} \right),$$



where  $B_{1/2}^+$  denotes the half-ball of radius  $\frac{1}{2}$

$$B_{1/2}^+ := \left\{ x \in B\left(0, \frac{1}{2}\right) : x_n \geq 0 \right\}.$$

To prove this estimate, let  $\eta \in C_c^\infty(B(0, 1))$  be a smooth cutoff function satisfying

$$\mathbb{1}_{B(0, \frac{1}{2})} \leq \eta \leq \mathbb{1}_{B(0, \frac{3}{4})} \quad \text{and} \quad |\nabla \eta| \leq 5.$$

As in the proof of the interior regularity estimate, Proposition A.2, for  $h > 0$  and  $k \in \{1, \dots, d\}$ , we denote by  $D_k^h$  the difference quotient, i.e.,

$$D_k^h v(x) = \frac{v(x + he_k) - v(x)}{h}.$$

The idea is then to apply formula (A.14) with the function  $v := D_k^{-h} \eta^2 D_k^h u_\Phi$ , for  $k \in \{1, \dots, d-1\}$ . Note that since  $u \in H^1 \Lambda^{r-1}(U)$  and  $\mathbf{t}u = 0$  on  $\partial U$ , we have  $u_\Phi \in H^1 \Lambda^{r-1}(B^+)$  and  $\mathbf{t}u = 0$  on  $\{x \in B(0, 1) : x_n = 0\}$ . This implies that  $v \in H^1 \Lambda^{r-1}(B^+)$  satisfies  $\mathbf{t}v = 0$  on  $\{x \in B(0, 1) : x_n = 0\}$ . Thus applying (A.14) yields

$$(A.15) \quad \underbrace{\int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi dv}_{\text{Term 1}} + \underbrace{\int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi v}_{\text{Term 2}} = \underbrace{\int_{B^+} g_\Phi \wedge v}_{\text{Term 3}}.$$

We then estimate the three terms of the previous equality.

*Estimate for Term 1.* We first compute

$$\begin{aligned} \int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi dv &= \int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi d(D_k^{-h} \eta^2 D_k^h u_\Phi) \\ &= \int_{B^+} \bar{\mathbf{a}}_\Phi du_\Phi \wedge D_k^{-h} d(\eta^2 D_k^h u_\Phi) \\ &= \int_{B^+} D_k^h (\bar{\mathbf{a}}_\Phi du_\Phi) \wedge d(\eta^2 D_k^h u_\Phi) \\ &= \int_{B^+} (\bar{\mathbf{a}}_\Phi(x + he_k) dD_k^h u_\Phi + (D_k^h \bar{\mathbf{a}}_\Phi) du_\Phi) \wedge d(\eta^2 D_k^h u_\Phi). \end{aligned}$$

Then note that by the ellipticity assumption (2.16) and by the definition of  $\bar{\mathbf{a}}_\Phi$ , there exists two constants  $c := c(\lambda, \Phi) > 0$  and  $C := C(\lambda, \Phi) < \infty$  such that

$$(A.16) \quad \begin{aligned} \int_{B^+} \bar{\mathbf{a}}_\Phi(x + he_k) dD_k^h u_\Phi \wedge d(\eta^2 D_k^h u_\Phi) \\ &= \int_{B^+} \bar{\mathbf{a}}_\Phi(x + he_k) dD_k^h u_\Phi \wedge (2\eta d\eta \wedge D_k^h u_\Phi + \eta^2 dD_k^h u_\Phi) \\ &\geq c \|\eta dD_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)}^2 - C \|\eta dD_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)} \|d\eta \wedge D_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)}. \end{aligned}$$

Using the inequality, for each  $\varepsilon > 0$ ,  $ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$  and [11, Lemma 7.23] as in (A.5), the previous display can be rewritten

$$\int_{B^+} \bar{\mathbf{a}}_\Phi(x + he_k) dD_k^h u_\Phi \wedge d(\eta^2 D_k^h u_\Phi) \geq c \|\eta dD_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)}^2 - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}^2.$$

Similarly, since  $\Phi$  is smooth, we have, for some constant  $C := C(d, \lambda, \Phi) < \infty$ ,

$$\begin{aligned} \int_{B^+} ((D_k^h \bar{\mathbf{a}}_\Phi) du_\Phi) \wedge d(\eta^2 D_k^h u_\Phi) \\ &\geq -C \|du_\Phi\|_{L^2 \Lambda^r(B^+)} \|d(\eta^2 D_k^h u_\Phi)\|_{L^2 \Lambda^r(B^+)} \\ &\geq -C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \left( \|\eta^2 dD_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)} + \|d\eta^2 \wedge D_k^h u_\Phi\|_{L^2 \Lambda^r(B^+)} \right). \end{aligned}$$

As in the previous computation, and using that  $\eta \leq 1$ , we obtain, for some  $C := C(d, \lambda, \Phi) < \infty$ ,

$$\int_{B^+} \left( (D_k^h \bar{\mathbf{a}}_\Phi) du_\Phi \right) \wedge d(\eta^2 D_k^h u_\Phi) \geq \frac{c}{2} \left\| \eta d D_k^h u_\Phi \right\|_{L^2 \Lambda^r(B^+)}^2 - C \left\| \nabla u_\Phi \right\|_{L^2 \Lambda^{r-1}(B^+)}^2,$$

where  $c$  is the constant which appears in (A.16). Combining the few previous displays yields

$$\int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi dv \geq c \left\| \eta d D_k^h u_\Phi \right\|_{L^2 \Lambda^r(B^+)}^2 - C \left\| \nabla u_\Phi \right\|_{L^2 \Lambda^{r-1}(B^+)}^2.$$

For a technical reason which will become clear later, we use the identity  $d(\eta d D_k^h u_\Phi) = \eta d(d D_k^h u_\Phi) + d\eta \wedge (D_k^h u_\Phi)$  to further refine

$$(A.17) \quad \int_{B^+} du_\Phi \wedge \bar{\mathbf{a}}_\Phi dv \geq c \left\| d(\eta D_k^h u_\Phi) \right\|_{L^2 \Lambda^r(B^+)}^2 - C \left\| \nabla u_\Phi \right\|_{L^2 \Lambda^{r-1}(B^+)}^2,$$

for some constants  $c := c(d, \lambda, \Phi) > 0$  and  $C := C(d, \lambda, \Phi) < \infty$ .

*Estimate for Term 2.* To estimate the second term, we first need to justify that there exists a constant  $C := C(d, \Phi) < \infty$  such that for each  $w \in H^1 \Lambda^{r-1}(B^+)$  supported in  $B(0, \frac{3}{4})$

$$(A.18) \quad \left| \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi D_k^{-h} w - \int_{B^+} \delta_\Phi D_k^h u_\Phi \wedge \star_\Phi \delta_\Phi w \right| \leq C \left\| \nabla u_\Phi \right\|_{H^1 \Lambda^{r-1}(B^+)} \left\| \nabla w \right\|_{H^1 \Lambda^{r-1}(B^+)}.$$

To prove this assumption, the idea is to expand everything. Consequently we write in coordinates  $u_\Phi = \sum_{|I|=r-1} u_{\Phi,I}(x) dx_I$  and  $w = \sum_{|J|=r-1} w_J(x) dx_J$ . Using this notation, the previous integral can be rewritten

$$\int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi D_k^{-h} w = \int_{B^+} \sum_{I,J,i,j} \frac{\partial u_{\Phi,I}}{\partial x_i}(x) \frac{\partial D_k^h w_J}{\partial x_j}(x) \phi_{I,J}^{i,j}(x) dx + \text{lower order},$$

where the sum is over every subset  $I, J$  of  $\{1, \dots, d\}$  of cardinality  $r-1$  and every  $i, j$  in  $\{1, \dots, d\}$ , and the functions  $\phi_{I,J}^{i,j}$  are smooth and depend only on  $\Phi$ . The "lower order" represents the terms of lower order, with zero or one partial derivative. These terms are easy to estimate and we have

$$(A.19) \quad |\text{lower order}| \leq C \left\| \nabla u_\Phi \right\|_{H^1 \Lambda^{r-1}(B^+)} \left\| \nabla w \right\|_{H^1 \Lambda^{r-1}(B^+)}.$$

We can then compute the discrete integration by parts

$$\begin{aligned} \int_{B^+} \sum_{I,J,i,j} \frac{\partial u_{\Phi,I}}{\partial x_i}(x) \frac{\partial D_k^{-h} w_J}{\partial x_j}(x) \phi_{I,J}^{i,j}(x) dx &= \int_{B^+} \sum_{I,J,i,j} \frac{\partial D_k^h u_{\Phi,I}}{\partial x_i}(x) \frac{\partial w_J}{\partial x_j}(x) \phi_{I,J}^{i,j}(x) dx \\ &+ \int_{B^+} \sum_{I,J,i,j} \frac{\partial u_{\Phi,I}}{\partial x_i}(x + h e_k) \frac{\partial w_J}{\partial x_j}(x) D_k^h \phi_{I,J}^{i,j}(x) dx. \end{aligned}$$

But we can factorize the first term on the right-hand side, this yields

$$\int_{B^+} \sum_{I,J,i,j} \frac{\partial D_k^h u_{\Phi,I}}{\partial x_i}(x) \frac{\partial w_J}{\partial x_j}(x) \phi_{I,J}^{i,j}(x) dx = \int_{B^+} \delta_\Phi D_k^h u_\Phi \wedge \star_\Phi \delta_\Phi w + \text{lower order}.$$

The terms "lower order" are as in (A.19) and can be estimated in a similar way. Since the functions  $\phi_{I,J}^{i,j}$  are smooth and depend only on  $\Phi$ , we can estimate the second term on the right-hand side crudely by, for some  $C := C(d, \Phi) < \infty$

$$\left| \int_{B^+} \sum_{I,J,i,j} \frac{\partial u_{\Phi,I}}{\partial x_i}(x + h e_k) \frac{\partial w_J}{\partial x_j}(x) D_k^h \phi_{I,J}^{i,j}(x) dx \right| \leq C \left\| u_\Phi \right\|_{H^1 \Lambda^{r-1}(B^+)} \left\| w \right\|_{H^1 \Lambda^{r-1}(B^+)}.$$

Combining the few previous displays shows (A.18). We then apply this estimate with  $w = \eta^2 D_k^h u_\Phi$ , this yields

$$\begin{aligned} \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi v &= \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi (D_k^{-h} \eta^2 D_k^h u_\Phi) \\ &\geq \int_{B^+} \delta_\Phi D_k^h u_\Phi \wedge \star_\Phi \delta_\Phi (\eta^2 D_k^h u_\Phi) - C \left\| u_\Phi \right\|_{H^1 \Lambda^{r-1}(B^+)} \left\| (\eta^2 D_k^h u_\Phi) \right\|_{H^1 \Lambda^{r-1}(B^+)}. \end{aligned}$$

Expanding the term  $\delta_\Phi(\eta^2 D_k^h u_\Phi)$  and using the same ideas as in the estimate of Term 1, we obtain

$$\int_{B^+} \delta_\Phi(D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta^2 D_k^h u_\Phi) \geq c \int_{B^+} \eta^2 \delta_\Phi(D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(D_k^h u_\Phi) - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}^2$$

One can put the  $\eta$  inside the derivative and further refine

$$\int_{B^+} \delta_\Phi(D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta^2 D_k^h u_\Phi) \geq c \int_{B^+} \delta_\Phi(\eta D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta D_k^h u_\Phi) - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}^2.$$

Combining the few previous displays shows

$$(A.20) \quad \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi v \geq c \int_{B^+} \delta_\Phi(\eta D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta D_k^h u_\Phi) - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}^2 \\ - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \|\nabla(\eta^2 D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)}.$$

Using the properties of  $\eta$ , one can show

$$\begin{aligned} \|\nabla(\eta^2 D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} &\leq C \|\eta^2 \nabla(D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} + C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \\ &\leq C \|\eta \nabla(D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} + C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \\ &\leq C \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} + C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}. \end{aligned}$$

This yields the following refinement of (A.20),

$$(A.21) \quad \int_{B^+} \delta_\Phi u_\Phi \wedge \star_\Phi \delta_\Phi v \geq c \int_{B^+} \delta_\Phi(\eta D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta D_k^h u_\Phi) - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)}^2 \\ - C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)}.$$

*Estimate for Term 3.* This term is the simpler to estimate, we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \int_{B^+} g_\Phi \wedge v &\leq \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)} \|v\|_{L^2 \Lambda^{r-1}(B^+)} \\ &\leq \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)} \|D_k^{-h} \eta^2 D_k^h u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \\ &\leq C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)} \|\nabla(\eta^2 D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)}. \end{aligned}$$

Using the same argument as in the previous step, we replace  $\eta^2$  by  $\eta$ ,

$$(A.22) \quad \int_{B^+} g_\Phi \wedge v \leq C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)} \left( \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} + \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B^+)} \right).$$

We then show one last estimate on the small but positive terms on the right-hand side of (A.17) and (A.21). Indeed by the Gaffney-Friedrich inequality applied to the form  $(\Phi^{-1})^*(\eta D_k^h u_\Phi)$  on the smooth domain  $U \cap \partial V$ , we have, for some constant  $C := C(d, V \cap U) < \infty$

$$\|\nabla(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)} \leq C \left( \|d(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^r(V)} + \|\delta(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-2}(V)} \right).$$

Since  $\Phi$  is assumed to be a smooth diffeomorphism, we have, for some  $C := C(d, \Phi) < \infty$ ,

$$\begin{aligned} \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} &\leq C \|\nabla(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)} + C \|(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)} \\ &\leq C \|\nabla(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)} + C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)}. \end{aligned}$$

Similarly, we have

$$\|d(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} \geq C \|(\Phi^{-1})^* d(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)} = C \|d(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(V)}.$$

And by the change of variable formula, we have

$$\|\delta(\Phi^{-1})^*(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-2}(V)}^2 = \int_{B^+} \delta_\Phi(\eta D_k^h u_\Phi) \wedge \star_\Phi \delta_\Phi(\eta D_k^h u_\Phi).$$

Combining the few previous displays with the estimates for the first term (A.17), the second term (A.21) and the third term (A.22), we eventually obtain, for some  $C := C(d, \lambda, \Phi) < \infty$

$$\begin{aligned} \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)}^2 &\leq C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)}^2 + C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)}^2 \\ &\quad + C \|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} \left( \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)} + \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)} \right). \end{aligned}$$

This implies

$$\|\nabla(\eta D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B^+)} \leq C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)} + C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)},$$

and in particular

$$\|\nabla(D_k^h u_\Phi)\|_{L^2 \Lambda^{r-1}(B_{1/2}^+)} \leq C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)} + C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)}.$$

The previous inequality is true for every  $h > 0$  small enough. Thus, by [11, Lemma 7.24], for each  $k \in \{1, \dots, d-1\}$ ,  $\partial_k \nabla u$  belongs to  $L^2 \Lambda^{r-1}(V)$  and

$$\|\partial_k \nabla u_\Phi\|_{L^2 \Lambda^{r-1}(B_{1/2}^+)} \leq C \|\nabla u_\Phi\|_{L^2 \Lambda^{r-1}(V)} + C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)}.$$

for some constant  $C := C(d, \lambda, \Phi) < \infty$ . This completes the proof of Step 3.

*Step 4.* From the result of Step 3, to prove that  $u_\Phi \in H^2 \Lambda^{r-1}(B_{1/2}^+)$ , there only remains to prove that  $\partial_d \partial_d u_\Phi$  belongs to  $L^2 \Lambda^{r-1}(B_{1/2}^+)$ . This is what is proved in this step, along with the estimate

$$(A.23) \quad \|\partial_d \partial_d u_\Phi\|_{L^2 \Lambda^{r-1}(B_{1/2}^+)} \leq C \|u_\Phi\|_{H^1 \Lambda^{r-1}(V)} + C \|g_\Phi\|_{L^2 \Lambda^{d-r+1}(B^+)},$$

for some constant  $C := C(d, \lambda, \Phi) < \infty$ . By (A.14), the function  $u_\Phi$  is a solution of the following equation

$$\Phi^*(d\bar{\mathbf{a}}d + \delta \star \delta)(\Phi^{-1})^* u_\Phi = g_\Phi \text{ in } B^+.$$

This second order differential operator can be written in the form

$$\Phi^*(d\bar{\mathbf{a}}d + \delta \star \delta)(\Phi^{-1})^* u = \sum_{j,k=1}^d A_{j,k} \partial_j \partial_k u + \sum_{j=1}^d A_j \partial_j u + Au,$$

where the coefficients  $A_{j,k}, A_j$  and  $A$  are smooth functions from  $B^+$  to the space of matrices of size  $\binom{d}{r-1} \times \binom{d}{r-1}$  (or equivalently the space of endomorphisms of  $\Lambda^r(\mathbb{R}^d)$ ). The idea to prove (A.23) is to show that this operator is strongly elliptic, i.e.,

$$(A.24) \quad \sum_{j,k=1}^d (\eta^\top A_{j,k}(x) \eta) \xi_j \xi_k \geq c |\eta|^2 |\xi|^2 \quad \forall x \in U, \forall \eta \in \mathbb{R}^{\binom{d}{r-1}}, \forall \xi \in \mathbb{R}^d.$$

To prove the strong ellipticity, it is enough, by [22, Theorem 4.6], to prove that for each  $w \in C_c^\infty \Lambda^{r-1}(B^+)$  that

$$(A.25) \quad \int_{B^+} \Phi^*(d\bar{\mathbf{a}}d + \delta \star \delta)(\Phi^{-1})^* w \wedge w \geq c \|w\|_{H^1 \Lambda^{r-1}(B^+)} - C \|w\|_{L^2 \Lambda^{r-1}(B^+)}.$$

This is a consequence of the following computation

$$\begin{aligned} \int_{B^+} \Phi^*(d\bar{\mathbf{a}}d + \delta \star \delta)(\Phi^{-1})^* w \wedge w &= \int_V (d\bar{\mathbf{a}}d + \delta \star \delta)(\Phi^{-1})^* w \wedge (\Phi^{-1}) w \\ &= \int_V \bar{\mathbf{a}}d(\Phi^{-1})^* w \wedge d(\Phi^{-1}) w + \int_V \delta(\Phi^{-1})^* w \wedge \delta(\Phi^{-1}) w \\ &\geq \lambda \left\| d(\Phi^{-1})^* w \right\|_{L^2 \Lambda^r(V)} + \left\| \delta(\Phi^{-1})^* w \right\|_{L^2 \Lambda^{r-2}(V)} \end{aligned}$$

Since  $w \in C_c^\infty \Lambda^{r-1}(B^+)$ , we also have  $(\Phi^{-1})^* w \in C_c^\infty \Lambda^{r-1}(V)$  and thus by the Gaffney-Friedrich inequality,

$$\left\| d(\Phi^{-1})^* w \right\|_{L^2 \Lambda^r(V)} + \left\| \delta(\Phi^{-1})^* w \right\|_{L^2 \Lambda^{r-2}(V)} \geq c \left\| (\Phi^{-1})^* w \right\|_{H^1 \Lambda^r(V)} - C \left\| (\Phi^{-1})^* w \right\|_{L^2 \Lambda^r(V)}.$$

We then note that

$$\|w\|_{H^1\Lambda^r(V)} \leq C \left\| (\Phi^{-1})^* w \right\|_{H^1\Lambda^r(B^+)}$$

and

$$\left\| (\Phi^{-1})^* w \right\|_{L^2\Lambda^r(B^+)} \leq C \|w\|_{L^2\Lambda^r(V)}$$

for some constant  $C := C(d, \Phi) < \infty$ . This implies (A.25). Now that we know that the operator is strongly elliptic, we have, by [22, Lemma 4.17], that the coefficient  $A_{n,n}$  has a uniformly bounded inverse. As a consequence, we have

$$\begin{aligned} \|\partial_d \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} &\leq \|A_{d,d} \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} \\ &\leq \sum_{j=1}^d \sum_{k=1}^{d-1} \|A_{j,k} \partial_j \partial_k u\|_{L^2\Lambda^r(B^+)} + \sum_{j=1}^d \|A_j \partial_j u\|_{L^2\Lambda^r(B^+)} + \|Au\|_{L^2\Lambda^r(B^+)}. \end{aligned}$$

Using the main result of Step 3, this gives, for some constant  $C := C(d, \lambda, \Phi) < \infty$ ,

$$\|\partial_d \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} \leq C \|u_\Phi\|_{H^1\Lambda^{r-1}(V)} + C \|g_\Phi\|_{L^2\Lambda^{d-r+1}(B^+)},$$

and the proof of Step 4 is complete.

*Step 5.* The main results of Steps 3 and 4 show that the function  $u_\Phi$  belongs to  $H^2\Lambda^{r-1}(B^+)$  and we have the estimate

$$\|u_\Phi\|_{H^2\Lambda^r(B^+)} \leq C \|u_\Phi\|_{H^1\Lambda^{r-1}(B^+)} + C \|g_\Phi\|_{L^2\Lambda^{d-r+1}(B^+)},$$

with  $C := C(d, \lambda, \Phi) < \infty$ . This implies

$$(A.26) \quad \|u\|_{H^2\Lambda^r(V \cap U)} \leq C \|u\|_{H^1\Lambda^{r-1}(V \cap U)} + C \|g\|_{L^2\Lambda^{d-r+1}(V \cap U)}.$$

Since  $\partial U$  is compact, we can cover  $\partial U$  with finitely many sets  $V_1, \dots, V_N$  as above. We sum the resulting estimates, along with the interior estimate from Proposition A.2 applied to the function  $f + u$ , and obtain  $u \in H^2\Lambda^r(U)$  with the estimate

$$\|du\|_{H^1\Lambda^r(U)} \leq C \|u\|_{H^1\Lambda^{r-1}(U)} + C \|g\|_{L^2\Lambda^{r-1}(U)},$$

for some  $C := C(d, \lambda, U) < \infty$ . We then simplify a little bit the right-hand side. Since we assumed  $u \in \mathcal{H}_D^{r-1}(U)^\perp$ , we have, by the Gaffney-Friedrich inequality, Proposition 4.3,

$$\|\nabla u\|_{L^2\Lambda^{r-1}(U)} \leq \|du\|_{L^2\Lambda^r(U)} + \|\delta u\|_{L^2\Lambda^{r-2}(U)}.$$

This inequality can be further refined, thanks to [27, Proposition 2.2.3] into

$$\|u\|_{H^1\Lambda^{r-1}(U)} \leq \|du\|_{L^2\Lambda^r(U)} + \|\delta u\|_{L^2\Lambda^{r-2}(U)}.$$

By (A.13) and the ellipticity assumption (2.16), we have

$$\|du\|_{L^2\Lambda^r(U)}^2 + \|\delta u\|_{L^2\Lambda^{r-2}(U)}^2 \leq C \|g\|_{L^2\Lambda^r(U)} \|u\|_{L^2\Lambda^{d-r+1}(U)}.$$

Combining the two previous displays with (A.26) shows

$$\|u\|_{H^2\Lambda^{r-1}(U)} \leq C \|g\|_{L^2\Lambda^{r-1}(U)}.$$

and the proof of Step 5 is complete.

*Step 6.* Applying the result of Step 5 with the specific function  $g = d\bar{\mathbf{a}}df$ , gives

$$\|u\|_{H^2\Lambda^{r-1}(U)} \leq C \|d\bar{\mathbf{a}}df\|_{L^2\Lambda^{r-1}(U)},$$

and consequently

$$\|du\|_{H^1\Lambda^{r-1}(U)} \leq C \|df\|_{H^1\Lambda^r(U)}.$$

Since two solutions of (A.9) differ by a form of  $C_{d,0}^{r-1}$ , they have the same exterior derivative. From this remark and the previous estimate, we obtain (A.10).  $\square$

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