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Nicolas Seguin, Magali Tournus

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Abstract

In this paper, we design and analyze a numerical scheme which approximates a Jin–Xin linear system with implicit equilibrium on a bounded domain. This scheme relaxes toward the asymptotic limit of the linear system. The main properties of the limiting scheme are that it does no require to invert the implicit function defining the manifold, and that it provides an accurate discretization of the boundary conditions.

Key-words: Asymptotic Preserving scheme, Hyperbolic Relaxation, Boundary layer.
Subject Classifications: 65N08, 65N12, 35L10, 35L65.

1 Introduction

The Jin–Xin model, introduced in [11], is a 2×2 linear hyperbolic system with a nonlinear dissipative source term which writes

$$\begin{cases} \partial_t s_\varepsilon + \partial_x w_\varepsilon = 0, \\ \partial_t w_\varepsilon + \partial_x s_\varepsilon = \dfrac{1}{\varepsilon}(f(s_\varepsilon) - w_\varepsilon). \end{cases}$$

When the source term becomes infinitely sharp, i.e. when $\varepsilon \to 0$, the conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0.$$

is obtained, under the subcharacteristic condition $|f'| \leqslant 1$. Then, the system can be viewed as a dissipative approximation of entropy weak solutions of conservation laws. A large literature is dedicated to this convergence, see for instance [14], [16], [1], [18]... In its usual formulation, the equilibrium manifold is explicit, and is given by $\{w = f(s)\}$.

In [19, 20], a model is introduced and analyzed for the evolution of the concentration of chemical species dissolved in a fluid moving along the loop of Henle in the human kidney. It corresponds to a countercurrent exchanger, i.e. a U-shaped circuit, made of two parallel tubes in which a fluid is flowing in opposite directions, connected at one of their ends. In the first tube, fluid moves with positive velocity 1 and has a concentration denoted by $u_\varepsilon(x,t)$, whereas in the other tube, fluid moves with negative velocity -1 and has a concentration denoted by $v_\varepsilon(x,t)$. The positive constant ε is the characteristic time associated with the chemical exchanges between the two tubes through

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the medullary interstitial region. A nonlinear function h encodes the dynamics of the exchange. The governing equations finally are

$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \dfrac{1}{\varepsilon}(h(v_\varepsilon) - u_\varepsilon), \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \dfrac{1}{\varepsilon}(u_\varepsilon - h(v_\varepsilon)). \end{cases} \tag{1}$$

This system also admits the alternative form, by defining $s_\varepsilon = u_\varepsilon + v_\varepsilon$ and $w_\varepsilon = u_\varepsilon - v_\varepsilon$,

$$\begin{cases} \partial_t s_\varepsilon + \partial_x w_\varepsilon = 0, \\ \partial_t w_\varepsilon + \partial_x s_\varepsilon = \dfrac{2}{\varepsilon}\left(\dfrac{h(s_\varepsilon - w_\varepsilon)}{2} - \dfrac{(s_\varepsilon + w_\varepsilon)}{2}\right). \end{cases} \tag{2}$$

Formally, this system converges when $\varepsilon \to 0$ towards the conservation law

$$\partial_t(h(v) + v) + \partial_x(h(v) - v) = 0. \tag{3}$$

This implicit equation is well-posed as soon as the flux $h(v)-v$ can be uniquely defined as a function of the unknown $h(v)+v$, following the Kruzhkov's theory [13]. This will be the case in our study, see assumptions below. However, from the numerical point of view, it is not straightforward to obtain a convergent and conservative numerical scheme for equation (3) starting with a classical scheme for (1) and letting ε go to 0, without inverting the flux $h(v) - v$ (with respect to v or $h(v) + v$). This will be the first goal of our study.

As mentioned above, the countercurrent exchanger model is completed by specific boundary conditions. Denoting the domain by $[0, L]$, with $L > 0$, the initial-boundary value problem (IBVP) writes

$$\begin{cases} \partial_t u_\varepsilon + \partial_x u_\varepsilon = \dfrac{1}{\varepsilon}(h(v_\varepsilon) - u_\varepsilon), & t > 0,\ x \in [0, L], \\ \partial_t v_\varepsilon - \partial_x v_\varepsilon = \dfrac{1}{\varepsilon}(u_\varepsilon - h(v_\varepsilon)), & t > 0,\ x \in [0, L], \\ u_\varepsilon(0, t) = u_b, \quad v_\varepsilon(L, t) = \alpha u_\varepsilon(L, t), & t > 0, \\ (u_\varepsilon, v_\varepsilon)(x, 0) = (u^0, v^0)(x), & x \in [0, L], \end{cases} \tag{S_ε}$$

where the reflection capacity α is assumed to be in $(0, 1)$, $u_b \in \mathbb{R}$ and initial conditions (u^0, v^0) are of bounded variations

$$u^0 \in BV([0, L]), \qquad v^0 \in BV([0, L]). \tag{4}$$

Some results of this paper are stated under the additional technical assumption that the initial conditions are at equilibrium

$$u^0(x) = h(v^0(x)), \qquad x \in [0, L]. \tag{5}$$

The IBVP for the Jin–Xin model has been studied by several authors, see [1], [5], [22], [23]... More specifically, the well-posedness and the asymptotic analysis of the IBVP (S_ε) are given in [17].

In order to understand the IBVP when $\varepsilon \to 0$, let us provide the assumptions on function h: there exists two positive constant $\beta \leqslant \mu$ such that

$$1 < \beta \leqslant h'(v) \leqslant \mu, \quad \text{and} \quad h(0) = 0. \tag{6}$$

As a consequence, the function

$$f : h(v) + v \mapsto h(v) - v \tag{7}$$

is increasing. Following the classical theory of IBVP for conservation laws provided in [2], only the boundary condition at $x = 0$ persists, and the limit IBVP is thus

$$\begin{cases} \partial_t(h(v) + v) + \partial_x(h(v) - v) = 0, & t > 0,\ x \in [0, L], \\ h(v(0, t)) = u_b, & t > 0, \\ (h(v) + v)(x, 0) = (h(v^0) + v^0)(x), & x \in [0, L]. \end{cases} \tag{S_0}$$

The convergence of solutions of (S_ε) to solutions of (S_0) is provided in [17]. We complement in the present paper the analysis of [17] showing the existence of a relaxation boundary layer at $x = L$ if the intersection between the equilibrium manifold

$$M_{eq} = \{(u, v) \in \mathbb{R}^2 \mid u = h(v)\}$$

and the boundary manifold

$$M_b = \{(u, v) \in \mathbb{R}^2 \mid v = \alpha u\}$$

is empty. In this paper, the goal is to obtain and analyze a numerical scheme which fits with the limit (S_0) and which is a accurate discretization of the boundary conditions of (S_0), using three-point schemes. At $x = 0$, the approximation of the boundary condition is classical, using a ghost cell and imposing inside the Dirichlet value. At $x = L$, in order to avoid any numerical boundary layer, the easiest way is to obtain the upwind scheme when $\varepsilon \to 0$, which does not depends on any ghost cell since $f' > 0$. We also show by numerical tests that this numerical treatment also provides an accurate approximation of the relaxation boundary layer on coarse mesh.

Let us sum up the requirements we presented on the approximation of the IBVP (S_ε):

1. Provide a first-order approximation of the relaxation system (S_ε) without any nonlinear inversion of the function h.

2. When $\varepsilon \to 0$, obtain a first-order approximation of the IBVP (S_0),

   - without the use of any nonlinear inversion of the function f defined by (7),
   - with an upwind discretisation of the flux f.

An asymptotic preserving scheme $(S_{\varepsilon,\Delta})$ is usually defined as a convergent scheme for the system (S_ε), which tends to become a convergent scheme (S_Δ) for the limiting equation as ε goes to zero. In other words, an asymptotic preserving scheme is a scheme $(S_{\varepsilon,\Delta})$ such that the following diagram is commutative.

$$\begin{tikzcd}
(S_\varepsilon) \arrow[r, "\boxed{\mathbf{2}}\ \varepsilon \to 0"'] & (S_0) \\
(S_{\varepsilon,\Delta}) \arrow[u, "\boxed{\mathbf{1}}\ \Delta \to 0"] \arrow[r, "\boxed{\mathbf{3}}\ \varepsilon \to 0"'] & (S_\Delta) \arrow[u, "\boxed{\mathbf{4}}\ \Delta \to 0"']
\end{tikzcd}$$

In the specific context of time-explicit numerical schemes, a necessary condition is that the CFL condition for $(S_{\varepsilon,\Delta})$ is uniform in ε.

A first idea to build an asymptotic preserving scheme is to use a splitting method [6]. The scheme (S_Δ) we obtain at the limit is highly diffusive, and generates a numerical boundary layer at $x = L$. The alternative method we use in the present paper is based on the use of well-balanced schemes, introduced by [8] and developed in [7] for the sake of asympptotic preserving scheme. The main idea is to cleverly approximate the source term in order to end up with the wanted discretized version of the flux at the limit. In our context, let us point out that

$$\text{the term } \frac{\Delta t}{\varepsilon + \Delta x} \quad \text{behaves like} \quad \begin{cases} \dfrac{\Delta t}{\varepsilon} \; \textit{as } \Delta x \text{ goes to zero,} \\ \\ \dfrac{\Delta t}{\Delta x} \text{ as } \varepsilon \text{ goes to zero.} \end{cases} \tag{8}$$

This work fits into the more general problem of building AP schemes with constraint on the limiting scheme (S). After the pionneering work [12], the authors of [3] developed a somewhat generic method to make optional the choice of the numerical scheme in the asymptotic regime $\varepsilon = 0$. Properties of stability and convergence are automatically given by the construction provided in [3], as the scheme they obtain at the limit can be seen as a convex combination of well-known schemes. Our specific problem cannot be directly solved using their method since it provides us with a scheme that requires to invert h. Since the scheme is built by hand and does not correspond to any classical scheme at the limit, we are left with analyzing its basic properties by hand as well.

The oultline of the paper is the following. In Section 2, we provide the definitions of the solutions of the IBVP's (S_ε) and (S_0) and the associated well-posedness and asymptotic results. We also describe the relaxation boundary layer which appears as soon as $M_{eq} \cap M_b = \emptyset$. In Section 3, we design a numerical scheme which fulfills all the above-mentioned requirements, and state the main results of convergence, showing the asymptotic compatibility of the approach (the so-called *asymptotic preserving* property [10]). Sections 4 and 5 are dedicated to the proofs of convergence of the scheme, respectively when $\varepsilon > 0$ and when $\varepsilon = 0$. The last section contains numerical results including comparison with the classical splitting method.

2 Well-posedness, zero-relaxation limit, and relaxation boundary layer

In all the following, we assume that assumption (6) is fulfilled, so that function f defined by (7) exists and is increasing.

2.1 Definitions and existing results

Let us provide the definition of weak solutions of the relaxation IBVP (S_ε), regardless of their smoothness.

Definition 1. *Consider any initial data* (u^0, v^0) *satisfying (4), and* $\varepsilon > 0$*. A* weak solution of the IBVP (S_ε) *is a couple of functions* $(u_\varepsilon, v_\varepsilon) \in C\left((0,T); \mathbf{L}^1[0,L]\right) \cap \mathbf{L}^\infty\left([0,T]; BV[0,L]\right)$ *such that for all* $(\Phi, \Psi) \in C^1([0,T] \times [0,L])^2$ *satisfying* $\Phi(x,T) = \Psi(x,T) = 0$ *and* $\Psi(0,t) = 0$*, the following* *equality holds*

$$\int_0^T\int_0^L \big[u_\varepsilon\partial_t\Phi + v_\varepsilon\partial_t\Psi + u_\varepsilon\partial_x\Phi - v_\varepsilon\partial_x\Psi\big]dx\ dt = -\frac{1}{\varepsilon}\int_0^T\int_0^L (h(v_\varepsilon)-u_\varepsilon)(\Phi-\Psi)\,dx\ dt$$
$$-\int_0^T u_b\Phi(0,t)\ dt - \int_0^T u_\varepsilon(L,t)\big[\alpha\Psi(L,t)-\Phi(L,t)\big]\ dt - \int_0^L \big[\Phi(x,0)u^0(x)+\Psi(x,0)v^0(x)\big]dx. \tag{9}$$

The first result is the well-posedness of the relaxation IBVP (S_ε).

Theorem 1 (Well-posedness of the relaxation IBVP [19])**.** *Under assumption (6), there is a unique weak solution to the relaxation IBVP (S_ε), in the sense of Definition 1.*

Now, let us define the entropy weak solutions of the zero-relaxation limit, following the theories provided in [13] and [2].

Definition 2. *Consider u^0, v^0 satisfying (4) and (5) and $v_b \in \mathbb{R}$. An* entropy weak solution *to (S_0) is a function $v \in C\left((0,T);\mathbf{L}^1[0,L]\right) \cap \mathbf{L}^\infty([0,T];BV[0,L])$ such that*

1. *for all non negative $\Phi \in C^1([0,T)\times(0,L))$, and for all $k\in\mathbb{R}$,*

$$\int_0^T\int_0^L \big[|h(v)+v-(h(k)+k)|\partial_t\Phi + |h(v)-v-(h(k)-k)|\partial_x\Phi\big]\ dx\ dt$$
$$+\int_0^L |h(v^0(x))+v^0(x)-(h(k)+k)|\Phi(x,0)\ dx \geqslant 0, \tag{10}$$

2. *for all k in the interval $I(v(0,t),u_b)$*

$$\text{sign}\Big(h(v(0,t))+v(0,t)-(h(u_b)+u_b)\Big)\Big(h(v(0,t))-v(0,t)-(h(k)-k)\Big) \leqslant 0. \tag{11}$$

Existence and uniqueness of such entropy solution follows from the theory developed in [2]. We state the following result of convergence partially proved in [17].

Theorem 2 (Convergence [17])**.** *We assume (4), (5) and (6). Consider a family of solutions $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ to the relaxation IBVP (S_ε). Then there exists a function v which is an entropy weak solution to (S_0) in the sense of Definition 2 such that*

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} h(v), \qquad v_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v, \qquad L^1([0,L]\times[0,T]).$$

The outline of the proof is as follows. First, a dissipative formulation for (S_ε) is obtained. Combined with $\mathbf{L}^\infty$ estimates for u_ε and v_ε, this proves that $(u_\varepsilon - h(v_\varepsilon))$ goes to zero in $\mathbf{L}^1([0,T]\times[0,L])$ (see [9]). The dissipative formulation also implies that the weak solution $(u_\varepsilon, v_\varepsilon)$ to the linear system (S_ε) satisfies the following entropy formulation: for all non negative $\Phi\in C^1([0,T]\times[0,L])$ such that $\Phi(.,T)=0$, and for all $k\in\mathbb{R}$,

$$\int_0^T\int_0^L \big[(|u_\varepsilon-h(k)|+|v_\varepsilon-k|)\partial_t\Phi + (|u_\varepsilon-h(k)|-|v_\varepsilon-k|)\partial_x\Phi\big]\ dx\ dt$$
$$+\frac{2}{\varepsilon}\int_0^T\int_0^L |u_\varepsilon-h(v_\varepsilon)|\Phi(x,t)\ dx\ dt + \int_0^L \big[|u^0(x)-h(k)|+|v^0(x)-k|\big]\Phi(x,0)\ dx$$
$$+\int_0^T \big[|u_b-h(k)|-|v_\varepsilon(0,t)-k|\big]\Phi(0,t)dt - \int_0^T \big[|u_\varepsilon(L,t)-h(k)|-|\alpha u_\varepsilon(L,t)-k|\big]\Phi(L,t)dt$$
$$\geqslant 0. \tag{12}$$

The non-linear formulation (12) is then passed to the limit: non linear-quantities $|u_\varepsilon - h(k)|$ and $|v_\varepsilon - k|$ converges towards $|h(v) - h(k)|$ and $|v - k|$ for some $v \in \mathbf{L}^\infty([0,T] \times [0,L])$ using BV estimates obtained in [17]. By considering test functions Φ satisfying $\Phi(0,t) = \Phi(L,t) = 0$, and letting ε go to zero in (12), we obtain that v satisfies the first item of Definition 2. Then, we use the same method as in [15], i.e. we consider for any $g \in C^1([0,T])$ sequences of test functions Φ_m such that $\partial_x \Phi_m(x,t)$ converges toward $g(t)\delta(x=0)$ and we let m go to infinity, which proves that v satisfies the second item of Definition 2.

2.2 Study of the relaxation boundary layer

This section is devoted to the existence of the boundary layer in the framework of continuous solutions. We first state that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) is uniformly bounded from above and below.

Proposition 1 (Uniform $\mathbf{L}^\infty$ bounds)**.** *We assume (4) and (6). Then, there exists $u_{min}, v_{min}, u_{max}$ and v_{max} which depend on $u_b, u^0, v^0, \alpha, \beta$ and μ such that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) satisfies the following estimates*

$$0 < u_{min} \leqslant u_\varepsilon(t,x) \leqslant u_{max}, \quad 0 < v_{min} \leqslant v_\varepsilon(t,x) \leqslant v_{max}, \qquad a.e.(x,t) \in [0,L] \times [0,T].$$

We prove here Proposition 1. For $U_b > 0$, we introduce the stationary system

$$\begin{cases} \dfrac{dU_\varepsilon}{dx}(x) = \dfrac{1}{\varepsilon}\Big[h(V_\varepsilon(x)) - U_\varepsilon(x)\Big], \\ -\dfrac{dV_\varepsilon}{dx}(x) = \dfrac{1}{\varepsilon}\Big[U_\varepsilon(x) - h(V_\varepsilon(x))\Big], \\ \\ \quad U_\varepsilon(0) = U_b, \qquad V_\varepsilon(L) = \alpha U_\varepsilon(L). \end{cases} \tag{13}$$

It was proved in [19] that (13) admits a unique solution and that this solution is continuously differentiable and non-negative. The proof of Proposition 1 is based on the following Lemma.

Lemma 1. *We assume (6). Then, there are four scalar numbers $u_{min}, u_{max}, v_{min}$ and v_{max} depending on U_b, α, β, μ such that the solution $(U_\varepsilon, V_\varepsilon)$ to (13) satisfies*

$$0 < u_{min} \leqslant U_\varepsilon(x) \leqslant u_{max}, \qquad 0 < v_{min} \leqslant V_\varepsilon(x) \leqslant v_{max}, \qquad x \in [0,L],\ \varepsilon > 0. \tag{14}$$

Let us prove Lemma 1.

Proof. **First Step. A bound from below for $U_\varepsilon - V_\varepsilon$.**

We add the two lines of (13) and obtain $\dfrac{d}{dx}(U_\varepsilon - V_\varepsilon)(x) = 0$ which implies that $U_\varepsilon(x) - V_\varepsilon(x)$ does not depend on x. Then, since $U_\varepsilon(0) - V_\varepsilon(0) = U_b - V_\varepsilon(0) \leqslant U_b$ and $U_\varepsilon(L) - V_\varepsilon(L) = (1-\alpha)U_\varepsilon(L) \geqslant 0$, we have

$$0 \leqslant U_\varepsilon - V_\varepsilon \leqslant U_b. \tag{15}$$

We now here that $U_\varepsilon - V_\varepsilon$ is uniformly bounded from below by some $k_{min} > 0$. Let us assume by contradiction that $\forall \varepsilon_0 > 0$, $\forall \delta > 0$, $\exists \varepsilon < \varepsilon_0$ such that $U_\varepsilon - V_\varepsilon < \delta$. We pick $0 < \delta < \min\left\{\dfrac{\beta - 1}{\mu} U_b,\ (1-\alpha)U_b\right\}$ and $\varepsilon_0 > 0$. Consider $\varepsilon < \varepsilon_0$ such that $U_\varepsilon - V_\varepsilon < \delta$. Then, for $\varepsilon < \varepsilon_0$, we have

$$(1-\alpha)U_\varepsilon(L) = U_\varepsilon(L) - V_\varepsilon(L) < \delta.$$

We also have

$$\begin{aligned}
\frac{dU_\varepsilon}{dx}(0) = \frac{1}{\varepsilon}\left\{h\left(V_\varepsilon(0)\right) - U_\varepsilon(0)\right\} &\geqslant \frac{1}{\varepsilon}\left\{h\left(V_\varepsilon(0)\right) - V_\varepsilon(0) - \delta\right\} \qquad \text{using } U_\varepsilon(0) - V_\varepsilon(0) < \delta \\
&\geqslant \frac{1}{\varepsilon}\left\{(\beta-1)V_\varepsilon(0) - \delta\right\}, \qquad \text{using (6)} \\
&\geqslant \frac{1}{\varepsilon}\left\{(\beta-1)(U_b - \delta) - \delta\right\} \geqslant 0 \qquad \text{since } \delta < \frac{\beta-1}{\beta}U_b.
\end{aligned}$$

The function U_ε is continuous, $U_\varepsilon(L) < \dfrac{\delta}{1-\alpha} \leqslant U_b$, $U_\varepsilon(0) = U_b$ and $\dfrac{dU_\varepsilon}{dx}(0) > 0$, then, there exists $x_\varepsilon \in (0, L)$ such that $U_\varepsilon(x_\varepsilon) = \max_{x\in[0,L]}\{U_\varepsilon(x)\}$. Then $\dfrac{dU_\varepsilon}{dx}(x_\varepsilon) = 0$, and the first line of (13) implies $h\left(V_\varepsilon(x_\varepsilon)\right) = U_\varepsilon(x_\varepsilon)$. Then we have

$$U_\varepsilon(x_\varepsilon) - V_\varepsilon(x_\varepsilon) = h\left(V_\varepsilon(x_\varepsilon)\right) - V_\varepsilon(x_\varepsilon) \geqslant (\beta-1)V_\varepsilon(x_\varepsilon) = (\beta-1)h^{-1}(U_\varepsilon(x_\varepsilon)) \geqslant \frac{\beta-1}{\mu}U_b,$$

which contradicts $U_\varepsilon(x_\varepsilon) - V_\varepsilon(x_\varepsilon) < \delta$ for $\delta < \dfrac{\beta-1}{\mu}U_b$. Then, by contradiction, there is $k_{min} > 0$ such that

$$k_{min} < U_\varepsilon - V_\varepsilon, \qquad \varepsilon > 0. \tag{16}$$

We denote $K_\varepsilon := U_\varepsilon - V_\varepsilon$.

Second Step. Uniform bounds for U_ε and V_ε.

Existence of u_{max}. We have $U_\varepsilon(0) = U_b$, and from (15) we deduce that $U_\varepsilon(L) \leqslant \dfrac{U_b}{1-\alpha}$. If we assume that U_ε reaches its maximal value at $x_\varepsilon \in (0, L)$, then,

$$0 = \frac{dU_\varepsilon}{dx}(x_\varepsilon) = \frac{1}{\varepsilon}\Big(h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) - U_\varepsilon(x_\varepsilon)\Big),$$

and thus, using (6) and $K_\varepsilon \leqslant U_b$,

$$U_\varepsilon(x_\varepsilon) = h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) \geqslant \beta\left(U_\varepsilon(x_\varepsilon) - K_\varepsilon\right) \geqslant \beta U_\varepsilon(x_\varepsilon) - \beta U_b,$$

which is $U_\varepsilon(x_\varepsilon) \leqslant \dfrac{\beta}{\beta-1}U_b$. Then in any case we can set $u_{max} = \max\left\{\dfrac{1}{1-\alpha},\ \dfrac{\beta}{\beta-1}\right\}U_b$.

Existence of u_{min}. Using (16) we have $U_\varepsilon(L) \geqslant \dfrac{k_{min}}{1-\alpha}$. If the minimum of U_ε is reached at $x_\varepsilon \in (0, L)$, equation (13) gives again directly $U_\varepsilon(x_\varepsilon) = h(U_\varepsilon(x_\varepsilon) - K_\varepsilon) \leqslant \mu U_\varepsilon(x_\varepsilon) - \mu k_{min}$, and $U_\varepsilon(x_\varepsilon) \geqslant \dfrac{\mu}{\mu-1}k_{min}$. We also recall that $U_\varepsilon(0) = U_b$. Then, in any case, we can then set $u_{min} = \min\left\{\dfrac{k_{min}}{1-\alpha},\ \dfrac{k_{min}\,\mu}{\mu-1},\ U_b\right\}$.

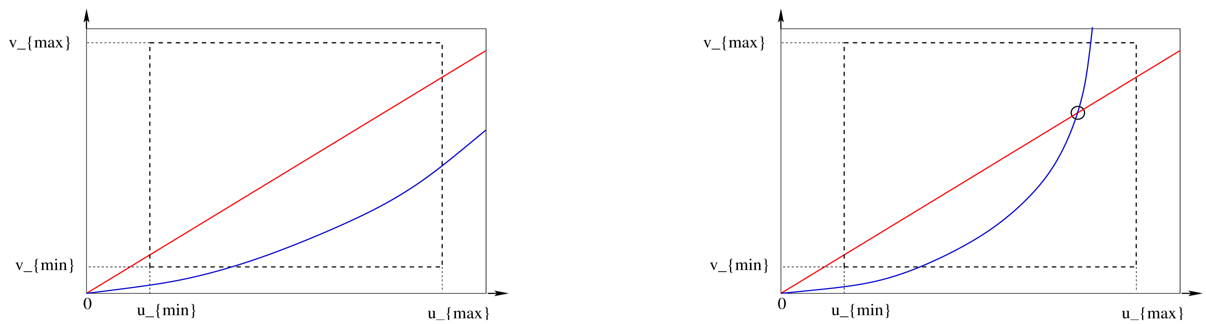
Existence of v_{max}. $V_\varepsilon(x) = U_\varepsilon(x) - K_\varepsilon \leqslant u_{max}$ and we can set $v_{max} = u_{max}$.

Existence of v_{min}. We have $V_\varepsilon(L) = \alpha U_\varepsilon(L) \geqslant \alpha u_{min}$. If we assume that V_ε reaches its minimum at $x = 0$, then, V_ε is increasing at $x = 0$. Equation (13) then implies $h(V_\varepsilon(0)) - U_b \geqslant 0$, and then $V_\varepsilon(0) \geqslant \dfrac{U_b}{\mu}$. Now if we assume that V_ε reaches its minimum at $x_\varepsilon \in (0, L)$, we have $\dfrac{d}{dx}V_\varepsilon(x_\varepsilon) = 0$ and then $h(V_\varepsilon(x_\varepsilon)) = U_\varepsilon(x_\varepsilon)$ which implies $V_\varepsilon(x_\varepsilon) \geqslant \dfrac{u_{min}}{\mu}$. In any case, we can set $v_{min} = \min\left\{\dfrac{u_{min}}{\mu}, \dfrac{U_b}{\mu}, \alpha u_{min}\right\}$. This ends the proof of Lemma 1. □

The comparison principle in [19] gives that $0 \leqslant u^0(x) \leqslant U_\varepsilon(x)$ and $0 \leqslant v^0(x) \leqslant V_\varepsilon(x)$ implies $0 \leqslant u_\varepsilon(t,x) \leqslant U_\varepsilon(x)$ and $0 \leqslant v_\varepsilon(t,x) \leqslant V_\varepsilon(x)$.

The choice $U_b = \max\{u_b, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}\}$ ends the proof of Proposition 1.

The solution $u_\varepsilon, v_\varepsilon$ is then uniformly contained in the rectangle $[u_{min}, u_{max}] \times [v_{min}, v_{max}]$. As depicted on Figure 2.2, depending on u_b, u^0, v^0, α, β, μ but not on ε, either $M_b \cap M_{eq} = \emptyset$, or there exists $(u_I, v_I) \in [u_{min}, u_{max}] \times [v_{min}, v_{max}]$ such that $M_b \cap M_{eq} = \{(u_I, v_I)\}$. In the first case, a boundary layer appears.



$M_b \cap M_{eq} = \emptyset$ $\qquad$ $M_b \cap M_{eq} = \{(u_I, v_I)\}$

Figure 1: Plot of the two equilibrium manifolds. Either M_{eq} (blue) intersects M_b (red) inside $[u_{min}, u_{max}] \times [v_{min}, v_{max}]$(right), either it does not (left).

Proposition 2 (Existence of the boundary layer)**.** *We assume (6). Assuming that $M_b \cap M_{eq} = \emptyset$, and that the solution $(u_\varepsilon, v_\varepsilon)$ to (S_ε) is continuous with respect to x, then there exists $D > 0$, independent of ε, and $\eta(\varepsilon) > 0$ such that*

$$|h(v_\varepsilon(x,t)) - u_\varepsilon(x,t)| > D, \qquad t \in [0,T], \quad x \in [L - \eta(\varepsilon), L].$$

Proof. Proposition 1 shows that M_b and M_{eq} are the graphs of two continuous functions, respectively $v = \alpha u$ and $v = h^{-1}(u)$, defined on the compact set $[u_{min}, u_{max}]$ and which do not intersect. Then there exists $m > 0$ which does not depend on ε such that

$$\forall (x_0, y_b) \in M_b,\ \forall (x_0, y_{eq}) \in M_{eq}, \qquad |y_{eq} - y_b| > m.$$

Thus for all $t \in [0,T]$, we have

$$|u_\varepsilon(L,t) - h(v_\varepsilon(L,t))| = |u_\varepsilon(L,t) - h(\alpha u_\varepsilon(L,t))| \geqslant \beta |h^{-1}(u_\varepsilon(L,t)) - \alpha u_\varepsilon(L,t)| \geqslant \beta m,$$

since $(u_{\varepsilon(L,t)}, h^{-1}(u_\varepsilon(L,t))) \in M_{eq}$ and $(u_\varepsilon(L,t), \alpha u_\varepsilon(L,t)) \in M_b$. Since u_ε, v_ε and h are continuous, this implies that there exists η that may depend on ε such that $|h(v_\varepsilon - u_\varepsilon)| > \beta m/2$ for $|x - L| < \eta$, and Proposition 2 holds for $D = \beta m/2$. $\square$

3 Construction of an Asymptotic Preserving scheme and main results

In the context of the finite volume schemes framework, we consider a mesh of N disjoint cells C_k, $k \in [\![1, N]\!]$. Let Δx be the size of each cell and let Δt be the time step. The final time is denoted by T, and the number of iterations is denoted by n_f, so that $n_f \Delta t = T$. The approximated value of the function $\rho(x,t)$ for $x \in C_k$ and $t \in [(n-1)\Delta t, n\Delta t]$ is denoted by ρ_k^n.

3.1 Construction of the scheme

We detail here the requirements we impose on the numerical schemes.

R-1. For simplicity, the scheme $(S_{\varepsilon,\Delta})$ is explicit, and its stencil contains 3 points.

R-2. The scheme $(S_{\varepsilon,\Delta})$ is upwind in the sense that the fluxes u and $-v$ are computed using only a one-sided approximation to the derivative.

R-3. The scheme (S_Δ) is upwind in the sense that at each time step, the updated value of the conservative quantity ρ_k^{n+1} only depends on $\{\rho_\ell^n,\ \ell \leqslant k\}$.

Based on remark (8), we consider a class of schemes of the form

$$\begin{cases} u_{\varepsilon,k}^{n+1} = u_{\varepsilon,k}^n - \dfrac{\Delta t}{\Delta x}\Big[u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n\Big] + \dfrac{\Delta t}{\varepsilon+\Delta x} S^u(u^n, v^n), \\ v_{\varepsilon,k}^{n+1} = v_{\varepsilon,k}^n - \dfrac{\Delta t}{\Delta x}\Big[-v_{\varepsilon,k+1}^n + v_{\varepsilon,k}^n\Big] - \dfrac{\Delta t}{\varepsilon+\Delta x} S^v(u^n, v^n), \end{cases} \tag{17}$$

where S^u and S^v are two ways to discretize the source term. Both S^u and S^v should be consistent with $h(v) - u$. The sum of the equations of (17) for $\varepsilon = 0$ is

$$u_{\varepsilon,k}^{n+1} + v_{\varepsilon,k}^{n+1} = u_{\varepsilon,k}^n + v_{\varepsilon,k}^n - \frac{\Delta t}{\Delta x}\Big[u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n - S^u + S^v\Big]. \tag{18}$$

The condition R-3 then leads us to impose that the discretization of the flux is

$$u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n - S^u + S^v = h(v_{\varepsilon,k}^n) - v_{\varepsilon,k}^n - (h(v_{\varepsilon,k-1}^n) - v_{\varepsilon,k-1}^n).$$

Since we restricted ourselves to linear upwind schemes with a three-point stencil, there exists 3 real numbers a, b, c such that

$$\begin{cases} S^u = h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n - a v_{\varepsilon,k+1}^n + b v_{\varepsilon,k}^n - c v_{\varepsilon,k-1}^n, \\ S^v = h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + (1-a) v_{\varepsilon,k+1}^n + (b-2) v_{\varepsilon,k}^n + (1-c) v_{\varepsilon,k-1}^n. \end{cases} \tag{19}$$

Among the class of numerical schemes (17) which satisfy (19), one can check that the ones which are stable in $\mathbf{L}^\infty \cap \mathrm{BV}$ are those where a, b and c satisfy $a = 0$, $b = c$, $1 \leqslant b \leqslant \beta$. The schemes we select are then written for $1 \leqslant b \leqslant \beta$

$$\begin{cases} \dfrac{u_{\varepsilon,k}^{n+1} - u_{\varepsilon,k}^n}{\Delta t} + \dfrac{u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n}{\Delta x} = \dfrac{1}{\varepsilon+\Delta x}\Big(h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n + b v_{\varepsilon,k}^n - b v_{\varepsilon,k-1}^n\Big), \\ \dfrac{v_{\varepsilon,k}^{n+1} - v_{\varepsilon,k}^n}{\Delta t} + \dfrac{v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n}{\Delta x} = -\dfrac{1}{\varepsilon+\Delta x}\Big(h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + v_{\varepsilon,k+1}^n + (b-2) v_{\varepsilon,k}^n + (1-b) v_{\varepsilon,k-1}^n\Big), \\ u_0^n = u_b, \qquad v_0^n = h^{-1}(u_b), \qquad v_{N+1}^n = \alpha u_N^n. \end{cases} \tag{20}$$

For simplicity, we focus on the case $b = 1$ and define the scheme for $k \in [\![1, N]\!]$ and $n \in \mathbb{N}$

$$\begin{cases} \dfrac{u_{\varepsilon,k}^{n+1} - u_{\varepsilon,k}^n}{\Delta t} + \dfrac{u_{\varepsilon,k}^n - u_{\varepsilon,k-1}^n}{\Delta x} = \dfrac{1}{\varepsilon+\Delta x}\Big(h(v_{\varepsilon,k-1}^n) - u_{\varepsilon,k-1}^n + v_{\varepsilon,k}^n - v_{\varepsilon,k-1}^n\Big), \\ \dfrac{v_{\varepsilon,k}^{n+1} - v_{\varepsilon,k}^n}{\Delta t} + \dfrac{v_{\varepsilon,k}^n - v_{\varepsilon,k+1}^n}{\Delta x} = -\dfrac{1}{\varepsilon+\Delta x}\Big(h(v_{\varepsilon,k}^n) - u_{\varepsilon,k}^n + v_{\varepsilon,k+1}^n - v_{\varepsilon,k}^n\Big), \\ u_0^n = u_b, \qquad v_0^n = h^{-1}(u_b), \qquad v_{N+1}^n = \alpha u_N^n. \end{cases} \tag{$S_{\varepsilon,\Delta}$}$$

The scheme $(S_{\varepsilon,\Delta})$ satisfy R-1, R-2 and R-3. The sequence of following results states that the AP diagram is commutative.

3.2 Convergence results

For all $\varepsilon > 0$, let us define the following functions

$$\begin{aligned} u_{\varepsilon,\Delta}(x,t) &= \sum_{n\in\mathbb{N}} \sum_{k\in[1,N]} u^n_{\varepsilon,k} \mathbb{1}_{[n\Delta t,(n+1)\Delta t)\times C_k}(x,t), \\ v_{\varepsilon,\Delta}(x,t) &= \sum_{n\in\mathbb{N}} \sum_{k\in[1,N]} v^n_{\varepsilon,k} \mathbb{1}_{[n\Delta t,(n+1)\Delta t)\times C_k}(x,t), \end{aligned} \tag{21}$$

where the sequence $(u_{\varepsilon,\Delta}, v_{\varepsilon,\Delta})$ is given by the scheme $(S_{\varepsilon,\Delta})$.

Theorem 3. *Assuming (4), (6), and the CFL condition $\mu\Delta t \leqslant \Delta x$, the approximate solution $(u_{\varepsilon,\Delta}, v_{\varepsilon,\Delta})$ defined in (21) satisfies*

$$\|u_{\varepsilon,\Delta} - u_\varepsilon\|_{L^1((0,T)\times[0,L])} \underset{\Delta\to 0}{\to} 0, \quad \|v_{\varepsilon,\Delta} - v_\varepsilon\|_{L^1((0,T)\times[0,L])} \underset{\Delta\to 0}{\to} 0.$$

where $(u_\varepsilon, v_\varepsilon)$ is the unique solution of (S_ε).

The scheme (S_Δ) is obtained by setting $\varepsilon = 0$ in $(S_{\varepsilon,\Delta})$, and enables us to build a sequence (u^n_k, v^n_k), $k \in [0,N]$, $n \geqslant 0$. Let us define

$$\begin{aligned} u_\Delta(x,t) &= \sum_{n\in\mathbb{N}} \sum_{k\in[1,N]} u^n_k \mathbb{1}_{[n\Delta t,(n+1)\Delta t)\times C_k}(x,t), \\ v_\Delta(x,t) &= \sum_{n\in\mathbb{N}} \sum_{k\in[1,N]} v^n_k \mathbb{1}_{[n\Delta t,(n+1)\Delta t)\times C_k}(x,t). \end{aligned} \tag{22}$$

Theorem 4. *Assuming (4), (5), (6), and the CFL condition $\mu\Delta t \leqslant \Delta x$, the approximate solution (u_Δ, v_Δ) defined in (22) satisfies*

$$\|u_\Delta - h(v)\|_{L^1((0,T)\times[0,L])} \underset{\Delta\to 0}{\to} 0, \quad \|v_\Delta - v\|_{L^1((0,T)\times[0,L])} \underset{\Delta\to 0}{\to} 0,$$

where v is the unique solution of (S_0).

We prove in the next section that the schemes $(S_{\varepsilon,\Delta})$ are convergent for all $\varepsilon > 0$, and that they relax toward an upwind convergent scheme when ε goes to zero. In Section 2, we stated the results that justify the arrow $\boxed{\mathbf{2}}$ of the diagram. We focus here on arrows $\boxed{\mathbf{1}}$, $\boxed{\mathbf{3}}$ and $\boxed{\mathbf{4}}$.

4 Convergence of the relaxation scheme $(S_{\varepsilon,\Delta})$ as $\Delta \to 0$

This section is devoted to the proof of Theorem 3. To avoid cumbersome notations, we drop the indices ε in the quantities $u^n_{\varepsilon,k}$ and $v^n_{\varepsilon,k}$. Throughout Propositions 3, 4, and 5, we prove uniform estimates on the functions $u_{\varepsilon,\Delta}$ and $v_{\varepsilon,\Delta}$ that enables us to pass to the limit using strong compactness.

Proposition 3 (Conservation, monotonicity and positivity).
We assume (4), (6). Then the sequence scheme $(S_{\varepsilon,\Delta})$ satisfies the following properties:

i) The quantity $u^n_k + v^n_k$ is preserved, i.e. there exists a numerical flux $(G^n_{k+\frac{1}{2}})_{i,n}$ such that

$$u^{n+1}_k + v^{n+1}_k = u^n_k + v^n_k - \frac{\Delta t}{\Delta x}\left(G^n_{k+\frac{1}{2}} - G^n_{k-\frac{1}{2}}\right), \qquad k = k \in [\![1,N]\!],\ n \in \mathbb{N}.$$

ii) Under the Courant-Friedrichs-Levy condition

$$\Delta t \leqslant \frac{\Delta x}{\mu}, \tag{23}$$

the scheme $(S_{\varepsilon,\Delta})$ is monotone in the sense that we can write

$$\begin{cases} u_k^{n+1} = G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n) \\ v_k^{n+1} = H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n), \end{cases}$$

where G and H are non-decreasing functions with respect to each of their variables.

iii) The scheme $(S_{\varepsilon,\Delta})$ preserves positivity:

$$\text{if } \ \forall k \in [1,N], \quad u_k^0 \geqslant 0,\ v_k^0 \geqslant 0, \qquad \text{then} \qquad \forall n \geqslant 0,\ \forall k \in [\![1,N]\!], \quad u_k^n \geqslant 0,\ v_k^n \geqslant 0.$$

Proof. We first write the scheme in a conservative form:

$$\begin{aligned} u_k^{n+1} = u_k^n - \frac{\Delta t}{\Delta x}\Big[&u_k^n - u_{k-1}^n + \frac{\Delta x}{2(\Delta x+\varepsilon)}\Big((h(v_k^n) - u_k^n + v_{k+1}^n - v_k^n) - (h(v_{k-1}^n) - u_{k-1}^n + v_k^n - v_{k-1}^n)\Big)\Big] \\ &+ \frac{\Delta x}{2(\Delta x+\varepsilon)}\Big[h(v_k^n) + h(v_{k-1}^n) - (u_k^n + u_{k-1}^n) + v_{k+1}^n - v_{k-1}^n\Big], \\ v_k^{n+1} = v_k^n - \frac{\Delta t}{\Delta x}\Big[&v_k^n - v_{k+1}^n + \frac{\Delta x}{2(\Delta x+\varepsilon)}\Big((h(v_{k-1}^n) - u_{k-1}^n + v_k^n - v_{k-1}^n) - (h(v_k^n) - u_k^n + v_{k+1}^n - v_k^n)\Big)\Big] \\ &- \frac{\Delta x}{2(\Delta x+\varepsilon)}\Big[h(v_k^n) + h(v_{k-1}^n) - (u_k^n + u_{k-1}^n) + v_{k+1}^n - v_{k-1}^n\Big], \end{aligned} \tag{24}$$

which proves (i). To prove the monotonicity property (ii), let us write the scheme under the form

$$\begin{aligned} u_k^{n+1} &= \Big[1 - \frac{\Delta t}{\Delta x}\Big]u_k^n + \Big[\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x+\varepsilon}\Big]u_{k-1}^n + \frac{\Delta t}{\Delta x+\varepsilon}\Big[h(v_{k-1}^n) - v_{k-1}^n\Big] \\ &\quad + \frac{\Delta t}{\Delta x+\varepsilon}v_k^n := G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n), \\ v_k^{n+1} &= \Big[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x+\varepsilon}\Big]v_k^n + \Big[\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x+\varepsilon}\Big]v_{k+1}^n - \frac{\Delta t}{\Delta x+\varepsilon}h(v_k^n) + \frac{\Delta t}{\Delta x+\varepsilon}u_k^n \\ &\quad := H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n). \end{aligned} \tag{25}$$

For any $\Delta x > 0,\ \Delta t > 0$, it is clear from the assumptions on h that G is non-decreasing with respect to u_{k-1}^n, v_{k-1}^n, v_k^n, and that H is non-decreasing with respect to v_{k-1}^n, v_{k+1}^n, v_k^n. By the CFL condition (23) and since $\mu > 1$, we have $\Delta t < \Delta x$, which also implies that G is non-decreasing with u_k^n and

$$\begin{aligned} \frac{\partial H}{\partial v_k^n}(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n) &= \frac{\partial}{\partial v_k^n}\Big[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x+\varepsilon}\Big]v_k^n - \frac{\Delta t}{\Delta x+\varepsilon}h(v_k^n) \\ &\geqslant \Big[1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x+\varepsilon} - \mu\frac{\Delta t}{\Delta x+\varepsilon}\Big]. \\ &> 1 - \mu\frac{\Delta t}{\Delta x} \geqslant 0. \end{aligned}$$

In conclusion the scheme is monotone provided that the stability condition (23) is satisfied.

Finally, we notice that $G(0,0,0,0,0,0) = H(0,0,0,0,0,0) = 0$, and the positivity (iii) follows directly from the monotonicity (ii). $\square$

We first prove $\mathbf{L}^\infty$ estimates that are useful to prove BV estimates.

Proposition 4 ($\mathbf{L}^\infty$ estimate)**.** *We assume (4), (6), the CFL condition (23) and $\Delta x < (1-\alpha)\varepsilon$. Then there exists a function M such that the solution (u_k^n, v_k^n) to the scheme $(S_{\varepsilon,\Delta})$ satisfies*

$$\forall n \geqslant 0,\ \forall k \in [\![1, N]\!], \qquad u_k^n \leqslant M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}), \qquad v_k^n \leqslant M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}).$$

We start with the following lemma which states the existence of a super-solution.

Lemma 2 (Existence of a super-solution)**.** *We assume (6) and we fix $\Delta x \leqslant (1-\alpha)\varepsilon$. For any $\delta > 0$, there exists $U_b \geqslant u_b$ depending on δ, a pair of vectors $(U, V) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+2}$ which may depend on ε, and a function $\bar{M} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\begin{cases} U_k - U_{k-1} = \dfrac{\Delta x}{\Delta x + \varepsilon}\Big[h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}\Big], & k \in [\![1, N]\!], \\ V_k - V_{k+1} = \dfrac{\Delta x}{\Delta x + \varepsilon}\Big[-h(V_k) + U_k - V_{k+1} + V_k\Big], & k \in [\![0, N]\!], \\ U_0 = U_b, \qquad V_{N+1} \geqslant \alpha U_N, \end{cases} \tag{26}$$

and

$$\delta \leqslant U_k \leqslant \bar{M}(\delta, \varepsilon), \quad \delta \leqslant V_k \leqslant \bar{M}(\delta, \varepsilon), \qquad k \in [\![1, N]\!]. \tag{27}$$

Proof of Lemma 2. We denote by $r = \Delta x/(\Delta x + \varepsilon)$. We have $0 < r < 1 - \alpha$. To prove Lemma 2, we are decoupling the difficulties. We first prove the existence of solutions for the system

$$\begin{cases} U_k - U_{k-1} = r\Big[h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}\Big], & k \in [\![1, N]\!], \\ V_k - V_{k+1} = r\Big[-h(V_k) + U_k - V_{k+1} + V_k\Big], & k \in [\![0, N]\!], \\ U_0 = U_b, \qquad V_{N+1} = V_b, \end{cases} \tag{28}$$

where $V_b \in \mathbb{R}^+$ and $U_b \in \mathbb{R}^+$ are given, using a fixed point argument. Then, we use a shooting method to prove that there is at least one value for V_b for which the solution to (28) satisfies $V_{N+1} \geqslant \alpha U_N$, which makes it a solution to (26) as well. In the last step, we prove that we can always find $U_b(\delta)$ the estimates (27).

Step 1. Existence of a solution for (28). We fix $U_b \geqslant u_b$ and $V_b \in \mathbb{R}^+$. We build here a solution (U, V) to (28) by defining U as a fixed point of the following operator $\Phi : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ (once the vector U is defined, the vector V is directly deduced from the second line of (28)). Given $\bar{U} \in \mathbb{R}^{N+1}$, we define $U := \Phi[\bar{U}]$ as the unique solution of the system

$$\begin{cases} U_k - U_{k-1} = r\Big[h(V_{k-1}) - U_{k-1} + V_k - V_{k-1}\Big], & k \in [\![1, N]\!], \\ V_k - V_{k+1} = r\Big[-h(V_k) + \bar{U}_k - V_{k+1} + V_k\Big], & k \in [\![0, N]\!], \\ U_0 = U_b, \qquad \bar{U}_0 = U_b, \qquad V_{N+1} = V_b. \end{cases} \tag{29}$$

We prove here that for any M such that $U_b \leqslant h(M)$ and $V_b \leqslant M$, we have $\Phi\Big([0, h(M)]^{N+1}\Big) \subset [0, h(M)]^{N+1}$. Indeed, let us assume $\bar{U} \in [0, h(M)]^{N+1}$. We define $(V_0, \dots, V_N)$ as the unique vector satisfying the second line of (29) and $V_{N+1} = V_b$. Let us assume that for some $k \in [\![0, N]\!]$, we have $0 \leqslant V_{k+1} \leqslant M$ (true for $k = N$), then the second line of (29) gives us

$$(rh + (1-r)Id)V_k = r\bar{U}_k + (1-r)V_{k+1}, \quad k \in [\![0, N]\!],$$

which implies

$$0 \leqslant V_k \leqslant M, \tag{30}$$

since $(rh + (1-r)Id)$ is invertible and monotone. By induction, estimate (30) holds for any $k \in [\![0, N]\!]$. We now define U by the first line of (29) and $U_0 = U_b$. For $k = 0$, we have the estimate $0 \leqslant U_k \leqslant h(M)$. The first line of (29) gives for any $k \in [\![0, N]\!]$

$$U_{k+1} = (1-r)U_k + r(h - Id)V_k + rV_{k+1}. \tag{31}$$

By induction, this directly implies, for all $k \in [\![0, N-1]\!]$,

$$0 \leqslant U_{k+1} \leqslant h(M).$$

We can now apply the Brouwer fixed point theorem to conclude that Φ admits at least one fixed point U. We notice that for any of these fixed points U, the couple (U, V) where V is defined by the second line of (28), is a solution to (28), which guarantees the existence of a solution to (28). This solution satisfies

$$0 \leqslant U_k \leqslant h(M), \quad 0 \leqslant V_k \leqslant M, \tag{32}$$

for any M such as $U_b \leqslant h(M)$ and $V_b \leqslant M$, and, moreover,

$$V_{k+1} - V_k = U_{k+1} - U_k, \quad k \in [\![0, N-1]\!]. \tag{33}$$

From now on, for each $U_b \geqslant u_b$, $V_b \in \mathbb{R}^+$, we choose a solution to (29) obtained through the process described in Step 1 and denote it by (U, V).

Step 2. An intermediate estimate on the solution. We prove here an estimate on the solution to (28) defined in Step 1. We have

$$V_{k+1} = V_k + \frac{r}{1-r}h(V_k) - \frac{r}{1-r}U_k \leqslant \frac{1-r+r\mu}{1-r}V_k.$$

By direct induction,

$$V_k \geqslant \left(\frac{1-r}{1-r+r\mu}\right)^{N+1-k} V_{N+1}. \tag{34}$$

Since

$$\left(\frac{1-r}{1-r+r\mu}\right)^{N+1-k} = \left(1 + \frac{\mu}{\varepsilon}\Delta x\right)^{N+1-k} \leqslant \left(1 + \frac{\mu}{\varepsilon}\Delta x\right)^{N+1} \leqslant \exp\left(\frac{\mu}{\varepsilon}\right), \tag{35}$$

we conclude combining (34) and (35) that

$$V_k \geqslant \exp\left(-\frac{\mu}{\varepsilon}\right) V_b, \quad k \in [\![0, N]\!]. \tag{36}$$

Step 3. The shooting method. We fix $U_b \geqslant u_b$. We define the shooting function P as $P : V_b \mapsto \alpha U_N - V_b$, where U_N is defined as the N-th component of the vector U selected in Step 1. We prove here that there exists $V_b \in \mathbb{R}^+$ such that $P(V_b) \leqslant 0$, i.e. such that the (selected) solution to (28) is a solution to (26) as well. We have

$$P(V_b) = \alpha U_N - V_{N+1} = \alpha U_N - U_N + U_N - V_N + V_N - V_{N+1}.$$

Using (33) and (28), we obtain

$$\begin{aligned} P(V_b) &= (\alpha - 1)\, U_N + U_0 - V_0 - \frac{r}{1-r}h(V_N) + \frac{r}{1-r}U_N \\ &= \left(\alpha - 1 + \frac{\Delta x}{\varepsilon}\right) U_N + U_0 - V_0 - \frac{r}{1-r}h(V_N). \end{aligned}$$

Since $-\dfrac{r}{1-r}h(V_N) < 0$, and since $\left(\alpha - 1 + \dfrac{\Delta x}{\varepsilon}\right) U_N \leqslant 0$ for $\Delta x \leqslant \varepsilon(1-\alpha)$, we have $P(V_b) < 0$ as soon as $U_0 - V_0 < 0$, i.e. as soon as

$$V_b > U_b \exp\left(\frac{\mu}{\varepsilon}\right),$$

using (36). As a conclusion, for any $U_b \geqslant u_b$ and V_b satisfying $V_b > U_b \exp\left(\dfrac{\mu}{\varepsilon}\right)$, any solution to (28) is a solution to (26).

Step 4. Estimates from below and from above for the super-solution. We fix $U_b \geqslant u_b$ and $V_b = U_b \exp\left(\dfrac{\mu}{\varepsilon}\right) + 1$ and denote by (U, V) the solution to (26) we selected in Step 1. We prove now that if we choose U_b large enough so that

$$\alpha \left(\frac{\varepsilon}{L+\varepsilon}\right)^2 \exp\left(-\frac{\mu}{\varepsilon}\right) U_b = \delta,$$

then (27) is guaranteed. Indeed, using the first line of (26), we obtain by direct induction

$$U_k \geqslant (1-r)^k U_b \geqslant (1-r)^{N+1} U_b, \quad k \in [\![1, N]\!], \tag{37}$$

and using (36) we have

$$V_{N+1} \geqslant \alpha U_{N+1}, \quad V_k \geqslant \exp\left(-\frac{\mu}{\varepsilon}\right) V_{N+1}, \quad k \in [\![0, N]\!]. \tag{38}$$

Combining (37) and (38), we deduce, since $\alpha < 1$ and $\exp\left(-\dfrac{\mu}{\varepsilon}\right) < 1$,

$$\min\left\{\min_{k\in[\![1,N]\!]} U_k, \min_{k\in[\![1,N+1]\!]} V_k\right\} \geqslant \alpha \exp\left(-\frac{\mu}{\varepsilon}\right)(1-r)^{N+1} U_b.$$

Using $N\Delta x = L$, we have

$$(1-r)^{N+1} = \left(1 - \frac{\Delta x}{\Delta x + \varepsilon}\right)^{N+1} = \left(\frac{N\varepsilon}{L + N\varepsilon}\right)^{N+1},$$

and since $N \to \left(\dfrac{N\varepsilon}{L+N\varepsilon}\right)^{N+1}$ is increasing, we have

$$(1-r)^{N+1} \geqslant \left(\frac{\varepsilon}{L+\varepsilon}\right)^2,$$

and thus

$$\min\left\{\min_{k\in[\![1,N]\!]} U_k, \min_{k\in[\![1,N+1]\!]} V_k\right\} \geqslant \delta.$$

According to Step 1, for M such that $V_b \leqslant M$ and $U_b \leqslant h(M)$, we have

$$0 \leqslant U_k \leqslant h(M), \quad 0 \leqslant V_k \leqslant M, \qquad k \in [\![1, N]\!]. \tag{39}$$

Then, estimate (27) holds for

$$\bar{M}(\delta, \varepsilon) = 1 + \left(\frac{L+\varepsilon}{\varepsilon}\right)^2 \exp\left(\frac{2\mu}{\varepsilon}\right)\delta.$$

This ends the proof of Lemma 2. □

Proof of Proposition 4. Consider the approximations $(u_k^n)_{k,n}$ and $(v_k^n)_{k,n}$ given by $(S_{\varepsilon,\Delta})$. We denote by $(U_k)_{k\in[0,N]}$ and $(V_k)_{k\in[0,N+1]}$ the vectors of $\mathbb{R}^N$ given by Lemma 2 and corresponding to $\delta := \max\{\|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}\}$. Let us assume that for some $n \geqslant 0$, we have for all $k \in [\![1, N]\!]$, $u_k^n \leqslant U_k$ and $v_k^n \leqslant V_k$. This is true for $n = 0$ since $u_k^0 = u^0(k\Delta x) \leqslant \delta \leqslant U_k$ and $v_k^0 = v^0(k\Delta x) \leqslant \delta \leqslant V_k$. Since the scheme $(S_{\varepsilon,\Delta})$ is monotone, we have

$$\begin{cases} u_k^{n+1} = G(u_{k-1}^n, u_k^n, v_{k-1}^n, v_k^n) \leqslant G(U_{k-1}, U_k, V_{k-1}, V_k) \\ v_k^{n+1} = H(u_k^n, v_{k-1}^n, v_k^n, v_{k+1}^n) \leqslant H(U_k, V_{k-1}, V_k, V_{k+1}), \end{cases} \tag{40}$$

where G and H are defined in (24). Since U_k, V_k satisfy (26), we have

$$\begin{cases} G(U_{k-1}, U_k, V_{k-1}, V_k) \leqslant U_k \\ H(U_k, V_{k-1}, V_k, V_{k+1}) \leqslant V_k. \end{cases} \tag{41}$$

The combination of (41) and (40) leads to

$$u_k^{n+1} \leqslant U_k, \qquad v_k^{n+1} \leqslant V_k, \qquad k \in [\![1, N]\!].$$

By induction on n, we have then

$$u_k^n \leqslant U_k, \qquad v_k^n \leqslant V_k, \qquad k \in [\![1, N]\!], \quad n \geqslant 0,$$

and thus the result follows from Lemma 2 with $M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) = \bar{M}(\max\{\|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}\}, \varepsilon)$. □

We define

$$TV(u^n) = \sum_{k=0}^{N-1} |u_{k+1}^n - u_k^n|, \qquad TV(v^n) = \sum_{k=0}^{N-1} |v_{k+1}^n - v_k^n|.$$

Proposition 5 (spatial BV estimate)**.** *We assume (4) and (6). Under the CFL condition (23) and assuming $\Delta x < (1-\alpha)\varepsilon$, for u_k^n and v_k^n given by the scheme $(S_{\varepsilon,\Delta})$, there exists K such that*

$$TV(u^n) + TV(v^n) \leqslant TV(u^0) + TV(v^0) + T\ K(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}).$$

Proof. We first write

$$\sum_{k=0}^{N-1} |u_{k+1}^{n+1} - u_k^{n+1}| + \sum_{k=0}^{N} |v_{k+1}^{n+1} - v_k^{n+1}| = \underbrace{\sum_{k=1}^{N-1} \left|u_{k+1}^{n+1} - u_k^{n+1}\right| + \sum_{k=1}^{N-1} \left|v_{k+1}^{n+1} - v_k^{n+1}\right|}_{M_{\sum_k}}$$
$$+ \underbrace{\left|u_1^{n+1} - u_b\right| + \left|v_1^{n+1} - v_0\right|}_{M_0}$$

We consider separately the terms $M_{\sum_k}$ and M_0.

Step 1: $M_{\sum_k}$. Using the numerical scheme $(S_{\varepsilon,\Delta})$ for $1 \leqslant k \leqslant N-1$, we have

$$
\begin{aligned}
|u_{k+1}^{n+1} - u_k^{n+1}| \leqslant & \left| \left(1 - \frac{\Delta t}{\Delta x}\right)(u_{k+1}^n - u_k^n) \right| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|u_k^n - u_{k-1}^n| + \frac{\Delta t}{\Delta x + \varepsilon}|v_{k+1}^n - v_k^n| \\
& + \frac{\Delta t}{\Delta x + \varepsilon}\left| h(v_k^n) - h(v_{k-1}^n) - (v_k^n - v_{k-1}^n) \right| \\
|v_{k+1}^{n+1} - v_k^{n+1}| \leqslant & \left| \left(1 - \frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)(v_{k+1}^n - v_k^n) - \frac{\Delta t}{\Delta x + \varepsilon}(h(v_{k+1}^n) - h(v_k^n)) \right| \\
& + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|v_{k+2}^n - v_{k+1}^n| + \frac{\Delta t}{\Delta x + \varepsilon}|u_{k+1}^n - u_k^n|.
\end{aligned}
$$

Under the CFL condition (23) which guarantees the positivity of the coefficients, the terms of $M_{\sum_k}$ can be reorganized the following way

$$
\begin{aligned}
M_{\sum_k} \leqslant & \sum_{k=1}^{N-1} \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right)|u_{k+1}^n - u_k^n| + \sum_{k=0}^{N-2} \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|u_{k+1}^n - u_k^n| \\
& + \sum_{k=1}^{N-1} \left(1 - \frac{\Delta t}{\Delta x} + (2-\mu)\frac{\Delta t}{\Delta x + \varepsilon}\right)|v_{k+1}^n - v_k^n| \\
& + \sum_{k=0}^{N-2} (\mu - 1)\frac{\Delta t}{\Delta x + \varepsilon}|v_{k+1}^n - v_k^n| + \sum_{k=2}^{N} \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|v_{k+1}^n - v_k^n|.
\end{aligned}
$$

which is

$$
\begin{aligned}
M_{\sum_k} \leqslant & \sum_{k=1}^{N-2} |u_{k+1}^n - u_k^n| + \sum_{k=2}^{N-2} |v_{k+1}^n - v_k^n| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|u_1^n - u_b| \\
& + \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right)|u_N^n - u_{N-1}^n| + \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right)|v_2^n - v_1^n| \\
& + \left(1 + (1-\mu)\frac{\Delta t}{\Delta x + \varepsilon}\right)|v_N^n - v_{N-1}^n| + (\mu - 1)\frac{\Delta t}{\Delta x + \varepsilon}|v_1^n - v_0| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|\alpha u_N^n - v_N^n|.
\end{aligned}
\tag{42}
$$

Step 2: M_0. The term corresponding to $k = 0$ is treated the following way

$$
\begin{aligned}
M_0 = & \left| \left(1 - \frac{\Delta t}{\Delta x}\right) u_1^n + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) u_b + \frac{\Delta t}{\Delta x + \varepsilon}[h(v_0) - v_0] + \frac{\Delta t}{\Delta x + \varepsilon} v_1^n - u_b \right| \\
& + \left| \left(1 - \frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) v_1^n + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) v_2^n - \frac{\Delta t}{\Delta x + \varepsilon} h(v_1^n) \right. \\
& \left. + \frac{\Delta t}{\Delta x + \varepsilon} u_1^n - v_0 \right|,
\end{aligned}
$$

we rearrange the terms and plug the equality $u_b = h(v_0)$

$$
\begin{aligned}
M_0 = & \left(1 - \frac{\Delta t}{\Delta x}\right)|u_1^n - u_b| + \frac{\Delta t}{\Delta x + \varepsilon}|v_1^n - v_0| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right)|v_2^n - v_1^n| + \frac{\Delta t}{\Delta x + \varepsilon}|u_1^n - u_b| \\
& + \left| \left(1 - \frac{\Delta t}{\Delta x}\right)(v_1^n - v_0) - \frac{\Delta t}{\Delta x + \varepsilon}(h(v_1^n) - h(v_0)) \right|.
\end{aligned}
$$

and thus

$$M_0 \leqslant \left(1 - \frac{\Delta t}{\Delta x} + \frac{\Delta t}{\Delta x + \varepsilon}\right) |u_1^n - u_b| + \left(\frac{\Delta t}{\Delta x} - \frac{\Delta t}{\Delta x + \varepsilon}\right) |v_2^n - v_1^n| \\ + \left(1 - \frac{\Delta t}{\Delta x} + (1-\mu)\frac{\Delta t}{\Delta x + \varepsilon}\right) |v_1^n - v_0|. \tag{43}$$

We combine now (42)and (43) to obtain

$$\sum_{k=0}^{N-1} \left[|u_{k+1}^{n+1} - u_k^{n+1}| + |v_{k+1}^{n+1} - v_k^{n+1}|\right] \leqslant \sum_{k=0}^{N-1} \left[|u_{k+1}^n - u_k^n| + |v_{k+1}^n - v_k^n|\right] + K(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})\Delta t, \tag{44}$$

where $K(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) = (\alpha+1)M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})$, since Proposition 4 implies $\frac{\varepsilon}{\varepsilon + \Delta x}|\alpha u_N^n - v_N^n| \leqslant (\alpha+1)M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})$. Proposition 5 follows by immediate induction.

□

Proof. (of Theorem 3) We use the following lemma.

Lemma 3 (Theorem 2.4. [4])**.** *Consider a sequence of functions $u_\Delta : [0, +\infty) \times [0, 1] \to \mathbb{R}$ with the following properties:*

1. $TV(u_\Delta(t, .)) \leqslant C, \qquad |u_\Delta(t, x)| \leqslant M, \qquad \forall (t, x) \in [0, T] \times [0, L]$
2. $\displaystyle\int_{[0,L]} |u_\Delta(t, x) - u_\Delta(s, x)|dx \leqslant L_1|t - s| + L_2\Delta, \qquad \forall s, t \geqslant 0,$

then, there exists a function $u \in \mathbf{L}^\infty([0, T] \times [0, L]) \cap \mathcal{C}((0, T); \mathbf{L}^1([0, L]))$, such that $TV(u(t, .)) \leqslant C$ and

$$\lim_{\Delta \to 0} u_\Delta = u, \qquad \mathbf{L}^1([0, T] \times [0, L]).$$

In [4], Theorem 2.4 is stated for $L_2 = 0$, but the proof can be easily adapted for $L_2 \neq 0$. We now prove Theorem 3. Consider the sequence of functions (u_Δ, v_Δ) defined in (21). Using the first item of Lemma 3 is satisfied. Fix $t < s$. There exists n and m natural numbers such that $t \in [n\Delta t, (n + 1)\Delta t)$ and $s \in [m\Delta t, (m + 1)\Delta t)$. Then, using $(S_{\varepsilon,\Delta})$ and Proposition 5, we have

$$\begin{aligned}
\int_{[0,L]} |u_\Delta(t, x) - u_\Delta(s, x)|dx &= \sum_{k=1}^{N} \Delta x|u_k^n - u_k^m| \leqslant \sum_{\ell=n}^{m-1}\sum_{k=1}^{N} \Delta x|u_k^{\ell+1} - u_k^\ell| \\
&\leqslant \Delta t \sum_{\ell=n}^{m-1}\sum_{k=1}^{N} \left\{|u_k^\ell - u_{k-1}^\ell| + \frac{\Delta x}{\varepsilon}\left|h(v_{k-1}^\ell) - u_{k-1}^\ell\right| + \frac{\Delta x}{\varepsilon}\left|v_k^\ell - v_{k-1}^\ell\right|\right\} \\
&\leqslant \Delta t \sum_{\ell=n}^{m-1} \left\{TV(u^0) + TV(v^0) + TK(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})\right. \\
&\left. + \frac{3+\mu}{\varepsilon} M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty})\right\} \\
&\leqslant L(|t - s| + \Delta t),
\end{aligned}$$

with

$$L = TV(u^0) + TV(v^0) + TK(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}) + \frac{3+\mu}{\varepsilon} M(\varepsilon, \|u^0\|_{\mathbf{L}^\infty}, \|v^0\|_{\mathbf{L}^\infty}).$$

Thus, the second item of Lemma 3 is satisfied as well, which guarantees the existence of u_ε and v_ε in $\mathbf{L}^\infty([0,T]\times[0,L])\cap\mathcal{C}((0,T);\mathbf{L}^1([0,L]))$, such that $TV(u_\varepsilon(t,.))$ and $TV(v_\varepsilon(t,.))$ are finite, and such that

$$\lim_{\Delta\to 0} u_{\varepsilon,\Delta} = u_\varepsilon, \quad \lim_{\Delta\to 0} v_{\varepsilon,\Delta} = v_\varepsilon, \qquad \mathbf{L}^1([0,T]\times[0,L]).$$

We do not detail the fact that the limit $(u_\varepsilon, v_\varepsilon)$ is a weak solution to the linear system (S_ε), but similar (and easier) arguments developed in next section when proving that (u,v) is the unique entropy solution to (S_0) can be applied.

□

5 Convergence of the equilibrium scheme (S_Δ)

In this section, we focus on the equilibrium scheme. The goal is to prove Theorem 4. It is allowed to allocate the value 0 to the parameter ε in the scheme $(S_{\varepsilon,\Delta})$. The scheme obtained for $\varepsilon = 0$ gives us two sequences u_k^n and v_k^n. We prove here that the sequences obtained are a good discretization of the solution to the system (S_0). We denote by

$$\lambda = \frac{\Delta t}{\Delta x}.$$

The scheme we obtain at the limit can be written, for $k \in [\![1,N]\!]$ and $n \geqslant 0$, as

$$\begin{cases} v_k^{n+1} = v_k^n - \lambda\big(h(v_k^n) - u_k^n\big), \\ u_k^{n+1} = u_k^n - \lambda\big(u_k^n - v_k^n - (h(v_{k-1}^n) - v_{k-1}^n)\big), \\ h(v_k^0) = u_k^0, \quad k \in [1,N], \qquad\qquad u_0^n = u_b, \quad h(v_0^n) = u_b, \end{cases} \tag{S_Δ}$$

or equivalently, using the intermediate variable $s_k^n := u_k^n + v_k^n$, as

$$s_k^{n+1} = s_k^n - \lambda\Big(h(v_k^n) - v_k^n - (h(v_{k-1}^n) - v_{k-1}^n)\Big), \tag{45}$$

$$v_k^{n+1} = v_k^n - \lambda(h(v_k^n) - u_k^n), \tag{46}$$

$$u_k^{n+1} = s_k^{n+1} - v_k^{n+1}, \tag{47}$$

$$h(v_k^0) = u_k^0, \quad k \in [1,N], \qquad\qquad u_0^n = u_b, \quad h(v_0^n) = u_0^n, \quad n \geqslant 0. \tag{48}$$

The scheme (S_Δ) provides a solver for the system S which does not require to invert the non-linear function h at each time step. The scheme can be interpreted the following way. Let us assume that the quantities u_k^n, v_k^n are given for a time step n and recall that $s_k^n = u_k^n + v_k^n$. The variable s_k^{n+1} is updated using (45), which discretizes the scalar equation $\partial_t(u+v)+\partial_x(h(v)-v) = 0$. Then, we aim to define $v_k^{n+1} := h^{-1}(u_k^n)$ without inverting h. This is approximately done with step (46), that may be seen as an approximation of the first iteration of the Newton iteration scheme solving $h(x) = u_k^n$ with the inital guess $x = v_k^n$. Note that this would be exactly the first iteration of the Newton iteration scheme in the case where $\lambda = (h'(v_k^n))^{-1}$.

Let us now prove that the solution to the scheme (S_Δ) converges toward the entropy solution of S. To do so, we first prove monotonicity and a priori BV bounds on u_Δ, v_Δ to guarantee the convergence of the sequence toward a couple (u,v) using Helly theorem. To make sure that the limit is the entropy solution to S, we then adapt the proof of the Lax–Wendroff theorem to our case.

Let us define the functions

$$\mathcal{U}_\lambda(u,\bar v,v) = u - \lambda(u - v - h(\bar v) + \bar v), \tag{49}$$
$$\mathcal{V}_\lambda(u,v) = v - \lambda(h(v) - u), \tag{50}$$

so that the numerical scheme (S_Δ) can be rewritten

$$u_k^{n+1} = \mathcal{U}_\lambda(u_k^n, v_{k-1}^n, v_k^n), \tag{51}$$
$$v_k^{n+1} = \mathcal{V}_\lambda(u_k^n, v_k^n). \tag{52}$$

Lemma 4 (Monotonicity of the numerical scheme)**.** *We assume (4) and (6). Under the CFL condition (23), the numerical scheme (S_Δ) is monotone, in the sense that*

1. *It preserves constant solutions at equilibrium: for all $v \in \mathbb{R}^+$,*

$$\begin{aligned} h(v) &= \mathcal{U}_\lambda(h(v), v, v),\\ v &= \mathcal{V}_\lambda(h(v), v). \end{aligned} \tag{53}$$

2. *The functions $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ are nondecreasing with respect to each variable.*

Proof. The first item of Lemma 4 is clear. To prove the second item, we rewrite $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ as

$$\mathcal{U}_\lambda(u,\bar v,v) = u(1-\lambda) + \lambda v + \lambda(h(\bar v) - \bar v), \tag{54}$$
$$\mathcal{V}_\lambda(u,v) = \lambda u + (v - \lambda h(v)). \tag{55}$$

Clearly $\lambda > 0$, $(h - Id)$by assumption and the CFL condition (23) can be written as $1 - \lambda > 0$. This implies that $\mathcal{U}_\lambda$ is increasing. Besides, assuming (23), one has $(Id - \lambda h)' = 1 - \lambda h' \geqslant 1 - \lambda\mu > 0$, then w$(Id - \lambda h)$ is increasing. This implies that $\mathcal{V}_\lambda$ is increasing.

□

Lemma 5 (A priori bounds)**.** *We assume (4), (6) and the CFL condition (23). Then the following estimates are satisfied for the sequences (u_k^n) and (v_k^n) defined by (S_Δ).*

1. $\mathbf{L}^\infty$ *bounds. For $n \geqslant 0$ and $k \in [\![1, N]\!]$,*

$$\begin{cases} m \leqslant v_k^0 \leqslant M \\ h(m) \leqslant u_k^0 \leqslant h(M) \end{cases} \Longrightarrow \begin{cases} m \leqslant v_k^n \leqslant M \\ h(m) \leqslant u_k^n \leqslant h(M). \end{cases} \tag{56}$$

2. *BV bounds. For $n \geqslant 0$,*

$$TV(u^n + v^n) \leqslant TV(u^n) + TV(v^n) \leqslant TV(u^0) + TV(v^0). \tag{57}$$

Proof. The $\mathbf{L}^\infty$ bounds follow directly from the monotonicity of the scheme. Indeed, assuming $m \leqslant v_k^0 \leqslant M$ and $h(m) \leqslant u_k^0 \leqslant h(M)$, we have

$$h(m) = \mathcal{U}_\lambda(h(m), m, m) \leqslant \mathcal{U}_\lambda(u_k^0, v_{k-1}^0, v_k^0) \leqslant \mathcal{U}_\lambda(h(M), M, M) = h(M)$$

and

$$m = \mathcal{V}_\lambda(h(m), m) \leqslant \mathcal{V}_\lambda(u_k^0, v_k^0) \leqslant \mathcal{V}_\lambda(h(M), M) = M.$$

The property is then true for $n = 1$ since $u_k^1 = \mathcal{U}_\lambda(u_k^0, v_{k-1}^0, v_k^0)$, $v_k^1 = \mathcal{V}_\lambda(u_k^0, v_k^0)$, and is easily generalized by induction. To obtain the BV bounds we notice that we have directly from the definition of the total variation and using a triangle inequality and (23) that

$$
\begin{aligned}
TV(u^{n+1}) &\leqslant (1-\lambda)\, TV(u^n) + \lambda\mu\, TV(v^n), \\
TV(v^{n+1}) &\leqslant \lambda\, TV(u^n) + (1-\lambda\mu)\, TV(v^n).
\end{aligned}
$$

Summing the two lines, we get

$$TV(u^{n+1}) + TV(v^{n+1}) \leqslant TV(u^n) + TV(v^n),$$

which proves Lemma 5. $\square$

Lemma 6 (Discrete entropy inequalities)**.** *We assume (4), (6) and the CFL condition (23). Then there exists $C > 0$ such that the numerical approximations u_k^n and v_k^n obtained by the scheme (S_Δ) satisfiy the following discrete entropy inequality for $n \geqslant 0$ and $k \in [\![1, N]\!]$*

$$\big[|u_k^{n+1} + v_k^{n+1} - (h(\kappa)+\kappa)| - |u_k^n + v_k^n - (h(\kappa)+\kappa)|\big] + \lambda\big[G_{k+1/2}^n - G_{k-1/2}^n\big] \leqslant 0 \tag{58}$$

where

$$G_{k+1/2} = h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_k^n \perp \kappa) - v_k^n \perp \kappa) \tag{59}$$

Proof. We recall that

$$a \top b = \max(a,b), \quad a \perp b = \min(a,b), \quad |a-b| = a \top b - a \perp b.$$

We have

$$(u_k^{n+1} + v_k^{n+1}) \top (h(\kappa)+\kappa) \leqslant u_k^{n+1} \top h(\kappa) + v_k^{n+1} \top \kappa$$

and the monotonicity of $\mathcal{U}_\lambda$ and $\mathcal{V}_\lambda$ implies that

$$u_k^{n+1} \top h(\kappa) + v_k^{n+1} \top \kappa \leqslant \mathcal{U}_\lambda(u_k^n \top h(\kappa), v_{k-1}^n \top \kappa, v_k^n \top \kappa) + \mathcal{V}_\lambda(u_k^n \top h(\kappa), v_k^n \top \kappa). \tag{60}$$

Rearranging the terms of (60) leads to

$$(u_k^{n+1} + v_k^{n+1}) \top (h(\kappa)+\kappa) \leqslant u_k^n \top h(\kappa) + v_k^n \top \kappa - \lambda\{h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_{k-1}^n \top \kappa) - v_{k-1}^n \top \kappa)\} \tag{61}$$

and for the same reason we have

$$(u_k^{n+1} + v_k^{n+1}) \perp (h(\kappa)+\kappa) \geqslant u_k^n \perp h(\kappa) + v_k^n \perp \kappa - \lambda\{h(v_k^n \perp \kappa) - v_k^n \perp \kappa - (h(v_{k-1}^n \perp \kappa) - v_{k-1}^n \perp \kappa)\} \tag{62}$$

The subtraction of (62) to (61) gives

$$\big[|u_k^{n+1} + v_k^{n+1} - (h(\kappa)+\kappa)| - |u_k^n + v_k^n - (h(\kappa)+\kappa)|\big] + \lambda\big[G_{k+1/2}^n - G_{k-1/2}^n\big] \leqslant 0,$$

The result of Lemma 6 thus holds. $\square$

We define for all vector $(u_k)_{k\in\mathbb{Z}}$

$$\|u\|_1 = \Delta x \sum_{k=1}^{N} |u_k|.$$

Lemma 7 (Control of the deviation with respect to equilibrium)**.** *We assume (4), (6) and the CFL condition (23). Then there is $C<1$ such that the discrepancy to equilibrium for the numerical approximations u_k^n and v_k^n obtained by the scheme (S_Δ) is controlled in the L^1 norm by*

$$\|h(u^n) - v^n\|_1 \leqslant C^n \|h(u^0) - v^0\|_1 + (\mu-1)\Delta x \frac{1-C^n}{1-C} \sup_{0\leqslant m\leqslant n-1} (TV(v^m) + 2|v^m|). \tag{63}$$

In particular, since $\lambda \leqslant \dfrac{1}{\mu}$ and $\mu > 1$, we have $-1 < C < 1$.

Proof. The Rolle's theorem gives us the existence of $\xi_k^n \in (v_k^n, v_k^{n+1})$ such that

$$h(v_k^{n+1}) = h(v_k^n) + h'(\xi_k^n)\,(v_k^{n+1} - v_k^n), \tag{64}$$

Combinig (64) with (S_Δ) gives us

$$h(v_k^{n+1}) - u_k^{n+1} = \{1 - \lambda(1 + h'(\xi_k^n))\}\,(h(v_k^n) - u_k^n) + \lambda\,\{h(v_k^n) - h(v_{k-1}^n) - (v_k^n - v_{k-1}^n)\}$$

We use a triangle inequality, multiply by Δx and sum from 1 to N to get

$$\|h(v^{n+1}) - u^{n+1}\|_1 \leqslant C\|h(v^n) - u^n\|_1 + (\mu-1)\Delta x\;(TV(v^n) + |v_1^n - v_0^n|),$$

which implies the estimate (63) by induction on n.

□

Proposition 6 (Convergence of the numerical scheme)**.** *We assume (4), (5), (6) and the CFL condition (23), then the sequence of numerical approximations $(u_k^n, v_k^n)_{k,n}$ obtained through Scheme (S_Δ) converges towards $(u,v) \in C\left((0,T);\mathbf{L}^1[0,L]\right) \cap \mathbf{L}^\infty\left([0,T];BV[0,L]\right)$ in $\mathbf{L}^1_{\mathrm{loc}}$ when $\Delta t \to 0$, up to a subsequence.*

Proof. of Proposition 6. We apply the same method used for Theorem 3. The sequences $(u_\Delta)_\Delta$ and $(v_\Delta)_\Delta$ satisfy the estimate 1 required for applying Lemma 3 (see Lemma 5) and we have, using (45) and Lemma 5,

$$\begin{aligned}
\int_{[0,L]} |s_\Delta(t,x) - s_\Delta(\tau,x)|dx &= \sum_{k=1}^{N} \Delta x |s_k^n - s_k^m| \leqslant \sum_{\ell=n}^{m-1}\sum_{k=1}^{N} \Delta x |s_k^{\ell+1} - s_k^{\ell}| \\
&\leqslant \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^{N} \left\{(h-Id)(v_k^n) - (h-Id)(v_{k-1}^n)\right\} \\
&\leqslant (\mu-1)\left(TV(u^0) + TV(v^0)\right) \sum_{\ell=n}^{m-1} \Delta t \\
&\leqslant L_s(|t-\tau| + \Delta t),
\end{aligned}$$

with $L_s = (\mu - 1)\left(TV(u^0) + TV(v^0)\right)$. We also have using (45) and Lemma 7

$$\begin{aligned}
\int_{[0,L]} |v_\Delta(t,x) - v_\Delta(\tau,x)|dx &= \sum_{k=1}^{N} \Delta x |v_k^n - v_k^m| \leqslant \sum_{\ell=n}^{m-1} \sum_{k=1}^{N} \Delta x |v_k^{\ell+1} - v_k^\ell| \\
&\leqslant \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^{N} \left\{ h(v_n^\ell) - u_k^\ell \right\} \\
&\leqslant \sum_{\ell=n}^{m-1} \Delta t \sum_{k=1}^{N} \frac{1}{\mu} \frac{2}{1-C} \left(TV(u^0) + TV(v^0)\right) \\
&\leqslant L_v(|t-\tau| + \Delta t),
\end{aligned}$$

with $L_v = \dfrac{1}{\mu}\dfrac{2}{1-C}\left(TV(u^0) + TV(v^0)\right)$. Thus, since $u_\Delta = s_\Delta - v_\Delta$, we have the same type of estimate for u_Δ and Lemma 3 guarantees the existence of u and v belonging to $\mathbf{L}^\infty([0,T]\times[0,L]) \cap \mathcal{C}((0,T);\mathbf{L}^1([0,L]))$, such that $TV(u(t,.)) \leqslant C$, $TV(v(t,.)) \leqslant C$ and such that

$$\lim_{\Delta\to 0} u_\Delta = u, \quad \lim_{\Delta\to 0} v_\Delta = v, \qquad \mathbf{L}^1([0,T]\times[0,L]).$$

We prove now that the limit v is an entropy solution to (S_0) and that $u = h(v)$ almost everywhere. To do so, we assume first that the problem is posed on the whole space line, i.e. that $x \in \mathbb{R}$. We begin by rewriting the discrete entropy inequality we proved on Lemma 6

$$\Delta x\big[|u_k^{n+1} + v_k^{n+1} - (h(\kappa) + \kappa)| - |u_k^n + v_k^n - (h(\kappa)+\kappa)|\big] + \Delta t\big[G_{k+1/2}^n - G_{k-1/2}^n\big] \leqslant 0. \tag{65}$$

Let us consider a nonnegative test function $\varphi \in \mathcal{C}_c^\infty([0,T]\times\mathbb{R})$, with $T>0$. We introduce

$$\varphi_k^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(n\Delta t, x)\, dx.$$

We multiply (65) by φ_k^n and sum over $n \in \mathbb{N}$ and $k \in [1,N]$. Therefore, if we define

$$A_{\Delta t} = \Delta x \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \big[|u_k^{n+1} + v_k^{n+1} - (h(\kappa)+\kappa)| - |u_k^n + v_k^n - (h(\kappa)+\kappa)|\big]\varphi_k^n, \tag{66}$$

$$B_{\Delta t} = \Delta t \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \big[G_{k+1/2}^n - G_{k-1/2}^n\big]\varphi_k^n, \tag{67}$$

the discrete inequality entropy becomes

$$A_{\Delta t} + B_{\Delta t} \leqslant 0. \tag{68}$$

The goal is to pass inequality (68) to the limit as Δt goes to zero and to check that v satisfies the classical continuous entropy inequality associated with (S_0). Let us begin with proving that

$$A_{\Delta t} \to A_0 =: -\int_{\mathbb{R}^+}\int_{\mathbb{R}} |h(v)+v-(h(\kappa)+\kappa)|\partial_t\varphi\, dx\, dt - \int_{\mathbb{R}} |u^0+v^0-(h(\kappa)+\kappa)|\varphi(0,x)\, dx \text{ when } \Delta t \to 0.$$

First, we split $A_{\Delta t}$ into two parts: $A_{\Delta t} = \bar{A}_{\Delta t} + \tilde{A}_{\Delta t}$, with

$$\bar{A}_{\Delta t} = \Delta x \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \big[|h(v_k^{n+1}) + v_k^{n+1} - (h(\kappa)+\kappa)| - |h(v_k^n) + v_k^n - (h(\kappa)+\kappa)|\big]\varphi_k^n,$$

$$\tilde{A}_{\Delta t} = \Delta x \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \big[R(u_k^{n+1}, v_k^{n+1}, \kappa) - R(u_k^n, v_k^n, \kappa)\big]\varphi_k^n,$$

with

$$R(u,v,\kappa) = |u+v-(h(\kappa)+\kappa)| - |h(v)+v-(h(\kappa)+\kappa)|.$$

The convergence of $\bar{A}_{\Delta t}$ is classical. Indeed, using the Abel rule (discrete integration by part),

$$\begin{aligned}\bar{A}_{\Delta t} &= -\Delta x \sum_{k\in\mathbb{Z}} \left[|h(v_k^0) + v_k^0 - (h(\kappa)+\kappa)|\varphi_k^0 + \sum_{n\in\mathbb{N}} |h(v_k^{n+1}) + v_k^{n+1} - (h(\kappa)+\kappa)|(\varphi_k^{n+1} - \varphi_k^n) \right] \\ &= -\int_{\mathbb{R}} \Big[|h(v_{\Delta t}(0,x)) + v_{\Delta t}(0,x) - (h(\kappa)+\kappa)|\varphi(0,x)\ dx \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}^+} |h(v(t+\Delta t,x)) + v(t+\Delta t,x) - (h(\kappa)+\kappa)| \frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t} dt\ dx\end{aligned}$$

Since the two terms converge strongly, we obtain

$$\bar{A}_{\Delta t} \to -\int_{\mathbb{R}^+}\int_{\mathbb{R}} |h(v)+v-(h(\kappa)+\kappa)|\partial_t\varphi\ dx\ dt - \int_{\mathbb{R}} |h(v^0)+v^0-(h(\kappa)+\kappa)|\varphi(0,x)\ dx$$

when $\Delta t \to 0$. We obtain for $\tilde{A}_{\Delta t}$ by similar calculations

$$\tilde{A}_{\Delta t} = -\Delta x \sum_{k\in\mathbb{Z}} R(u_k^0, v_k^0, \kappa)\varphi_k^0 - \Delta x \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}} R(u_k^{n+1}, v_k^{n+1}, \kappa)(\varphi_k^{n+1} - \varphi_k^n)$$

which is equal to

$$-\int_{\mathbb{R}} R(u_{\Delta t}(0,x), v_{\Delta t}(0,x), \kappa)\varphi(0,x) dx - \int_{\mathbb{R}}\int_{\mathbb{R}^+} R(u_{\Delta t}(t+\Delta t), v_{\Delta t}(t+\Delta t), \kappa)\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t} dt\ dx.$$

It is straightforward that

$$-\int_{\mathbb{R}} R(u_{\Delta t}(0,x), v_{\Delta t}(0,x), \kappa)\varphi(0,x) dx \to -\int_{\mathbb{R}} \big[|u^0+v^0-(h(\kappa)+\kappa)| - |h(v^0)+v^0-(h(\kappa)+\kappa)| \big] \varphi(0,x)\ dx$$

while for the last term, we have

$$\begin{aligned}&\left| \int_{\mathbb{R}}\int_{\mathbb{R}^+} R(u_{\Delta t}(t+\Delta t), v_{\Delta t}(t+\Delta t), \kappa)\frac{\varphi(t+\Delta t,x)-\varphi(t,x)}{\Delta t} dt\ dx \right| \\ &\qquad\qquad \leqslant \|\partial_t\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+\times\mathbb{R})}(\mu+1)T \sup_{t\geqslant 0} \|u_{\Delta t} - h(v_{\Delta t})\|_{\mathbf{L}^1(\mathbb{R})}(t),\end{aligned}$$

which tends to 0 when $\Delta t \to 0$ using Lemma 7. Gathering all these results gives us $A_{\Delta t} \to A_0$.

Let us now focus on $B_{\Delta t}$. One may first remark that

$$G_{k+1/2}^n = h(v_k^n \top \kappa) - v_k^n \top \kappa - (h(v_k^n \perp \kappa) - v_k^n \perp \kappa)$$

since h is increasing. Therefore, we have

$$B_{\Delta t} = -\Delta t \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{Z}} G_{k-1/2}\ (\varphi_k^n - \varphi_{k-1}^n)$$

which is equal to

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}}\Big\{(h(v_{\Delta t}(n\Delta t,x)\top\kappa)-v_{\Delta t}(n\Delta t,x)\top\kappa-(h(v_{\Delta t}(n\Delta t,x)\perp\kappa)-v_{\Delta t}(n\Delta t,x)\perp\kappa) \\ \frac{\varphi(n\Delta t,x)-\varphi(n\Delta t,x-\Delta x)}{\Delta x}\Big\}dx\ dt$$

and tends toward

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}}(h(v_{\Delta t}\top\kappa)-v_{\Delta t}\top\kappa-(h(v_{\Delta t}\perp\kappa)-v_{\Delta t}\perp\kappa)(t,x)\ \partial_x\varphi(t,x)\ dx\ dt.$$

As this stage, we have proved that any limit of the numerical scheme (S_Δ) satisfies the entropy inequalities (10), restricting the support of φ to $[0,T)\times(0,L)$. Concerning the second point of Definition 2, one can use the Otto's formalism and invoke a more general result of convergence, see for instance [21]. □

This ends the proof to Theorem 4.

6 Numerical illustrations

We now present some numerical results which illustrate the good behavior of our numerical scheme. We only focus on the case of a linear source term

$$h(v)=\mu v,$$

with $\mu=3$, and the boundary conditions are

$$u_\varepsilon(0,t)=1 \text{ and } v_\varepsilon(L,t)=\alpha u_\varepsilon(L,t),$$

with $\alpha=0.1$ and $L=1$. We choose the same initial data for all experiments:

$$\forall x\in(0,L),\quad u_\varepsilon(x,0)=u_\varepsilon(x,0)=1.$$

Let us note that this initial data is neither compatible with the left boundary condition, nor with the equilibrium $u=h(v)$. As a result, we expect to see a right-going wave initiated by the left boundary condition and also a boundary layer at the right boundary.

We compare our numerical scheme (called the AP scheme in the sequel) with the classical splitting method, using an implicit Euler method for the source term:

$$\begin{cases} u_k^{n+1/2}=u_k^n-\lambda(u_k^n-u_{k-1}^n)\\ v_k^{n+1/2}=v_k^n+\lambda(v_{k+1}^n-v_k^n)\end{cases}$$
$$\begin{cases} u_k^{n+1}=u_k^{n+1/2}+\frac{\Delta t}{\varepsilon}(h(v_k^{n+1})-u_k^{n+1})\\ v_k^{n+1}=v_k^{n+1/2}+\frac{\Delta t}{\varepsilon}(u_k^{n+1}-h(v_k^{n+1}))\end{cases} \tag{69}$$

where the second part can be explicitly solved since the source term is linear. In all the numerical tests, we used

$$\Delta t=\Delta x/3.$$

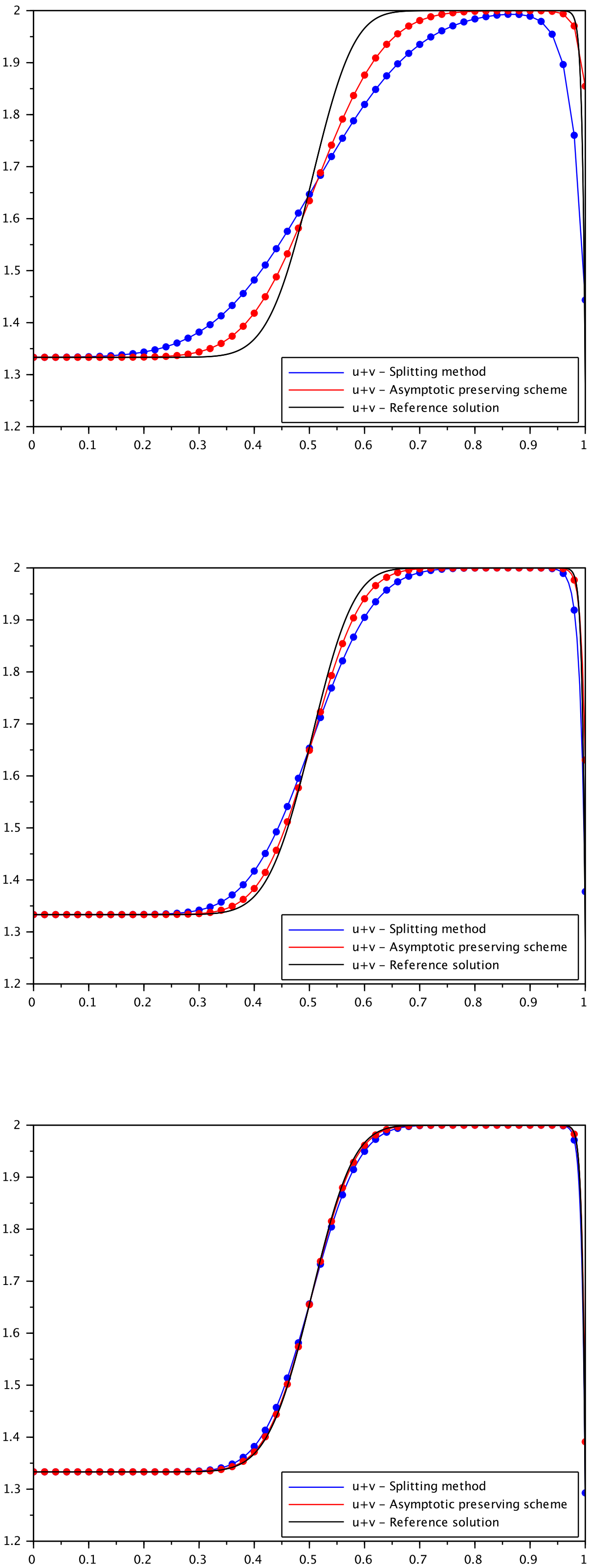


Figure 2: Comparison of the AP scheme and of the splitting method with a reference solution for several mesh sizes: 50 (up), 200 (center), 800 (bottom) — $u + v$ *vs.* space.

6.1 Different mesh sizes for the relaxation model

The first test illustrates the accuracy of both schemes according to the number of cells: we use successively 50, 200 and 800 cells, while $\varepsilon = 10^{-2}$ and $T = 1$.

In Figure 2, one can check that both schemes seem to converge towards the same profile, the reference solution, which has been computed using the AP scheme with 3000 cells. The splitting method is clearly more diffusive than the AP scheme, in particular when the number of cells is small. The wave and the boundary layer are better approximated by the AP scheme.

6.2 Behavior of the relaxation boundary layer

In this test, we study how the numerical schemes approximate the boundary layer at the right boundary. To do so, we use a final time T equal to 5 which corresponds to a stationary solution. Three values of ε are used: 10^{-1}, 10^{-2} and 10^{-5}. The number of cells is 100 for all the tests, so that when $\varepsilon = 10^{-5}$, one may expect an under-resolved boundary layer.

In Figure 3, we have plotted the results provided by the schemes, and a reference solution computed by the AP scheme with 1000 cells. Let us mention that we represent only the right part of the domain in order to better see the differences. For $\varepsilon = 10^{-1}$, the profiles provided by the two numerical schemes are similar but one can note that the point at the right boundary given by the AP scheme is significantly greater than the points obtained by the splitting method and the reference solution. When ε is equal to 10^{-2}, this difference increases. However, the shape of the boundary layer is much better approximated by the AP scheme than by the splitting method. The case of $\varepsilon = 10^{-5}$ leads to much larger discrepancies. The boundary layer of the reference solution is so tiny that it cannot be seen, so that in the figure, one can only see a constant state. The AP scheme provides the same constant state, this is due to its upwind nature when ε is very small. On the contrary, the splitting method leads to a large numerical boundary layer, which would remain even for $\varepsilon = 0$. The only way to make it disappear would be to let the number of cells tend to infinity.

6.3 Numerical results for the equilibrium case

We now investigate the behavior of both schemes when $\varepsilon = 0$, using different numbers of cells. Results are plotted at time $T = 1$. Up and at the center of Figure 4, the results of the splitting method and of the AP scheme are shown, for the unknown $u + v$. One can check that no boundary layer is present with the AP scheme. The splitting method, which still suffer from a dependence of the right boundary condition, provides a numerical boundary layer at the right, which reduces when the number of cells increases. Moreover, since the AP scheme is an upwind scheme when $\varepsilon = 0$, it is less diffusive and more accurate that the splitting method.

The figure at the bottom represents $|h(v) - u|$ in order to understand of the results are far from the equilibrium. Since the AP scheme is fully explicit and do not use the inverse of function h, one cannot expect to be exactly at the equilibrium. The gap from the equilibrium appears near the discontinuity and disappears when the number of cells increases.

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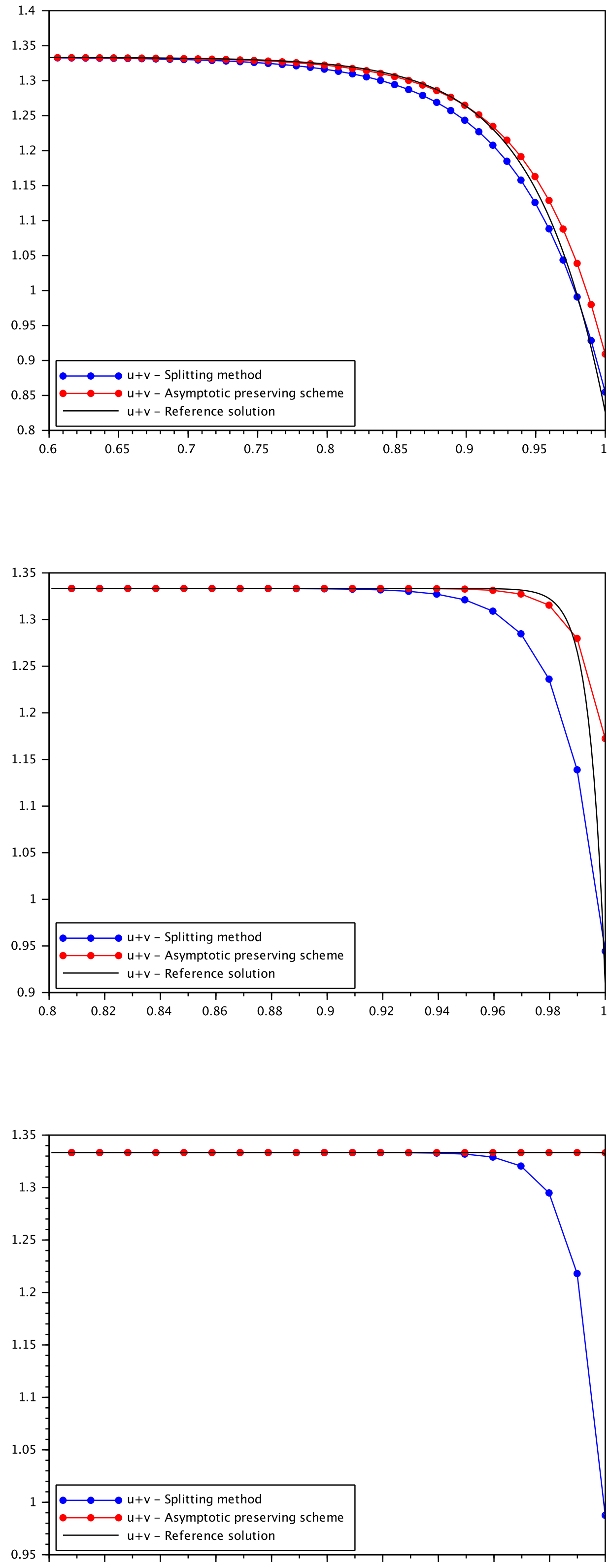

Figure 3: Comparison of the AP scheme and of the splitting method with a reference solution for several values of ε: 10^{-1} (up), 10^{-2} (center), 10^{-5} (bottom) — $u+v$ *vs.* space.

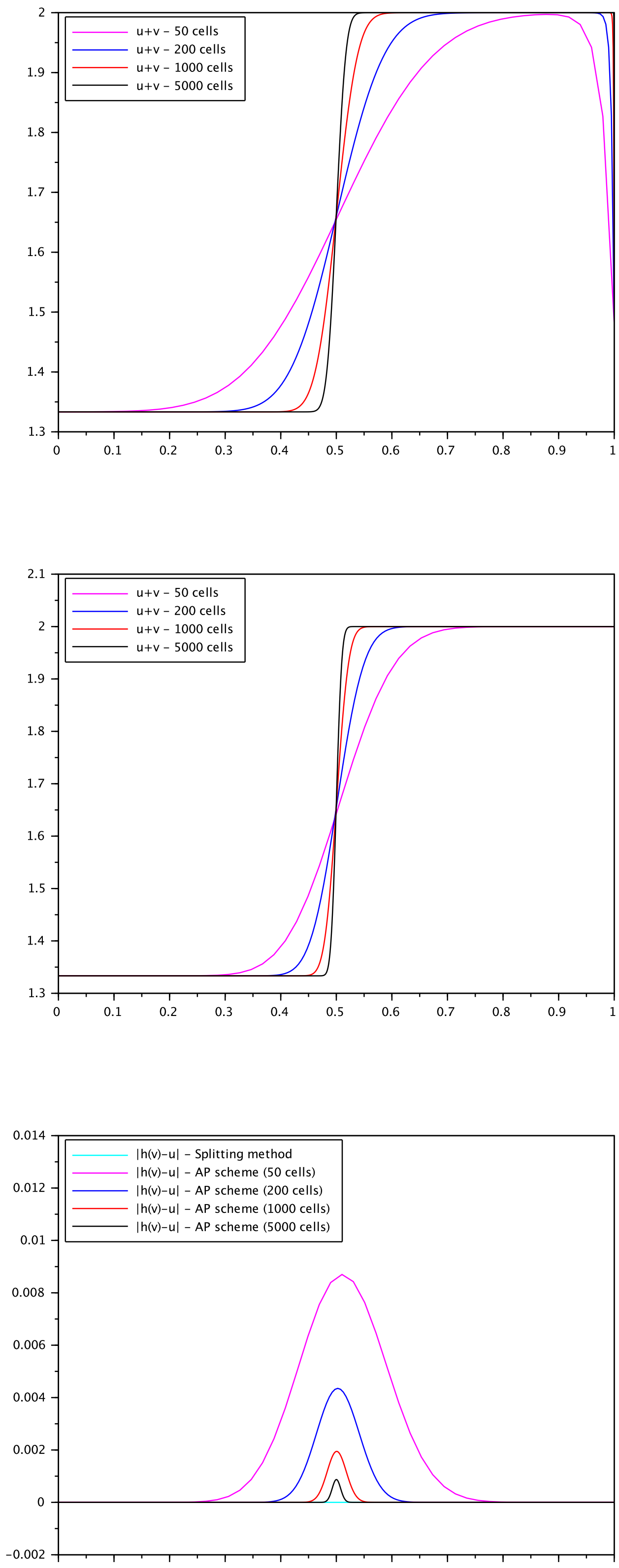

Figure 4: Results provided by the splitting method (left: $u + v$) and by AP scheme (center: $u + v$, right: $|h(v) - u|$), for several mesh sizes: 50, 200, 1000, 5000 cells.