Mass functions of a compact manifold

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MASS FUNCTIONS OF A COMPACT MANIFOLD

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ABSTRACT. Let M be a compact manifold of dimension n. In this paper, we introduce the *Mass Function* $a \geq 0 \mapsto X_+^M(a)$ (resp. $a \geq 0 \mapsto X_-^M(a)$) which is defined as the supremum (resp. infimum) of the masses of all metrics on M whose Yamabe constant is larger than a and which are flat on a ball of radius 1 and centered at a point $p \in M$. Here, the mass of a metric flat around p is the constant term in the expansion of the Green function of the conformal Laplacian at p. We show that these functions are well defined and have many properties which allow to obtain applications to the Yamabe invariant (i.e. the supremum of Yamabe constants over the set of all metrics on M).

CONTENTS

1. INTRODUCTION

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ and denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g : \quad C^\infty(M) \to C^\infty(M)$$

the conformal Laplace operator of g, where s_g is the scalar curvature of g and Δ_g is the Laplace-Beltrami operator with non-negative spectrum. Assume that the metric g is flat on an open neighborhood of a point $p \in M$ and that all eigenvalues of L_g are strictly positive. Then it is well-known that there exists a unique Green function G_g of L_g at p, i. e. in the sense of distributions we have $L_g G_g = \delta_p$, the function G_g is smooth and strictly positive on $M \setminus \{p\}$ and as $x \to p$ we have

$$G_g(x) = \frac{1}{(n-2)\omega_{n-1} r(x)^{n-2}} + m(g, p) + o(1)$$

Key words and phrases. Yamabe operator; Yamabe invariant; surgery; positive mass theorem.

where ω_{n-1} is the volume of the standard sphere of dimension $n-1$, the function r denotes the Riemannian distance from p and $m(g,p) \in \mathbb{R}$ is a number called the mass of g at p. This quantity is related to the so-called ADM mass of an asymptotically flat Riemannian manifold. The study of the mass has led to many interesting results in geometric analysis and General Relativity. An example is an application to the so-called conformal Yamabe constant of (M,g) defined by

$$Y(M,g) := \inf \int_M s_g \, dv^g,$$

where the inf is taken over the set of all Riemannian metrics on M which have unit volume and are conformal to g. Namely, in a famous article [8], Richard Schoen used positivity of the mass $m(g,p)$ to prove that $Y(M,g) < Y(S^n, g_{\mathrm{can}})$ if (M,g) satisfies the assumptions above and is not conformally diffeomorphic to the standard sphere (S^n, g_{can}).

In this article we consider the dependence of the mass on the Yamabe constant $Y(M,g)$. We define two functions $a \mapsto X_+^M(a)$ and $a \mapsto X_-^M(a)$ whose values are a sup and an inf of masses $m(g,p)$ respectively taken over the set of all Riemannian metrics g with $Y(M,g) > a$ which are flat on a ball of radius 1 centered at $p \in M$ (see Definition 3.1). We prove that for small values of a the values $X_+^M(a)$ and $X_-^M(a)$ decrease and increase respectively under surgery of codimension at least 3 (see Theorem 5.1). Finally, we give an application to the smooth Yamabe invariant of M defined by

$$\sigma(M) := \sup Y(M,g)$$

where the sup is taken over the set of all Riemannian metrics on M. The question of whether for a given smooth manifold M one has $\sigma(M) < \sigma(S^n)$ is open in general. We prove that if $X_+^M(\sigma(M)) > 0$ then we have $\sigma(M) < \sigma(S^n)$. The precise statement is given in Theorem 6.1.

In the proofs of these theorems we use a surgery result obtained by the second author together with Ammann and Dahl [1] and a variational characterization of the mass $m(g,p)$ obtained by the two authors of the present article [5].

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2. Notation

Let M be a closed manifold of dimension $n \geq 3$. The set of Riemannian metrics on M will be denoted by $\mathcal{M}_M$. For $g \in \mathcal{M}_M$, we denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g : \quad C^\infty(M) \to C^\infty(M)$$

the conformal Laplace operator of g, where s_g is the scalar curvature of g and Δ_g is the Laplace-Beltrami operator with non-negative spectrum. Moreover, we write $N := \frac{2n}{n-2}$ and we denote by

$$Y(g) := Y(M,g) = \inf \left\{ \frac{\int_M u L_g u \, dv^g}{\|u\|_{L^N}^2} \,\middle|\, u \in C^\infty(M) \setminus \{0\} \right\}$$

the (conformal) Yamabe constant of g and by $\sigma(M) := \sup_{g \in \mathcal{M}_M} Y(g)$ the (smooth) Yamabe invariant of M. We will write $Y(g)$ instead of $Y(M,g)$ since M will always be clear from the context. We have $Y(g) > 0$ if and only if all eigenvalues of L_g

are strictly positive. In the following, we will always assume that $\sigma(M) > 0$. We define, for any $a \in [0, \sigma(M)[$,

$$Z_M(a) := \{g \in \mathcal{M}_M \mid Y(g) > a\}$$

$$\Omega_M^a := \left\{(g,p) \in Z_M(a) \times M \mid B_p^g(1) \text{ is isometric to } \mathbb{B}\right\}.$$

where $B_p^g(1)$ stands for the ball with center p and radius 1 with respect to the metric g and where $\mathbb{B}$ is the standard Euclidean unit ball of dimension n. Note that these sets are not empty as soon as $\sigma(M) > 0$ (see the first item of Proposition 4.1).

Let η be a smooth function on M such that $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B_p^g(\frac{1}{2})$ and $\mathrm{supp}(\eta) \subset B_p^g(1)$, where ω_{n-1} denotes the volume of S^{n-1} with the standard metric. If $(g,p) \in \Omega_M^a$ for some $a \geq 0$ then there exists a unique Green function G_g of L_g at p and we have for all $x \in M \setminus \{p\}$:

$$G_g(x) = \eta(x) r(x)^{2-n} + m(g,p) + \alpha(x),$$

where $r(x)$ denotes the Riemannian distance of x and p with respect to g, α is a smooth function defined on all of M which is harmonic on $B_p^g(\frac{1}{2})$ and satisfies $\alpha(p) = 0$ and $m(g,p) \in \mathbb{R}$ is a number called the mass of g at p.
We recall that $m(g,p)$ has a variational characterization established in [5]. Namely, the function $F_\eta \colon M \to \mathbb{R}$ defined by

$$F_\eta(x) := \begin{cases} \Delta_g(\eta r^{2-n})(x), & x \neq p \\ 0, & x = p \end{cases}$$

is smooth on M. For every $u \in C^\infty(M)$ we define

$$J_p^g(u) := \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g + 2\int_M u F_\eta \, dv^g + \int_M u L_g u \, dv^g.$$

Then, it was proven in [5] that

$$-m(g,p) = \inf\{J_p^g(u) \mid u \in C^\infty(M)\} \tag{1}$$

and that the infimum is attained for the smooth function β defined by

$$\beta(x) := m(g,p) + \alpha(x).$$

We say that a closed manifold satisfies PMT (for Positive Mass Theorem) if for every metric g on M and for all points $p \in M$ such that g is flat on an open neighborhood of p and $Y(g) > 0$ we have $m(g,p) \geq 0$. It is conjectured that every closed manifold satisfies PMT. This conjecture has been proved in some special cases (see e. g. [7], [11], [9]). A complete proof has been announced by Lohkamp [6] and Schoen-Yau [10].

3. Upper and lower mass functions of M

Definition 3.1. The *upper* (resp. *lower*) mass function $X_+^M : [0, \sigma(M)] \to \mathbb{R} \cup \{\pm\infty\}$ (resp. $X_-^M : [0, \sigma(M)] \to \mathbb{R} \cup \{\pm\infty\}$) are defined by : for all $a \in [0, \sigma(M)]$

$$X_+^M(a) := \limsup_{\varepsilon \to 0} \sup_{(g,p) \in \Omega_M^{\max(a-\varepsilon,0)}} m(g,p)$$

(resp.

$$X_-^M(a) := \liminf_{\varepsilon \to 0} \inf_{(g,p) \in \Omega_M^{\max(a-\varepsilon,0)}} m(g,p).)$$

Note that the maximum in the definitions above is only to ensure that $X_-^M(a)$ and $X_+^M(a)$ are well defined when $a = 0$. If $a > 0$, one can just replace $\Omega_M^{\max(a-\varepsilon,0)}$ by $\Omega_M^{a-\varepsilon}$ in these defintions. The goal of this paper is to establish several properties of $X_\pm^M(a)$.

4. Basic properties of $X_\pm^M$

Proposition 4.1. *It holds that*

1. *X_+^M and X_-^M are well defined for all $a \in [0, \sigma(M)]$ as soon as $\sigma(M) > 0$;*
2. *For all $a \in [0, \sigma(M)]$, $X_+^M(a) \geq X_-^M(a)$;*
3. *X_+^M is a decreasing function of a and X_-^M is an increasing function of a and they are both left continuous;*
4. *For all $a \in [0, \sigma(M)]$, $0 \geq X_-^M(a) > -\infty$;*
5. *For all $a > 0$, $X_+^M(a) < +\infty$;*
6. *$X_-^M(0) = 0$ if and only if M possesses the property PMT ;*
7. *Let M, N be compact manifolds of dimension $n \geq 3$ with positive Yamabe invariant. Then, for all $a > 0$ we have*

$$X_+^{M\amalg N}(a) = \max(X_+^M(a), X_+^N(a)) \quad \text{and} \quad X_-^{M\amalg N}(a) = \min(X_-^M(a), X_-^N(a)).$$

Proof. (1) It suffices to show that $\Omega^a(M)$ is not empty if $0 < a < \sigma(M)$. We fix $a' \in (a, \sigma(M))$. First, it is clear that there exists a metric g with $Y(g) = a'$. Hence, let $\xi = \sum_i dx_i^2$, where $(x_1, \cdots, x^n)$ is a system of normal coordinates at some $p \in M$, be a flat metric around p and let $g_\varepsilon := (1-\eta_\varepsilon)g + \eta_\varepsilon \xi$, where $\eta_\varepsilon : M \to [0,1]$ is a cut-off function equal to 1 on $B_p^g(\varepsilon)$, equal to 0 outside $B_p^g(2\varepsilon)$ and such that $|d\eta_\varepsilon| \leq \frac{2}{\varepsilon}$ and $|\nabla^2 \eta_\varepsilon| \leq \frac{2}{\varepsilon^2}$. By Lemma A.1 we have $\lim_{\varepsilon\to 0} Y(g_\varepsilon) = Y(g)$. Now, the metric $h_\varepsilon = \frac{1}{\varepsilon^2} g_\varepsilon$ is flat on $B_p^{h_\varepsilon}(1)$. If ε is small enough then $Y(h_\varepsilon) = Y(g_\varepsilon) > a$ which implies that $h_\varepsilon \in \Omega^a(M)$.

(2) and (3) are clear from the definitions.

(4) Let u be any nonzero smooth function compactly supported in the Euclidean ball $\mathbb{B}$. Let $(g,p) \in \Omega_M^a$ for some a. From the definition of Ω_M^a, we can identify $(B_p(1), g)$ with $\mathbb{B}$ so that u can be considered as a test function in the variational characterization (1) which provides

$$-m(g,p) \leq J_g^p(u).$$

The inequality $X_-^M(a) > -\infty$ follows by noticing that $J_g^p(u)$ does not depend on the choice of $a \geq 0$ nor on the choice of $(g,p) \in \Omega_M^a$.

Let us prove now that $X_-^M(a) \leq 0$. It comes from the facts that if $(g,p) \in \Omega_M^a$ then $(bg, p) \in \Omega_M^a$ for any $b \geq 1$, and also that for any such b

$$m(bg, p) = b^{1-\frac{n}{2}} m(g,p).$$

(5) Let t$a > 0$ and let $(g,p) \in \Omega_M^a$. We have to show that $m(g,p)$ is bounded by a constant which depends only on a but not on (g,p). Let $u \in C^\infty(M)$. In what follows, $C > 0$ denotes a positive constant which might depend on a but not on (g,p). By the variational characterization (1), choose u so that

$$-m(g,p) + 1 \geq J_g^p(u)$$

From the definition of J^p_g, one has

$$J^p_g(u) \geq -C + 2\int_M uF_\eta \, dv^g + \int_M uL_g u \, dv^g.$$

Using that fact that $Y(g) \geq a$ and using Hölder inequality, one gets

$$J^p_g(u) \geq -C - 2\|F_\eta\|_{L^\infty} \left(\int_{B^g_p(1)} |u|^N dv^g\right)^{\frac{1}{N}} \mathrm{vol}(B^g_p(1))^{\frac{N-1}{N}} + a\left(\int_M |u|^N, dv^g\right)^{\frac{2}{N}}.$$

Set now

$$X_g = \left(\int_M |u|^N dv^g\right)^{\frac{1}{N}},$$

we obtain that there exists some $C', C'' > 0$ independent of (g, p) such that

$$J^p_g(u) \geq C - C' X_g + C'' X_g^2. \tag{2}$$

This quantity is bounded from below independently of (g, p). This show that $m(g, p)$ is bounded from above by a constant independent of $(g, p) \in \Omega^a_M$. This implies that for all $a > 0$, $X^M_+(a) < +\infty$.

(6) Clearly the property PMT for a manifold is equivalent to $X^M_-(0) \geq 0$. Since $X^M_-(0) \leq 0$ by item (4), the result follows.

(7) Let $a \in (0, \sigma(M \amalg N)] = (0, \min(\sigma(M), \sigma(N))]$ and $\varepsilon > 0$. On the one hand, let $(g, p) \in \Omega^{a-\varepsilon}_{M\amalg N}$ where $p \in M$. Then g decomposes as $g = g_M \amalg g_N$ where $g_M \in \mathcal{M}_M$ and $g_N \in \mathcal{M}_N$. We have $a - \varepsilon < Y(g) = \min(Y(g_M), Y(g_N))$. Since $p \in M$, this implies that $(g_M, p) \in \Omega^{a-\varepsilon}_M$. Let $\rho > 0$. If ε is small enough, it follows from the definition of $X^M_+(a)$ that $m(g, p) = m(g_M, p) \leq X^M_+(a) + \rho \leq \max(X^M_+(a), X^N_+(a)) + \rho$. In the same way, if $p \in N$, $m(g, p) \leq \max(X^M_+(a), X^N_+(a)) + \rho$. From these inequalities, we obtain

$$X^{M\amalg N}_+(a) \leq \max(X^M_+(a), X^N_+(a)) + \rho$$

and since ρ is arbitrary

$$X^{M\amalg N}_+(a) \leq \max(X^M_+(a), X^N_+(a)).$$

On the other hand, let $(g_M, p) \in \Omega^{a-\varepsilon}_M$ and let g_N any metric on N with $Y(g_N) \geq a - \varepsilon$. If ε is small enough, then $m(g_M, p) = m(g_M \amalg g_N, p) \leq X^{M\amalg N}_+(a) + \rho$. Hence $X^M_+(a) \leq X^{M\amalg N}_+(a) + \rho$. The same holds for N and the result follows.

The proof for $X^M_-(a)$ is similar.

□

5. $X^M_\pm(a)$ and surgery

In this section, we first establish the following theorem, whose proof is a consequence of the results in [5] and [1]

Theorem 5.1. *Let M be a compact manifold of dimension $n \geq 3$ and $M^\sharp$ be obtained from M by a surgery of dimension $k \leq n-3$. Then, for all $a \in [0, \sigma(M)]$, one has*

$$X^M_+(a) \leq X^{M^\sharp}_+(\min(a, \Lambda_{n,k})) \quad \text{and} \quad X^M_-(a) \geq X^{M^\sharp}_-(\min(a, \Lambda_{n,k})).$$

where $\Lambda_{n,0} = +\infty$ and where $\Lambda_{n,k} > 0$ depends only on n and k.

A consequence of Theorem 5.1 is

Corollary 5.2. *Let M_0 be any compact non spin (resp. spin) simply connected manifold of dimension $n \geq 5$ such that $\sigma(M_0) > 0$ and let $a > 0$. Then, for all compact (resp. compact spin) manifolds M of the same dimension one has*

$$0 \geq X_-^M(\min(a, \Lambda_n)) \geq X_-^{M_0}(\min(a, \Lambda_n)) \quad \text{and}$$
$$X_+^M(\min(a, \Lambda_n)) \leq X_+^{M_0}(\min(a, \Lambda_n))$$

where $\Lambda_n = \min_{1 \leq k \leq n-3} \Lambda_{n,k} > 0$.

This has the following obvious consequence:

Corollary 5.3. *Let M_0, M_1 be two compact non spin (resp. spin) simply connected manifolds of dimension $n \geq 5$ such that $\sigma(M_0), \sigma(M_1) > 0$ and let $a \in (0, \Lambda_n)$. Then we have*

$$X_\pm^{M_0}(a) = X_\pm^{M_1}(a).$$

Remark 5.4. By Corollary C in [4], if M is a compact simply connected non-spin manifold of dimension at least 5 then $\sigma(M) > 0$. By [1], when M is simply connected and $\sigma(M) > 0$, it holds that

$$\sigma(M) \geq \min\{\Lambda_n, \sigma(W_1), ..., \sigma(W_k)\}$$

where $W_1, ..., W_k$ are generators of the oriented cobordism group in dimension n.

Remark 5.5. This corollary allows to recover a result in [5]: if M_0 not spin, simply connected of dimension $n \geq 5$ satisfies PMT, then all the manifolds of the same dimension satisfy PMT. Indeed, assume that M_0 satisfies PMT then $X_-^{M_0}(0) = 0$ (see Proposition 4.1) and hence $X_-^M(0) = 0$ which implies PMT. Note that Lohkamp [6] and Schoen and Yau [10] recently announced a complete proof of the Positive Mass Theorem (i.e. all manifolds satisfy PMT).

Another consequence is the following:

Corollary 5.6. *Assume that M is simply connected, that $\sigma(M) > 0$ and that $a < \Lambda_n$. Then, $X_+^M(a) > 0$.*

Remark 5.7. Again, if the proof of the Positive Mass Theorem by Lohkamp in [6] or by Schoen and Yau announced in [10] is confirmed then, for all M and all $a < \sigma(M)$, $X_+^M(a) > 0$.

5.1. **Proof of Theorem 5.1.** Let $g \in \Omega_M^a$. In [5], we constructed a sequence of metrics g_k on $M^\sharp$ such that $\lim_k m(g_k, p) = m(g, p)$. In the construction, the metric g_k can be made isometric to g in $B_p^g(1)$ (as soon as $B_p^g(1)$ is topologically trivial). Moreover, we used exactly the same metrics as in the main result of [1] where it was proved that

$$\lim_k Y(g_k) \geq \min(\Lambda_{n,k}, Y(g))$$

where $\Lambda_{n,0} = +\infty$ and $\Lambda_{n,k} > 0$ depends only on n and k. This proves that for all $\varepsilon > 0$ we have $g_k \in \Omega_{M^\sharp}^{\min(a, \Lambda_{n,k}) - \varepsilon}$ as soon as k is large enough. Theorem 5.1 easily follows.

5.2. **Proof of Corollary 5.2.** (1) Let M_0 be a compact non-spin (resp. spin) simply connected manifold with $\sigma(M_0) > 0$ and M any compact (resp. compact spin) manifold of the same dimension. By Proposition 4.1,

$$X_+^{M\amalg(-M)}(a) = X_+^M(a),$$

where $(-M)$ is M equipped with the opposite orientation. Theorem 5.1 then shows that

$$X_+^M(a) = X_+^{M\amalg(-M)}(a) \leq X_+^{M\sharp(-M)}(\min(a, \Lambda_{n,k})) \tag{3}$$

where $\sharp$ denotes the connected sum. Here, we used that the connected sum is a surgery of dimension 0.

(2) The manifolds $M\sharp(-M)$ and $M_0\sharp(-M_0)$ are oriented (resp. spin) cobordant since they are both oriented (resp. spin) cobordant to S^n. Since $M_0\sharp(-M_0)$ is simply connected and not spin (resp. spin), it is obtained from $M\sharp(-M)$ by a finite sequence of surgeries of dimension $k \leq n-3$ (see the proofs of Theorem B and Theorem C in the article [4] by Gromov-Lawson). Theorem 5.1 then implies that

$$X_+^{M\sharp(-M)}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0\sharp(-M_0)}(\min(a, \Lambda_{n,k})). \tag{4}$$

Inequality (3) remains true when M is replaced by M_0. As a consequence, we get from Proposition 4.1 that

$$X_+^{M_0\sharp(-M_0)}(a) = \max(X_+^{M_0\sharp(-M_0)}(a), X_+^{M_0}(a)) = X_+^{M_0\sharp(-M_0)\amalg M_0}(a).$$

Using Theorem 5.1, we obtain

$$X_+^{M_0\sharp(-M_0)}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0\sharp(-M_0)\sharp M_0}(\min(a, \Lambda_{n,k}))$$

Now, $M_0\sharp(-M_0)\sharp M_0$ is oriented (resp. spin) cobordant to M_0 and M_0 is simply connected and not spin (resp. spin): by the same argument as above, M_0 is obtained from M by a finite sequence of surgeries of dimension $k \leq n-3$. This proves that

$$X_+^{M_0\sharp(-M_0)}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0\sharp(-M_0)\sharp M_0}(\min(a, \Lambda_{n,k})) \leq X_+^{M_0}(\min(a, \Lambda_{n,k})).$$

Together with Inequalities (3) and (4), we obtain the desired inequality

$$X_+^M(\min(a, \Lambda_{n,k})) \leq X_+^{M_0}(\min(a, \Lambda_{n,k})).$$

(3) The argument for $X_-^M(a)$ is similar.

6. Application to the Yamabe invariant

Theorem 6.1.

1. For any compact manifold M with $\sigma(M) > 0$, one has $X_+^M(0) = +\infty$.

2. For every $n \geq 3$ there exists a constant $d_n > 0$ such that for all compact manifolds M of dimension n with $\sigma(M) > 0$ and for all $a \in (0, \sigma(M))$ we have

$$X_+^M(a) \leq d_n \, \frac{(\sigma(S^n) - a)^{1/n}}{a}.$$

Corollary 6.2. *Let d_n be the constant in part 1 of Theorem 6.1 and suppose that M is a compact manifold of dimension n with $\sigma(M) > 0$ such that*

$$\frac{d_n}{\sigma(S^n)} < \limsup_{\varepsilon \to 0} \frac{X_+^M(\sigma(M) - \varepsilon)}{\varepsilon^{1/n}}.$$

Then we have $\sigma(M) < \sigma(S^n)$.

Note that the hypothesis of Corollary 6.2 is satisfied if $X_+^M(\sigma(M)) > 0$ since the function $a \mapsto X_+^M(a)$ is continuous from the left. This fact leads to a natural question: is this possible that $X_+^M(\sigma(M)) > 0$? The answer is given by

Proposition 6.3. *It holds that*

$$X_+^{\mathbb{R}P^3}(\sigma(\mathbb{R}P^3)) > 0 \quad \textit{and} \quad X_+^{S^n}(\sigma(S^n)) = 0.$$

Proof of Proposition 6.3. The fact that $X_+^{S^n}(\sigma(S^n)) = 0$ is an immediate consequence of Corollary 6.2. Beside, it was proven by Bray and Neves [3] that $\sigma(\mathbb{R}P^3) < \sigma(S^3)$ is attained by the standard metric. Since the standard metric of $\mathbb{R}P^3$ is locally conformally flat, one can choose a metric g in its conformal class such that $B_p^g(1)$ is flat where $p \in \mathbb{R}P^3$ is fixed. Then, since $\mathbb{R}P^3$ satisfies PMT, one has

$$X_+^{\mathbb{R}P^3}(\sigma(\mathbb{R}P^3)) \geq m(g, p) > 0.$$

□

Proof of Corollary 6.2. Assume that $\sigma(M) = \sigma(S^n)$. Using Theorem 6.1 we get

$$\frac{d_n}{\sigma(S^n)} < \limsup_{\varepsilon \to 0} \frac{d_n\, \varepsilon^{1/n}}{(\sigma(S^n) - \varepsilon)\varepsilon^{1/n}} = \limsup_{\varepsilon \to 0} \frac{d_n}{\sigma(S^n) - \varepsilon}$$

which is a contradiction. □

Proof of Theorem 6.1. For the first statement, let $p \in M$ and $g_m \in \Omega_M^0$ be a sequence of metrics converging in C^2 to some metric g_∞ such that $Y(g_\infty) = 0$. To see the existence of such a sequence, it suffices to construct g_∞. For this, just consider any metric $(g, p) \in \Omega_M^a$ for some $a > 0$. It is standard that one can modify g locally outside $B_p^g(1)$ to get a metric h such that $Y(h) < 0$. Then, set $g_t = tg + (1 - t)h$ for $t \in [0, 1]$. Let

$$t_\infty := \max\{t \in [0, 1] \mid Y(g_t) \leq 0\}.$$

We can then set $g_\infty := g_{t_\infty}$ and $g_m := g_{t_\infty + \frac{1}{m}}$. It was then proven in [5] and in [2] that $\lim_m m(g_m, p) = +\infty$ which proves that $X_+^M(0) = +\infty$.

The second statement is much harder to prove. Let $a \in (0, \sigma(M))$ and let $(g_\varepsilon)_{\varepsilon \in (0,a)}$ be a sequence of Riemannian metrics on M such that for all ε we have $g_\varepsilon \in \Omega_M^{a-\varepsilon}$ and

$$A_\varepsilon := m(g_\varepsilon, p) \geq X_+^M(a) + \varepsilon.$$

For every ε let β_ε be the smooth function on M such that $-A_\varepsilon = J_p^g(\beta_\varepsilon)$. We put

$$X_\varepsilon := \Big(\int_M |\beta_\varepsilon|^N \, dv^{g_\varepsilon} \Big)^{\frac{1}{N}}.$$

By Hölder's inequality and by the definition of $Y(g_\varepsilon)$ we have

$$\begin{aligned}-A_\varepsilon = J_p^{g_\varepsilon}(\beta_\varepsilon) &= \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^{g_\varepsilon} + 2\int_M \beta F_\eta \, dv^{g_\varepsilon} + \int_M \beta L_g \beta \, dv^{g_\varepsilon} \\ &\geq \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^{g_\varepsilon} - 2\Big(\int_M |F_\eta|^{\frac{N}{N-1}} \, dv^{g_\varepsilon}\Big)^{\frac{N-1}{N}} X_\varepsilon + (a-\varepsilon) X_\varepsilon^2.\end{aligned}$$

We put

$$C_n := \int_{M\setminus\{p\}} \eta r^{2-n} F_\eta \, dv^g, \quad D_n := 2\Big(\int_M |F_\eta|^{\frac{N}{N-1}} \, dv^g\Big)^{\frac{N-1}{N}}.$$

The numbers C_n, D_n are independent of (M, g) since g is flat on the supports of η and F_η. We get

$$-A_\varepsilon \geq C_n - D_n X_\varepsilon + (a-\varepsilon) X_\varepsilon^2.$$

We have $C_n \geq 0$ since for $M = S^n$ with the standard metric g_{can} we have

$$0 = -m(g_{\mathrm{can}}, p) \leq J_p^{g_{\mathrm{can}}}(0) = C_n.$$

We also have $C_n + A_\varepsilon \geq 0$ for all ε since

$$-A_\varepsilon = m(g_\varepsilon, p) \leq J_p^{g_\varepsilon}(0) = C_n.$$

The quadratic function $f(x) := A_\varepsilon + C_n - D_n x + (a-\varepsilon)x^2$ satisfies $f(X_\varepsilon) \leq 0$ and attains its minimum value at $x_0 = -\frac{D_n}{2(a-\varepsilon)}$. We get

$$0 \geq f(x_0) = A_\varepsilon + C_n - \frac{D_n^2}{4(a-\varepsilon)}$$

and thus

$$X_+^M(a) + \varepsilon \leq A_\varepsilon \leq \frac{D_n^2}{4(a-\varepsilon)} - C_n \leq \frac{D_n^2}{4(a-\varepsilon)}.$$

As $\varepsilon \to 0$ we obtain

$$X_+^M(a) \leq \frac{D_n^2}{4a} \tag{5}$$

Moreover, since $f(X_\varepsilon) \leq 0$, the number X_ε is less than or equal to the largest root of the equation $f(x) = 0$. Using that $A_\varepsilon + C_n \geq 0$ we get

$$X_\varepsilon \leq \frac{D_n + \sqrt{D_n^2 - 4(A_\varepsilon + C_n)(a-\varepsilon)}}{2(a-\varepsilon)} \leq \frac{D_n}{a-\varepsilon}.$$

Recall that the function β_ε is harmonic on $B_p^{g_\varepsilon}(\frac{1}{2})$. Thus for all $x \in B_p^{g_\varepsilon}(\frac{1}{3})$ we have

$$\beta_\varepsilon(x) = \frac{6^n}{\mathrm{vol}(\mathbb{B})} \int_{B_x^{g_\varepsilon}(\frac{1}{6})} \beta_\varepsilon(y) \, dv^{g_\varepsilon},$$

where $\mathbb{B}$ denotes the Euclidean unit ball, since $B_x^{g_\varepsilon}(\frac{1}{6}) \subset B_p^{g_\varepsilon}(\frac{1}{2})$. Using Hölder's inequality we get for all $x \in B_p^{g_\varepsilon}(\frac{1}{3})$:

$$\begin{aligned}|\beta_\varepsilon(x)| &\leq \frac{6^n}{\mathrm{vol}(\mathbb{B})}\mathrm{vol}\Big(B_x^{g_\varepsilon}\Big(\frac{1}{6}\Big)\Big)^{\frac{N-1}{N}}\Big(\int_{B_x^{g_\varepsilon}(\frac{1}{6})}|\beta_\varepsilon|^N\Big)^{\frac{1}{N}}\\ &\leq 6^{\frac{n}{N}}\mathrm{vol}(\mathbb{B})^{-\frac{1}{N}}\,X_\varepsilon\\ &\leq 6^{\frac{n}{N}}\mathrm{vol}(\mathbb{B})^{-\frac{1}{N}}\,\frac{D_n}{a-\varepsilon}\\ &=: \frac{B_n}{a-\varepsilon}. \end{aligned} \tag{6}$$

For every ε we define $\rho_\varepsilon > 0$ such that

$$\rho_\varepsilon^{n-2} = \delta_n \frac{|A_\varepsilon|}{(B_n + |A_\varepsilon|)^2} \tag{7}$$

where $\delta_n > 0$ can be chosen such that for all ε the number ρ_ε is as small as we want since the function $x \mapsto \frac{|x|}{(B_n+|x|)^2}$ is bounded. We choose δ_n such that $\rho_\varepsilon < \frac{1}{6}$ for all ε. Then for every ε we choose $h_\varepsilon \in C^\infty(M)$ such that $0 \leq h_\varepsilon \leq 1$, $h_\varepsilon \equiv 1$ on $B_p^{g_\varepsilon}(\rho_\varepsilon)$, $h_\varepsilon \equiv 0$ on $M \setminus B_p^{g_\varepsilon}(2\rho_\varepsilon)$ and $|dh_\varepsilon| \leq \frac{2}{\rho_\varepsilon}$. Moreover, for every ε we write the Green function G_ε of L_{g_ε} as

$$G_\varepsilon(x) = \eta(x)r(x)^{2-n} + A_\varepsilon + \alpha_\varepsilon(x)$$

where $\alpha_\varepsilon \in C^\infty(M)$ is harmonic on $B_p^{g_\varepsilon}(\frac{1}{2})$ and satisfies $\alpha_\varepsilon(p) = 0$.

Step 1. *For ε close enough to 0 we have*

$$\int_M |d(h_\varepsilon\alpha_\varepsilon)|^2 dv^{g_\varepsilon} \leq \frac{(n-2)\omega_{n-1}}{4}\,|A_\varepsilon|.$$

Since α_ε is harmonic on the support of dh_ε, we can use Identity (3) in [5] and Hölder inequality to write

$$\int_M |d(h_\varepsilon\alpha_\varepsilon)|^2 dv^{g_\varepsilon} = \int_{C_\varepsilon} |dh_\varepsilon|^2\alpha_\varepsilon^2 dv^{g_\varepsilon} \leq \left(\int_{C_\varepsilon}|dh_\varepsilon|^n dv_{g_\varepsilon}\right)^{\frac{2}{n}}\left(\int_{C_\varepsilon}|\alpha_\varepsilon|^N dv_{g_\varepsilon}\right)^{\frac{2}{N}}$$

where $C_\varepsilon := B_p^{g_\varepsilon}(2\rho_\varepsilon) \setminus B_p^{g_\varepsilon}(\rho_\varepsilon)$ is the support of dh_ε. Observe that the definition of h_ε and the fact that the volume of the support of dh_ε is bounded by $C\rho_\varepsilon^n$ with C independent of ε imply that there exists $\alpha_0 > 0$ which is independent of ε such that

$$\left(\int_{C_\varepsilon}|dh_\varepsilon|^n dv_{g_\varepsilon}\right)^{\frac{2}{n}} \leq \alpha_0.$$

Hence, for all ε small enough,

$$\int_M |d(h_\varepsilon\alpha_\varepsilon)|^2 dv^{g_\varepsilon} \leq \alpha_0\left(\int_{C_\varepsilon}|\alpha_\varepsilon|^N dv_{g_\varepsilon}\right)^{\frac{2}{N}}. \tag{8}$$

Now, with $\alpha_\varepsilon = \beta_\varepsilon - A_\varepsilon$ and using the equations (6) and (7) we get for ε close to 0:

$$\begin{aligned}\Big(\int_{C_\varepsilon} |\alpha_\varepsilon|^N \, dv^{g_\varepsilon}\Big)^{\frac{1}{N}} &\leq \Big(\int_{C_\varepsilon} |\beta_\varepsilon|^N \, dv^{g_\varepsilon}\Big)^{\frac{1}{N}} + |A_\varepsilon| \mathrm{vol}(C_\varepsilon)^{\frac{1}{N}} \\ &\leq \Big(\frac{B_n}{a-\varepsilon} + |A_\varepsilon|\Big)\mathrm{vol}(C_\varepsilon)^{\frac{1}{N}} \\ &= \Big(\frac{B_n}{a-\varepsilon} + |A_\varepsilon|\Big)(2^n-1)^{\frac{1}{N}} \rho_\varepsilon^{\frac{n-2}{2}} \mathrm{vol}(\mathbb{B})^{\frac{1}{N}} \\ &\leq \Big(\frac{B_n}{a-\varepsilon} + |A_\varepsilon|\Big)(2^n-1)^{\frac{1}{N}} \delta_n^{\frac{n-2}{2}} \frac{\sqrt{|A_\varepsilon|}}{B_n + |A_\varepsilon|} \mathrm{vol}(\mathbb{B})^{\frac{1}{N}} \\ &\leq E_n \sqrt{|A_\varepsilon|}\end{aligned}$$

where $E_n > 0$ is independent of ε. By choosing δ_n in equation (7) smaller we may assume that $E_n^2 \alpha_0 \leq \frac{(n-2)\omega_{n-1}}{4}$. Therefore the assertion of Step 1 follows from the equation (8).

For every ε we define

$$\beta_\varepsilon := \zeta_n |A_\varepsilon|^{\frac{1}{2}} \rho_\varepsilon^{\frac{n}{2}} \tag{9}$$

where $\zeta_n > 0$ will be fixed later. We have for all ε:

$$\beta_\varepsilon = \zeta_n |A_\varepsilon|^{\frac{1}{2}} \rho_\varepsilon^{\frac{n-2}{2}} \rho_\varepsilon = \zeta_n \delta_n^{\frac{1}{2}} \frac{|A_\varepsilon|}{B_n + |A_\varepsilon|} \rho_\varepsilon \leq \zeta_n \delta_n^{\frac{1}{2}} \rho_\varepsilon. \tag{10}$$

We define $u_\varepsilon \in C^\infty(M)$ by

$$u_\varepsilon(r) = \left(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + r^2}\right)^{\frac{n-2}{2}}$$

and $\psi_\varepsilon \in C^\infty(M)$ by

$$\psi_\varepsilon = \left| \begin{array}{ccc} u_\varepsilon & \text{if} & r \leq \rho_\varepsilon \\ \ell_\varepsilon(G_\varepsilon - h_\varepsilon \alpha_\varepsilon) & \text{if} & \rho_\varepsilon \leq r \leq 2\rho_\varepsilon \\ \ell_\varepsilon G_\varepsilon & \text{if} & r \geq 2\rho_\varepsilon \end{array} \right.$$

where $\ell_\varepsilon = u_\varepsilon(\rho_\varepsilon)(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon)^{-1}$ so that ψ_ε is continuous.

Step 2. *Conclusion.*

We set

$$E_\varepsilon = \int_M \Big(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2\Big) dv^{g_\varepsilon}$$

and

$$D_\varepsilon = \left(\int_M |\psi_\varepsilon|^N \, dv^{g_\varepsilon}\right)^{\frac{2}{N}}$$

so that

$$Q_\varepsilon(\psi_\varepsilon) = \frac{E_\varepsilon}{D_\varepsilon}. \tag{11}$$

We write

$$E_\varepsilon = E_1 + E_2$$

where

$$E_1 = \int_{B_\varepsilon} \Big(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2\Big) dv^{g_\varepsilon}$$

and

$$E_2 = \int_{M\setminus B_\varepsilon} \Big(|d\psi_\varepsilon|^2 + \frac{n-2}{4(n-1)} s_{g_\varepsilon} |\psi_\varepsilon|^2 \Big) dv^{g_\varepsilon}$$

where $B_\varepsilon := B_p^{g_\varepsilon}(\rho_\varepsilon)$ which is isometric to the Euclidean ball of radius ρ_ε. On B_ε, it holds that

$$\Delta_{g_\varepsilon} u_\varepsilon = n(n-2) u_\varepsilon^{N-1}$$

and we get from multiplying this equation by u_ε and integrating by parts that

$$\int_{B_\varepsilon} |du_\varepsilon|^2 dv^{g_\varepsilon} - \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r}\, da^{g_\varepsilon} = n(n-2) \int_{B_\varepsilon} |u_\varepsilon|^N dv^{g_\varepsilon}. \tag{12}$$

One also has

$$\sigma(S^n) = \frac{\int_{\mathbb{R}^n} |du_\varepsilon|^2 dx}{\left(\int_{\mathbb{R}^n} |u_\varepsilon|^N dx\right)^{\frac{2}{N}}} = n(n-2) \left(\int_{\mathbb{R}^n} |u_\varepsilon|^N dx \right)^{\frac{2}{n}}$$

which leads to

$$\left(\int_{B_\varepsilon} |u_\varepsilon|^N dx \right)^{\frac{2}{n}} \leq \frac{\sigma(S^n)}{n(n-2)}$$

Plugging this estimate into equation (12), we obtain

$$E_1 \leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} + \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r}\, da^{g_\varepsilon}. \tag{13}$$

Now, we evaluate E_2. For this, we integrate by parts:

$$E_2 = \int_{M\setminus B_\varepsilon} \psi_\varepsilon L_{g_\varepsilon} \psi_\varepsilon \, dv_{g_\varepsilon} + E_3$$

where E_3 is a boundary term which will be computed below. Since $L_{g_\varepsilon} G_\varepsilon = 0$ on $M \setminus B_\varepsilon$, we obtain

$$E_2 = \ell_\varepsilon^2 \int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon)(G_\varepsilon - h_\varepsilon \alpha_\varepsilon) dv_{g_\varepsilon} + E_3$$

where $B' := B_p^{g_\varepsilon}(2\rho_\varepsilon)$. Note that since α_ε is harmonic and h_ε is constant on B_ε, one has on B_ε,

$$L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) = \Delta_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) = 0.$$

Hence, by definition of the Green function G_ε, one has:

$$\int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) G_\varepsilon dv^{g_\varepsilon} = \int_M L_{g_\varepsilon}(-h_\varepsilon \alpha_\varepsilon) G_\varepsilon dv^{g_\varepsilon} = -\alpha_\varepsilon(p) = 0.$$

We also have

$$\int_{B'_\varepsilon \setminus B_\varepsilon} L_{g_\varepsilon}(h_\varepsilon \alpha_\varepsilon)(h_\varepsilon \alpha_\varepsilon) dv^{g_\varepsilon} = \int_M L_{g_\varepsilon}(h_\varepsilon \alpha_\varepsilon)(h_\varepsilon \alpha_\varepsilon) = \int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv^{g_\varepsilon}.$$

Since, by Step 1 we have

$$\int_M |d(h_\varepsilon \alpha_\varepsilon)|^2 dv^{g_\varepsilon} \leq \frac{(n-2)\omega_{n-1}}{4} |A_\varepsilon| =: \gamma_\varepsilon \tag{14}$$

we obtain:

$$E_2 \leq \ell_\varepsilon^2 \gamma_\varepsilon + E_3.$$

So let us evaluate E_3:

$$\begin{aligned} E_3 &= -\ell_\varepsilon^2 \int_{\partial B_\varepsilon} (G_\varepsilon - h_\varepsilon \alpha_\varepsilon) \frac{\partial (G_\varepsilon - h_\varepsilon \alpha_\varepsilon)}{\partial r} \\ &= -\ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \Big(-\frac{1}{\omega_{n-1}} \rho_\varepsilon^{1-n} \Big) \mathrm{vol}(\partial B_\varepsilon) \\ &= \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big). \end{aligned}$$

Combining this with (13)

$$\begin{aligned} E_1 + E_2 \le \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} \\ + \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} da^{g_\varepsilon} + \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) + \ell_\varepsilon^2 \gamma_\varepsilon. \end{aligned} \tag{15}$$

It remains to compute

$$E_4 := \int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} da^{g_\varepsilon}.$$

Using the fact that on ∂B_ε we have

$$u_\varepsilon = \ell_\varepsilon \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big)$$

and that

$$\frac{\partial u_\varepsilon}{\partial r} = -(n-2) \frac{\rho_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} u_\varepsilon$$

we obtain

$$E_4 = -(n-2)\omega_{n-1} \frac{\rho_\varepsilon^n}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big)^2$$

which together with the definition (9) of β_ε leads to

$$\begin{aligned} &\int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} da^{g_\varepsilon} + \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \\ &= \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \Big(1 - \frac{\rho_\varepsilon^n}{\beta_\varepsilon^2 + \rho_\varepsilon^2} (\rho_\varepsilon^{2-n} + (n-2)\omega_{n-1} A_\varepsilon) \Big) \\ &= \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \frac{\beta_\varepsilon^2 - \rho_\varepsilon^n (n-2)\omega_{n-1} A_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \\ &= \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \frac{(\zeta_n^2 |A_\varepsilon| - (n-2)\omega_{n-1} A_\varepsilon) \rho_\varepsilon^n}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^n + \rho_\varepsilon^2} \\ &= \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} + A_\varepsilon \rho_\varepsilon^{n-2} \Big) \frac{\zeta_n^2 |A_\varepsilon| - (n-2)\omega_{n-1} A_\varepsilon}{\zeta_n^2 |A_\varepsilon| \rho_\varepsilon^{n-2} + 1} \end{aligned}$$

By assumption we have

$$A_\varepsilon \ge X_+^M(a) + \varepsilon.$$

If $X_+^M(a) \le 0$, then the assertion 1 of Theorem 6.1 holds trivially. Thus we may assume that $A_\varepsilon > 0$ for all ε. We put

$$\zeta_n := \frac{\sqrt{(n-2)\omega_{n-1}}}{2}.$$

If ρ_ε is small enough, we obtain

$$\begin{aligned}&\int_{\partial B_\varepsilon} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} da^{g_\varepsilon} + \ell_\varepsilon^2 \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big) \\ &\leq \ell_\varepsilon^2 \frac{1}{(n-2)\omega_{n-1}} \Big(- \frac{(n-2)\omega_{n-1} A_\varepsilon}{2} \Big).\end{aligned}$$

Inserting this into equation (15) and using the definition (14) of γ_ε we get

$$\begin{aligned}E_1 + E_2 &\leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} \\ &\quad - \ell_\varepsilon^2 \frac{1}{(n-2)\omega_{n-1}} \Big(- \frac{(n-2)\omega_{n-1} A_\varepsilon}{2} + \frac{(n-2)\omega_{n-1} A_\varepsilon}{4} \Big) \\ &= \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - \frac{\ell_\varepsilon^2 A_\varepsilon}{4}.\end{aligned}$$

Moreover, if ρ_ε is small enough, we obtain

$$\begin{aligned}\ell_\varepsilon^2 &= \Big(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + \rho_\varepsilon^2} \Big)^{n-2} \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big)^{-2} \\ &= \Big(\frac{\zeta_n |A_\varepsilon|^{1/2} \rho_\varepsilon^{n/2}}{\zeta_n^2 A_\varepsilon \rho_\varepsilon^n + \rho_\varepsilon^2} \Big)^{n-2} \Big(\frac{1}{(n-2)\omega_{n-1}} \rho_\varepsilon^{2-n} + A_\varepsilon \Big)^{-2} \\ &= \Big(\frac{\zeta_n |A_\varepsilon|^{1/2} \rho_\varepsilon^{n/2}}{\zeta_n^2 A_\varepsilon \rho_\varepsilon^{n-2} + 1} \Big)^{n-2} \rho_\varepsilon^{4-2n} \Big(\frac{1}{(n-2)\omega_{n-1}} + A_\varepsilon \rho_\varepsilon^{n-2} \Big)^{-2} \rho_\varepsilon^{2n-4} \\ &\geq \Big(\frac{\zeta_n |A_\varepsilon|^{1/2}}{2} \Big)^{n-2} \rho_\varepsilon^{\frac{n(n-2)}{2}} \frac{(n-2)^2 \omega_{n-1}^2}{4}.\end{aligned}$$

From this and the definition (7) of ρ_ε it follows that

$$\begin{aligned}E_1 + E_2 &\leq \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - 4^{-n} \big((n-2)\omega_{n-1} \big)^{\frac{n+2}{2}} |A_\varepsilon|^{\frac{n}{2}} \rho_\varepsilon^{\frac{n(n-2)}{2}} \\ &= \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - 4^{-n} \big((n-2)\omega_{n-1} \big)^{\frac{n+2}{2}} \delta_n^{\frac{n}{2}} \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n} \\ &=: \sigma(S^n) \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N dv^{g_\varepsilon} \right)^{\frac{2}{N}} - E_n' \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}.\end{aligned}$$

We have

$$D_\varepsilon \geq \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N \, dv^{g_\varepsilon} \right)^{\frac{2}{N}}$$

and therefore

$$Q_\varepsilon(\psi_\varepsilon) = \frac{E_1 + E_2}{D_\varepsilon} \leq \sigma(S^n) - E_n' \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n} \left(\int_{B_\varepsilon} |\psi_\varepsilon|^N \, dv^{g_\varepsilon} \right)^{-\frac{2}{N}}$$

Moreover, using the substitution $r = \beta_\varepsilon s$ we get

$$\begin{aligned}\int_{B_\varepsilon} |\psi_\varepsilon|^N \, dv^{g_\varepsilon} &= \omega_{n-1} \int_0^{\rho_\varepsilon} \Big(\frac{\beta_\varepsilon}{\beta_\varepsilon^2 + r^2}\Big)^n r^{n-1} \, dr \\ &= \omega_{n-1} \int_0^{\rho_\varepsilon/\beta_\varepsilon} \Big(\frac{1}{1+s^2}\Big)^n s^{n-1} \, ds \\ &\leq \omega_{n-1} \int_0^{\infty} \Big(\frac{1}{1+s^2}\Big)^n s^{n-1} \, ds \\ &=: F_n.\end{aligned}$$

It follows that

$$Q_\varepsilon(\psi_\varepsilon) \leq \sigma(S^n) - \frac{E_n'}{F_n^{2/N}} \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}$$

and therefore with $G_n := (E_n' F_n^{-2/N})^{1/n}$ we get

$$a - \varepsilon < Y(g_\varepsilon) \leq \sigma(S^n) - (G_n)^n \frac{|A_\varepsilon|^n}{(B_n + |A_\varepsilon|)^n}.$$

We take the limit $\varepsilon \to 0$ and we get

$$a \leq \sigma(S^n) - (G_n)^n \frac{X_+^M(a)^n}{(B_n + X_+^M(a))^n} = \sigma(S^n) - (G_n)^n \frac{1}{(\frac{B_n}{X_+^M(a)} + 1)^n}$$

It follows that

$$\frac{B_n}{X_+^M(a)} + 1 \geq \frac{G_n}{(\sigma(S^n) - a)^{1/n}}$$

We put $\alpha_n := \sigma(S^n) - \frac{(G_n)^n}{2^n}$ and we distinguish two cases.
a) If $a \geq \alpha_n$ we get

$$\frac{B_n}{X_+^M(a)} \geq \frac{1}{2} \frac{G_n}{(\sigma(S^n) - a)^{1/n}}$$

and since $a < \sigma(S^n)$ we get

$$X_+^M(a) \leq \frac{2B_n}{G_n} (\sigma(S^n) - a)^{1/n} \leq \frac{2B_n}{G_n} \, \frac{\sigma(S^n)}{a} (\sigma(S^n) - a)^{1/n}$$

b) If $a \leq \alpha_n$ we have

$$\sigma(S^n) - a \geq \sigma(S^n) - \alpha_n$$

and therefore using equation (5)

$$X_+^M(a) \leq \frac{D_n^2}{4a} \leq \frac{D_n^2 (\sigma(S^n) - a)^{1/n}}{4(\sigma(S^n) - \alpha_n)^{1/n} a} = \frac{D_n^2 (\sigma(S^n) - a)^{1/n}}{2 G_n a}.$$

This shows the second statement. □

Appendix A. A lemma on the Yamabe constant

Let M be a compact manifold of dimension $n \geq 3$. For any Riemannian metric g on M we denote by

$$L_g := \Delta_g + \frac{n-2}{4(n-1)} s_g$$

the Yamabe operator of g and by

$$Q_g : \quad C^\infty(M) \to C^\infty(M), \quad Q_g(u) := \frac{\int_M u L_g u \, dv^g}{(\int_M |u|^N)^{2/N}}$$

the Yamabe functional of g where $N := \frac{2n}{n-2}$. We denote by

$$Y(g) := \inf\{Q_g(u) \mid u \in C^\infty(M),\, u \neq 0\}$$

the Yamabe constant of (M, g). In this Appendix, we prove the following Lemma which is used several times in the paper.

Lemma A.1. *Let (M, g) be a compact Riemannian manifold with $Y(g) > 0$. Let $p \in M$, let $\varepsilon > 0$ and let h be a Riemannian metric defined on $B_p^g(2\varepsilon)$. We assume that g and h coincide at p. Let $\chi_\varepsilon \in C^\infty(M)$, $0 \leq \chi_\varepsilon \leq 1$ be a cut-off function equal to 1 on $B_p^g(\varepsilon)$, equal to 0 outside $B_p^g(2\varepsilon)$ and which satisfies $|d\chi_\varepsilon|_g \leq \frac{C}{\varepsilon}$ and $|\nabla^2\chi_\varepsilon| \leq \frac{C}{\varepsilon^2}$. Define*

$$g_\varepsilon := \chi_\varepsilon h + (1 - \chi_\varepsilon)g.$$

Then we have $\lim_{\varepsilon\to 0} Y(g_\varepsilon) = Y(g)$.

Proof. Note that by Taylor expansion we have on $B_p^g(2\varepsilon)$

$$|g - h|_g \leq C\varepsilon \quad \text{and thus} \quad |g - g_\varepsilon|_g \leq C\varepsilon$$

for some C independent of ε. Thus for all $u \in C^\infty(M)$ we have

$$\left|\int_M u\, dv^{g_\varepsilon} - \int_M u\, dv^g\right| \leq C\varepsilon \int_{B_p^g(2\varepsilon)} |u|\, dv^g,$$
$$\left||du|^2_{g_\varepsilon} - |du|^2_g\right| = \left|(g_\varepsilon^{ij} - g^{ij})\partial_i u \partial_j u\right| \leq C\varepsilon |du|^2_g$$

and using that $|\nabla^2\chi_\varepsilon|\,|g - h|_g \leq \frac{C}{\varepsilon}$ we get

$$|s_{g_\varepsilon} - s_g| \leq \frac{C}{\varepsilon} \quad \text{and} \quad |s_{g_\varepsilon}| \leq \frac{C}{\varepsilon}.$$

Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions on M such that $\int_M |u_\varepsilon|^N dv^g$ and $Q_g(u_\varepsilon)$ are bounded independently of ε. We prove that

$$\lim_{\varepsilon\to 0}(Q_{g_\varepsilon}(u_\varepsilon) - Q_g(u_\varepsilon)) = 0. \tag{16}$$

By the triangle inequality we have

$$\left|\int_M s_{g_\varepsilon} u_\varepsilon^2\, dv^{g_\varepsilon} - \int_M s_g u_\varepsilon^2\, dv^g\right|$$
$$\leq \left|\int_M s_{g_\varepsilon} u_\varepsilon^2\, dv^{g_\varepsilon} - \int_M s_{g_\varepsilon} u_\varepsilon^2\, dv^g\right| + \int_M |s_{g_\varepsilon} - s_g||u_\varepsilon|^2\, dv^g$$

By the Hölder inequality we get

$$\int_M |s_{g_\varepsilon} - s_g||u_\varepsilon|^2 dv^g \leq \frac{C}{\varepsilon}\mathrm{vol}(B_p^g(2\varepsilon))^{\frac{2}{n}} \left(\int_M |u_\varepsilon|^N dv^g\right)^{\frac{2}{N}} \leq C\varepsilon \left(\int_M |u_\varepsilon|^N dv^g\right)^{\frac{2}{N}},$$

and

$$\left|\int_M s_{g_\varepsilon} u_\varepsilon^2\, dv^{g_\varepsilon} - \int_M s_{g_\varepsilon} u_\varepsilon^2\, dv^g\right| \leq C\varepsilon \int_{B_p^g(2\varepsilon)} |s_{g_\varepsilon}||u_\varepsilon|^2\, dv^g$$
$$\leq C\mathrm{vol}(B_p^g(2\varepsilon))^{\frac{2}{n}} \left(\int_M |u_\varepsilon|^N dv^g\right)^{\frac{2}{N}} \leq C\varepsilon^2 \left(\int_M |u_\varepsilon|^N dv^g\right)^{\frac{2}{N}}.$$

Moreover, we get

$$\begin{aligned}
&\left|\int_M |du_\varepsilon|^2_{g_\varepsilon}\, dv^{g_\varepsilon} - \int_M |du_\varepsilon|^2_g\, dv^g\right| \\
&\leq \left|\int_M |du_\varepsilon|^2_{g_\varepsilon}\, dv^{g_\varepsilon} - \int_M |du_\varepsilon|^2_{g_\varepsilon}\, dv^g\right| + \int_M \left||du_\varepsilon|^2_{g_\varepsilon} - |du_\varepsilon|^2_g\right|\, dv^g \\
&\leq C\varepsilon \int_M |du_\varepsilon|^2_g\, dv^g \\
&= C\varepsilon \left(\int_M u_\varepsilon L_g u_\varepsilon\, dv^g - \int_M s_g |u_\varepsilon|^2\, dv^g\right) \\
&\leq C\varepsilon \left(\int_M u_\varepsilon L_g u_\varepsilon\, dv^g + C\left(\int_M |u_\varepsilon|^N\, dv^g\right)^{\frac{2}{N}}\right).
\end{aligned}$$

The equation (16) follows easily from these estimates. Now let $(u_\varepsilon)_\varepsilon$ be a sequence in $C^\infty(M)$ such that $\int_M |u_\varepsilon|^N\, dv^g = 1$ and

$$Q_{g_\varepsilon}(u_\varepsilon) = Y(g_\varepsilon) + \varepsilon$$

for all ε. Using (16) we get

$$Y(g) \leq \liminf_{\varepsilon\to 0} Q_g(u_\varepsilon) = \liminf_{\varepsilon\to 0} Q_{g_\varepsilon}(u_\varepsilon) = \liminf_{\varepsilon\to 0} Y(g_\varepsilon).$$

On the other hand let $u \in C^\infty(M)$ such that $\int_M |u|^N\, dv^g = 1$ and

$$Q_g(u) = Y(g) + \delta$$

where $\delta > 0$. Using (16) we get

$$\limsup_{\varepsilon\to 0} Y(g_\varepsilon) \leq \limsup_{\varepsilon\to 0} Q_{g_\varepsilon}(u) = \limsup_{\varepsilon\to 0} Q_g(u) = Y(g) + \delta.$$

As $\delta \to 0$ we get $\limsup_{\varepsilon\to 0} Y(g_\varepsilon) \leq Y(g)$. □

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