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# Random walk approximation of BSDEs with Hölder continuous terminal condition

Christel Geiss<sup>1</sup>, Céline Labart<sup>2</sup>, Antti Luoto<sup>3</sup>

## Abstract

In this paper we consider the random walk approximation of the solution of a Markovian BSDE whose terminal condition is a locally Hölder continuous function of the Brownian motion. We state the rate of the  $L_2$ -convergence of the approximated solution to the true one. The proof relies in part on growth and smoothness properties of the solution  $u$  of the associated PDE. Here we improve existing results by showing some properties of the second derivative of  $u$  in space.

**Keywords :** Backward stochastic differential equations, numerical scheme, random walk approximation, speed of convergence

**MSC codes :** 65C30 60H35 60G50 65G99

## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space carrying the standard Brownian motion  $B = (B_t)_{t \geq 0}$  and assume  $(\mathcal{F}_t)_{t \geq 0}$  is the augmented natural filtration. We consider the following backward stochastic differential equation (BSDE for short)

$$Y_s = g(B_T) + \int_s^T f(r, B_r, Y_r, Z_r) dr - \int_s^T Z_r dB_r, \quad 0 \leq s \leq T, \quad (1)$$

where  $f$  is Lipschitz continuous and  $g$  is a locally  $\alpha$ -Hölder continuous and polynomially bounded function (see (3)). In this paper we are interested in the  $L_2$ -convergence of the numerical approximation of (1) by using a random walk. First results dealing with the numerical approximation of BSDEs date back to the late 1990s. Bally (see [2]) was the first to consider this problem by introducing random discretization, namely the jump times of a Poisson process. In his PhD thesis, Chevance (see [17]) proposed the following discretization

$$y_k = \mathbb{E}(y_{k+1} + hf(y_{k+1}) | \mathcal{F}_k^n), \quad k = n-1, \dots, 0, \quad n \in \mathbb{N}^*$$

and proved the convergence of  $(Y_t^n)_t := (y_{[t/h]})_t$  to  $Y$ . At the same time, Coquet, Mackevičius and Mémin [18] proved the convergence of  $Y^n$  by using convergence of filtrations, still in the case of

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a generator independent from  $z$ . The general case ( $f$  depends on  $z$ , terminal condition  $\xi \in L_2$ ) has been studied by Briand, Delyon and Mémmin (see [5]). In that paper the authors define an approximated solution  $(Y^n, Z^n)$  based on random walk and prove weak convergence to  $(Y, Z)$  using convergence of filtrations. We also refer to [27], [29], [30], [31] for other numerical methods for BSDEs which use a random walk approach. The rate of convergence of this method was left as an open problem.

Introducing instead of random walk an approach based on the dynamic programming equation, Bouchard and Touzi in [8] and Zhang in [35] managed to establish a rate of convergence. However, to be fully implementable, this algorithm requires to have a good approximation of its associated conditional expectation. For this, various methods have been developed (see [24], [19], [15]). Forward methods have also been introduced to approximate (1) : a branching diffusion method (see [26]), a multilevel Picard approximation (see [34]) and Wiener chaos expansion (see [7]). Many extensions of (1) have also been considered : high order schemes (see [11], [10]), schemes for reflected BSDEs (see [3], [14]), for fully-coupled BSDEs (see [21], [9]), for quadratic BSDEs (see [13]), for BSDEs with jumps (see [23]) and for McKean-Vlasov BSDEs (see [1], [16], [12]).

From a numerical point of view, the random walk is of course not competitive with recent methods listed above. We emphasize that the aim of this paper is to give the convergence rate of the initial method based on random walk, which, to the best of our knowledge, has not been done so far.

As in [5], let us introduce the following approximation of  $B$ , based on a random walk:

$$B_t^n = \sqrt{h} \sum_{i=1}^{\lfloor t/h \rfloor} \varepsilon_i, \quad 0 \leq t \leq T,$$

where  $h = \frac{T}{n}$  ( $n \in \mathbb{N}^*$ ) and  $(\varepsilon_i)_{i=1,2,\dots}$  is a sequence of i.i.d. Rademacher random variables. Consider the following approximated solution  $(Y^n, Z^n)$  of  $(Y, Z)$

$$Y_{t_k}^n = g(B_T^n) + h \sum_{m=k}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) - \sqrt{h} \sum_{m=k}^{n-1} Z_{t_m}^n \varepsilon_{m+1}, \quad 0 \leq k \leq n-1. \quad (2)$$

The main result of our paper gives the rate of convergence in  $L_2$ -norm of  $Y_v^n - Y_v$  and  $Z_v^n - Z_v$  for each  $v \in [0, T)$  (see Theorem 3.1). Basically, we get that the  $L_2$ -norm of the error on  $Y$  is of order  $h^{\frac{\alpha}{4}}$  and the  $L_2$ -norm of the error on  $Z$  is of order  $\frac{h^{\frac{\alpha}{4}}}{\sqrt{T-v}}$ . The proof of this result is based on several ingredients. In particular, we need some estimates on the bound of the first and second derivatives of the solution of the PDE associated to the BSDE (1). We establish these bounds in the case of a forward backward SDE (FBSDE for short) whose terminal condition satisfies the Hölder continuity condition (3). This result extends Zhang [36, Theorem 3.2].

The rest of the paper is organized as follows. Section 2 introduces notations, assumptions and the representation for  $Z$  and  $Z^n$  based on the Malliavin weights. Section 3 states the rate of convergence of the error on  $Y$  and  $Z$  in  $L_2$ -norm, which is the main result of the paper. Section 4 presents numerical simulations and Section 5 recalls some properties of Malliavin weights, of the regularity of solutions to FBSDEs with a locally Hölder continuous terminal condition function and states some properties of the solutions to the PDEs associated to these FBSDEs.

## 2 Preliminaries

This section is dedicated to notations, assumptions and the representation of  $Z$  and  $Z^n$  using the Malliavin weights.

**Notation:**

- $\mathcal{G}_k := \sigma(\varepsilon_i : 1 \leq i \leq k)$  and  $\mathcal{G}_0 = \{\emptyset, \Omega\}$ . The associated discrete-time random walk  $(B_{t_k}^n)_{k=0}^n$  is  $(\mathcal{G}_k)_{k=0}^n$ -adapted.
- $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{P})}$  for  $p \geq 1$  and for  $p = 2$  simply  $\|\cdot\|$ . constant.

**Assumption 2.1.**

- $g$  is locally Hölder continuous with order  $\alpha \in (0, 1]$  and polynomially bounded ( $p_0 \geq 0, C_g > 0$ ) in the following sense

$$\forall (x, y) \in \mathbb{R}^2, \quad |g(x) - g(y)| \leq C_g(1 + |x|^{p_0} + |y|^{p_0})|x - y|^\alpha. \quad (3)$$

- The function  $[0, T] \times \mathbb{R}^3 : (t, x, y, z) \mapsto f(t, x, y, z)$  satisfies

$$|f(t, x, y, z) - f(t', x', y', z')| \leq L_f(\sqrt{t - t'} + |x - x'| + |y - y'| + |z - z'|). \quad (4)$$

Notice that (3) implies

$$|g(x)| \leq K(1 + |x|^{p_0+1}) =: \Psi(x). \quad (5)$$

In the rest of the paper, the study of the error  $(Y^n - Y, Z^n - Z)$  will either rely on (2) or on its integral version:

$$Y_s^n = g(B_T^n) + \int_{(s, T]} f(r, B_{r-}^n, Y_{r-}^n, Z_{r-}^n) d[B^n, B^n]_r - \int_{(s, T]} Z_{r-}^n dB_r^n, \quad 0 \leq s \leq T, \quad (6)$$

where the backward equation (6) arises from (2) by setting  $Y_r^n := Y_{t_m}^n$  and  $Z_r^n := Z_{t_m}^n$  for  $r \in [t_m, t_{m+1})$ . For  $n$  large enough, (6) has a unique solution  $(Y^n, Z^n)$ , and  $(Y_{t_m}^n, Z_{t_m}^n)_{m=0}^{n-1}$  is adapted to the filtration  $(\mathcal{G}_m)_{m=0}^{n-1}$ . Let us now introduce the Malliavin representations for  $Z$  and  $Z^n$ . They are the cornerstone of our study of the error on  $Z$ .

## 2.1 Representations for $Z$ and $Z^n$

We will use the representation (see Ma and Zhang [28, Theorem 4.2])

$$Z_t = \mathbb{E}_t \left( g(B_T) N_T^t + \int_t^T f(s, B_s, Y_s, Z_s) N_s^t ds \right), \quad 0 \leq t \leq T, \quad (7)$$

where  $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ , and for all  $s \in (t, T]$  we have

$$N_s^t := \frac{B_s - B_t}{s - t}.$$

**Lemma 2.2.** *Suppose that Assumption 2.1 holds. Then the process  $Z^n$  given by (6) has the representation*

$$Z_{t_k}^n = \mathbb{E}_k \left( g(B_T^n) \frac{B_{t_n}^n - B_{t_k}^n}{t_n - t_k} \right) + \mathbb{E}_k \left( h \sum_{m=k+1}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right) \quad (8)$$

for  $k = 0, 1, \dots, n-1$ , where  $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot | \mathcal{G}_k]$ .

*Proof.* We multiply equation (2) by  $\varepsilon_{k+1}$  and take the conditional expectation with respect to  $\mathcal{G}_k$ . Since  $(Y_{t_k}^n, Z_{t_k}^n)$  is  $\mathcal{G}_k$ -measurable, it holds for  $0 \leq k \leq n-1$  that

$$\begin{aligned} & \mathbb{E}_k \left( Y_{t_k}^n \varepsilon_{k+1} \right) \\ &= \mathbb{E}_k (g(B_T^n) \varepsilon_{k+1}) + h \mathbb{E}_k \left( \sum_{m=k}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \varepsilon_{k+1} \right) - \sqrt{h} \mathbb{E}_k \left( \sum_{m=k}^{n-1} Z_{t_m}^n \varepsilon_{m+1} \varepsilon_{k+1} \right) \\ &= \sqrt{h} \mathbb{E}_k \left( g(B_T^n) \frac{B_{t_n}^n - B_{t_k}^n}{t_n - t_k} \right) + h^{3/2} \sum_{m=k+1}^{n-1} \mathbb{E}_k \left( f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right) - \sqrt{h} Z_{t_k}^n, \end{aligned} \quad (9)$$

where the l.h.s. is equal to zero. Indeed, for  $m \geq k+1$ , we have

$$\mathbb{E}_k (Z_{t_m}^n \varepsilon_{m+1} \varepsilon_{k+1}) = \mathbb{E}_k (Z_{t_m}^n \varepsilon_{k+1} \mathbb{E}_m \varepsilon_{m+1}) = 0,$$

and for  $m = k$  it holds  $\mathbb{E}_k (Z_{t_k}^n \varepsilon_{k+1}^2) = Z_{t_k}^n$ . Moreover, the fact that  $B_T^n = \sqrt{h} \sum_{m=0}^{n-1} \varepsilon_{m+1}$ , where  $(\varepsilon_m)_{m=1,2,\dots}$  are i.i.d., yields

$$\mathbb{E}_k (g(B_T^n) \varepsilon_{k+1}) = \mathbb{E}_k \left( g(B_T^n) \sum_{m=k}^{n-1} \frac{\varepsilon_{k+1}}{n-k} \right) = \mathbb{E}_k \left( g(B_T^n) \sum_{m=k}^{n-1} \frac{\varepsilon_{m+1}}{n-k} \right) = \sqrt{h} \mathbb{E}_k \left( g(B_T^n) \frac{B_{t_n}^n - B_{t_k}^n}{t_n - t_k} \right).$$

Similarly, for  $m \geq k+1$ , we get (using [5, Proposition 5.1], where it is stated that both  $Y_{t_m}^n$  and  $Z_{t_m}^n$  can be represented as functions of  $t_m$  and  $B_{t_m}^n$ )

$$\mathbb{E}_k (f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \varepsilon_{k+1}) = \sqrt{h} \mathbb{E}_k \left( f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right).$$

It remains to divide (9) by  $\sqrt{h}$  and rearrange. □

### 3 Main result

This section is devoted to the main result of the paper: the rate of the  $L_2$ -convergence of  $(Y^n, Z^n)$  to  $(Y, Z)$ . The proof will rely on the fact that the random walk  $B^n$  can be constructed from the Brownian motion  $B$  by Skorohod embedding. Let  $\tau_0 := 0$  and define

$$\tau_k := \inf\{t > \tau_{k-1} : |B_t - B_{\tau_{k-1}}| = \sqrt{h}\}, \quad k \geq 1.$$

Then  $(B_{\tau_k} - B_{\tau_{k-1}})_{k=1}^\infty$  is a sequence of i.i.d. random variables with

$$\mathbb{P}(B_{\tau_k} - B_{\tau_{k-1}} = \pm\sqrt{h}) = \frac{1}{2},$$

which means that  $\sqrt{h}\varepsilon_k \stackrel{d}{=} B_{\tau_k} - B_{\tau_{k-1}}$ . We will use this random walk for our approximation, i.e. we will require

$$B_t^n = \sum_{k=1}^{\lfloor t/h \rfloor} (B_{\tau_k} - B_{\tau_{k-1}}), \quad 0 \leq t \leq T. \quad (10)$$

Properties satisfied by  $\tau_k$  and  $B_{\tau_k}$  are stated in Lemma A.1. We will denote by  $\mathbb{E}_{\tau_k}$  the conditional expectation w.r.t.  $\mathcal{F}_{\tau_k}$ .

**Theorem 3.1.** *Let Assumption 2.1 hold. If  $B^n$  satisfies (10) then we have (for sufficiently large  $n$ ) that*

$$\begin{aligned}\mathbb{E}|Y_v - Y_v^n|^2 &\leq C_0 h^{\frac{\alpha}{2}} \quad \text{for } v \in [0, T), \\ \mathbb{E}|Z_v - Z_v^n|^2 &\leq C_0 \frac{h^{\frac{\alpha}{2}}}{T - t_k} + C_1 \frac{h^{\frac{\alpha}{2}}}{(T - v)^{1 - \frac{\alpha}{2}}} \mathbf{1}_{v \neq t_k} \quad \text{for } v \in [t_k, t_{k+1}), \quad k = 0, \dots, n-1,\end{aligned}$$

where we have the dependencies  $C_0 = C(T, p_0, L_f, C_g, C_{5.3}^y, C_{5.3}^z, K_f, c_{5.4}, \alpha)$ ,  $C_1 = C(T, p_0, C_{5.3}^z, \alpha)$  and  $K_f := \sup_{0 \leq t \leq T} |f(t, 0, 0, 0)|$ .

**Remark 3.2.** *Theorem 3.1 implies that*

$$\sup_{v \in [0, T)} \mathbb{E}|Y_v - Y_v^n|^2 \leq C_0 h^{\frac{\alpha}{2}} \quad \text{and} \quad \mathbb{E} \int_0^T |Z_v - Z_v^n|^2 dv \leq C(C_0, C_1, \beta) h^\beta \quad \text{for } \beta \in (0, \frac{\alpha}{2}).$$

*Proof of Theorem 3.1.* Let  $u : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$  be the solution of the PDE associated to (1). Since by Theorem 5.4

$$Y_s = u(s, B_s), \quad Z_s = u_x(s, B_s), \quad a.s.$$

we introduce

$$F(s, x) := f(s, x, u(s, x), u_x(s, x)),$$

so that  $F(s, B_s) = f(s, B_s, Y_s, Z_s)$ . We first give some properties satisfied by  $F$ .

**Lemma 3.3.** *If Assumption 2.1 holds then  $F$  is a Lipschitz continuous and polynomially bounded function in  $x$  :*

$$\begin{aligned}|F(t, x_1) - F(t, x_2)| &\leq C(T, L_f, c_{5.4}^{2,3}) (1 + |x_1|^{p_0+1} + |x_2|^{p_0+1}) \frac{|x_1 - x_2|}{(T - t)^{1 - \frac{\alpha}{2}}}, \\ |F(t, x)| &\leq C(T, L_f, c_{5.4}^{1,2}, K_f) \frac{\Psi(x)}{(T - t)^{\frac{1-\alpha}{2}}},\end{aligned}$$

where  $\Psi(x)$  is given in (5).

*Proof of Lemma 3.3.* Thanks to the mean value theorem and Theorem 5.4-(ii-c) and (iii-b) we have for  $x_1, x_2 \in \mathbb{R}$  that there exist  $\xi_1, \xi_2 \in [\min\{x_1, x_2\}, \max\{x_1, x_2\}]$  such that

$$\begin{aligned}|F(t, x_1) - F(t, x_2)| &= |f(t, x_1, u(t, x_1), u_x(t, x_1)) - f(t, x_2, u(t, x_2), u_x(t, x_2))| \\ &\leq L_f (|x_1 - x_2| + |u(t, x_1) - u(t, x_2)| + |u_x(t, x_1) - u_x(t, x_2)|) \\ &\leq L_f \left( 1 + \frac{c_{5.4}^2 \Psi(\xi_1)}{(T - t)^{\frac{1-\alpha}{2}}} + \frac{c_{5.4}^3 \Psi(\xi_2)}{(T - t)^{1 - \frac{\alpha}{2}}} \right) |x_1 - x_2| \\ &\leq C(T, L_f, c_{5.4}^{2,3}) (1 + |x_1|^{p_0+1} + |x_2|^{p_0+1}) \frac{|x_1 - x_2|}{(T - t)^{1 - \frac{\alpha}{2}}}.\end{aligned}$$

The second inequality can be shown similarly. □

For the estimate of  $\mathbb{E}|Y_{t_k} - Y_{t_k}^n|^2$  we will use (1) and (2): Since  $Y_{t_k}^n$  is  $\mathcal{F}_{\tau_k}$ -measurable we have

$$\begin{aligned} \|Y_{t_k} - Y_{t_k}^n\| &\leq \|\mathbb{E}_{t_k} g(B_T) - \mathbb{E}_{\tau_k} g(B_T^n)\| \\ &\quad + \left\| \mathbb{E}_{t_k} \int_{t_k}^T f(s, B_s, Y_s, Z_s) ds - h \mathbb{E}_{\tau_k} \sum_{m=k}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \right\|. \end{aligned} \quad (11)$$

We frequently express conditional expectations with the help of an independent copy of  $B$  denoted by  $\tilde{B}$ , for example  $\mathbb{E}_t g(B_T) = \tilde{\mathbb{E}} g(B_t + \tilde{B}_{T-t})$ .

By (3) and Lemma A.1,

$$\begin{aligned} \|\mathbb{E}_{t_k} g(B_T) - \mathbb{E}_{\tau_k} g(B_T^n)\|^2 &= \mathbb{E} |\tilde{\mathbb{E}} g(B_{t_k} + \tilde{B}_{T-t_k}) - \tilde{\mathbb{E}} g(B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{n-k}})|^2 \\ &\leq (\mathbb{E} \tilde{\mathbb{E}}(\Psi_1)^4)^{\frac{1}{2}} (\mathbb{E} |\tilde{\mathbb{E}} |B_{t_k} - B_{\tau_k} + \tilde{B}_{T-t_k} - \tilde{B}_{\tilde{\tau}_{n-k}}|^{4\alpha})^{\frac{1}{2}} \\ &\leq C(C_g, T, p_0) ((\mathbb{E} |B_{t_k} - B_{\tau_k}|^{4\alpha})^{\frac{1}{2}} + (\mathbb{E} |B_{T-t_k} - B_{\tau_{n-k}}|^{4\alpha})^{\frac{1}{2}}) \\ &\leq C(C_g, T, p_0) h^{\frac{\alpha}{2}}, \end{aligned} \quad (12)$$

where  $\Psi_1 := C_g(1 + |B_{t_k} + \tilde{B}_{T-t_k}|^{p_0} + |B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{n-k}}|^{p_0})$ . To estimate the other term in (11) we consider the decomposition

$$\begin{aligned} &\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) - \mathbb{E}_{\tau_k} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \\ &= (\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) - \mathbb{E}_{t_k} f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m})) + (\mathbb{E}_{t_k} F(t_m, B_{t_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{\tau_m})) \\ &\quad + (\mathbb{E}_{\tau_k} F(t_m, B_{\tau_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{t_m})) + (\mathbb{E}_{\tau_k} f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m}) - \mathbb{E}_{\tau_k} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n)) \\ &=: D_1(s, m) + D_2(m) + \dots + D_4(m) \end{aligned}$$

so that

$$\begin{aligned} &\left\| \mathbb{E}_{t_k} \int_{t_k}^T f(s, B_s, Y_s, Z_s) ds - h \mathbb{E}_{\tau_k} \sum_{m=k}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \right\| \\ &\leq \sum_{m=k}^{n-1} \left( \left\| \int_{t_m}^{t_{m+1}} D_1(s, m) ds \right\| + h \sum_{i=2}^4 \|D_i(m)\| \right). \end{aligned}$$

For  $D_1$  we have by Theorem 5.3 that

$$\begin{aligned} \|D_1(s, m)\| &\leq L_f(\sqrt{s-t_m} + \|B_s - B_{t_m}\| + \|Y_s - Y_{t_m}\| + \|Z_s - Z_{t_m}\|) \\ &\leq C(T, L_f, C_{5.3}^y, C_{5.3}^z, p_0) (T-s)^{\frac{\alpha-2}{2}} h^{\frac{1}{2}}, \end{aligned} \quad (13)$$

where the last inequality follows from  $\|B_s - B_{t_m}\| = \sqrt{s-t_m} \leq h^{\frac{1}{2}}$  for  $s \in [t_m, t_{m+1}]$  and

$$\begin{aligned} \|Y_s - Y_{t_m}\| + \|Z_s - Z_{t_m}\| &\leq (\mathbb{E} \Psi(B_{t_m})^2)^{\frac{1}{2}} \left( C_{5.3}^y \left( \int_{t_m}^s (T-r)^{\alpha-1} dr \right)^{\frac{1}{2}} + C_{5.3}^z \left( \int_{t_m}^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}} \right) \\ &\leq C(T, C_{5.3}^y, C_{5.3}^z, p_0) \sqrt{s-t_m} ((T-s)^{\frac{\alpha-1}{2}} + (T-s)^{\frac{\alpha-2}{2}}). \end{aligned}$$

We bound  $D_2$  using Lemma 3.3 and Lemma A.1. Similar to (12) we conclude (setting  $\Psi_2 := 1 + |B_{t_k} + \tilde{B}_{t_{m-k}}|^{p_0+1} + |B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{m-k}}|^{p_0+1}$ ) that

$$\|D_2(m)\| = \left( \mathbb{E} |\mathbb{E}_{t_k} F(t_m, B_{t_m}) - \mathbb{E}_{\tau_k} F(t_m, B_{\tau_m})|^2 \right)^{\frac{1}{2}}$$

$$\begin{aligned}
&\leq C(T, L_f, c_{5.4}^{2,3}) (\mathbb{E} \tilde{\mathbb{E}} \Psi_2^4)^{\frac{1}{4}} \frac{1}{(T - t_m)^{1 - \frac{\alpha}{2}}} (t_k h + t_{m-k} h)^{\frac{1}{4}} \\
&\leq C(T, p_0, L_f, c_{5.4}^{2,3}) \frac{1}{(T - t_m)^{1 - \frac{\alpha}{2}}} h^{\frac{1}{4}}.
\end{aligned}$$

For  $D_3$  we apply again Lemma 3.3 and Lemma A.1,

$$\begin{aligned}
\|D_3(m)\| &\leq \|F(t_m, B_{t_m}) - F(t_m, B_{\tau_m})\| \leq C(T, L_f, c_{5.4}^{2,3}) \frac{1}{(T - t_m)^{1 - \frac{\alpha}{2}}} \|\Psi_3 |B_{t_m} - B_{\tau_m}|\| \\
&\leq C(T, p_0, L_f, c_{5.4}^{2,3}) \frac{1}{(T - t_m)^{1 - \frac{\alpha}{2}}} h^{\frac{1}{4}},
\end{aligned}$$

where  $\Psi_3 := 1 + |B_{t_m}|^{p_0+1} + |B_{\tau_m}|^{p_0+1}$ . For the last term  $D_4$  we get

$$\|D_4(m)\| \leq L_f (h^{\frac{1}{2}} + \|B_{t_m} - B_{t_m}^n\| + \|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|).$$

Finally, using the estimates for the terms  $D_1(s, m), D_2(m), \dots, D_4(m)$  we arrive at

$$\begin{aligned}
\|Y_{t_k} - Y_{t_k}^n\| &\leq C(C_g, T, p_0) h^{\frac{\alpha}{4}} + C(T, L_f, C_{5.3}^y, C_{5.3}^z, p_0) h^{\frac{1}{2}} \int_{t_k}^T (T - s)^{\frac{\alpha-2}{2}} ds \\
&\quad + C(T, p_0, L_f, c_{5.4}^{2,3}) h^{\frac{1}{4}} \sum_{m=k}^{n-1} \frac{h}{(T - t_m)^{1 - \frac{\alpha}{2}}} + h L_f \sum_{m=k}^{n-1} (\|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|) \\
&\leq C(C_g, T, p_0, L_f, c_{5.4}^{2,3}, C_{5.3}^y, C_{5.3}^z) h^{\frac{\alpha}{4}} + h L_f \sum_{m=k}^{n-1} (\|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|). \quad (14)
\end{aligned}$$

For  $\|Z_{t_k} - Z_{t_k}^n\|$  we exploit the representations (7) and (8) and estimate

$$\begin{aligned}
\|Z_{t_k} - Z_{t_k}^n\| &\leq \frac{1}{T - t_k} \|\mathbb{E}_{t_k} g(B_T)(B_T - B_{t_k}) - \mathbb{E}_{\tau_k} g(B_{\tau_n})(B_{\tau_n} - B_{\tau_k})\| \\
&\quad + \left\| \mathbb{E}_{t_k} \left( \int_{t_{k+1}}^T f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k} ds \right) \right. \\
&\quad \quad \left. - \mathbb{E}_{\tau_k} \left( h \sum_{m=k+1}^{n-1} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n) \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right) \right\| \\
&\quad + \left\| \mathbb{E}_{t_k} \int_{t_k}^{t_{k+1}} f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k} ds \right\|.
\end{aligned}$$

Then, similar to (12), we have for the terminal condition by Lemma A.1 that

$$\begin{aligned}
&\|\mathbb{E}_{t_k} [g(B_T)(B_T - B_{t_k})] - \mathbb{E}_{\tau_k} [g(B_{\tau_n})(B_{\tau_n} - B_{\tau_k})]\| \\
&= \|\tilde{\mathbb{E}}[g(B_{t_k} + \tilde{B}_{T-t_k}) - g(B_{t_k})](\tilde{B}_{T-t_k} - \tilde{B}_{\tilde{\tau}_{n-k}}) + \tilde{\mathbb{E}}[g(B_{t_k} + \tilde{B}_{T-t_k}) - g(B_{\tau_k} + \tilde{B}_{\tilde{\tau}_{n-k}})] \tilde{B}_{\tilde{\tau}_{n-k}}\| \\
&\leq C(C_g, T, p_0) h^{\frac{1}{4}} (T - t_k)^{\frac{\alpha}{2} + \frac{1}{4}} + C(C_g, T, p_0) h^{\frac{\alpha}{4}} (T - t_k)^{\frac{1}{2}} \leq C(C_g, T, p_0) h^{\frac{\alpha}{4}} (T - t_k)^{\frac{1}{2}}.
\end{aligned}$$

Here we have used that  $\tilde{\mathbb{E}}[g(B_{t_k})(\tilde{B}_{T-t_k} - \tilde{B}_{\tilde{\tau}_{n-k}})] = 0$ . The term  $\tilde{\mathbb{E}}[g(B_{t_k} + \tilde{B}_{T-t_k}) - g(B_{t_k})](\tilde{B}_{T-t_k} - \tilde{B}_{\tilde{\tau}_{n-k}})$  provides us with the factor  $(T - t_k)^{\frac{\alpha}{2}} ((T - t_k)h)^{\frac{1}{4}}$ . For the next term of the estimate of  $\|Z_{t_k} - Z_{t_k}^n\|$  we use for  $s \in [t_m, t_{m+1})$ , where  $m \geq k + 1$ , the decomposition

$$\frac{\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s)(B_s - B_{t_k})}{s - t_k} - \frac{\mathbb{E}_{\tau_k} f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n)(B_{t_m}^n - B_{t_k}^n)}{t_m - t_k}$$



$$\begin{aligned}
&= \frac{\mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s)(B_s - B_{t_k})}{s - t_k} - \frac{\mathbb{E}_{t_k} f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m})(B_{t_m} - B_{t_k})}{t_m - t_k} \\
&\quad + \frac{\mathbb{E}_{t_k} F(t_m, B_{t_m})(B_{t_m} - B_{t_k})}{t_m - t_k} - \frac{\mathbb{E}_{\tau_k} F(t_m, B_{\tau_m})(B_{\tau_m} - B_{\tau_k})}{t_m - t_k} \\
&\quad + \mathbb{E}_{\tau_k} \left[ [F(t_m, B_{\tau_m}) - F(t_m, B_{t_m})] \frac{B_{\tau_m} - B_{\tau_k}}{t_m - t_k} \right] \\
&\quad + \mathbb{E}_{\tau_k} \left[ [f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m}) - f(t_{m+1}, B_{t_m}^n, Y_{t_m}^n, Z_{t_m}^n)] \frac{B_{t_m}^n - B_{t_k}^n}{t_m - t_k} \right] \\
&=: T_1(s, m) + T_2(m) + \dots + T_4(m).
\end{aligned}$$

Then by the conditional Hölder inequality and by (13) as well as by Lemma 3.3 we have

$$\begin{aligned}
\|T_1(s, m)\| &\leq \|D_1(s, m)\| \frac{\|B_s - B_{t_k}\|}{s - t_k} + \|f(t_m, B_{t_m}, Y_{t_m}, Z_{t_m})\| \left\| \frac{B_s - B_{t_k}}{s - t_k} - \frac{B_{t_m} - B_{t_k}}{t_m - t_k} \right\| \\
&\leq C(T, L_f, C_{5.3}^y, C_{5.3}^z, p_0) (T - s)^{\frac{\alpha-2}{2}} \frac{h^{\frac{1}{2}}}{\sqrt{s - t_k}} \\
&\quad + C(T, L_f, c_{5.4}^{1,2}, K_f) \frac{(\mathbb{E}\Psi(B_{t_m})^2)^{\frac{1}{2}}}{(T - t_m)^{\frac{1-\alpha}{2}}} \\
&\quad \times \left( \frac{\|B_s - B_{t_m}\|}{s - t_k} + \|B_{t_m} - B_{t_k}\| \left| \frac{1}{s - t_k} - \frac{1}{t_m - t_k} \right| \right) \\
&\leq C(T, L_f, K_f, C_{5.3}^y, C_{5.3}^z, c_{5.4}^{1,2}, p_0) (T - s)^{\frac{\alpha-2}{2}} \frac{h^{\frac{1}{4}}}{(s - t_k)^{\frac{3}{4}}}.
\end{aligned}$$

Indeed,

$$\frac{\|B_s - B_{t_m}\|}{s - t_k} + \|B_{t_m} - B_{t_k}\| \left| \frac{1}{s - t_k} - \frac{1}{t_m - t_k} \right| \leq \frac{\sqrt{s - t_m}}{s - t_k} + \frac{\sqrt{t_m - t_k}(s - t_m)}{(s - t_k)(t_m - t_k)} \leq C \frac{h^{\frac{1}{4}}}{(s - t_k)^{\frac{3}{4}}},$$

where the last inequality follows from  $s - t_m \leq t_{m+1} - t_m = h$  and  $h \leq t_m - t_k \leq s - t_k$ . We estimate  $T_2$  with the help of Lemma 3.3 and Lemma A.1 as follows :

$$\begin{aligned}
\|T_2(m)\| &\leq \|\widehat{D}_2(m)\| \frac{\|B_{t_m} - B_{t_k}\|}{t_m - t_k} + \|F(t_m, B_{\tau_m})\| \frac{\|B_{t_{m-k}} - B_{\tau_{m-k}}\|}{t_m - t_k} \\
&\leq C(T, p_0, L_f, K_f, c_{5.4}) \frac{1}{(T - t_m)^{1-\frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{3}{4}}}.
\end{aligned}$$

Here  $\widehat{D}_2(m) := (\mathbb{E}|F(t_m, B_{t_k} + \tilde{B}_{t_{m-k}}) - F(t_m, B_{\tau_k} + \tilde{B}_{\tau_{m-k}})|^2)^{\frac{1}{2}}$  which can be estimated as  $D_2(m)$ . For  $T_3$  the conditional Hölder inequality and Lemma A.1 yield

$$\|T_3(m)\| \leq \|\widehat{D}_3(m)\| \left\| \frac{B_{\tau_m} - B_{\tau_k}}{t_m - t_k} \right\| \leq C(T, p_0, L_f, c_{5.4}^{2,3}) \frac{1}{(T - t_m)^{1-\frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{1}{2}}},$$

where  $\widehat{D}_3(m) := F(t_m, B_{\tau_m}) - F(t_m, B_{t_m})$  is estimated as  $D_3(m)$ . Finally,

$$\|T_4(m)\| \leq L_f(h^{\frac{1}{2}} + \|B_{t_m} - B_{t_m}^n\| + \|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|) \frac{1}{\sqrt{t_m - t_k}}.$$

For the estimate of  $\left\| \mathbb{E}_{t_k} \int_{t_k}^{t_{k+1}} f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k} ds \right\|$  one notices that by the conditional Hölder inequality,

$$\begin{aligned} \left\| \mathbb{E}_{t_k} f(s, B_s, Y_s, Z_s) \frac{B_s - B_{t_k}}{s - t_k} \right\| &= \left\| \mathbb{E}_{t_k} [(f(s, B_s, Y_s, Z_s) - f(s, B_{t_k}, Y_{t_k}, Z_{t_k})) \frac{B_s - B_{t_k}}{s - t_k}] \right\| \\ &\leq \|f(s, B_s, Y_s, Z_s) - f(s, B_{t_k}, Y_{t_k}, Z_{t_k})\| \frac{1}{\sqrt{s - t_k}} \\ &\leq C(T, L_f, C_{5.3}^y, C_{5.3}^z, p_0) (T - s)^{\frac{\alpha-2}{2}} \frac{h^{\frac{1}{2}}}{\sqrt{s - t_k}}, \end{aligned}$$

where the last inequality follows in the same way as in (13). Consequently, we have

$$\begin{aligned} \|Z_{t_k} - Z_{t_k}^n\| &\leq \frac{C(C_g, T, p_0)}{(T - t_k)^{\frac{1}{2}}} h^{\frac{\alpha}{4}} + C(T, L_f, K_f, C_{5.3}^y, C_{5.3}^z, c_{5.4}^{1,2}, p_0) \int_{t_k}^T \frac{ds}{(T - s)^{1 - \frac{\alpha}{2}} (s - t_k)^{\frac{3}{4}}} h^{\frac{1}{4}} \\ &\quad + C(T, p_0, L_f, K_f, c_{5.4}) h \sum_{m=k+1}^{n-1} \frac{1}{(T - t_m)^{1 - \frac{\alpha}{2}}} \frac{h^{\frac{1}{4}}}{(t_m - t_k)^{\frac{3}{4}}} \\ &\quad + L_f h \sum_{m=k+1}^{n-1} (\|B_{t_m} - B_{t_m}^n\| + \|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|) \frac{1}{\sqrt{t_{m-k}}}. \end{aligned}$$

Lemma A.2 enables to bound the second and third term of the r.h.s. by  $C \frac{h^{\frac{1}{4}}}{(T - t_k)^{\frac{3}{4} - \frac{\alpha}{2}}} B(\frac{\alpha}{2}, \frac{1}{4})$ , which is bounded by  $C \frac{h^{\frac{\alpha}{4}}}{(T - t_k)^{\frac{1}{2} - \frac{\alpha}{4}}}$ . Thus we get

$$\|Z_{t_k} - Z_{t_k}^n\| \leq \frac{C_0 h^{\frac{\alpha}{4}}}{(T - t_k)^{\frac{1}{2}}} + L_f h \sum_{m=k+1}^{n-1} (\|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|) \frac{1}{\sqrt{t_{m-k}}}.$$

Then we use (14) and the above estimate to get

$$\|Y_{t_k} - Y_{t_k}^n\| + \|Z_{t_k} - Z_{t_k}^n\| \leq \frac{C_0 h^{\frac{\alpha}{4}}}{(T - t_k)^{\frac{1}{2}}} + C(L_f) h \sum_{m=k+1}^{n-1} (\|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|) \frac{1}{\sqrt{t_{m-k}}}.$$

If this inequality is iterated, one gets a shape where the Gronwall lemma applies. Indeed, setting  $a_m := (\|Y_{t_m} - Y_{t_m}^n\| + \|Z_{t_m} - Z_{t_m}^n\|)$  one has to consider the double sum

$$\sum_{m=k+1}^{n-1} \left( \sum_{l=m+1}^{n-1} \frac{a_l h}{\sqrt{t_{l-m}}} \right) \frac{h}{\sqrt{t_{m-k}}} = h \sum_{l=k+1}^{n-1} \left( \sum_{m=k+1}^{l-1} \frac{h}{\sqrt{t_{m-k}} \sqrt{t_{l-m}}} \right) a_l \leq Ch \sum_{l=k+1}^{n-1} a_l.$$

Consequently,

$$\|Y_{t_k} - Y_{t_k}^n\| + \|Z_{t_k} - Z_{t_k}^n\| \leq \frac{C_0 h^{\frac{\alpha}{4}}}{(T - t_k)^{\frac{1}{2}}}$$

which gives the bound on the error on  $Z$ . Moreover, (14) yields

$$\|Y_{t_k} - Y_{t_k}^n\| \leq C_0 h^{\frac{\alpha}{4}}.$$

If  $v \in [t_k, t_{k+1})$ , we have by Theorem 5.3 that

$$\|Y_v - Y_v^n\| \leq \|Y_v - Y_{t_k}\| + \|Y_{t_k} - Y_{t_k}^n\| \leq C(C_{5.3}^y, T, p_0) \left( \int_{t_k}^v (T - r)^{\alpha-1} dr \right)^{\frac{1}{2}} + \|Y_{t_k} - Y_{t_k}^n\|,$$

$$\|Z_v - Z_v^n\| \leq \|Z_v - Z_{t_k}\| + \|Z_{t_k} - Z_{t_k}^n\| \leq C(C_{5.3}^z, T, p_0) \left( \int_{t_k}^v (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}} + \|Z_{t_k} - Z_{t_k}^n\|,$$

where

$$\int_{t_k}^v (T-r)^{\alpha-1} dr \leq \frac{1}{\alpha} (v-t_k)^\alpha \leq \frac{1}{\alpha} h^\alpha$$

and

$$\int_{t_k}^v (T-r)^{\alpha-2} dr \leq \frac{1}{(T-v)^{1-\frac{\alpha}{2}}} \int_{t_k}^v (T-r)^{\frac{\alpha}{2}-1} dr \leq \frac{1}{(T-v)^{1-\frac{\alpha}{2}}} \frac{2}{\alpha} (v-t_k)^{\frac{\alpha}{2}} \leq \frac{2}{\alpha} \frac{h^{\frac{\alpha}{2}}}{(T-v)^{1-\frac{\alpha}{2}}}.$$

□

## 4 Numerical simulations

This section deals with the algorithm used to compute  $(Y_{t_k}^n, Z_{t_k}^n)_{k=0, \dots, n}$  and numerical experiments for three different terminal conditions. In each case the exact solution is available and we are able to compute the error  $(Y^n - Y, Z^n - Z)$  in  $L_2$ -norm.

### 4.1 Simulation of $(\tau_1, \dots, \tau_n)$ and $B^n$

In order to simulate  $(\tau_1, \dots, \tau_n)$ , we use the fact that

$$\tau_0 = 0 \quad \text{and} \quad \forall k \geq 1, \quad \tau_k = \tau_{k-1} + \sigma_k,$$

where  $(\sigma_k)_{1 \leq k \leq n}$  is an i.i.d. sequence whose common law  $\sigma$  represents the first exit time of the Brownian motion  $B$  of the interval  $[-\sqrt{h}, \sqrt{h}]$ ,

$$\sigma := \inf\{t > 0 : |B_t| = \sqrt{h}\}$$

From the book of Borodin and Salminen [4], we have that the Laplace transform of  $\sigma$  is given by  $\mathbb{E}(e^{-\lambda\sigma}) = \frac{1}{\cosh(\sqrt{2\lambda h})}$ .

Let  $F$  denote the cumulative distribution function of  $\sigma$ . It holds  $\mathbb{E}(e^{-\lambda\sigma}) = \lambda \hat{F}(\lambda)$ , where  $\hat{F}$  is the Laplace transform of  $F$ . Then, to obtain  $F$ , it remains to inverse numerically its Laplace transform. Once we have  $F$ , we simulate the sequence  $(\sigma_k)_{1 \leq k \leq n}$  by following the steps of Algorithm 1.

---

**Algorithm 1** Simulation of the sequence  $(\tau_1, \dots, \tau_n)$

---

Simulate one vector with uniform law  $(U_1, \dots, U_n)$

$\tau_0 = 0$

**for**  $k = 1 : n$  **do**

    Compute  $\sigma_k := F^{-1}(U_k)$

    Define  $\tau_k = \tau_{k-1} + \sigma_k$

**end for**

---

## 4.2 Simulation of $B^n$

In order to get the trajectory  $B_{t_1}^n, \dots, B_{t_n}^n$  ( $B_{t_0}^n = 0$ ), we simulate an i.i.d. Bernoulli sequence  $(\xi_k)_{1 \leq k \leq n}$  i.e.  $\mathbb{P}(\xi_k = \pm 1) = \frac{1}{2}$ . Then

$$B_{t_{k+1}}^n = \begin{cases} B_{t_k}^n + \sqrt{h} & \text{if } \xi_k = 1 \\ B_{t_k}^n - \sqrt{h} & \text{otherwise.} \end{cases} \quad (15)$$

## 4.3 Simulation of $(Y^n, Z^n)$

Since  $B^n$  is built using the random walk (15), it can be represented by a recombining binomial tree. Both  $(Y_{t_k}^n)_{0 \leq k \leq n}$  and  $(Z_{t_k}^n)_{0 \leq k \leq n-1}$  can then also be represented as a recombining binomial tree. Since  $Y_{t_n}^n = g(B_{t_n}^n)$ , we solve backward in time the BSDE by following these equalities, ensuing from (2) ( $Y_{t_k}^n$  has been replaced by  $Y_{t_{k+1}}^n$  in the generator term, but the error induced by this modification is smaller than the ones we consider)

$$\begin{aligned} Z_{t_k}^n &= \frac{1}{\sqrt{h}} \mathbb{E}_{\tau_k}(Y_{t_{k+1}}^n \varepsilon_{k+1}), \\ Y_{t_k}^n &= \mathbb{E}_{\tau_k}(Y_{t_{k+1}}^n + hf(t_{k+1}, B_{t_k}^n, Y_{t_{k+1}}^n, Z_{t_k}^n)). \end{aligned}$$

## 4.4 Study of the error $\mathbb{E}|Y_{t_k}^n - Y_{t_k}|^2$ and $\mathbb{E}|Z_{t_k}^n - Z_{t_k}|^2$

In this subsection we assume that we are able to compute the exact solution  $(Y, Z)$ . We want to study numerically the convergence in  $n$  of  $\mathbb{E}|Y_{t_k}^n - Y_{t_k}|^2$  and  $\mathbb{E}|Z_{t_k}^n - Z_{t_k}|^2$ , where  $(Y, Z)$  solves (1) and  $(Y^n, Z^n)$  solves (6). To do so, we approximate the error  $\mathbb{E}|A_{t_k}^n - A_{t_k}|^2$  ( $A = Y$  or  $A = Z$ ) by Monte Carlo:

$$\mathbb{E}|A_{t_k}^n - A_{t_k}|^2 \sim \frac{1}{M} \sum_{m=1}^M |A_{t_k}^{n,m} - A_{t_k}^m|^2 := E_A \quad (16)$$

1. For each Monte Carlo simulation, we pick at random one sequence  $(\xi_1, \dots, \xi_n)$  (which gives the value of  $(B_{t_1}^n, \dots, B_{t_n}^n)$ ) and one sequence  $(\tau_1, \dots, \tau_n)$ .
2. From the sequence  $(\xi_1, \dots, \xi_n)$  we get the trajectory of  $Y^n$ , including  $Y_{t_k}^n$ .
3. From the sequence  $(B_{\tau_1}, \dots, B_{\tau_n})$  (which is equal to  $(B_{t_1}^n, \dots, B_{t_n}^n)$ ), we compute  $B_{t_k}$  by using the Brownian bridge method. We deduce  $(Y_{t_k}, Z_{t_k})$  as functions of  $B_{t_k}$ .

In the following experiments, we plot the logarithm of the errors  $E_Y$  and  $E_Z$  (defined in (16)) w.r.t.  $\log(n)$ . From Theorem 3.1, we get that  $\log(E_Y)$  and  $\log(E_Z)$  decrease as  $-\frac{\alpha}{2} \log(n)$ . By using a linear regression, we compute the slope of the line solving the least square problem and compare it to  $-\frac{\alpha}{2}$ .

## 4.5 Numerical Experiment

### 4.5.1 Case $g(x) = e^{T+x}$ and $f(y, z) = y + z$

We consider the BSDE with terminal condition  $g(x) = e^{T+x}$  and driver  $f(y, z) = y + z$ . In this case, we know that  $Y_t = e^{T+B_t + \frac{\alpha}{2}(T-t)}$ . We run  $M = 20000$  Monte Carlo simulations.

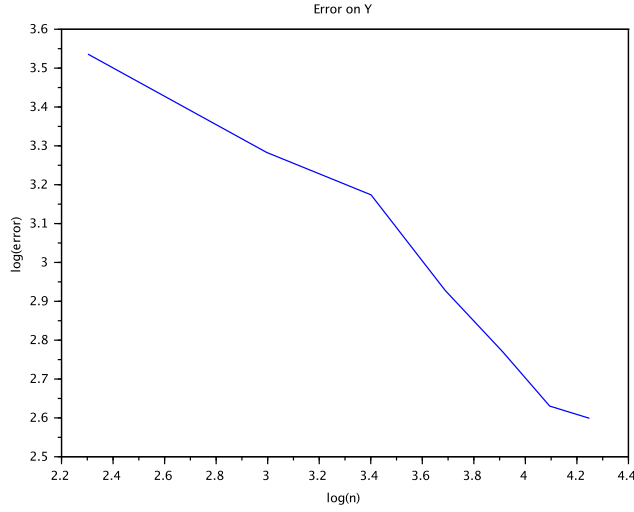


Figure 1:  $\log(\text{error on } Y)$  w.r.t.  $\log(n)$  -  $f(y, z) = y + z - g(x) = e^{T+x}$

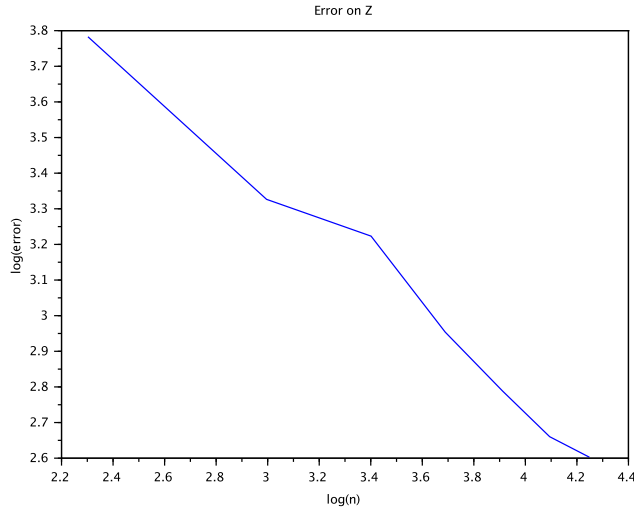


Figure 2:  $\log(\text{error on } Z)$  w.r.t.  $\log(n)$  -  $f(y, z) = y + z - g(x) = e^{T+x}$

Figure 1 (resp. Figure 2) represents  $\log(\text{error on } Y)$  (the error is defined by (16)) (resp.  $\log(\text{error on } Z)$ ) with respect to  $\log(n)$ . For the  $Y$  case, the slope ensuing from the linear regression is  $-0.53$ . Even though  $g(x) = e^{T+x}$  does not satisfy (3),  $g$  is locally Lipschitz continuous, and the outcome seems to be consistent with Theorem 3.1 for  $\alpha = 1$ . For the  $Z$  case, we get the slope  $-0.61$ .

#### 4.5.2 Case $g(x) = x^2$ and $f(y, z) = y + z$

In that case, we know that  $Y_t = e^{T-t}((B_t - (T-t))^2 + T-t)$  and  $Z_t = 2e^{T-t}(B_t - (T-t))$ . We run  $M = 20000$  Monte Carlo simulations.

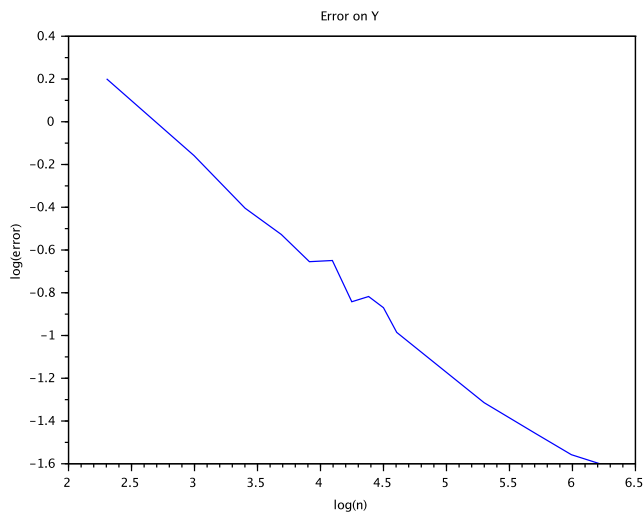


Figure 3:  $\log(\text{error on } Y)$  as a function of  $\log(n)$  -  $f(y, z) = y + z - g(x) = x^2$

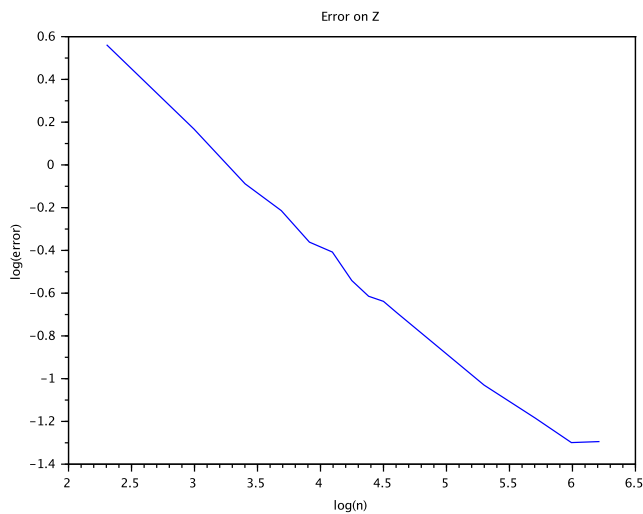


Figure 4:  $\log(\text{error on } Z)$  as a function of  $\log(n)$  -  $f(y, z) = y + z - g(x) = x^2$

Figure 3 represents  $\log(\text{error on } Y)$  with respect to  $\log(n)$ . The slope of the linear regression is  $-0.465$ . Figure 4 represents  $\log(\text{error on } Z)$  with respect to  $\log(n)$ . The slope of the linear regression is  $-0.48$ . The results are then consistent with Theorem 3.1.

#### 4.5.3 Case $g(x) = \sqrt{|x|}$ and $f(y, z) = y + z$

In that case, we know that  $Y_t = e^{\frac{T-t}{2}} \tilde{\mathbb{E}}(\sqrt{|\tilde{B}_{T-t} + B_t|} e^{\tilde{B}_{T-t}})$ . We run  $M = 20000$  Monte Carlo simulations.

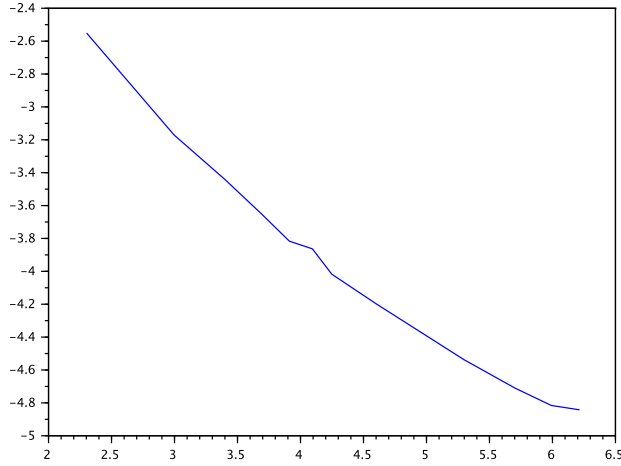


Figure 5:  $\log(\text{error on } Y)$  as a function of  $\log(n)$  -  $f(y, z) = y + z - g(x) = \sqrt{|x|}$

Figure 5 represents  $\log(\text{error on } Y)$  with respect to  $\log(n)$ . The slope of the linear regression  $-0.56$ . Here we notice that the modulus of the slope we get is larger than  $\frac{1}{4}$ , the upper bound obtained in that case in Theorem 3.1.

## 5 Some properties of solutions to PDEs and BSDEs

In the following we recall and prove results for FBSDEs with a general forward process, even though we apply them in the present paper only for the case where the forward process is just the Brownian motion. Restricting ourselves to the case of Brownian motion would not shorten the proofs considerably. Let us consider the following SDE started in  $(t, x)$ ,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \quad 0 \leq t \leq s \leq T, \quad (17)$$

where  $b$  and  $\sigma$  satisfy

### Assumption 5.1.

1.  $b, \sigma \in C_b^{0,2}([0, T] \times \mathbb{R})$ , in the sense that the derivatives of order  $k = 0, 1, 2$  w.r.t. the space variable are continuous and bounded on  $[0, T] \times \mathbb{R}$ ,
2. the first and second derivatives of  $b$  and  $\sigma$  w.r.t. the space variable are assumed to be  $\gamma$ -Hölder continuous (for some  $\gamma \in (0, 1]$ , w.r.t. the parabolic metric  $d((x, t), (x', t')) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}}$  on all compact subsets of  $[0, T] \times \mathbb{R}$ ,
3.  $b, \sigma$  are  $\frac{1}{2}$ -Hölder continuous in time, uniformly in space,
4.  $\sigma(t, x) \geq \delta > 0$  for all  $(t, x)$ .

## 5.1 Malliavin weights

In this section we recall the Malliavin weights and their properties from [22, Subsection 1.1 and Remark 3].

**Lemma 5.2.** *Let  $H : \mathbb{R} \rightarrow \mathbb{R}$  be a polynomially bounded Borel function. If Assumption 5.1 holds and  $X^{t,x}$  is given by (17), then setting*

$$G(t, x) := \mathbb{E}H(X_R^{t,x})$$

*implies that  $G \in C^{1,2}([0, R] \times \mathbb{R})$ . Especially it holds for  $0 \leq t \leq r < R \leq T$  that*

$$\partial_x G(r, X_r^{t,x}) = \mathbb{E}[H(X_R^{t,x})N_R^{r,1,(t,x)}|\mathcal{F}_r^t], \quad \text{and} \quad \partial_x^2 G(r, X_r^{t,x}) = \mathbb{E}[H(X_R^{t,x})N_R^{r,2,(t,x)}|\mathcal{F}_r^t],$$

*where  $(\mathcal{F}_r^t)_{r \in [t, T]}$  is the augmented natural filtration of  $(B_r^{t,0})_{r \in [t, T]}$ ,*

$$N_R^{r,1,(t,x)} = \frac{1}{R-r} \int_r^R \frac{\nabla X_s^{t,x}}{\sigma(s, X_s^{t,x}) \nabla X_r^{t,x}} dB_s \quad \text{and} \quad N_R^{r,2,(t,x)} = \frac{N_R^{\rho,1,(t,x)} \nabla X_R^{t,x} N_R^{\rho,1,(t,x)} + \nabla N_R^{\rho,1,(t,x)}}{\nabla X_r^{t,x}},$$

*with  $\rho := \frac{r+R}{2}$ . Moreover, for  $q \in (0, \infty)$  it holds a.s.*

$$(\mathbb{E}[|N_R^{r,i,(t,x)}|^q | \mathcal{F}_r^t])^{\frac{1}{q}} \leq \frac{\kappa_q}{(R-r)^{\frac{i}{2}}}, \quad (18)$$

*and  $\mathbb{E}[N_R^{r,i,(t,x)} | \mathcal{F}_r^t] = 0$  a.s. for  $i = 1, 2$ . Finally, we have*

$$\|\partial_x G(r, X_r^{t,x})\|_{L_p(\mathbb{P})} \leq \kappa_q \frac{\|H(X_R^{t,x}) - \mathbb{E}[H(X_R^{t,x}) | \mathcal{F}_r^t]\|_{L_p(\mathbb{P})}}{\sqrt{R-r}}$$

*and*

$$\|\partial_x^2 G(r, X_r^{t,x})\|_{L_p(\mathbb{P})} \leq \kappa_q \frac{\|H(X_R^{t,x}) - \mathbb{E}[H(X_R^{t,x}) | \mathcal{F}_r^t]\|_{L_p(\mathbb{P})}}{R-r}$$

*for  $1 < q, p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .*

## 5.2 Regularity of solutions to BSDEs

Let us now consider the FBSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dB_r, \quad 0 \leq t \leq s \leq T, \quad (19)$$

where  $X^{t,x}$  is the process satisfying (17). The following result is taken from [22, Theorem 1]. We reformulate it here for the simple situation where we need it. On the other hand, we will use  $\mathbb{P}_{t,x}$  and are interested in an estimate for all  $(t, x) \in [0, T] \times \mathbb{R}$ .

**Theorem 5.3.** *Let Assumption 2.1 and 5.1 hold. Then for any  $p \in [2, \infty)$  the following assertions are true.*

(i) *There exists a constant  $C_{5.3}^y > 0$  such that for  $0 \leq t < s < T$  and  $x \in \mathbb{R}$ ,*

$$\|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \leq C_{5.3}^y \Psi(x) \left( \int_t^s (T-r)^{\alpha-1} dr \right)^{\frac{1}{2}},$$



(ii) There exists a constant  $C_{5.3}^z > 0$  such that for  $0 \leq t < s < T$  and  $x \in \mathbb{R}$ ,

$$\|Z_s - Z_t\|_{L_p(\mathbb{P}_{t,x})} \leq C_{5.3}^z \Psi(x) \left( \int_t^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}}.$$

The constants  $C_{5.3}^y$  and  $C_{5.3}^z$  depend on  $K_f, L_f, C_g, c_{5.4}^{1,2}, T, p_0, b, \sigma, \kappa_q$  and  $p$ .

*Proof of Theorem 5.3.* (i) First we follow the step [22, Theorem 1, proof of  $(C2_l) \implies (C3_l)$ ]. We conclude from the linear growth  $|f(r, x, y, z)| \leq L_f(|x| + |y| + |z|) + K_f$  and from the Burkholder-Davis-Gundy inequality with constant  $a_p > 0$  that

$$\begin{aligned} & \|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \\ &= \left\| \int_t^s f(r, X_r, Y_r, Z_r) dr - \int_t^s Z_r dB_r \right\|_{L_p(\mathbb{P}_{t,x})} \\ &\leq K_f(s-t) + L_f \int_t^s \|X_r\|_{L_p(\mathbb{P}_{t,x})} + \|Y_r\|_{L_p(\mathbb{P}_{t,x})} + \|Z_r\|_{L_p(\mathbb{P}_{t,x})} dr + a_p \left( \int_t^s \|Z_r\|_{L_p(\mathbb{P}_{t,x})}^2 dr \right)^{\frac{1}{2}}. \end{aligned}$$

We then use (i) and (ii) of Theorem 5.4 below to get

$$\begin{aligned} & \|Y_s - Y_t\|_{L_p(\mathbb{P}_{t,x})} \\ &\leq K_f(s-t) + C(T, L_f, c_{5.4}^{1,2}, p, b, \sigma, p_0) \Psi(x) \left[ \int_t^s \left(1 + (T-r)^{\frac{\alpha-1}{2}}\right) dr + \left( \int_t^s (T-r)^{\alpha-1} dr \right)^{\frac{1}{2}} \right]. \end{aligned}$$

(ii) Here one can follow [22, Theorem 1, proof of  $(C4_l) \implies (C1_l)$ ].

**Step 1:** We first assume additionally that  $f : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is continuously differentiable in  $x$ ,  $y$ , and  $z$  with uniformly bounded derivatives as it was assumed for [22, Theorem 1]. To take the dependency on  $x$  into consideration which arises since we use  $\mathbb{P}_{t,x}$ , it suffices to replace everywhere in the proof in [22] the constant  $c_{B_{p,\infty}^\Theta}$  by  $C(T, C_g, \sigma, b, p, p_0) \Psi(x)$ . The constant  $C_{5.3}^z$  depends moreover on  $L_f$  and  $\kappa_q$ .

**Step 2:** Now let  $f$  be as in Assumption 5.1. In [22, Theorem 1, proof of  $(C4_l) \implies (C1_l)$ ] a linear BSDE is used which describes the behaviour of the process  $Z$  minus its counterpart where the generator is identically 0. Here the partial derivatives of  $f_x, f_y, f_z$  appear but only their uniform bound is needed in the estimates. Hence if  $f$  satisfies (4), we can use mollifying as explained in (25) below (one may choose  $N = \infty$ ). Since  $|\partial_x f^\varepsilon(t, x, y, z)|, |\partial_y f^\varepsilon(t, x, y, z)|$  and  $|\partial_z f^\varepsilon(t, x, y, z)|$  are bounded by  $L_f$  we conclude from Step 1 that for all  $\varepsilon > 0$  the process  $Z^\varepsilon$  corresponding to  $f^\varepsilon$  satisfies

$$\|Z_s^\varepsilon - Z_t^\varepsilon\|_{L_p(\mathbb{P}_{t,x})} \leq C_{5.3}^z \Psi(x) \left( \int_t^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}} \quad (20)$$

for  $p \geq 2$ . Especially, the family  $\{|Z_s^\varepsilon - Z_t^\varepsilon|^q : \varepsilon > 0\}$  is then uniformly integrable provided that  $q < p$ . By an a priori estimate (cf. [6, Lemma 3.1]) we have that

$$\mathbb{E} \int_0^T |Z_r - Z_r^\varepsilon|^2 dr \leq C \int_0^T \sup_{x,y,z} |f(r, x, y, z) - f^\varepsilon(r, x, y, z)|^2 dr \leq C \varepsilon^2 T L_f^2.$$

Fubini's theorem implies that there exists a sequence  $\varepsilon_m \rightarrow 0$  and a measurable set  $N \subseteq [0, T]$  of Lebesgue measure zero, such that  $\lim_{m \rightarrow \infty} \mathbb{E} |Z_r - Z_r^{\varepsilon_m}|^2 = 0$  for all  $r \in [0, T] \setminus N$ . Consequently, for any  $q < p$  and all  $t, s \in [0, T] \setminus N$  with  $t < s$ ,

$$\|Z_s - Z_t\|_{L_q(\mathbb{P}_{t,x})} \leq C_{5.3}^z \Psi(x) \left( \int_t^s (T-r)^{\alpha-2} dr \right)^{\frac{1}{2}}.$$

The assertion follows for all  $q \geq 2$  since (20) holds for all  $p \in [2, \infty)$ . Since by Theorem 5.4 (ii) the process  $Z$  does have a continuous version, we finally get the assertion for all  $t < s$ .  $\square$

### 5.3 Properties of the associated PDE

We collect in the theorem below properties of the solution to the PDE which are mainly known. The new part concerns  $\partial_x^2 u$ . For Lipschitz continuous  $g$ , the behaviour of  $\partial_x^2 u$  has been studied in [37]. General results related to this topic can be found in [20].

**Theorem 5.4.** *Consider the FBSDE (19) and let Assumptions 2.1 and 5.1 hold. Then for the solution  $u$  of the associated PDE*

$$\begin{cases} u_t(t, x) + \frac{\sigma^2(t, x)}{2} u_{xx}(t, x) + b(t, x) u_x(t, x) + f(t, x, u(t, x), \sigma(t, x) u_x(t, x)) = 0, \\ u(T, x) = g(x), \quad x \in \mathbb{R} \end{cases} \quad t \in [0, T), x \in \mathbb{R},$$

we have

(i)  $Y_t = u(t, X_t)$  where  $u(t, x) = \mathbb{E}_{t,x} \left( g(X_T) + \int_t^T f(r, X_r, Y_r, Z_r) dr \right)$  and  $|u(t, x)| \leq c_{5.4}^1 \Psi(x)$  with  $\Psi$  given in (5), where  $c_{5.4}^1$  depends on  $C_g, T, p_0, L_f, K_f$  and on the bounds and Lipschitz constants of  $b$  and  $\sigma$ .

(ii)  $u_x$  exists,

$$u_x(t, x) = \mathbb{E}_{t,x} \left( g(X_T) N_T^{t,1} + \int_t^T f(r, X_r, Y_r, Z_r) N_r^{t,1} dr \right), \quad (21)$$

and

- (a)  $u_x$  is continuous in  $[0, T) \times \mathbb{R}$ ,
- (b)  $Z_s^{t,x} = u_x(s, X_s^{t,x}) \sigma(s, X_s^{t,x})$ ,
- (c)  $|u_x(t, x)| \leq \frac{c_{5.4}^2 \Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}$ ,

where  $c_{5.4}^2$  depends on  $C_g, T, p_0, \kappa_2, L_f, K_f$  and on the bounds and Lipschitz constants of  $b$  and  $\sigma$ .

(iii)  $u_{xx}$  exists,

$$u_{xx}(t, x) = \mathbb{E}_{t,x} \left( g(X_T) N_T^{t,2} + \int_t^T [f(r, X_r, Y_r, Z_r) - f(r, X_t, Y_t, Z_t)] N_r^{t,2} dr \right), \quad (22)$$

and

- (a)  $u_{xx}$  is continuous in  $[0, T) \times \mathbb{R}$ ,
- (b)  $|u_{xx}(t, x)| \leq \frac{c_{5.4}^3 \Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}$ ,

where  $c_{5.4}^3$  depends on  $C_g, T, p_0, \kappa_2, L_f, C_{5.3}^y, C_{5.3}^z$  and on the bounds and Lipschitz constants of  $b$  and  $\sigma$ .

In the following  $c_{5.4}$  represents  $(c_{5.4}^1, c_{5.4}^2, c_{5.4}^3)$  and  $c_{5.4}^{i,j}$  ( $i \neq j$ ) represents  $(c_{5.4}^i, c_{5.4}^j)$ ,  $(i, j) \in \{1, 2, 3\}$ .

*Proof.* (i): This follows from [36, Theorem 3.2].

(ii): From the proof of [36, Theorem 3.2], we get (21). The points (ii)(a) and (b) ensue from [36, Theorem 3.2 (i)]. It remains to prove (c).

**Proof of (ii) (c):** We show the assertion for a generator not depending on  $X$ , since the terms arising from that dependency would be easy to treat. Since  $\mathbb{E}_{t,x}(\mathbb{E}_{t,x}(g(X_T))N_T^{t,1}) = 0$  we can subtract it from the right hand side of (21) and get

$$\partial_x u(t, x) = \mathbb{E}_{t,x} \left( [g(X_T) - \mathbb{E}_{t,x}(g(X_T))]N_T^{t,1} + \int_t^T f(r, Y_r, Z_r)N_r^{t,1} dr \right).$$

It holds

$$\mathbb{E}_{t,x}|g(X_T) - \mathbb{E}_{t,x}g(X_T)|^2 = \mathbb{E}_{t,x}|g(X_T) - \tilde{\mathbb{E}}g(\tilde{X}_T^{t,X_t})|^2 \leq \mathbb{E}_{t,x}\tilde{\mathbb{E}}|g(X_T) - g(\tilde{X}_T^{t,X_t})|^2,$$

and thanks to the Cauchy-Schwarz inequality with  $\Psi_1 = C_g(1 + |X_T|^{p_0} + |\tilde{X}_T^{t,X_t}|^{p_0})$  and equation (3),

$$\begin{aligned} \mathbb{E}_{t,x}\tilde{\mathbb{E}}|g(X_T) - g(\tilde{X}_T^{t,X_t})|^2 &\leq \mathbb{E}_{t,x}\tilde{\mathbb{E}}(\Psi_1^2|X_T - \tilde{X}_T^{t,X_t}|^{2\alpha}) \\ &\leq \left[\mathbb{E}_{t,x}\tilde{\mathbb{E}}\Psi_1^4\right]^{\frac{1}{2}} \left[\mathbb{E}_{t,x}\tilde{\mathbb{E}}|X_T - \tilde{X}_T^{t,X_t}|^{4\alpha}\right]^{\frac{1}{2}} \\ &\leq C(C_g, T, p_0, b, \sigma)\Psi^2(x)(T-t)^\alpha. \end{aligned} \quad (23)$$

Relation (18) and the Lipschitz continuity of  $f$  imply

$$\begin{aligned} |\partial_x u(t, x)| &\leq \frac{C(C_g, T, p_0, \kappa_2, b, \sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}} \\ &\quad + C(L_f, K_f)\mathbb{E}_{t,x} \int_t^T (1 + |u(r, X_r)| + |\partial_x u(r, X_r)\sigma(r, X_r)|)|N_r^{t,1}|dr. \end{aligned} \quad (24)$$

Since we have  $|g(x)| \leq \Psi(x)$ , [36, Theorem 3.2 (ii)] gives  $|u(t, x)| \leq c\Psi(x)$  and  $|\partial_x u(t, x)| \leq c\Psi(x)(T-t)^{-1/2}$ , where  $c$  depends on  $T, L_f, K_f, \kappa_2, b, \sigma$  and  $p_0$ . Hence inequality (24) becomes

$$\begin{aligned} |\partial_x u(t, x)| &\leq \frac{C(C_g, T, p_0, \kappa_2, b, \sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}} + C(L_f, K_f, c, \sigma)\mathbb{E}_{t,x} \left( \int_t^T \left( 1 + \Psi(X_r) + \frac{\Psi(X_r)}{(T-r)^{\frac{1}{2}}} \right) |N_r^{t,1}|dr \right) \\ &\leq \frac{C(C_g, T, p_0, \kappa_2, b, \sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}} + C(T, L_f, K_f, \kappa_2, b, \sigma, p_0) \int_t^T \frac{\Psi(x)}{(T-r)^{\frac{1}{2}}(r-t)^{\frac{1}{2}}} dr \\ &\leq \frac{C(C_g, T, p_0, \kappa_2, L_f, K_f, b, \sigma)\Psi(x)}{(T-t)^{\frac{1-\alpha}{2}}}. \end{aligned}$$

(iii): We start with an approximation of  $g$  and  $f$  by smooth and bounded functions. Let  $\phi$  be a non-negative  $C^\infty$  function with support  $[-1, 1]$ , such that  $\int_{\mathbb{R}} \phi(u)du = 1$ , and  $\varepsilon \in (0, 1]$ . For  $N \in \mathbb{N}$  let  $b_N : \mathbb{R} \rightarrow [-N-1, N+1]$  be a monotone  $C^\infty$  function such that  $0 \leq b'_N(x) \leq 1$  and

$$b_N(x) := \begin{cases} N+1, & x > N+2, \\ x, & |x| \leq N, \\ -N-1, & x < -N-2. \end{cases}$$

Define

$$g^{\varepsilon, N}(x) = \int_{-1}^1 \phi(u)g(b_N(x) - \varepsilon u)du$$

and

$$f^{\varepsilon, N}(r, y, z) = \int_{-1}^1 \int_{-1}^1 \phi(u)\phi(v)f(r, b_N(y) - \varepsilon u, b_N(z) - \varepsilon v)dudv. \quad (25)$$

**Lemma 5.5.**  $g^{\varepsilon, N}$  and  $f^{\varepsilon, N}$  satisfy

- (a)  $\|g^{\varepsilon, N}\|_{\infty} + \|f^{\varepsilon, N}\|_{\infty} \leq C = C(\varepsilon, N)$  for some  $C(\varepsilon, N) > 0$ ,
- (b)  $g^{\varepsilon, N}$  and  $f^{\varepsilon, N}$  are  $C^{\infty}$  functions, with bounded derivatives (the bounds depend on  $\varepsilon$  and  $N$ ).  
Moreover,  $f^{\varepsilon, N}$  is a Lipschitz function in  $y$  and  $z$ , with Lipschitz constant  $L_f$ ,
- (c)  $g^{\varepsilon, N}$  satisfies (3), uniformly in  $\varepsilon \in (0, 1)$  and  $N \geq 1$ ,
- (d) for all  $x \in \mathbb{R}$  and  $\varepsilon \in [0, 1]$ , we have  $|g^{\varepsilon, N}(x) - g(x)| \leq C(C_g)\Psi(x)(\varepsilon^{\alpha} + \frac{|x|^{\alpha+1}}{N})$ ,
- (e) for all  $r \in [0, T]$  and for all  $(y, z) \in \mathbb{R}^2$ , we have

$$|f^{\varepsilon, N}(r, y, z) - f(r, y, z)| \leq L_f(2\varepsilon + |b_N(y) - y| + |b_N(z) - z|).$$

*Proof.* (a) Since  $g$  is locally Hölder continuous in the sense of (3),  $|g(x)| \leq C_g(1 + |x|^{p_0+1})$ . Then, we get  $|g^{\varepsilon, N}(x)| \leq C_g(1 + (N + 1 + \varepsilon)^{p_0+1})$ , and for  $f$  being Lipschitz continuous in  $y$  and  $z$ , uniformly in time, the same type of result applies.

(b) Since  $\phi$  is a  $C^{\infty}$  function and  $f$  and  $g$  are of polynomial growth, we get the result.

(c) Since  $g$  is locally Hölder continuous, we get

$$\begin{aligned} |g^{\varepsilon, N}(x) - g^{\varepsilon, N}(y)| &\leq \int_{-1}^1 |\phi(u)|C_g(1 + |b_N(x) - \varepsilon u|^{p_0} + |b_N(y) - \varepsilon u|^{p_0})|b_N(x) - b_N(y)|^{\alpha}du \\ &\leq \int_{-1}^1 C_g|\phi(u)|(1 + (|x| + \varepsilon)^{p_0} + (|y| + \varepsilon)^{p_0})|x - y|^{\alpha}du \\ &\leq C(C_g)(1 + |x|^{p_0} + |y|^{p_0})|x - y|^{\alpha}. \end{aligned}$$

(d) We have

$$\begin{aligned} |g^{\varepsilon, N}(x) - g(x)| &= \left| \int_{-1}^1 \phi(u)(g(b_N(x) - \varepsilon u) - g(x))du \right| \\ &\leq C_g \int_{-1}^1 |\phi(u)|(1 + |b_N(x) - \varepsilon u|^{p_0} + \varepsilon^{p_0} + |x|^{p_0})(|b_N(x) - x|^{\alpha} + \varepsilon^{\alpha})du \\ &\leq C(C_g)(1 + |x|^{p_0})(\varepsilon^{\alpha} + |x|^{\alpha}\mathbf{1}_{|x| \geq N}), \end{aligned}$$

and the result follows.

(e) We simply have to apply the Lipschitz property of  $f$  to get the result. □

We put now  $\varepsilon := \frac{1}{N}$  and write  $(g^N, f^N)$  instead of  $(g^{\frac{1}{N}, N}, f^{\frac{1}{N}, N})$  in order to simplify the notation and consider the BSDE

$$Y_t^N = g^N(X_T) + \int_t^T f^N(r, Y_r^N, Z_r^N) dr - \int_t^T Z_r^N dB_r.$$

**Representation for**  $\partial_x^2 u^N(t, x)$ .

By (i) we have that

$$u^N(t, x) = \mathbb{E}_{t,x} g^N(X_T^{t,x}) + \int_t^T \mathbb{E}_{t,x} f^N(r, Y_r^N, Z_r^N) dr.$$

According to Lemma 5.2 it holds that  $\partial_x^2 \mathbb{E}_{t,x} g^N(X_T) = \mathbb{E}_{t,x}[g^N(X_T) N_T^{t,2}]$  and

$$\partial_x^2 \mathbb{E}_{t,x} f^N(r, Y_r^N, Z_r^N) = \mathbb{E}_{t,x}[f^N(r, Y_r^N, Z_r^N) N_r^{t,2}],$$

because

$$f^N(r, Y_r^N, Z_r^N) = f^N(r, u^N(r, X_r), \sigma(r, X_r) u_x^N(r, X_r)),$$

and  $f^N(r, y, z)$  is continuous and bounded. Moreover, [25, Proposition 4] (or [21, Theorem 2.1]) implies that  $u^N(r, x)$  is  $C^{1,2}$  and it holds that  $|u^N(r, x)| + |\partial_x u^N(r, x)| + |\partial_x^2 u^N(r, x)| \leq C^N$  for some  $C^N > 0$ . Since  $\sigma$  is continuous,

$$(r, x) \mapsto f^N(r, u^N(r, x), \sigma(r, x) u_x^N(r, x))$$

is a bounded Borel function. Notice that by Lemma 5.2

$$\mathbb{E}_{t,x}[N_r^{t,2}] = 0 \quad \text{and} \quad \mathbb{E}_{t,x}[(N_r^{t,2})^2] \leq \frac{\kappa_2^2}{(r-t)^2}, \quad (26)$$

so that

$$\mathbb{E}_{t,x}[f^N(r, Y_r^N, Z_r^N) N_r^{t,2}] = \mathbb{E}_{t,x}([f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_t^N, Z_t^N)] N_r^{t,2}).$$

Using the Lipschitz continuity of  $f^N$  (see Lemma 5.5), the inequality of Cauchy-Schwarz and Theorem 5.3 one can derive the upper bound

$$\begin{aligned} & |\partial_x^2 \mathbb{E}_{t,x} f^N(r, Y_r^N, Z_r^N)| \\ & \leq \mathbb{E}_{t,x}[|f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_t^N, Z_t^N)| |N_r^{t,2}|] \\ & \leq C(L_f, \kappa_2) (\mathbb{E}_{t,x}(|Y_r^N - Y_t^N|^2 + |Z_r^N - Z_t^N|^2))^{\frac{1}{2}} \frac{1}{r-t} \\ & \leq C(L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \left[ \left( \int_t^r (T-s)^{\alpha-1} ds \right)^{\frac{1}{2}} + \left( \int_t^r (T-s)^{\alpha-2} ds \right)^{\frac{1}{2}} \right] \frac{\Psi(x)}{r-t} \\ & \leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(x) \frac{1}{(T-r)^{1-\frac{\alpha}{2}} (r-t)^{\frac{1}{2}}}. \end{aligned} \quad (27)$$

By this we do have an integrable bound for the derivative, and by dominated convergence we get

$$\begin{aligned} \partial_x^2 \int_t^T \mathbb{E}_{t,x} f^N(r, Y_r^N, Z_r^N) dr &= \int_t^T \partial_x^2 \mathbb{E}_{t,x} f^N(r, Y_r^N, Z_r^N) dr \\ &= \int_t^T \mathbb{E}_{t,x} \{ [f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_t^N, Z_t^N)] N_r^{t,2} \} dr. \end{aligned}$$

Hence we can write (using Fubini's theorem for the integral)

$$\partial_x^2 u^N(t, x) = \mathbb{E}_{t,x} \left( g^N(X_T) N_T^{t,2} + \int_t^T [f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_t^N, Z_t^N)] N_r^{t,2} dr \right).$$

**Convergence of  $\partial_x^2 u^N(t, x)$ .** Since  $\mathbb{E}_{t,x}[\mathbb{E}_{t,x}(g^N(X_T))N_T^{t,2}] = 0$ , Cauchy-Schwarz's inequality and the local Hölder continuity of  $g^N$  (see Lemma 5.5) give like in (23) that

$$\begin{aligned} |\mathbb{E}_{t,x}(g^N(X_T)N_T^{t,2})| &= \left| \mathbb{E}_{t,x} \left( [g^N(X_T) - \mathbb{E}_{t,x}(g^N(X_T))] N_T^{t,2} \right) \right| \\ &\leq \left( \mathbb{E}_{t,x}(|g^N(X_T) - \mathbb{E}_{t,x}(g^N(X_T))|^2) \right)^{\frac{1}{2}} \frac{\kappa_2}{T-t} \\ &\leq C(C_g, T, p_0, \kappa_2, b, \sigma) \frac{\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}, \end{aligned}$$

for all  $N \in \mathbb{N}$ . For the second term we can use the upper bound (27) and Lemma A.2 to get

$$\begin{aligned} \mathbb{E}_{t,x} \int_t^T \left| [f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_t^N, Z_t^N)] N_r^{t,2} \right| dr \\ \leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \int_t^T \frac{\Psi(x)}{(T-r)^{1-\frac{\alpha}{2}}(r-t)^{\frac{1}{2}}} dr, \\ \leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(x) \frac{B(\frac{\alpha}{2}, \frac{1}{2})}{(T-t)^{\frac{1}{2}-\frac{\alpha}{2}}}, \end{aligned}$$

which implies

$$|\partial_x^2 u^N(t, x)| \leq C(C_g, T, L_f, p_0, \kappa_2, C_{5.3}^y, C_{5.3}^z, b, \sigma) \frac{\Psi(x)}{(T-t)^{1-\frac{\alpha}{2}}}. \quad (28)$$

According to [21, Theorem 2.1]  $\partial_x^2 u^N(t, x)$  is continuous. Let

$$v(t, x) := \mathbb{E}_{t,x} \left( g(X_T) N_T^{t,2} + \int_t^T [f(r, Y_r, Z_r) - f(r, Y_t, Z_t)] N_r^{t,2} dr \right).$$

We show that for any  $(t, x) \in [0, T) \times \mathbb{R}$  it holds  $\partial_x^2 u^N(t, x) \rightarrow v(t, x)$  if  $N \rightarrow \infty$ , and that  $v$  is continuous on  $[0, T) \times \mathbb{R}$ . The idea to show continuity of  $v$  is as follows: If  $(t_n, x_n) \rightarrow (t, x)$ , then we may assume that we can find a  $\delta > 0$  such that  $x_n \in (x - \delta, x + \delta)$  and  $t_n \in (t - \delta, t + \delta) \subseteq [0, T)$  for each sufficiently large  $n$ . We consider

$$\begin{aligned} |v(t_n, x_n) - v(t, x)| &\leq |v(t_n, x_n) - \partial_x^2 u^N(t_n, x_n)| + |\partial_x^2 u^N(t_n, x_n) - \partial_x^2 u^N(t, x)| \\ &\quad + |\partial_x^2 u^N(t, x) - v(t, x)|. \end{aligned}$$

Since  $\partial_x^2 u^N$  is continuous, the term  $|\partial_x^2 u^N(t_n, x_n) - \partial_x^2 u^N(t, x)|$  is small for large  $n$ . Hence it suffices to show that  $\sup_{s \in (t-\delta, t+\delta), y \in (x-\delta, x+\delta)} |\partial_x^2 u^N(s, y) - v(s, y)|$  is small for large  $N$ . Let  $(s, y) \in (t - \delta, t + \delta) \times (x - \delta, x + \delta)$ . It holds

$$|\partial_x^2 u^N(s, y) - v(s, y)| \leq \mathbb{E}_{s,y} |[g^N(X_T) - g(X_T)] N_T^{s,2}| + \int_s^T D^{\frac{1}{2}}(r, s) \frac{\kappa_2}{r-s} dr := D_1 + D_2,$$

where (setting  $\|\cdot\|_{\mathbb{P}_{s,y}} := \|\cdot\|_{L_2(\mathbb{P}_{s,y})}$ )

$$D(r, s) := \|f^N(r, Y_r^N, Z_r^N) - f^N(r, Y_s^N, Z_s^N) - [f(r, Y_r, Z_r) - f(r, Y_s, Z_s)]\|_{\mathbb{P}_{s,y}}^2$$

$$\begin{aligned} &\leq L_f(\|Y_r^N - Y_s^N\|_{\mathbb{P}_{s,y}} + \|Z_r^N - Z_s^N\|_{\mathbb{P}_{s,y}} + \|Y_r - Y_s\|_{\mathbb{P}_{s,y}} + \|Z_r - Z_s\|_{\mathbb{P}_{s,y}}) \\ &\quad \times (\|f^N(r, Y_r^N, Z_r^N) - f(r, Y_r, Z_r)\|_{\mathbb{P}_{s,y}} + \|f^N(r, Y_s^N, Z_s^N) - f(r, Y_s, Z_s)\|_{\mathbb{P}_{s,y}}). \end{aligned}$$

First, let us bound  $D_1$ . According to Cauchy-Schwarz's inequality, (30) below and (26) we get

$$D_1 \leq \delta_1 \sqrt{\mathbb{E}_{s,y}(|N_T^{s,2}|^2)} \leq \frac{\delta_1 \kappa_2}{T-s} \leq \frac{\delta_1 \kappa_2}{T-t-\delta}.$$

Now let us bound  $D_2$ . According to Theorem 5.3 it holds

$$\begin{aligned} D^{\frac{1}{2}}(r, s) &\leq C(T, L_f, C_{5.3}^y, C_{5.3}^z) \Psi^{\frac{1}{2}}(y) \frac{(r-s)^{\frac{1}{4}}}{(T-r)^{\frac{1}{2}-\frac{\alpha}{4}}} \\ &\quad \times (\|f^N(r, Y_r^N, Z_r^N) - f(r, Y_r, Z_r)\|_{\mathbb{P}_{s,y}} + \|f^N(r, Y_s^N, Z_s^N) - f(r, Y_s, Z_s)\|_{\mathbb{P}_{s,y}})^{\frac{1}{2}}. \end{aligned}$$

Then, using (31), (33), (34) and Proposition 5.6 below gives

$$D^{\frac{1}{2}}(r, s) \leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(y) \frac{(r-s)^{\frac{1}{4}}}{(T-r)^{\frac{1}{2}-\frac{\alpha}{4}}} \frac{\delta_1}{(T-r)^{\frac{1}{4}}}.$$

Hence we have shown that

$$\begin{aligned} D_2 &\leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(y) \int_s^T \frac{\delta_1}{(r-s)^{\frac{3}{4}} (T-r)^{\frac{1}{2}-\frac{\alpha}{4}+\frac{1}{4}}} dr \\ &\leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(y) \frac{\delta_1}{(T-s)^{\frac{1}{2}-\frac{\alpha}{4}}}, \\ &\leq C(T, L_f, \kappa_2, C_{5.3}^y, C_{5.3}^z) \Psi(x+\delta) \frac{\delta_1}{(T-t-\delta)^{\frac{1}{2}-\frac{\alpha}{4}}} \quad \forall (s, y) \in (t-\delta, t+\delta) \times (x-\delta, x+\delta). \end{aligned}$$

Consequently,  $\sup_{y \in (x-\delta, x+\delta), s \in (t-\delta, t+\delta)} |\partial_x^2 u^N(s, y) - v(s, y)|$  is small for large  $N$ , hence  $v$  is continuous. Since

$$\partial_x u^N(t, x) - \partial_x u^N(t, y) = \int_y^x \partial_x^2 u^N(t, z) dz$$

converges to

$$\partial_x u(t, x) - \partial_x u(t, y) = \int_y^x v(t, z) dz,$$

it follows that  $\partial_x^2 u(t, x) = v(t, x)$ . Then point (iii-a) and (22) are proved. Since  $\partial_x^2 u^N$  converges to  $v$  for  $N \rightarrow \infty$ , we deduce point (iii-b) from (28).  $\square$

**Proposition 5.6.** *Let Assumptions 5.1 and 2.1 hold. Then for any  $(s, y) \in (t-\delta, t+\delta) \times (x-\delta, x+\delta)$  with  $t+\delta < T$  and  $r$  such that  $s \leq r < T$  we have*

$$\|Y_r^N - Y_r\|_{L_2(\mathbb{P}_{s,y})} + \|Z_r^N - Z_r\|_{L_2(\mathbb{P}_{s,y})} \leq \frac{\delta_1}{\sqrt{T-r}},$$

where  $\delta_1$  denotes a generic constant which tends to 0 when  $N$  tends to  $+\infty$ .

*Proof.* Let here  $\|\cdot\|$  stand for  $\|\cdot\|_{L_2(\mathbb{P}_{s,y})}$ . We will use for the  $Y$  differences the inequality

$$\|Y_r^N - Y_r\| \leq \|g^N(X_T) - g(X_T)\| + \int_r^T \|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\| dw.$$

For the  $Z$  differences we get by (21) and (ii-b)

$$\begin{aligned} & \|Z_r^N - Z_r\| \\ & \leq C(\sigma) \left( \left\| \mathbb{E}_r \left( g^N(X_T) - g(X_T) \right) N_T^{r,1} \right\| + \left\| \mathbb{E}_r \int_r^T (f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)) N_w^{r,1} dw \right\| \right) \\ & \leq C(\kappa_2, \sigma) \left( \frac{\|g^N(X_T) - g(X_T)\|}{\sqrt{T-r}} + \int_r^T \|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\| \frac{1}{\sqrt{w-r}} dw \right). \end{aligned}$$

Let  $S(r) := \|Y_r^N - Y_r\| + \|Z_r^N - Z_r\|$ . Using the inequality  $(1 + \frac{1}{\sqrt{w-r}}) \leq C(T) \frac{1}{\sqrt{w-r}}$  for  $r < w \leq T$  gives

$$S(r) \leq C(T, \kappa_2, \sigma) \left( \|g^N(X_T) - g(X_T)\| \frac{1}{\sqrt{T-r}} + \int_r^T \|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\| \frac{1}{\sqrt{w-r}} dw \right). \quad (29)$$

Let us bound  $\|g^N(X_T) - g(X_T)\|$ . By Lemma 5.5 we get the estimate

$$\begin{aligned} \mathbb{E}_{s,y} |g^N(X_T) - g(X_T)|^2 & \leq C(C_g) \mathbb{E}_{s,y} \left( \Psi(X_T)^4 \right)^{\frac{1}{2}} \left( \mathbb{E}_{s,y} \left( \frac{1}{N^\alpha} + \frac{|X_T|^{\alpha+1}}{N} \right)^4 \right)^{\frac{1}{2}} \\ & \leq C(C_g, T, b, \sigma, p_0) \Psi(y)^2 \left( \frac{1}{N^{2\alpha}} + \frac{|y|^{2\alpha+2}}{N^2} \right) \\ & \leq C(C_g, T, b, \sigma, p_0) \Psi(x + \delta)^2 \left( \frac{1}{N^{2\alpha}} + \frac{|x + \delta|^{2\alpha+2}}{N^2} \right) \leq \delta_1^2, \quad (30) \end{aligned}$$

for any arbitrarily small  $\delta_1 > 0$ , provided that  $N$  is sufficiently large. Let us now bound  $\|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\|$ . Using again Lemma 5.5 yields to

$$\begin{aligned} & \|f^N(w, Y_w^N, Z_w^N) - f(w, Y_w, Z_w)\| \\ & \leq \|f^N(w, Y_w^N, Z_w^N) - f^N(w, Y_w, Z_w)\| + \|f^N(w, Y_w, Z_w) - f(w, Y_w, Z_w)\| \\ & \leq L_f (\|Y_w^N - Y_w\| + \|Z_w^N - Z_w\| + \frac{2}{N} + \|b_N(Y_w) - Y_w\| + \|b_N(Z_w) - Z_w\|). \quad (31) \end{aligned}$$

Then, plugging (30) and (31) into (29) gives

$$\begin{aligned} S(r) & \leq \frac{C(T, \kappa_2, \sigma) \delta_1}{\sqrt{T-r}} + C(T, \kappa_2, \sigma) L_f \int_r^T \frac{S(w)}{\sqrt{w-r}} dw \\ & \quad + C(T, \kappa_2, \sigma) L_f \int_r^T \frac{\frac{1}{N} + \|b_N(Y_w) - Y_w\| + \|b_N(Z_w) - Z_w\|}{\sqrt{w-r}} dw. \quad (32) \end{aligned}$$

To estimate  $\|b_N(Z_w) - Z_w\|$  we use  $Z_w = \sigma(w, X_w) u_x(w, X_w)$  and choose a small  $a > 0$  such that  $\beta := \frac{(2+a)(1-\alpha)}{2} < 1$ . Then

$$\|b_N(Z_w) - Z_w\|^2 = \mathbb{E}_{s,y} |b_N(Z_w) - Z_w|^2 \mathbf{1}_{|Z_w| \geq N} \leq \frac{\mathbb{E}_{s,y} |Z_w|^{2+a}}{N^a} = \frac{\mathbb{E}_{s,y} |\sigma(w, X_w) u_x(w, X_w)|^{2+a}}{N^a}.$$



Using Theorem 5.4 (ii-c) yields

$$\begin{aligned} \mathbb{E}_{s,y}|b_N(Z_w) - Z_w|^2 &\leq \frac{C(c_{5.4}^2, \sigma)\mathbb{E}_{s,y}\Psi(X_w)^{(2+a)}}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}N^a} \leq \frac{C(T, p_0, c_{5.4}^2, \sigma, b)\Psi(y)^{(2+a)}}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}N^a} \\ &\leq \frac{\delta_1}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}}, \quad \forall (s, y) \in (t-\delta, t+\delta) \times (x-\delta, x+\delta). \end{aligned} \quad (33)$$

Similarly,

$$\mathbb{E}_{s,y}|b_N(Y_w) - Y_w|^2 \leq \frac{C(T, p_0, c_{5.4}^1, b, \sigma)\Psi(y)^{(2+a)}}{N^a} \leq \delta_1, \quad \forall (s, y) \in (t-\delta, t+\delta) \times (x-\delta, x+\delta). \quad (34)$$

Plugging (33) and (34) into (32) gives

$$\begin{aligned} S(r) &\leq \frac{C(T, \kappa_2)\delta_1}{\sqrt{T-r}} + C(T, \kappa_2)L_f \int_r^T \frac{S(w)}{\sqrt{w-r}} dw \\ &\quad + C(T, \kappa_2)L_f \int_r^T \frac{\frac{1}{N} + \delta_1}{\sqrt{w-r}} + \frac{\delta_1}{(T-w)^{\frac{(2+a)(1-\alpha)}{2}}\sqrt{w-r}} dw \\ &\leq C(T, \kappa_2, L_f) \left( \frac{\delta_1}{\sqrt{T-r}} + \int_r^T \frac{S(w)}{\sqrt{w-r}} dw \right), \end{aligned}$$

where the last inequality comes from Lemma A.2 ( $\beta < 1$ ). It remains to apply a version of Gronwall's Lemma (see e.g. [32, Lemma 3.1]) to see that  $S(r) \leq \frac{C(T, \kappa_2, L_f)\delta_1}{\sqrt{T-r}}$ . Since  $C(T, \kappa_2, L_f)\delta_1$  becomes arbitrarily small for  $N$  large, we will slightly abuse the notation and write  $S(r) \leq \frac{\delta_1}{\sqrt{T-r}}$ .  $\square$

## A Technical results and estimates

**Lemma A.1.** *For all  $0 \leq k \leq m \leq n$  and  $p > 0$ , it holds for  $h = \frac{T}{n}$  that*

(i)  $\mathbb{E}\tau_k = kh$ ,

(ii)  $\mathbb{E}|\tau_1|^p \leq C(p)h^p$ ,

(iii)  $\mathbb{E}|B_{\tau_m} - B_{\tau_k}|^2 = t_m - t_k$ ,

(iv)  $\mathbb{E}|B_{\tau_k} - B_{t_k}|^{2p} \leq C(p)\mathbb{E}|\tau_k - t_k|^p \leq C(p)(t_k h)^{\frac{p}{2}}$ .

*Proof.* The strong Markov property of the Brownian motion implies that  $(\tau_i - \tau_{i-1})_{i=1}^\infty$  is an i.i.d. sequence. According to [33, Proposition 11.1 (iii)], we have that  $\mathbb{E}\tau_1 = \frac{T}{n}$ , and (i) follows. Item (ii) follows by [33, Proposition 11.1 (iv)] and Jensen's inequality. To prove item (iii), recall that  $(B_{\tau_i} - B_{\tau_{i-1}})_{i=1}^\infty$  is a centered i.i.d. sequence with  $\mathbb{E}(B_{\tau_i} - B_{\tau_{i-1}})^2 = h$ ,  $i \geq 1$ . (iv): The BDG inequality implies that for each  $p > 0$ ,

$$\begin{aligned} \mathbb{E}|B_{\tau_k} - B_{t_k}|^p &= \mathbb{E} \left| \int_0^{\tau_k \vee t_k} (\mathbf{1}_{[0, \tau_k]}(r) - \mathbf{1}_{[0, t_k]}(r)) dB_r \right|^p \\ &\leq C(p) \mathbb{E} \left( \int_0^{\tau_k \vee t_k} \mathbf{1}_{[0, \tau_k] \Delta [0, t_k]}(r) dr \right)^{p/2} = \mathbb{E}|\tau_k - t_k|^{p/2}. \end{aligned}$$

To prove the second inequality of (iv), a generalization of [33, Proposition 11.1 (iv)], we first assume that  $p \geq 1$ . Let us rewrite  $\tau_k - t_k = \sum_{i=1}^k \eta_i$  where  $(\eta_i)_{1 \leq i \leq k}$  is an i.i.d. centered sequence of random variables distributed as  $\tau_1 - h$ . Burkholder's and Hölder's inequalities, and finally item (ii) yield

$$\mathbb{E}|\tau_k - t_k|^p \leq C(p) \mathbb{E} \left( \sum_{i=1}^k \eta_i^2 \right)^{\frac{p}{2}} \leq k^{\frac{p}{2}-1} \sum_{i=1}^k \mathbb{E}(\eta_i^p) \leq C(p)(t_k h)^{\frac{p}{2}},$$

which proves the claim for  $p \geq 1$ . The case  $p < 1$  follows from this result by Jensen's inequality.  $\square$

**Lemma A.2.** *For all  $t \in [0, T)$  and for all  $\alpha < 1, \beta < 1$  we have*

$$\int_t^T \frac{1}{(T-r)^\alpha (r-t)^\beta} dr = \frac{1}{(T-t)^{\alpha+\beta-1}} B(1-\alpha, 1-\beta),$$

where  $B$  denotes the beta function.

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