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PROOF OF THE MAGNETIC GAP-LABELLING CONJECTURE FOR PRINCIPAL SOLENOIDAL TORI

MOULAY TAHAR BENAMEUR AND VARGHESE MATHAI

ABSTRACT. In this note, we prove the magnetic spectral gap-labelling conjecture as stated in [7], in all dimensions, for principal solenoidal tori.

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1. INTRODUCTION

In a recent paper [7], the authors formulated for the first time the magnetic gap-labelling conjecture (MGL). More precisely, given a constant magnetic field on Euclidean space \mathbb{R}^p determined by a skew-symmetric $(p \times p)$ matrix Θ , and a \mathbb{Z}^p -invariant probability measure μ on the disorder set Σ which is by hypothesis a Cantor set, where the action of \mathbb{Z}^p is assumed to be minimal, the corresponding Integrated Density of States of any self-adjoint operator affiliated to the twisted crossed product algebra $C(\Sigma) \rtimes_{\sigma} \mathbb{Z}^p$, where σ is the multiplier on \mathbb{Z}^p associated to Θ , takes on values on spectral gaps in the *magnetic gap-labelling group* cf. Definition 1. The *magnetic frequency group* cf. Definition 1 is defined as an explicit countable subgroup of \mathbb{R} involving Pfaffians of Θ and its sub-matrices. Our the magnetic gap-labelling conjecture states that the magnetic gap labelling group is a subgroup of the magnetic frequency group. In [7], we gave evidence for the validity of our conjecture in 2D, 3D, the Jordan block diagonal case and the periodic case in all dimensions. The goal of this

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note is to give a complete proof of the magnetic gap-labelling conjecture in all dimensions, for principal solenoidal tori, see Theorem 2 in section 5. This result is obtained as a byproduct of a stronger result relating the magnetic gap-labelling conjecture with the integrality of the Chern character.

Historically speaking, the gap-labelling theorem was originally conjectured by Bellissard [2] in the late 1980s. It concerns the labelling of gaps in the spectrum of a Schrödinger operator (in the absence of a magnetic field) by the elements of a subgroup of \mathbb{R} which results from pairing the K_0 -group of the noncommutative analog for the Brillouin zone with the tracial state defined by the probability measure on the hull. The problem arises in a mathematical version of solid state physics in the context of aperiodic tilings. Its three proofs, discovered independently by the authors of [8, 13, 5] all concern the proof of a statement in K-theory. See also [9]. Earlier results include the proof of the gap-labelling conjecture in 1D [3], 2D [4, 18] and in 3D [6]. We remark that Bellissard's conjecture requires that the gap labelling group is equal to the frequency group, as opposed to the subgroup relationship in the case when the magnetic field is non-zero.

The main results of the paper are in sections 3 and 5. In section 3, a proof of MGL is given assuming the integrality of the Chern character, an independent result, which extends the results of [9] and which was expected by many experts in the area. In section 5, the magnetic gap-labelling conjecture for principal solenoidal tori is proved in full generality as an application of the previous result, by showing that the Chern character is indeed integral in this principal case. The MGL conjecture remains open for general solenoidal tori, see comments after the statement of Theorem 2.

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2. CONJECTURES ABOUT THE MAGNETIC GAP-LABELLING GROUP

We review in this first section the magnetic gap-labelling conjecture as stated in [7], that we shall call MGL.

2.1. Some notations. Assume that \mathbb{Z}^p acts minimally on a Cantor set Σ and let μ be a \mathbb{Z}^p -invariant probability measure on Σ . The subgroup of \mathbb{R} which is generated by μ -measures of clopen subspaces of Σ is denoted $\mathbb{Z}[\mu]$. This is known as the group of frequencies of the aperiodic potential associated with the quasi-crystal, i.e. appearing in the Fourier expansion of that potential. It can also be seen as the image under (the integral associated with) the probability measure μ of $C(\Sigma, \mathbb{Z})$, the group of continuous integer valued functions on Σ .

That is,

$$\mathbb{Z}[\mu] = \left\{ \int_{\Sigma} f(z) d\mu(z) \mid f \in C(\Sigma, \mathbb{Z}) \right\} = \mu(C(\Sigma, \mathbb{Z}))$$

Recall that if the group \mathbb{Z}^p acts on a module M , then the *coinvariants* of M is defined as the quotient

$$M_{\mathbb{Z}^p} := M / \{m - gm \mid m \in M, g \in \mathbb{Z}^p\},$$

and the *invariants* of M is defined as

$$M^{\mathbb{Z}^p} := \{m \in M \mid m = gm \text{ for all } g \in \mathbb{Z}^p\}.$$

Let I be an ordered subset of $\{1, \dots, p\}$ with an even number of elements, and let, according to the previous notations, $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ denote the coinvariants of $C(\Sigma, \mathbb{Z})$ under the action of the subgroup \mathbb{Z}^{I^c} of \mathbb{Z}^p , where I^c denotes the index set that is the complement to I . Let similarly $(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}})^{\mathbb{Z}^I}$ denote the subset of $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ composed of those \mathbb{Z}^{I^c} -coinvariant classes in $C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}}$ which are invariant under the induced action of the subgroup \mathbb{Z}^I , or equivalently invariant under the action of the whole group \mathbb{Z}^p . Define

$$\mathbb{Z}_I[\mu] = \mu \left((C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}})^{\mathbb{Z}^I} \right).$$

Notice that

$$\mathbb{Z}_{\{1, \dots, p\}}[\mu] = \mathbb{Z} \subset \mathbb{Z}_I[\mu] \subset \mathbb{Z}[\mu] = \mathbb{Z}_{\emptyset}[\mu].$$

2.2. Labelling the gaps. Let σ be a multiplier of \mathbb{Z}^p which is associated with the skew symmetric matrix Θ . The \mathbb{Z}^p invariant probability measure μ yields a regular trace τ^μ on the twisted crossed product C^* -algebra $C(\Sigma) \rtimes_{\sigma} \mathbb{Z}^p$, which is by fiat the operator norm completion of the $*$ -algebra of compactly supported continuous functions $C_c(\mathbb{Z}^p \times \Sigma)$ acting via the left regular representation on the Hilbert space $L^2(\Sigma, d\mu) \otimes \ell^2(\mathbb{Z}^d)$. The trace τ^μ is defined on the dense subalgebra $C_c(\mathbb{Z}^p \times \Sigma)$ by the equality

$$(1) \quad \tau^\mu(f) = \langle \mu, f_0 \rangle = \int_{\Sigma} f_0(z) d\mu(z), \quad \text{where } f_0 : z \mapsto f(0, z),$$

with 0 the zero element of \mathbb{Z}^p . Hence τ^μ induces a trace

$$\tau^\mu (= \tau_*^\mu) : K_0(C(\Sigma) \rtimes_{\sigma} \mathbb{Z}^p) \longrightarrow \mathbb{R}.$$

The C^* -algebra $C(\Sigma) \rtimes_{\sigma} \mathbb{Z}^p$ (or rather a Morita equivalent one) is a receptacle for the spectral projections onto spectral gaps of the magnetic Schrödinger operator associated with our system, as explained earlier. Any gap in its spectrum may therefore be labelled by the trace of the corresponding projection.

Definition 1. The *magnetic gap-labelling group* is defined to be the range of the trace τ^μ

$$\text{Range}(\tau^\mu) := \tau^\mu(K_0(C(\Sigma) \rtimes_{\sigma} \mathbb{Z}^p)) \subset \mathbb{R}.$$

The *magnetic frequency group* is defined to be the countable subgroup of \mathbb{R} given by:

$$\sum_{0 \leq |I| \leq p, |I| \text{ even}} \text{Pf}(\Theta_I) \mathbb{Z}_I[\mu],$$

where $\text{Pf}(\Theta_I)$ denotes the Pfaffian of the skew symmetric $(|I| \times |I|)$ submatrix Θ_I of Θ , whose (i, j) entry is $\Theta_{(i,j)}$ for all $i, j \in I$.

We next recall the formulation of our magnetic gap labelling conjecture from [7], and later give evidence for its validity.

Conjecture 1 (Magnetic gap-labelling conjecture (MGL)). *Let Σ be a Cantor set with a minimal action of \mathbb{Z}^p that preserves a Borel probability measure μ . Let σ be the multiplier on \mathbb{Z}^p associated to a skew-symmetric $(p \times p)$ matrix Θ . Then the magnetic gap-labelling group is contained in the magnetic frequency group.*

When the magnetic field is trivial, the magnetic gap-labelling group coincides with the now classical gap-labelling group, while the magnetic frequency group reduces to the group $\mathbb{Z}[\mu] = \mathbb{Z}_0[\mu]$ recalled in Subsection 2.1. In this case, our conjecture does reduce to the Bellissard conjecture [2] which predicts the equality of these two groups. Indeed, $\mathbb{Z}[\mu]$ is then obviously contained in the gap-labelling group, see [8, 9].

3. PROOF OF MGL, ASSUMING INTEGRALITY OF THE CHERN CHARACTER

The integrality hypothesis: (IH) *The range of the Chern character $Ch : K^p(X) \rightarrow H^{[p]}(X, \mathbb{Q}) \simeq \bigoplus_{k \geq 0} H^{p+2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Q}))$ is contained in $H^{p+2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Z}))$.*

We will show in the last section that (IH) is satisfied for principal solenoidal tori in all dimensions.

Theorem 1. *Suppose that the integrality hypothesis (IH) stated above is satisfied, then Conjecture 1, is true.*

Proof. By using the twisted index theorem (cf. [7]), we are reduced to proving that for a given multi-index $I := (i_1, \dots, i_{2k})$, the range of the map

$$\cup \psi_I := \cup(\psi_{i_1} \cup \dots \cup \psi_{i_{2k}}) : H^{p-2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Z})) \rightarrow H^p(\mathbb{Z}^p, C(\Sigma, \mathbb{Z})) \simeq C(\Sigma, \mathbb{Z})_{\mathbb{Z}^p},$$

lies inside $(C(\Sigma, \mathbb{Z})_{\mathbb{Z}^{I^c}})^I$.

Our method is an induction on p . We denote by $(T_i)_{1 \leq i \leq p}$ the generators of the action of \mathbb{Z}^p on Σ . We have the following exact sequences (cf. [10])

$$0 \rightarrow H^{p-2k-1}(\langle T_i, i \neq i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_2} \rangle} \rightarrow H^{p-2k}(\mathbb{Z}^p, C(\Sigma, \mathbb{Z})) \rightarrow H^{p-2k}(\langle T_i, i \neq i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_2} \rangle} \rightarrow 0$$

and

$$0 \rightarrow H^{p-2k-1}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_1} \rangle} \longrightarrow H^{p-2k}(\langle T_i, i \neq i_2 \rangle, C(\Sigma, \mathbb{Z})) \longrightarrow H^{p-2k}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_1} \rangle} \rightarrow 0.$$

It is then clear that the map $\cup\psi_I$ vanishes on the range of $H^{p-2k-1}(\langle T_{i_j}, j \neq 2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_2} \rangle}$ in the first exact sequence and yields a well defined morphism

$$\alpha_I : H^{p-2k}(\langle T_i, i \neq i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_2} \rangle} \longrightarrow C(\Sigma, \mathbb{Z})_{\mathbb{Z}^p}.$$

Moreover, this latter map is the restriction of $\cup(\psi_{i_2} \cup \dots \cup \psi_{i_{2k}})$ to the invariants $H^{p-2k}(\langle T_i, i \neq i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_2} \rangle}$. Now a careful inspection of the second exact sequence allows to deduce that $\cup(\psi_{i_2} \cup \dots \cup \psi_{i_{2k}})$ vanishes on the range of $H^{p-2k-1}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_1} \rangle}$ in this second exact sequence. Therefore, again the map $\cup(\psi_{i_2} \cup \dots \cup \psi_{i_{2k}})$ induces a well defined morphism from $(H^{p-2k}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_1} \rangle})_{\langle T_{i_1} \rangle}$ to $C(\Sigma, \mathbb{Z})_{\langle T_i, i \neq i_2 \rangle}$, which is easily seen to coincide with the restriction to

$$(H^{p-2k}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z}))_{\langle T_{i_1} \rangle})_{\langle T_{i_1} \rangle}$$

of the morphism

$$\cup(\psi_{i_3} \cup \dots \cup \psi_{i_{2k}}) : H^{p-2k}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z})) \longrightarrow H^{p-2}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z})).$$

Notice that $H^{p-2}(\langle T_i, i \neq i_1, i_2 \rangle, C(\Sigma, \mathbb{Z})) \simeq C(\Sigma, \mathbb{Z})_{\langle T_i, i \neq i_1, i_2 \rangle}$. Applying the induction hypothesis, we can deduce that the range of this latter factorizes through

$$(C(\Sigma, \mathbb{Z})_{\langle T_i, i \neq i_1, \dots, i_{2k} \rangle})_{\langle T_{i_j}, j=3, \dots, 2k \rangle}.$$

So the restriction to the invariants under T_{i_1} and T_{i_2} , allows to deduce that the range of the original map $\cup\psi_I$ is contained in

$$(C(\Sigma, \mathbb{Z})_{\langle T_i, i \neq i_1, \dots, i_{2k} \rangle})_{\langle T_{i_j}, j=1, \dots, 2k \rangle},$$

hence the conclusion. □

4. INTEGRALITY OF THE CHERN CHARACTER: THE CASE OF THE TORUS

The integrality of the Chern character for the torus is standard. To be self-contained, and in the spirit of our approach to the gap-labelling problem, we give a proof based on the index theorem which could be known to the experts. For n even, let $E \rightarrow \mathbb{T}^n$ be a complex vector bundle over the torus, and $\not\partial_E$ denote the Dirac operator twisted by a connection on E . Then since the torus is flat, the Atiyah-Singer index theorem [1] says that

$$index(\not\partial_E) = \int_{\mathbb{T}^n} Ch(E).$$

We deduce that the top degree component of the Chern character $Ch_n(E)$ is integral.

Let $L_{ij} \rightarrow \mathbb{T}^n$ denote the line bundle over the torus such that $c_1(L_{ij}) = dx_i \wedge dx_j$. Then

$$\text{index}(\not{D}_{E \otimes L_{ij}}) = \int_{\mathbb{T}^n} Ch(E)(1 + dx_i \wedge dx_j)$$

since $(dx_i \wedge dx_j)^k = 0$ for $k > 1$. Since this is true for all $1 \leq i, j \leq n$, we conclude that the $(n-2)$ degree component of the Chern character, $Ch(E)_{n-2}$ is integral. By induction, we see that all the components of $Ch(E)$ are integral.

In the case when n is odd, consider the path of self adjoint Dirac operators \not{D}_t , $t \in [0, 1]$, where $\not{D}_t = (1-t)\not{D} \otimes I_N + tU(\not{D} \otimes I_N)U^*$, where $U : \mathbb{T}^n \rightarrow U(N)$ is a smooth map that represents an element in $K^1(\mathbb{T}^n)$. Then the spectral flow of the family \not{D}_t , $t \in [0, 1]$ is given by the Atiyah-Singer index theorem [1], [11]

$$SF(\not{D}_t) = \int_{\mathbb{T}^n} Ch(U)$$

Again we conclude that the top degree component of the Chern character $Ch_n(U)$ is integral.

Let $\not{D}_{L_{ij}t}$, $t \in [0, 1]$ denote the path of self adjoint Dirac operators

$$(1-t)\not{D}_{L_{ij}} \otimes I_N + tU(\not{D}_{L_{ij}} \otimes I_N)U^*.$$

Then the spectral flow of the family $\not{D}_{L_{ij}t}$, $t \in [0, 1]$ is given by the Atiyah-Singer index theorem,

$$SF(\not{D}_{L_{ij}t}) = \int_{\mathbb{T}^n} Ch(U)(1 + dx_i \wedge dx_j)$$

Since this is true for all $1 \leq i, j \leq n$, we conclude that the $(n-2)$ degree component of the Chern character, $Ch(U)_{n-2}$ is integral. By induction, we see that all the components of $Ch(U)$ are integral.

5. INTEGRALITY OF THE CHERN CHARACTER FOR PRINCIPAL SOLENOIDAL TORI

For each $j \in \mathbb{N}$, let $X_j = \mathbb{T}^n$. Define the finite regular covering map $f_{j+1} : X_{j+1} \rightarrow X_j$ to be a finite covering map such that degree of f_{j+1} is greater than 1 for all j . For example, let $f_{j+1}(z_1, \dots, z_n) = (z_1^{p_1}, \dots, z_n^{p_n})$, where each $p_j \in \mathbb{N} \setminus \{1\}$. Set (X_∞, f_∞) to be the inverse limit, $\varprojlim (X_j, f_j)$. Then X_∞ is a solenoid manifold, and $X_\infty \subset \prod_{j \in \mathbb{N}} X_j$, where the right hand side is a compact space when given the Tychonoff topology, therefore X_∞ is also compact. Let $G_j = \mathbb{Z}^n / \Gamma_j$ be the finite covering space group of the finite cover $p_j : X_j \rightarrow X_1 = \mathbb{T}^n$. Then the inverse limit $G_\infty = \varprojlim G_j = \varprojlim \mathbb{Z}^n / \Gamma_j$ is a profinite Abelian group that is homeomorphic to the Cantor set, cf. Lemma 5.1 in [14]. Moreover $G_\infty \rightarrow X_\infty \rightarrow \mathbb{T}^n$ is a principal fibre bundle, cf. Theorem 5.6 in [14]. We call such an X a *principal solenoidal torus*.

Now K-theory is continuous under taking inverse limits in the category of compact Hausdorff spaces, which follows from Proposition 6.2.9 in [21], see also [15],

$$K^\bullet(X_\infty) \cong \varprojlim K^\bullet(X_j).$$

Now the Chern character

$$Ch : K^\bullet(X_j) \rightarrow H^\bullet(X_j, \mathbb{Z})$$

maps to integral cohomology, as shown earlier, since X_j is a torus. Therefore

$$\varinjlim Ch : K^\bullet(X_\infty) \cong \varinjlim K^\bullet(X_j) \rightarrow \varinjlim H^\bullet(X_j, \mathbb{Z}) \cong H^\bullet(X_\infty, \mathbb{Z})$$

by the continuity for Čech cohomology under taking inverse limits in the category of compact Hausdorff spaces, cf. [16]. Now $H^\bullet(X_j, \mathbb{Z})$ are torsionfree Abelian groups, therefore the direct limit $\varinjlim H^\bullet(X_j, \mathbb{Z})$ is again a torsionfree Abelian group. So we have proved the following,

Theorem 2 (Integrality of the Chern character). *Let X be a principal solenoidal manifold over \mathbb{T}^n as above. Then the Chern character,*

$$\varinjlim Ch : K^\bullet(X_\infty) \rightarrow H^\bullet(X_\infty, \mathbb{Q})$$

is integral, that is, the range is contained in $H^\bullet(X_\infty, \mathbb{Z})$.

In particular, the integrality hypothesis (IH) of section 3 is true in this case.

Conversely, let Σ be a Cantor set. Then Σ can be made into a profinite abelian group G (in many ways, cf. [19]). Let $\rho : \mathbb{Z}^n \rightarrow G$ be a homomorphism with dense range. Then we can form the principal G bundle, $G \rightarrow X \rightarrow \mathbb{T}^n$, where $X = \mathbb{R}^n \times_\rho G$.

Remark 1. Consider the countable nested family of finite index normal subgroups of \mathbb{Z} defined by $p^j\mathbb{Z}$, $j \in \mathbb{N}$. Then the inverse limit $\varprojlim \mathbb{Z}/p^j\mathbb{Z} = \mathbb{Z}_p$ is the profinite group of p -adic integers and hence is a Cantor set. Consider $f_j : S^1 \rightarrow S^1$ defined by $f_j(z) = z^{p^j}$, whose kernel is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}$. Let $X = \varprojlim (S^1, f_j)$. Then X is a principal \mathbb{Z}_p -bundle over S^1 , cf. [14], and is also an explicit example of a principal solenoidal torus.

Remark 2. There is an interesting connection between the general case and *number theory*. The Cantor group is the profinite completion $\widehat{\mathbb{Z}}^n$ of the Abelian lattice \mathbb{Z}^n . In number theory, $\widehat{\mathbb{Z}}^n$ is identified with the Galois group of infinite separable field extensions, cf. [20].

Therefore understanding the outer automorphism group of the Cantor group is equivalent to understanding the outer automorphism group of the Galois group $\widehat{\mathbb{Z}}^n$.

A general minimal action of \mathbb{Z}^n on Σ is however more general, being a homomorphism $\phi : \mathbb{Z}^n \rightarrow \text{Homeo}(\Sigma) = \text{Aut}(G)$ with dense orbits on Σ . It gives rise to a general solenoidal torus, that isn't necessarily principal. A solenoidal torus is an example of a solenoidal manifold as defined in [17, 19] where some interesting results are proved regarding these. It remains an open question whether the Chern character is integral in this general case.

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