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On pathwise quadratic variation for càdlàg functions

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Abstract

We revisit Föllmer's concept of quadratic variation of a càdlàg function along a sequence of time partitions and discuss its relation with the Skorokhod topology. We show that in order to obtain a robust notion of pathwise quadratic variation applicable to sample paths of càdlàg processes, one must reformulate the definition of pathwise quadratic variation as a limit in Skorokhod topology of discrete approximations along the partition. The definition then simplifies and one obtains the Lebesgue decomposition of the pathwise quadratic variation as a result, rather than requiring it as an extra condition.

Keywords: Quadratic variation; semimartingale; pathwise calculus; Ito formula; pathwise integration; cadlag functions; Skorokhod topology.

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1 Quadratic variation along a sequence of partitions

In his seminal paper *Calcul d'Itô sans probabilités* [13], Hans Föllmer introduced a pathwise concept of quadratic variation and used it to provide a pathwise proof of the Itô formula. Föllmer showed that if a function $x \in D([0,T],\mathbb{R})$ has quadratic variation along a sequence $\pi^n = (t_0^n = 0 < .. < t_j^n < ... < t_{m(n)}^n = T)$ of time partitions of $[0,T]$ in the sense that for each $t \in [0,T]$ the limit

$$\mu^n := \sum_{[t_j^n, t_{j+1}^n] \in \pi_n} \delta(\cdot - t_j) \quad |x(t_{j+1}^n) - x(t_j^n)|^2 \tag{1}$$

converges weakly to a Radon measure μ_π such that

$$[x]_\pi^c : t \mapsto \mu_\pi([0,t]) - \textstyle\sum_{s \leq t} |\Delta x(s)|^2 \text{ is a continuous increasing function,} \tag{L}$$

then a pathwise Itô formula may be obtained for functions of x [13]: for any $f \in C^2(\mathbb{R},\mathbb{R})$,

$$f(x(t)) - f(x(0)) = \int_0^t f'(x) d^\pi x + \frac{1}{2} \int_0^t f"(x(s)) d[x]_\pi(s) \tag{2}$$

where $\int_0^t f'(x) d^\pi x$ is a pathwise integral defined as a limit of left Riemann sums computed along the partition:

$$\int_0^t g(x) d^\pi x := \lim_{n \to \infty} \sum_{\pi_n} g(x(t_j^n)). \left(x(t_{j+1}^n) - x(t_j^n)\right). \tag{3}$$

The quantity

$$[x]_\pi(t) = \mu_\pi([0,t]) = [x]_\pi^c(t) + \sum_{s \leq t} |\Delta x(s)|^2$$

is called the quadratic variation of x along π. This result has many interesting applications and has been extended to less regular functions [2, 8, 10, 17] and path-dependent functionals [1, 4, 5, 6, 8]. Most of these works, with the exception of [6, 16], these extensions have focused on continuous paths.

Föllmer's definition [13] contains the condition (L) on the Lebesgue decomposition of the limit μ_π: the atoms of μ should correspond exactly to the jumps of x and their mass should be $|\Delta x(t)|^2$ or, equivalently, the discontinuity points of $[x]_\pi$ should coincide with those of x, with $\Delta[x]_\pi(t) = |\Delta x(t)|^2$. This condition can not be removed: as shown by Coquet et al. [9], there are counterexamples of continuous functions x such that (1) converges to a limit with atoms. Conversely, one can give examples of discontinuous functions for which (1) converges to an atomless measure. If this condition is not satisfied, then the pathwise integral (3) fails to satisfy

$$\Delta \left(\int_0^t x(s-) dx(s) \right)(t) = x(t-) \Delta x(t),$$

at each t where condition (L) is not met. On the other hand, this condition (L) is not easy to check and seems to require a link between the path x and

the sequence of partitions π, making it difficult to apply to sample paths of stochastic processes.

In this work, we revisit Föllmer's concept of pathwise quadratic variation along a sequence of partitions and show that it has hitherto unsuspected links with the Skorokhod topology. In particular, we show that in order to obtain a robust notion of pathwise quadratic variation applicable to sample paths of càdlàg processes, one must reformulate the definition of the quadratic variation as a limit, in Skorokhod topology, of discrete approximations defined along the partition. This leads to a simpler definition of pathwise quadratic variation and, rather than requiring the Lebesgue decomposition of the pathwise quadratic variation as an extra condition, we obtain it as a consequence of the definition.

Outline We begin by recalling Föllmer's definition of pathwise quadratic variation and variations of it which have been used in the literature. Then, we introduce a new definition of quadratic variation for real-valued càdlàg functions based on the Skorokhod topology and prove equivalence among the various definitions. Section 3 extends the results to vector-valued functions. In Section 4, we show that our approach leads to simple proofs for various properties of pathwise quadratic variation.

2 Pathwise quadratic variation for cadlag functions

Let $\pi := (\pi_n)_{n\geq 1}$ be a sequence of partitions $\pi_n = (t_0^n, ..., t_{k_n}^n)$ of $[0,\infty)$ into intervals $0 = t_0^n < ... < t_{k_n}^n < \infty$; $t_{k_n}^n \uparrow \infty$ with vanishing mesh $|\pi_n| \downarrow 0$ on compacts. By convention, $\max(\emptyset \cap \pi_n) := 0$, $\min(\emptyset \cap \pi_n) := t_{k_n}^n$.

Denote $\mathcal{D} := \mathcal{D}([0,\infty),\mathbb{R})$ the space of cadlag functions and $\mathcal{C} := \mathcal{C}([0,\infty),\mathbb{R})$ the subspace of real-valued continuous functions. We equip $\mathcal{D}$ with a metric d which induces the Skorokhod (a.k.a J_1) topology [14, Ch. VI]. Denote $\mathcal{D}_0^+ \subset \mathcal{D}$ to be the subset of non-negative increasing right-continuous functions null at 0.

Definition 2.1 (Föllmer1981)**.** *We say that $x \in \mathcal{D}$ has (finite) quadratic variation $[x]_\pi$ along π if the sequence of measures*

$$\mu_n := \sum_{t_i \in \pi_n} (x_{t_{i+1}} - x_{t_i})^2 \delta_{t_i} \tag{4}$$

converges vaguely to a Radon measure μ on $[0,\infty)$ with $[x]_\pi(t) = \mu([0,t])$, such that $[x]_\pi^c$ defined by $[x]_\pi^c(t) := \mu([0,t]) - \sum_{s\leq t}(\Delta x_s)^2$ is a continuous increasing function. $[x]_\pi(t) = \mu([0,t])$ is then called the quadratic variation of x along π, and admits the following Lebesgue decomposition:

$$[x]_\pi(t) = [x]_\pi^c(t) + \sum_{s\leq t}(\Delta x_s)^2. \tag{5}$$

We denote $\mathcal{Q}_0^\pi$ the set of $x \in \mathcal{D}$ satisfying these properties.

At this point let us draw a link between vague and weak convergence of Radon measures on $[0,\infty)$, the link of which, is well known in the special case where the measures are sub-probability measures:

Lemma 2.1. *Let v_n and v be non-negative Radon measures on $[0,\infty)$ and $J \subset [0,\infty)$ be the set of atoms of v, the followings are equivalent:*

(i) $v_n \to v$ vaguely on $[0,\infty)$.
(ii) $v_n \to v$ weakly on $[0,T]$ for every $T \notin J$.

Proof. Let $f \in C_K([0,\infty))$ be a compactly supported continuous function. Since J is countable, $\exists\, T \notin J$; $\mathrm{supp}(f) \subset [0,T]$. Now (ii) $\Rightarrow \int_0^\infty f dv_n = \int_0^T f dv_n \longrightarrow \int_0^T f dv = \int_0^\infty f dv \Rightarrow$ (i). Suppose (i) holds, let $T \notin J$ and $f \in \mathcal{C}\left([0,T], \|\cdot\|_\infty\right)$. Since $f = (f)^+ - (f)^-$, we may take $f \geq 0$ and define the following extensions:

$$\begin{aligned}
\overline{f}^\epsilon(t) &:= f(t)\mathbb{1}_{[0,T]}(t) + f(T)\left(1 + \frac{T-t}{\epsilon}\right)\mathbb{1}_{(T,T+\epsilon]}(t) \\
\underline{f}^\epsilon(t) &:= f(t)\mathbb{1}_{[0,T-\epsilon]}(t) + f(T)\left(\frac{T-t}{\epsilon}\right)\mathbb{1}_{(T-\epsilon,T]}(t),
\end{aligned}$$

then $\overline{f}^\epsilon$, $\underline{f}^\epsilon \in \mathcal{C}_K([0,\infty))$, $0 \leq \underline{f}^\epsilon \leq f\mathbb{1}_{[0,T]} \leq \overline{f}^\epsilon \leq \|f\|_\infty$ and we have

$$\int_0^\infty \underline{f}^\epsilon dv_n \leq \int_0^T f dv_n \leq \int_0^\infty \overline{f}^\epsilon dv_n.$$

Since $v_n \to v$ vaguely and $T \notin J$, thus, as $n \to \infty$ we obtain

$$\begin{aligned}
0 &\leq \limsup_n \int_0^T f dv_n - \liminf_n \int_0^T f dv_n \leq \int_0^\infty \overline{f}^\epsilon - \underline{f}^\epsilon dv \\
&\leq \|f\|_\infty v\left((T-\epsilon, T+\epsilon]\right) \xrightarrow{\epsilon} 0,
\end{aligned}$$

hence by monotone convergence

$$\lim_n \int_0^T f dv_n = \lim_\epsilon \int_0^\infty \underline{f}^\epsilon dv = \int_0^T f dv$$

and (ii) follows. □

Proposition 2.1. *If $x \in \mathcal{Q}_0^\pi$, then the pointwise limit s of*

$$s_n(t) := \sum_{t_i \in \pi_n} (x_{t_{i+1}\wedge t} - x_{t_i \wedge t})^2 \tag{6}$$

exists and s admits the Lebesgue decomposition:

$$s(t) = s^c(t) + \sum_{s \leq t} (\Delta x_s)^2, \tag{7}$$

and $s = [x]$.

Proof. If $x \in Q_0^\pi$, define

$$q_n(t) := \sum_{\pi_n \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2,$$

the distribution function of μ_n in (4). Since $\mu_n \to \mu$ vaguely, we have $q_n \to [x]$ pointwise at all continuity points of $[x]$ (Lemma 2.1 & [11, X.11]). Let I be

the set of continuity points of $[x]$. Observe (q_n) is monotonic in $[0,\infty)$ and I is dense in $[0,\infty)$, if $t \notin I$, it follows [11, X.8] that

$$[x]_{t-} \leq \liminf_n q_n(t) \leq \limsup_n q_n(t) \leq [x]_{t+} = [x]_t.$$

Thus, we may take any sub-sequence (n_k) such that $\lim_k q_{n_k}(t) =: q(t)$. Since $x \in Q_0^\pi$ and the Lebesgue decomposition (5) holds on $[x]$, we have

$$([x](t+\epsilon) - q(t)) + (q(t) - [x](t-\epsilon)) = [x](t+\epsilon) - [x](t-\epsilon) \xrightarrow{\epsilon} |\Delta x(t)|^2. \quad (8)$$

If $t \pm \epsilon \in I$, $\tilde{\pi}_k := \pi_{n_k}$ and $t_j^{(k)} := \max\{\tilde{\pi}_k \cap [0,t)\}$, the second sum in (8) is

$$\lim_k \sum_{\substack{t_i \in \tilde{\pi}_k; \\ t-\epsilon < t_i \leq t}} (x_{t_{i+1}} - x_{t_i})^2 = \lim_k \sum_{\substack{t_i \in \tilde{\pi}_k; \\ t-\epsilon < t_i < t_j^{(k)}}} (x(t_{i+1}) - x(t_i)^2 + (\Delta x_t)^2 \geq (\Delta x_t)^2$$

by the fact that x is càdlàg and that $t \notin I$.

We see from (8) that $q(t) = [x](t)$ as $\epsilon \to 0$. Since the choice of the convergent subsequence, we conclude that $q_n \to [x]$ pointwise on $[0,\infty)$. Observe that the pointwise limits of (s_n) and (q_n) coincide (11) by the right-continuity of x and that $x \in Q_0^\pi$, Prop. 2.1 follows. □

Denote $\mathcal{Q}_1^\pi$ the set of $x \in \mathcal{D}$ such that (s_n) defined in (6) has a pointwise limit s with Lebesgue decomposition given by (7). Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi$ and we have:

Proposition 2.2. *If $x \in \mathcal{Q}_1^\pi$, then the pointwise limit q of*

$$q_n(t) := \sum_{\pi_n \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2 \quad (9)$$

exists and q admits the Lebesgue decomposition:

$$q(t) = q^c(t) + \sum_{s \leq t} (\Delta x_s)^2, \quad (10)$$

and $q = s$.

Proof. Define $t_i^{(n)} := \max\{\pi_n \cap [0,t]\}$. Since the pointwise limits of (s_n) and (q_n) coincide i.e.

$$|s_n(t) - q_n(t)| = (x_{t_{i+1}^{(n)}} - x_t)^2 + 2(x_{t_{i+1}^{(n)}} - x_t)(x_t - x_{t_i^{(n)}}) \quad (11)$$

converges to 0 by the right-continuity of x. Prop. 2.2 now follows from $x \in \mathcal{Q}_1^\pi$. □

Denote $\mathcal{Q}_2^\pi$ the set of $x \in \mathcal{D}$ such that the quadratic sums (q_n) defined in (9) have a pointwise limit q with Lebesgue decomposition (10). Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi \subset \mathcal{Q}_2^\pi$ and we have:

Proposition 2.3. *If $x \in \mathcal{Q}_2^\pi$, then $q_n \to q$ in the Skorokhod topology.*

Proof. Since $x \in \mathcal{Q}_2^\pi$, we have $q_n \to q$ pointwise on $[0,\infty)$ and that (q_n), q are elements in $\mathcal{D}_0^+$. By [14, Thm.VI.2.15], it remains to show that

$$\sum_{s\le t}(\Delta q_n(s))^2 \xrightarrow{n} \sum_{s\le t}(\Delta q(s))^2$$

on a dense subset of $[0,\infty)$. Let $t>0$, define $J^\epsilon := \{s \ge 0 | (\Delta X_s)^2 \ge \frac{\epsilon}{2}\}$, $J_n^\epsilon := \{t_i \in \pi_n | \exists s \in (t_i, t_{i+1}]; (\Delta X_s)^2 \ge \frac{\epsilon}{2}\} \subset \pi_n$ and observe that

$$\begin{aligned}\sum_{s\le t}(\Delta q_n(s))^2 &= \sum_{\pi_n \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^4 \\ &= \sum_{J_n^\epsilon \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^4 + \sum_{(J_n^\epsilon)^c \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^4. \qquad (12)\end{aligned}$$

Since x is càdlàg and that $|\pi_n| \downarrow 0$ on compacts, the first sum in (12) converges to $\sum_{J^\epsilon \ni s \le t}(\Delta x_s)^4$ and the second sum in (12)

$$\sum_{(J_n^\epsilon)^c \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^4 \le \left(\sup_{(J_n^\epsilon)^c \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^2\right) \sum_{(J_n^\epsilon)^c \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^2 \le \epsilon q(t)$$

for sufficiently large n [6, Appendix A.8] hence

$$\lim_n \sum_{s\le t}(\Delta q_n(s))^2 = \sum_{J^\epsilon \ni s \le t}(\Delta x_s)^4 + \overbrace{\limsup_n \sum_{(J_n^\epsilon)^c \ni t_i \le t}(x_{t_{i+1}} - x_{t_i})^4}^{\le \epsilon q(t)}.$$

By the Lebesgue decomposition (10), we observe $\sum_{J^\epsilon \ni s \le t}(\Delta x_s)^4 \le q(t)^2$ and that

$$\lim_n \sum_{s\le t}(\Delta q_n(s))^2 = \sum_{s\le t}(\Delta x_s)^4 = \sum_{s\le t}(\Delta q(s))^2$$

as $\epsilon \to 0$. □

Denote $\mathcal{Q}^\pi$ the set of càdlàg functions $x \in \mathcal{D}$ such that the limit $\tilde{q}$ of (q_n) exists in $(\mathcal{D}, d)$. Then $\mathcal{Q}_0^\pi \subset \mathcal{Q}_1^\pi \subset \mathcal{Q}_2^\pi \subset \mathcal{Q}^\pi$ and we have:

Proposition 2.4. *$\mathcal{Q}^\pi \subset \mathcal{Q}_0^\pi$ and $\tilde{q} = [x]$.*

Proof. Let $x \in \mathcal{Q}^\pi$ and I be the set of continuity points of $\tilde{q}$. [14, VI.2.1(b.5)] implies that $q_n \to \tilde{q}$ pointwise on I. Since $q_n \in \mathcal{D}_0^+$ and I is dense on $[0,\infty)$, it follows $\tilde{q} \in \mathcal{D}_0^+$. Denote μ to be the Radon measure of $\tilde{q}$ on $[0,\infty)$, observe the set of atoms of μ is $J := [0,\infty)\backslash I$ and that (q_n) are the distribution functions of the discrete measures (μ_n) in (4). Thus, by (Lemma 2.1 & [11, X.11]), we see that $\mu_n \longrightarrow \mu$ vaguely on $[0,\infty)$.

If $t>0$, put $t_i^{(n)} := \max\{\pi_n \cap [0,t)\}$. Since $|\pi_n| \downarrow 0$ on compacts, we have $t_i^{(n)} < t$, $t_i^{(n)} \uparrow t$ and $t_{i+1}^{(n)} \downarrow t$. Observe that

$$\Delta q_n(t) = \begin{cases}\left(x_{t_{i+1}} - x_{t_i}\right)^2, & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.}\end{cases} \qquad (13)$$

If $\Delta\tilde{q}(t) = 0$, [14, VI.2.1(b.5)] implies that $\Delta q_n(t_i^{(n)}) \to \Delta\tilde{q}(t)$. Hence, by the fact that x is càdlàg , $(\Delta x_t)^2 = \lim_n \Delta q_n(t_i^{(n)}) = \Delta\tilde{q}(t)$. If $\Delta\tilde{q}(t) > 0$, there exists [14, VI.2.1(a)] a sequence $t'_n \to t$ such that $\Delta q_n(t'_n) \to \Delta\tilde{q}(t) > 0$. Using the fact that x is càdlàg , $t'_n \to t$ and (13), we deduce that (t'_n) must coincide with $(t_i^{(n)})$ for all n sufficiently large, else we will contradict $\Delta\tilde{q}(t) > 0$. Thus, $(\Delta x_t)^2 = \lim_n \Delta q_n(t_i^{(n)}) = \lim_n \Delta q_n(t'_n) = \Delta\tilde{q}(t)$ and the Lebesgue decomposition (5) holds on $\tilde{q}$.

By Def.2.1, we have $\tilde{q} = [x]$ hence $\mathcal{Q}^\pi \subset \mathcal{Q}_0^\pi$. □

Theorem 1. *Let*

- $\mathcal{Q}_0^\pi$ *be the set of* $x \in \mathcal{D}$ *satisfying Definition 2.1.*
- $\mathcal{Q}_1^\pi$ *the set of* $x \in \mathcal{D}$ *such that* (s_n) *defined in* (6) *has a pointwise limit* s *with Lebesgue decomposition given by* (7).
- $\mathcal{Q}_2^\pi$ *be the set of* $x \in \mathcal{D}$ *such that the quadratic sums* (q_n) *defined by* (9) *have a pointwise limit* q *with Lebesgue decomposition* (10).
- $\mathcal{Q}^\pi$ *the set of càdlàg functions* $x \in \mathcal{D}$ *such that the limit* $\tilde{q}$ *of* (q_n) *exists in* $(\mathcal{D}, d)$.

Then:

i $\mathcal{Q}_0^\pi = \mathcal{Q}_1^\pi = \mathcal{Q}_2^\pi = \mathcal{Q}^\pi$.

ii If $x \in \mathcal{Q}_0^\pi$, *then* $[x]_\pi = s(x) = q(x) = \tilde{q}(x)$.

iii x *has finite quadratic variation along* π *if and only if*

$$q_n(t) := \sum_{\pi_n \ni t_i \le t} (x_{t_{i+1}} - x_{t_i})^2$$

converges in $(\mathcal{D}, d)$.

iv If (q_n) *defined by* (9) *converges in* $(\mathcal{D}, d)$, *the limit is equal to* $[x]$.

Proof. These results are a consequence of Prop. 2.1, 2.2, 2.3 and 2.4. □

We see that the two defining properties of $[x]$ in Def. 2.1 are consequences per Thm. 1. The following corollary treats the special case when the underlying space is $\mathcal{C}$.

Corollary 2.1. *Let* $x \in \mathcal{Q}^\pi$, s_n *be defined as in in (6).*

i $q_n \to [x]$ *uniformly on compacts in* $[0, \infty)$ *if and only if* $x \in \mathcal{C}$.

ii If $q_n \to [x]$ *uniformly on compacts in* $[0, \infty)$, *then* $q_n \to [x]$.

Proof. (i): This is a consequence of Prop. 2.4, (5) and [14, VI.1.17(b)].
(ii): Let $T > 0$, $\|\cdot\|_T$ the supremum norm on $\mathcal{D}([0, T])$ and observe that

$$\|s_n - [x]\|_T \le \|q_n - [x]\|_T + \|s_n - q_n\|_T.$$

Since (i) implies $x \in \mathcal{C}$, (ii) now follows from uniform continuity of x and (11). □

Remark 2.1. *The converse of (ii) is not true in general.*

Remark 2.2. *We note that some references have used the pointwise limit of the sequence*

$$p_n(t) := \sum_{\pi_n \ni t_{i+1} \leq t} (x_{t_{i+1}} - x_{t_i})^2$$

together with the Lebesgue decomposition (5)), to define $[x]$*. To see why this is not the correct choice, take* $t_0 \notin \pi$*, put* $x_t := 1\!\!1_{[t_0,\infty)}(t)$ *then obviously* $[x](t_0) = \lim s_n(t_0) = \lim q_n(t_0) = 1$ *but* $\lim p_n(t_0) = 0$*.*

3 Quadratic variation for multidimensional functions

Denote $\mathcal{D}^n := \mathcal{D}([0,\infty), \mathbb{R}^n)$ and $\mathcal{D}^{n\times n} := \mathcal{D}([0,\infty), \mathbb{R}^{n\times n})$ to be the Skorokhod spaces [3, 19, 15], each of which equipped with a metric d which induces the corresponding Skorokhod J_1 topology [14, Ch. VI]. $\mathcal{C}^n := \mathcal{C}([0,\infty), \mathbb{R}^n)$ the subspace of continuous functions in $\mathcal{D}^n$. We recall Theorem 1 from the one dimensional case $n = 1$ that (Def. 2.1) and (Thm. 1.iii) are equivalent.

It is known thqt $x, y \in \mathcal{Q}^\pi$ does not imply $x + y \in \mathcal{Q}^\pi$ [7, 18]. Therefore, the notion of quadratic variation in the multidimensional setting was originally defined in [13] as follows:

Definition 3.1 (Föllmer1981)**.** *We say that* $\mathbf{x} := (\mathbf{x^1}, \ldots, \mathbf{x^n})^\mathbf{T} \in \mathcal{D}^\mathbf{n}$ *has finite quadratic variation along* π *if all* x^i*,* $x^i + x^j$ $(1 \leq i, j \leq n)$ *have finite quadratic variation.*

The quadratic (co)variation $[x^i, x^j]$ is then defined as

$$[x^i, x^j]_t := \frac{1}{2}\left([x^i + x^j]_t - [x^i]_t - [x^j]_t\right), \tag{14}$$

which admits the following Lebesgue decomposition:

$$[x^i, x^j]_t = [x^i, x^j]^c_t + \sum_{s\leq t} \Delta x^i_s \Delta x^j_s. \tag{15}$$

We call $[\mathbf{x}] := ([x^i, x^j])_{1\leq i\leq j\leq n}$ the quadratic (co)variation of $\mathbf{x}$.

Note that, in Def. 3.1, it is impossible to give a multidimensional definition without first introducing the case $n = 1$. The following definition avoids this and directly defines the concept of multidimensional quadratic variation:

Definition 3.2. *We say that* $\mathbf{x} \in \mathcal{D}^n$ *has finite quadratic variation along* π *if*

$$\mathbf{q}_n(t) := \sum_{\pi_n \ni t_i \leq t} (\mathbf{x}_{t_{i+1}} - \mathbf{x}_{t_i})(\mathbf{x}_{t_{i+1}} - \mathbf{x}_{t_i})^T$$

converges in $(\mathcal{D}^{n\times n}, d)$*. The limit, denoted by* $[\mathbf{x}]$ *is called the quadratic variation of* $\mathbf{x}$*.*

We shall now prove the equivalence of these definitions.

For $u, v, w \in \mathcal{D}$, let us write

$$q_n^{(u,v)}(t) := \sum_{\pi_n \ni t_i \leq t} (u_{t_{i+1}} - u_{t_i})(v_{t_{i+1}} - v_{t_i})$$

and $q_n^{(w)} := q_n^{(w,w)}$. Note that the Skorokhod topology on $(\mathcal{D}^n, d)$ is strictly finer than the product topology on $(\mathcal{D}, d)^n$ [14, VI.1.21] and that $(\mathcal{D}, d)$ is not a topological vector space [14, VI.1.22], hence the following is essential:

Lemma 3.1. *Let $t > 0$, there exists a sequence $t_n \to t$ such that*

$$\lim_n \left(\Delta q_n^{(u,v)}(t_n) \right) = \Delta \left(\lim_n q_n^{(u,v)} \right)(t),$$

$\forall u, v \in \mathcal{D}$; $(q_n^{(u,v)})$ converges in $(\mathcal{D}, d)$.

Proof. Define $t_i^{(n)} := \max\{\pi_n \cap [0, t)\}$. Since $|\pi_n| \downarrow 0$ on compacts, we have $t_i^{(n)} < t$, $t_i^{(n)} \uparrow t$ and $t_{i+1}^{(n)} \downarrow t$. Observe that

$$\Delta q_n^{(u,v)}(t) = \begin{cases} \left(u_{t_{i+1}} - u_{t_i}\right)\left(v_{t_{i+1}} - v_{t_i}\right), & \text{if } t = t_i \in \pi_n. \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Put $\tilde{q} := \lim_n q_n^{(u,v)}$. If $\Delta\tilde{q}(t) = 0$, [14, VI.2.1(b.5)] implies that $\Delta q_n^{(u,v)}(t_i^{(n)}) \to \Delta\tilde{q}(t)$. If $\Delta\tilde{q}(t) > 0$, there exists [14, VI.2.1(a)] a sequence $t'_n \to t$ such that $\Delta q_n^{(u,v)}(t'_n) \to \Delta\tilde{q}(t) > 0$. Using the fact that u, v are càdlàg , $t'_n \to t$ and (16), we deduce that (t'_n) must coincide with $(t_i^{(n)})$ for n sufficiently large, else we will contradict $\Delta\tilde{q}(t) > 0$. Put $t_n := t_i^{(n)}$. □

Proposition 3.1. *Let $x, y \in \mathcal{Q}^\pi$, then $(q_n^{(x+y)})$ converges in $(\mathcal{D}, d)$ if and only if $(q_n^{(x,y)})$ does. In this case, $x + y \in \mathcal{Q}^\pi$ and $\lim_n q_n^{(x,y)} = \frac{1}{2}([x + y] - [x] - [y])$.*

Proof. Since

$$q_n^{(x+y)} = q_n^{(x)} + q_n^{(y)} + 2q_n^{(x,y)}$$

and that $x, y \in \mathcal{Q}^\pi$, Prop. 3.1 follows from Lemma 3.1 and [14, VI.2.2(a)]. □

Proposition 3.2. *$(\mathbf{q}_n)$ converges in $(\mathcal{D}^{n\times n}, d)$ if and only if it converges in $(\mathcal{D}, d)^{n\times n}$.*

Proof. Since the Skorokhod topology on $(\mathcal{D}^{n\times n}, d)$ is strictly finer than the product topology on $(\mathcal{D}, d)^{n\times n}$ [14, VI.1.21], we have $(\mathcal{D}^{n\times n}, d)$ convergence implies $(\mathcal{D}, d)^{n\times n}$ convergence. The other direction follows from the observation that

$$\mathbf{q}_n = \left(q_n^{(x^i, x^j)} \right)_{1 \leq i \leq j \leq n}, \tag{17}$$

Lemma 3.1 and [14, VI.2.2(b)]. □

Theorem 2. *Definition 3.1 and 3.2 are equivalent.*

Proof. This is a consequence of (14), (17), Prop. 3.1 & 3.2 and Thm. 1 □

Corollary 3.1. *If $\mathbf{x} \in \mathcal{D}^n$ has finite quadratic variation, then*

(i) $\mathbf{q_n} \to [\mathbf{x}]$ (local) uniformly on $[0, \infty)$ if and only if $\mathbf{x} \in \mathcal{C}^n$.
(ii) $F(\mathbf{q_n}) \to F([\mathbf{x}])$ for all functionals F J_1-continuous at $[\mathbf{x}]$.

Proof. This is a consequence of Thm.2, (15) and [14, VI.1.17.b]. □

Remark 3.1. *For $\mathbf{x}$ to have finite quadratic variation, it is sufficient that $(\mathbf{q_n})$ converges in $(\mathcal{D}, d)^{n\times n}$ due to Prop. 3.2. (i.e. component-wise convergence)*

4 Some applications

We now show that our approach yields simple proofs for some properties of pathwise quadratic variation.

Denote $\mathcal{D} := \mathcal{D}([0, \infty), \mathbb{R})$ and $\mathcal{D}^{d\times d} := \mathcal{D}([0, \infty), \mathbb{R}^{d\times d})$ to be the Skorokhod spaces, each of which equipped with a *complete* metric δ which induces the corresponding Skorokhod (a.k.a J_1) topology. Denote $\mathcal{F}$ to be the J_1 Borel sigma algebra of $\mathcal{D}$ (a.k.a the canonical sigma algebra generated by coordinates). Recall that $\mathcal{Q}_0^\pi$ is the set of paths with finite quadratic variation along π in the sense of Def. 2.1.

Given $x \in \mathcal{D}$, when do we know that x has finite quadratic variation? By Def. 2.1, it is a pre-requisite to verify the condition on the Lebesgue decomposition (5) on the limit measure μ. The following characterization, by contrast, does not refer to μ:

Property 1. *$x \in Q_0^\pi$ if and only if (q_n) defined by (9) is a Cauchy sequence in $(\mathcal{D}, \delta)$.*

Proof. This is a consequence of Thm. 1 and that $(\mathcal{D}, \delta)$ is complete □

One of the main advantages of having convergence in the J_1 topology is that it ensures convergence of jumps in a regulated manner. It comes in handy when accessing the limit of $q_n(t_n)$ as $n \to \infty$.

Property 2. *Let $x \in Q_0^\pi$, for each $t \geq 0$, we define $t'_n := \max\{t_i < t | t_i \in \pi_n\}$, then*

$$\begin{aligned}
t_n \longrightarrow t; t_n \leq t'_n &\Longrightarrow q_n(t_n-) \longrightarrow [x](t-),\\
t_n \longrightarrow t; t_n < t'_n &\Longrightarrow q_n(t_n) \longrightarrow [x](t-),\\
t_n \longrightarrow t; t_n \geq t'_n &\Longrightarrow q_n(t_n) \longrightarrow [x](t),\\
t_n \longrightarrow t; t_n > t'_n &\Longrightarrow q_n(t_n-) \longrightarrow [x](t).
\end{aligned}$$

In particular, the sequence (t'_n) is asymptotically unique in the sense that any other sequence (t''_n) meeting the above properties coincides with (t'_n) for n sufficiently large.

Proof. This is a consequence of Thm. 1 and [14, VI.2.1]. □

Given a càdlàg process X (i.e. a $(\mathcal{D}, \mathcal{F})$-measurable random variable), a natural quantity to consider is $\mathbb{P}(X \in \mathcal{Q}_0^\pi)$. This only makes sense however if $\mathcal{Q}_0^\pi$ is $\mathcal{F}$-measurable. This 'natural' property, not easy to show using the original definition (Def. 2.1), becomes simple thanks to Theorem. 1:

Property 3 (Measurability of $\mathcal{Q}_0^\pi$)**.** *$\mathcal{Q}_0^\pi$ is $\mathcal{F}$-measurable.*

Proof. By Thm. 1, $\mathcal{Q}_0^\pi = \mathcal{Q}^\pi$ and by definition, $\mathcal{Q}^\pi$ is the J_1 convergence set of

$$x \longmapsto \sum_{\pi_n \ni t_i \leq \cdot} (x(t_{i+1}) - x(t_i))^2,$$

$n \geq 1$ on $\mathcal{D}$. Since $\mathcal{D}$ is completely metrisable, the claim follows from [11, V.3]. □

In [13], Föllmer introduced the class of *Dirichlet processes*, defined as càdlàg processes with finite quadratic variation i.e. such that the sequence

$$S_n(t) := \sum_{\pi_n \ni t_i \leq t} (X(t_{i+1}) - X(t_i))^2$$

converges in probability for every t and the limit $[X]$ is an increasing process whose paths have Lebesgue decomposition of the form (5):

$$[X]_t = [X]_t^c + \sum_{s \leq t} (\Delta X_s)^2.$$

This class of processes is strictly larger than the class of semimartingales and upon which the Itô formula can be applied pathwise.

Theorem 3. *Let $\mathbf{X}$ be an $\mathbb{R}^d$-valued càdlàg process, define a sequence of $(\mathcal{D}^{d\times d}, \delta)$-valued random variables $(\mathbf{q}_n)$ by*

$$\mathbf{q}_n(t) := \sum_{\pi_n \ni t_i \leq t} (\mathbf{X}(t_{i+1}) - \mathbf{X}(t_i))(\mathbf{X}(t_{i+1}) - \mathbf{X}(t_i))^T$$

then the followings are equivalent:

i $\mathbf{X}$ *is a Dirichlet process.*

ii $(\mathbf{q}_n)$ *converges in probability.*

iii $(\mathbf{q}_n)$ *is a Cauchy sequence in probability.*

In addition,

iv If $(\mathbf{q}_n)$ *converges in probability, the limit is* $[\mathbf{X}]$.

v The convergence of $(\mathbf{q}_n)$ *is UCP if and only if* $\mathbf{X}$ *is a continuous process of quadratic variation* $[\mathbf{X}]$.

vi $(\mathbf{q}_n)$ *converges (resp. is a Cauchy sequence) in probability if and only if each component sequence of* $(\mathbf{q}_n)$ *converges (resp. is a Cauchy sequence) in probability.*

Proof. We first remark that $(\mathcal{D}^{d\times d}, \delta)$ is a complete seperable metric space [14], hence by [12, Lemma 9.2.4], Cauchy is equivalent to convergence in probability. By [12, Thm. 9.2.1], we can pass to subsequences and apply Prop. 3.2, Thm. 2 & Cor. 3.1 pathwise to $\mathbf{X}$, the claims follow. □

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