

The elastocapillary ridge as a non-integer disclination: Supplementary Information

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DERIVATION OF THE INCREMENTAL DEFORMATION FIELD

We recall that the equilibrium equation can be written as $\text{div}(\mathbf{P}\mathbf{F}') = \mathbf{0}$ where the first Piola-Kirchoff tensor is $\mathbf{P} = \mu\mathbf{F} - P\mathbf{F}^{-1}$ and $\mathbf{F}' = \partial\mathbf{r}'/\partial\mathbf{R}$, where the div operator is evaluated in the prestretched configuration \mathcal{B}' . Everywhere at the free boundary $z' = h$ except at the triple line, Nanson's formula gives $\mathbf{P}\mathbf{F}' \cdot \mathbf{n}' = \mathbf{f}^C + \gamma_s \mathbf{n} \cdot (\nabla \mathbf{n})$ where $\mathbf{n}' = (0, 1)$ is the outward unit vector normal to the free surface in \mathcal{B}' and \mathbf{n} is the outward unit normal vector in \mathcal{B} . We linearize the equilibrium condition and boundary conditions *around the finitely prestretched configuration* given by the mapping $\mathbf{r}' = (r', z') = (\lambda R, Z/\lambda^2)$ in order to obtain the incremental solution [1]. To this end, it is convenient to separate the pressure into the form $P = P^* + \epsilon p$ where P^* is a constant pressure in \mathcal{B}' and p is the incremental pressure from \mathcal{B}' to \mathcal{B} . By applying the boundary condition $\mathbf{P}\mathbf{F}'\mathbf{n}' = \mathbf{0}$ which holds in \mathcal{B}' before the deposition of the drop, it is easily found that $P^* = \mu/\lambda^4$. Following this decomposition, the equilibrium equations and boundary conditions can be linearized with respect to ϵ and are solved easily by shifting into Fourier space, leading to the following solution for the vertical displacement of the free surface $\zeta(r') = u_z(r', h)$:

$$\zeta(r') = \gamma_\ell \sin \alpha \int_0^\infty ds \frac{J_0(sr') \left(\rho J_0(s\rho) - \frac{2J_1(s\rho)}{s} \right)}{2\mu g(hs, \lambda) + s\gamma_s} \quad (1)$$

where the function $g(hs, \lambda)$ is:

$$g(hs, \lambda) = \frac{- (\lambda^{12} + 6\lambda^6 + 1) \sinh(hs) \sinh(h\lambda^3 s) + (\lambda^{12} + 2\lambda^6 + 5) \lambda^3 \cosh(hs) \cosh(h\lambda^3 s) - 4(\lambda^9 + \lambda^3)}{2\lambda^4 (\lambda^6 - 1) (\lambda^3 \sinh(hs) \cosh(h\lambda^3 s) - \cosh(hs) \sinh(h\lambda^3 s))} \quad (2)$$

Here we have assumed that the lower surface of the elastic layer is bonded to an infinitely rigid surface, $u_r(r', 0) = u_z(r', 0) = 0$. Note that when there is no prestretch, i.e when $\lambda = 1$, the solution (1) reduces to the well know solution of the elastowetting problem where the surfaces force distribution is damped both by the elasticity and the surface energy of the substrate. In absence of surface tension, we recover a known result of incremental elasticity [2]. Interestingly, it should be noted that the introduction of a finite prestretch only affect the elastic term $2\mu g(hs, \lambda)$ but not the term associated with the surface energy of the solid $s\gamma_s$. This result could in fact be anticipated because the cost of creating a unit of area is independent of the underlying deformation in the absence of any Shuttleworth effect, and thus the surface energy term is independent of λ , as seen in (1). Furthermore, it is known that the incremental response of a prestretched elastic half-space is that of a transversely isotropic linear half-space. Indeed, for very thick sample, i.e in the limit $h \rightarrow \infty$, we find that $g_\infty(\lambda) = \lim_{h \rightarrow \infty} g(hs, \lambda) = (\lambda^9 + \lambda^6 + 3\lambda^3 - 1)/(2(\lambda^7 + \lambda^4))$, implying that the incremental response of a prestretched Neo-Hookean elastic half-space with elastic modulus μ is identical to that of a linear elastic half-space without prestretch but with an effective shear modulus $\mu g_\infty(\lambda)$. This increase of the apparent rigidity is a purely nonlinear effect. As a consequence, the incremental deformation theory predicts that: i) the overall profile of the ridge, and in particular its height $d = \zeta(\rho)$, depend on the prestretch λ ; ii) regarding the opening angle of the ridge however, we recover the classical result $\theta = \pi - 2\gamma_s \zeta'(0^-) = \pi - \gamma_\ell/\gamma_s$ for all thicknesses H and prestretch λ . Therefore the incremental theory predicts that the opening angle is constant, for any prestretch λ , in the limit of small deformations ($\gamma_\ell/2\gamma_s \ll 1$).

DERIVATION OF THE NONLINEAR ELASTIC FORCE AT THE TIP OF THE ELASTOCAPILLARY RIDGE

We wish to evaluate here the vertical elastic force f_z^E acting at the tip of the elastocapillary ridge. In this section we only consider a drop of infinite radius and focus on a single symmetric contact line. Within the framework of linear elasticity, the elastic force at the tip vanishes. Here we calculate the first order correction due to the nonlinearities and show that this leads to a nonlinear force $f_z^E \approx -2[T_{rz}u_z]$ acting on the ridge. Let us first recall the expression given in the main text for the force acting on an elastic singularity:

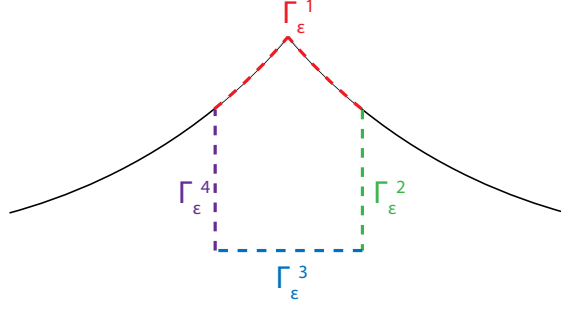


FIG. 1: Definition of the contour $\Gamma_\epsilon = \bigcup_{i=1}^4 \Gamma_\epsilon^i$ enclosing the singularity

$$\mathbf{f}^E = \lim_{\epsilon \rightarrow 0} \int_{\Gamma'_\epsilon} (\mathcal{W}_e \mathbf{I} - \mathbf{F}^T \mathbf{P}) \boldsymbol{\nu}' d\ell' \quad (3)$$

which can be rewritten as [3]

$$\mathbf{f}^E = \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} \mathbf{T} \cdot \boldsymbol{\nu} d\ell \quad (4)$$

where \mathbf{T} is the Cauchy stress tensor and Γ_ϵ is now a contour enclosing the tip of the ridge *in the current configuration*. The vector $\boldsymbol{\nu}$ is outward vector normal to Γ_ϵ . We consider the composed contour $\Gamma_\epsilon = \bigcup_{i=1}^4 \Gamma_\epsilon^i$ enclosing the tip of the elastocapillary ridge as shown in figure 1.

Let us now calculate the integral (4). At the free surface, i.e. on Γ_ϵ^1 , the boundary conditions of the incremental elastic problem imply, at first order in the parameter γ_ℓ/γ_s , that:

$$-T_{rr}^{(0)} \frac{\partial \zeta}{\partial r'} + T_{rz}^{(1)} = 0 \quad \text{along } \mathbf{e}_r \quad (5)$$

$$T_{zz}^{(1)} = \gamma_s \delta(r') - \gamma_s \frac{\partial^2 \zeta}{\partial r'^2} \quad \text{along } \mathbf{e}_z \quad (6)$$

Where $T_{rr}^{(0)} = (\lambda^2 - 1/\lambda^4)$ is the radial stress in the prestretched configuration while the quantities labeled with a superscript (1) refer to the incremental stress in the current configuration. When $\lambda \neq 1$, the radial stress $T_{rr}^{(0)} \neq 0$ and therefore the shear stress $T_{rz}^{(1)}$ at the free surface is non-zero and discontinuous (owing to the discontinuity of $\partial \zeta / \partial r'$). At second-order in γ_ℓ/γ_s , this non-zero shear stress will induce a stress $-T_{rz}^{(1)} \frac{\partial \zeta}{\partial r}$ in the vertical direction (i.e. in equation 6). Therefore we have $\mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z \sim (T_{zz}^{(1)} - T_{rz}^{(1)} \frac{\partial \zeta}{\partial r'})$ up to second order on Γ_ϵ^1 . Note that in absence of pre-stretch (i.e. when $\lambda = 1$), the second-order contribution vanishes.

We now evaluate the integral of this quantity along the element of contour Γ_ϵ^1 in the limit $\epsilon \rightarrow 0$. In the current configuration, the infinitesimal length element is $d\ell = (1 + \partial u / \partial r') dr'$ and thus:

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^1} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \left(T_{zz}^{(1)} - T_{rz}^{(1)} \frac{\partial \zeta}{\partial r'} \right) \left(1 + \frac{\partial u}{\partial r'} \right) dr' + h.o.t \quad (7)$$

$$= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} T_{zz}^{(1)} dr' + \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} \left(T_{zz}^{(1)} \frac{\partial u}{\partial r'} - T_{rz}^{(1)} \frac{\partial \zeta}{\partial r'} \right) dr' + h.o.t \quad (8)$$

Because the stress field $T_{zz}^{(1)}$ has a logarithmic integrable singularity the first integral of (8) vanishes (in the limit $\epsilon \rightarrow 0$), a known result in the context of linear elastocapillarity. Using the incompressibility relation and the balance of linear momentum, the second integral can be transformed into:

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^1} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = - \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dr' \left. \frac{\partial (T_{rz}^{(1)} \zeta)}{\partial r'} \right|_{z'=0} - \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dr' \left. \frac{\partial (T_{zz}^{(1)} \zeta)}{\partial z'} \right|_{z'=0} + h.o.t \quad (9)$$

Turning now to the integral along the boundary Γ_ϵ^3 (located say, at $z' = -\epsilon$), we have, again using the incompressibility relation and the balance of linear momentum

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^3} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dr' \zeta \left. \frac{\partial T_{rz}^{(1)}}{\partial r'} \right|_{z'=-\epsilon} + \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dr' \left. \frac{\partial (T_{zz}^{(1)} \zeta)}{\partial z'} \right|_{z'=-\epsilon} + h.o.t \quad (10)$$

Now, $T_{rz}^{(1)}$ and its derivative are continuous for all r' at $z' = -\epsilon$ so that the first integral in the r.h.s of (10) vanishes. We now add (9) and (10) and, noting that the field $\partial(T_{zz}^{(1)}\zeta)/\partial z'$ is also continuous, the second integrals of the r.h.s of (9) cancels the the second integrals of the r.h.s of (10). We thus obtain:

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^1 \cup \Gamma_\epsilon^3} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = - \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dr' \left. \frac{\partial(T_{rz}^{(1)}\zeta)}{\partial r'} \right|_{z'=-\epsilon} + h.o.t \quad (11)$$

which does not vanish because $T_{rz}^{(1)}(r', z' = 0)$ is discontinuous at $r' = 0$, leading to

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^1 \cup \Gamma_\epsilon^3} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = - \left[T_{rz}^{(1)}\zeta \right] + h.o.t \quad (12)$$

where the symbol $[\cdot]$ denote the jump of the quantity inside the bracket across the ridge at $r = 0$. A similar reasoning can be used to show that

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon^2 \cup \Gamma_\epsilon^4} \mathbf{T} \cdot \boldsymbol{\nu} \cdot \mathbf{e}_z d\ell = - \left[T_{rz}^{(1)}\zeta \right] + h.o.t \quad (13)$$

Note that the result of integral (13) is just the classical Peach-Koehler force acting on a disclination inside a bulk solid while (12) is the additional force due to the free surface, which adds an equal contribution to the vertical force (thus showing that the free surface acts as a mirror disclination). Summing up (12)-(13) we obtain the total vertical force f_z^E acting at the tip of a symmetric elastocapillary ridge:

$$f_z^E = \mathbf{f}^E \cdot \mathbf{e}_z = -2 \left[T_{rz}^{(1)}\zeta \right] + h.o.t \quad (14)$$

□



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