

OPTIMAL CORRECTOR ESTIMATES ON PERCOLATION CLUSTERS

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ABSTRACT. We prove optimal quantitative estimates on the first-order correctors on supercritical percolation clusters: we show that they are bounded in $d \geq 3$ and have logarithmic growth in $d = 2$, in the sense of stretched exponential moments. The main ingredients are a renormalization scheme of the supercritical percolation cluster, following the works of Pisztora [29] and Barlow [10]; large-scale regularity estimates developed in the previous paper [7]; and a nonlinear concentration inequality of Efron-Stein type which is used to transfer quantitative information from the environment to the correctors.

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1. INTRODUCTION

1.1. Motivation and informal summary of results. The main goal of this paper is to prove optimal bounds on the first-order correctors for the random conductance model on a supercritical percolation cluster. We show, in the sense of stretched exponential moments, that the correctors are bounded in dimensions $d \geq 3$, and have increments which grow like the square root of the logarithm of the distance in $d = 2$.

As explained in [5], tight bounds on the correctors are the crucial ingredient for the derivation of optimal error and two-scale expansion estimates for the homogenization of general boundary value problems. They can also inform the performance of numerical algorithms for the computation of the homogenized diffusivity [26] and of solutions to the heterogeneous equation [2]. The need for quantitative estimates was also fundamental to the derivation of the quenched central limit theorem for the corresponding random walk, see [30, 24, 10, 23, 11]. The bounds we present here allow in particular to obtain much more precise quantitative central limit theorems for this random walk.

Our approach is inspired by recent developments in the quantitative homogenization of uniformly elliptic random environments, in particular by the works of Armstrong, Kuusi, Mourrat and Smart [9, 8, 3, 4, 5] and the works of Gloria, Neukamm and Otto [16, 18, 17, 19, 20, 21, 22]. The main challenge faced in the previous article [7] and in the present one is to adapt the various tools and proofs, available in the uniformly elliptic setting, to the supercritical percolation cluster. To this end, a renormalization argument was developed in [7], the main results of which will be recorded in Section 2, where \mathbb{Z}^d was partitioned into triadic cubes of different random sizes, well connected

in the sense of Antal and Pisztora [1]. This partition allows to distinguish regions of \mathbb{Z}^d where the infinite cluster is well-behaved, its geometry looks like the geometry of the lattice \mathbb{Z}^d , from regions where the infinite cluster is badly-behaved. In the first case, it is rather straightforward to adapt the theory developed in the uniformly elliptic setting. Problems arise where the infinite cluster is badly-behaved. In this situation the theory cannot be adapted. Fortunately there are few regions where the cluster is badly-behaved, and the theory of stochastic homogenization in the uniformly elliptic setting is robust enough to be adapted to the supercritical cluster.

Our strategy to prove the optimal scaling limit of the corrector relies on a concentration inequality (cf Proposition 2.18), which gives us a convenient way to transfer quantitative information from the coefficients to the correctors. This idea originates in an unpublished paper from Naddaf and Spencer [27], and was then developed by Gloria and Otto [19, 20] and Gloria, Neukamm and Otto [18] (see also Mourrat [25]) to study stochastic homogenization. More precisely, thanks to this inequality we are able to obtain quantitative estimates on the spatial average of the gradient of the corrector.

We then need one last ingredient to transfer bounds on the spatial average of the gradient of the corrector to the oscillation of the correctors. This will be achieved by the multiscale Poincaré inequality, Proposition 2.19. This inequality is a refinement of the Poincaré inequality, more suited to the study of rapidly oscillating functions such as the corrector.

The model is the following: consider the random conductance model on the infinite percolation cluster for supercritical bond percolation on the graph $(\mathbb{Z}^d, \mathcal{B}_d)$ in dimension $d \geq 2$. Here \mathcal{B}_d is the set of *bonds*, that is, unordered pairs $\{x, y\}$ with $x, y \in \mathbb{Z}^d$ satisfying $|x - y| = 1$. We are given $\lambda \in (0, 1)$ and a function

$$\mathbf{a} : \mathcal{B}_d \longrightarrow \{0\} \cup [\lambda, 1].$$

We call $\mathbf{a}(\{x, y\})$ the *conductance* of the bond $\{x, y\} \in \mathcal{B}_d$ and we assume that $\{a(e)\}_{e \in \mathcal{B}_d}$ is an i.i.d. ensemble. We assume that the Bernoulli random variable $\mathbb{1}_{\{\mathbf{a}(e) \neq 0\}}$ has parameter $\mathbf{p} > \mathbf{p}_c(d)$, where $\mathbf{p}_c(d)$ is the bond percolation threshold for the lattice \mathbb{Z}^d . It follows that the graph $(\mathbb{Z}^d, \mathcal{E}(\mathbf{a}))$, where $\mathcal{E}(\mathbf{a})$ is the set of edges $e \in \mathcal{B}_d$ for which $\mathbf{a}(e) \neq 0$, has a unique infinite connected component, which we denote by $\mathcal{C}_\infty = \mathcal{C}_\infty(\mathbf{a})$.

Our interest in this paper is the elliptic finite difference equation

$$(1.1) \quad -\nabla \cdot \mathbf{a} \nabla u = 0 \quad \text{in } \mathcal{C}_\infty.$$

Here the elliptic operator $-\nabla \cdot \mathbf{a} \nabla$ is defined on functions $u : \mathcal{C}_\infty \rightarrow \mathbb{R}$ by

$$(1.2) \quad (-\nabla \cdot \mathbf{a} \nabla u)(x) := \sum_{y \sim x} \mathbf{a}(\{x, y\}) (u(x) - u(y)).$$

In [7], we proved that, almost surely, each u solution of (1.1) with at most linear growth can be written

$$u(x) = c + p \cdot x + \chi_p(x),$$

where $c \in \mathbb{R}, p \in \mathbb{R}^d$ and χ_p is a function, called the corrector, with sublinear growth: there exists $\delta > 0$ such that,

$$\lim_{R \rightarrow \infty} \frac{1}{R^{1-\delta}} \operatorname{osc}_{x \in \mathcal{C}_\infty \cap B_R} \chi_p(x) = 0,$$

where we used the notation, for any subset $A \subseteq \mathbb{Z}^d$ and any function $f : A \rightarrow \mathbb{R}$

$$\operatorname{osc}_A f := \sup_A f - \inf_A f.$$

The sublinear growth of the corrector is a very important property which was proven quantitatively (as stated above) in [7] and qualitatively in [13]. The main goal of this paper is then to derive the *optimal* scaling of the corrector.

1.2. Notation and assumptions.

1.2.1. *General notation for the probabilistic model.* We denote by \mathbb{Z}^d the standard d -dimensional hypercubic lattice. A point $x \in \mathbb{Z}^d$ will often be called a *vertex*. The set of edges of \mathbb{Z}^d (i.e the set of unoriented pairs of nearest neighbors) is denoted by $\mathcal{B}_d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. More specifically, given a subset $U \subseteq \mathbb{Z}^d$, we denote by $\mathcal{B}_d(U)$ the set of the edges of U , i.e, $\mathcal{B}_d(U) := \{\{x, y\} : x, y \in U, |x - y|_1 = 1\}$. The canonical basis of \mathbb{R}^d is denoted by $\mathbf{e}_1, \dots, \mathbf{e}_d$. For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if $\{x, y\} \in \mathcal{B}_d$. For some fixed parameter $\lambda \in (0, 1]$, we define the probability space $\Omega := (\{0\} \cup [\lambda, 1])^{\mathcal{B}_d}$ and we equip this probability space with the Borel σ -algebra $\mathcal{F} := \mathcal{B}(\{0\} \cup [\lambda, 1])^{\otimes \mathcal{B}_d}$. Given an edge $e \in \mathcal{B}_d$, we denote by $\mathbf{a}(e)$ the projection

$$\mathbf{a}(e): \begin{array}{ccc} \Omega & \rightarrow & \{0\} \cup [\lambda, 1], \\ (\omega_{e'})_{e' \in \mathcal{B}_d} & \mapsto & \omega_e. \end{array}$$

We denote by \mathbf{a} the collection $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ and we refer to this mapping as the *environment*. For every $U \subseteq \mathbb{Z}^d$, we denote by $\mathcal{F}(U) \subseteq \mathcal{F}$ the σ -algebra generated by the mappings $(\mathbf{a}(e))_{e \in \mathcal{B}_d(U)}$.

We fix a probability measure \mathbb{P}_0 supported in $\{0\} \cup [\lambda, 1]$ satisfying the property,

$$(1.3) \quad \mathbf{p} := \mathbb{P}_0([\lambda, 1]) > \mathbf{p}_c(d).$$

where $\mathbf{p}_c(d)$ is the bond percolation threshold for the lattice \mathbb{Z}^d . We then equip the measurable space (Ω, \mathcal{F}) with the i.i.d. probability measure $\mathbb{P} = \mathbb{P}_0^{\otimes \mathcal{B}_d}$ such that the sequence of random variables $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ is an i.i.d. family of random variables of law \mathbb{P}_0 . The expectation with respect to \mathbb{P} is denoted by \mathbb{E} .

Given an environment \mathbf{a} , we say that an edge $e \in \mathcal{B}_d$ is *open* if $\mathbf{a}(e) > 0$ and *closed* if $\mathbf{a}(e) = 0$. Given two vertices $x, y \in \mathbb{Z}^d$, we say that there is a *path connecting x and y* if there exists a sequence of open edges of the form $\{x, z_1\}, \dots, \{z_n, z_{n+1}\}, \dots, \{z_N, y\}$. The two vertices x and y are then said to be *connected*, we denote $x \leftrightarrow_{\mathbf{a}} y$, if there exists a path connecting x and y . A *cluster* is a connected subset $\mathcal{C} \subseteq \mathbb{Z}^d$. Thanks to (1.3), we know that, \mathbb{P} -almost surely, there exists a unique maximal infinite cluster [12]. This cluster is denoted by $\mathcal{C}_\infty := \mathcal{C}_\infty(\mathbf{a})$.

We also denote by $E_d := \{(x, y) : x, y \in \mathbb{Z}^d, x \sim y\}$ the set of oriented edges. More generally, we define, for a subset $U \subseteq \mathbb{Z}^d$, $E_d(U) := \{(x, y) : x, y \in U, x \sim y\}$.

For $x \in \mathbb{Z}^d$, we define the translation τ_x on Ω to be the application

$$\tau_x: \begin{array}{ccc} \Omega & \rightarrow & \Omega, \\ (\omega_e)_{e \in \mathcal{B}_d} & \mapsto & (\omega_{e+x})_{e \in \mathcal{B}_d}. \end{array}$$

Note that the measure \mathbb{P} is stationary with respect to the \mathbb{Z}^d -translations: for each $x \in \mathbb{Z}^d$,

$$(1.4) \quad (\tau_x)_* \mathbb{P} = \mathbb{P},$$

where $(\tau_x)_* \mathbb{P}$ is the pushforward measure defined by, for each $A \in \mathcal{F}$, $(\tau_x)_* \mathbb{P}(A) = \mathbb{P}(\tau_x^{-1}(A))$.

1.2.2. *Notation for functions.* We define a *vector field* to be a function $G : E_d \rightarrow \mathbb{R}$ satisfying the following antisymmetry property: for each $(x, y) \in E_d$,

$$G(x, y) = -G(y, x).$$

Given a function $u : \mathbb{Z}^d \rightarrow \mathbb{R}$, we define its gradient ∇u to be the vector field

$$(\nabla u)(x, y) := u(x) - u(y).$$

For a random function defined on a cluster \mathcal{C} , $u : \mathcal{C} \rightarrow \mathbb{R}$, we define ∇u to be the vector field defined by

$$(1.5) \quad (\nabla u)(x, y) := \begin{cases} u(x) - u(y) & \text{if } x, y \in \mathcal{C} \text{ and } \mathbf{a}(\{x, y\}) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathbf{a}\nabla u$ to be the vector field defined by

$$(\mathbf{a}\nabla u)(x, y) := \mathbf{a}(\{x, y\})(\nabla u)(x, y).$$

We typically think of \mathcal{C} as being the infinite cluster \mathcal{C}_∞ .

For $q \in \mathbb{R}^d$, we denote by q the constant vector field, defined according to the formula

$$q(x, y) := q \cdot (x - y).$$

For a given vector field G , we define, for every $x \in \mathbb{Z}^d$,

$$(1.6) \quad |G|(x) := \left(\frac{1}{2} \sum_{(x,y) \in E_d} |G(x, y)|^2 \right)^{\frac{1}{2}}.$$

Given a subset $U \subseteq \mathbb{Z}^d$, we equip the space of vector fields with a scalar product, defined by

$$\langle F, G \rangle_U := \sum_{(x,y) \in E_d(U)} F(x, y)G(x, y).$$

We will also frequently make use of the following notation, given a vector field G , we define

$$\langle G \rangle_U = \sum_{(x,y) \in E_d(U)} G(x, y)(x - y).$$

Given an environment \mathbf{a} , two functions $u, v : \mathbb{Z}^d \rightarrow \mathbb{R}$, and a subset $U \subseteq \mathbb{Z}^d$, the Dirichlet form can be written with the previous notation as

$$\langle \nabla u, \mathbf{a}\nabla v \rangle_U = \sum_{(x,y) \in E_d(U)} (u(x) - u(y)) \mathbf{a}(\{x, y\})(v(x) - v(y)).$$

We then define the elliptic operator $-\nabla \cdot \mathbf{a}\nabla$ by, for each $u : \mathbb{Z}^d \rightarrow \mathbb{R}$ and $x \in \mathbb{Z}^d$,

$$(-\nabla \cdot \mathbf{a}\nabla u)(x) := \sum_{x \sim y} \mathbf{a}(\{x, y\})(u(x) - u(y)).$$

For a given a subset $U \subseteq \mathbb{Z}^d$, we define the random set of \mathbf{a} -harmonic functions in U by,

$$\mathcal{A}(U) := \{u : U \mapsto \mathbb{R} : (-\nabla \cdot \mathbf{a}\nabla u)(x) = 0, x \in \text{int}_{\mathbf{a}}U\},$$

where $\text{int}_{\mathbf{a}}U$ is the interior of U with respect to the environment \mathbf{a} , defined according to

$$\text{int}_{\mathbf{a}}U := \{x \in U : \forall y \in \mathbb{Z}^d, (y \sim x \text{ and } \mathbf{a}(\{x, y\}) \neq 0) \implies y \in U\}.$$

Given a subset $U \subseteq \mathbb{Z}^d$ and a function $w : U \rightarrow \mathbb{R}$, we generally denote sums by integrals; for instance,

$$(1.7) \quad \text{we write } \int_U w(x) dx \text{ instead of } \sum_{x \in U} w(x).$$

If U is a finite (resp. a continuous) set, we denote its cardinality (resp. its Lebesgue measure) by $|U|$. It will always be clear from context whether we are referring to the continuous integral (resp. to the Lebesgue measure) or to the discrete integral (resp. the cardinality). The normalized integral for a discrete (resp. continuous) function $w : U \rightarrow \mathbb{R}$ defined on a discrete (resp. continuous) subset $U \subseteq \mathbb{Z}^d$ (resp. $U \subseteq \mathbb{R}^d$) is denoted

$$\int_U w(x) dx = \frac{1}{|U|} \int_U w(x) dx.$$

To shorten the notation, we sometimes write

$$(w)_U := \int_U w(x) dx.$$

We denote by $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ (resp. $C^\infty(\mathbb{R}^d, \mathbb{R})$) the set of smooth compactly supported (resp. smooth) functions in \mathbb{R}^d and by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space, i.e.,

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \forall (k, \alpha_1, \dots, \alpha_d) \in \mathbb{N}^{d+1}, \quad \sup_{x \in \mathbb{R}^d} |x|^k |\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f(x)| < \infty \right\}$$

and by $\mathcal{S}'(\mathbb{R}^d)$ (or \mathcal{S}') its topological dual, the space of tempered distribution. Given $\Omega \subseteq \mathbb{R}^d$ a domain, we denote by $C_c^\infty(\Omega, \mathbb{R})$ (resp. $C^\infty(\Omega, \mathbb{R})$) the set of smooth compactly supported (resp. smooth) functions in Ω .

For $q \in [1, \infty)$, we denote the L^q and normalized L^q norms by

$$\|w\|_{L^q(U)} := \left(\int_U |w(x)|^q dx \right)^{\frac{1}{q}} \quad \text{and} \quad \|w\|_{\underline{L}^q(U)} := \left(\int_U |w(x)|^q dx \right)^{\frac{1}{q}}.$$

Moreover, we write $\|w\|_{L^\infty(U)} := \sup_{x \in U} |w(x)|$. For $k \in \mathbb{N}$, we denote by $W^{k,q}(\Omega)$ the Sobolev space, by $W_0^{k,q}(\Omega)$ the closure of $C_c^\infty(\Omega, \mathbb{R})$ in $W^{k,q}(\Omega)$, and by $W_{\text{loc}}^{k,q}(\Omega)$ the space of local Sobolev functions. For $k \in \mathbb{Z}$ with $k < 0$, we denote by $W^{k,q}(\Omega)$ the topological dual of $W_0^{-k,p}(\Omega)$, with $p = \frac{q}{q-1}$.

For vectors of \mathbb{R}^d , we denote by $|\cdot|$ the ℓ_∞ norm, i.e., $|x| = \max_{i=1, \dots, d} |x_i|$. This distance can then be extended to a pseudometric on the subsets of \mathbb{Z}^d by $\text{dist}(U, V) = \inf_{x \in U, y \in V} |x - y|$.

We also use the notation $B_R(x)$ to denote the closed ball centered in $x \in \mathbb{Z}^d$ with radius $R > 0$ with respect to the ℓ_∞ norm. The ball $B_R(0)$ is simply denoted B_R .

1.2.3. *Notation for cubes.* A *cube* is a subset of \mathbb{Z}^d of the form

$$\left(z + (-N, N)^d \right) \cap \mathbb{Z}^d, \quad N \in \mathbb{N}, z \in \mathbb{Z}^d.$$

For the cube given in the previous display, which we denote by \square , we define its *center* and its *size* to be the point $z \in \mathbb{Z}^d$ and the integer $2N - 1$. We denote its size by $\text{size}(\square)$. In particular, with this convention, we have $|\square| = (\text{size}(\square))^d$. For a nonnegative real number $r > 0$ and a cube \square , of center $z \in \mathbb{Z}^d$ and size $N \in \mathbb{N}$, we denote by $r\square$ the cube

$$r\square := \left(z + (-rN, rN)^d \right) \cap \mathbb{Z}^d.$$

This notation is nonstandard because the multiplication by r only affects the size of the cube, indeed the cube $r\square$ has size $\lceil r \text{size}(\square) \rceil$, but the center of the cube remains unchanged. We now introduce a specific category of cubes, namely the *triadic cubes*. A triadic cube is a cube of the form

$$(1.8) \quad \square_n(z) := \left(z + \left(-\frac{1}{2}3^n, \frac{1}{2}3^n \right)^d \right) \cap \mathbb{Z}^d, \quad n \in \mathbb{N}, z \in 3^n \mathbb{Z}^d.$$

To simplify the notation, we also write $\square_n = \square_n(0)$. This collection of cubes enjoys a number of very convenient properties. First, any two triadic cubes (of possibly different sizes) are either disjoint or else one is included in the other. Moreover, for every $m, n \in \mathbb{N}$ with $n \leq m$, the triadic cube \square_m can be uniquely partitioned into $3^{d(m-n)}$ disjoint triadic cubes of size 3^n , i.e., cubes of the form $\square_n(z)$ with $z \in 3^n \mathbb{Z}^d$. We denote by \mathcal{T} the collection of triadic cubes and by \mathcal{T}_n the collection of triadic cubes of size 3^n , i.e., $\mathcal{T}_n := \{z + \square_n : z \in 3^n \mathbb{Z}^d\}$.

For each $n \in \mathbb{N}$ and each $\square \in \mathcal{T}_n$, we define the predecessor of \square , to be the unique triadic cube $\tilde{\square} \in \mathcal{T}_{n+1}$ such that $\square \subseteq \tilde{\square}$. If $\tilde{\square}$ is the predecessor of \square , then we also say that \square is a successor $\tilde{\square}$. In particular, a cube of \mathcal{T}_0 does not have any successor, while each cube of $\mathcal{T} \setminus \mathcal{T}_0$ has exactly 3^d successors.

1.2.4. *The \mathcal{O}_s notation.* We next introduce a series of notation and properties which will be useful to measure the stochastic integrability and sizes of random variables. Given two parameters $s, \theta > 0$ and a nonnegative random variable X , we denote by

$$X \leq \mathcal{O}_s(\theta) \text{ if and only if } \mathbb{E} \left[\exp \left(\left(\frac{X}{\theta} \right)^s \right) \right] \leq 2.$$

Note that by Markov's inequality, the tail of a random variable X satisfying $X \leq \mathcal{O}_s(\theta)$ decreases exponentially fast, that is to say, for every $t > 0$,

$$\mathbb{P}[X \geq \theta t] \leq 2 \exp(-t^s).$$

For a given sequence $(Y_i)_{i \in \mathbb{N}}$ of nonnegative random variables and a sequence $(\theta_i)_{i \in \mathbb{N}}$ of nonnegative real numbers, we write

$$X \leq \sum_{i \in \mathbb{N}} Y_i \mathcal{O}_s(\theta_i),$$

if there exists a sequence of nonnegative random variables $(Z_i)_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$, $Z_i \leq \mathcal{O}_s(\theta_i)$ and

$$X \leq \sum_{i \in \mathbb{N}} Y_i Z_i.$$

We now record some properties pertaining to this notation. All these properties are proved in [5, Appendix A] and we refer to this reference for the proofs. This notation is compatible with the addition, meaning that, for any $s > 0$, there exists a constant C depending only on s , which may be taken to be 1 if $s \geq 1$,

$$(1.9) \quad X_1 \leq \mathcal{O}_s(\theta_1) \text{ and } X_2 \leq \mathcal{O}_s(\theta_2) \implies X + Y \leq \mathcal{O}_s(C(\theta_1 + \theta_2)).$$

More generally, for any $s > 0$, there exists a constant $C(s) < \infty$ such that, for every measure space (X, \mathcal{F}, μ) , every jointly measurable family $\{X(x)\}_{x \in E}$ of nonnegative random variables and every measurable function $\theta : E \rightarrow \mathbb{R}_+$, we have

$$(1.10) \quad \forall x \in E, X(x) \leq \mathcal{O}_s(\theta(x)) \implies \int_E X(x) d\mu(x) \leq \mathcal{O}_s \left(C \int_E \theta(x) d\mu(x) \right).$$

Moreover the constant can be chosen to be

$$(1.11) \quad \begin{cases} C(s) = \left(\frac{1}{s \ln 2} \right)^{\frac{1}{s}} & \text{if } s < 1 \\ C(s) = 1 & \text{if } s \geq 1. \end{cases}$$

From the definition, we have, for each $\lambda \in \mathbb{R}_+$,

$$X \leq \mathcal{O}_s(\theta) \implies \lambda X \leq \mathcal{O}_s(\lambda \theta).$$

This notation is also compatible with the multiplication in the sense that

$$(1.12) \quad |X_1| \leq \mathcal{O}_{s_1}(\theta_1) \text{ and } |X_2| \leq \mathcal{O}_{s_2}(\theta_2) \implies |XY| \leq \mathcal{O}_{\frac{s_1 s_2}{s_1 + s_2}}(\theta_1 \theta_2).$$

Moreover, it is easy to check that one can decrease the integrability exponent s , i.e., for each $0 < s' < s$, there exists a constant $C := C(s') < \infty$ such that

$$(1.13) \quad X \leq \mathcal{O}_s(\theta_1) \implies X \leq \mathcal{O}_{s'}(C\theta_1).$$

1.3. Statement of the main results. Denote by \mathcal{A}_1 the (random) vector space of \mathbf{a} -harmonic functions with at most linear growth, i.e.,

$$\mathcal{A}_1 := \left\{ u : \mathcal{C}_\infty \rightarrow \mathbb{R} \mid \lim_{R \rightarrow \infty} \frac{1}{R^2} \|u\|_{L^2(\mathcal{C}_\infty \cap B_R)} = 0 \right\}$$

By Theorem 2 of [7], we know that, \mathbb{P} -almost surely, the space \mathcal{A}_1 has dimension $d + 1$ and that every function $u \in \mathcal{A}_1$ can be written $u = c + p \cdot x + \chi_p(x)$, with $c \in \mathbb{R}$ and $p \in \mathbb{R}^d$. The family $\{\chi_p\}_{p \in \mathbb{R}^d}$ is called the correctors. We already proved the sublinear growth of the correctors, indeed by [7,

(1.22)], we know that there exist two exponents $\delta := \delta(d, \mathbf{p}, \lambda) > 0$, $s := s(d, \mathbf{p}, \lambda) > 0$ and a constant $C := C(d, \mathbf{p}, \lambda)$ such that, for each $R \geq 1$,

$$(1.14) \quad \operatorname{osc}_{\mathcal{C}_\infty \cap B_R} \chi_p \leq \mathcal{O}_s(C|p|R^{1-\delta}).$$

The sublinear growth of the corrector can also be expressed with a minimal scale, indeed by [7, (1.18)], there exists a random variable \mathcal{X} satisfying

$$\mathcal{X} \leq \mathcal{O}_s(C),$$

such that for each $R \geq \mathcal{X}$,

$$(1.15) \quad \left\| \chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R} \right\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)} \leq C|p|R^{1-\delta}.$$

Moreover, the corrector satisfies the following stationarity property, for each $x, y \in \mathbb{Z}^d$, each $p \in \mathbb{R}^d$ and each $z \in \mathbb{Z}^d$

$$(1.16) \quad (\chi_p(x) - \chi_p(y)) \mathbf{1}_{\{x, y \in \mathcal{C}_\infty\}}(\mathbf{a}) = (\chi_p(x+z) - \chi_p(y+z)) \mathbf{1}_{\{z+x, z+y \in \mathcal{C}_\infty\}}(\tau_z \mathbf{a})$$

The first main theorem of this article gives optimal scaling bounds of the correctors in the L^q norm.

Theorem 1 (Optimal L^q estimates for first-order correctors). *There exist two exponents $s := s(d, \mathbf{p}, \lambda) > 0$, $k := k(d, \mathbf{p}, \lambda) < \infty$ and a constant $C(d, \mathbf{p}, \lambda) < \infty$ such that for each $R \geq 1$, each $q \geq 1$, and each $p \in \mathbb{R}^d$,*

$$(1.17) \quad \left(R^{-d} \int_{\mathcal{C}_\infty \cap B_R} \left| \chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s(C|p|q^k \log^{\frac{1}{2}} R) & \text{if } d = 2, \\ \mathcal{O}_s(C|p|q^k) & \text{if } d \geq 3. \end{cases}$$

In Section 5, we improve this L^q bound into an L^∞ bound.

Theorem 2 (Optimal L^∞ estimates for first-order correctors). *There exist an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a constant $C(d, \mathbf{p}, \lambda) < \infty$ such that for each $x, y \in \mathbb{Z}^d$ and each $p \in \mathbb{R}^d$,*

$$|\chi_p(x) - \chi_p(y)| \mathbf{1}_{\{x, y \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s(C|p| \log^{\frac{1}{2}} |x-y|) & \text{if } d = 2, \\ \mathcal{O}_s(C|p|) & \text{if } d \geq 3. \end{cases}$$

1.4. Outline of the paper. The rest of the paper is organised as follows. In Section 2, we recall (mostly without proof) some properties of the infinite cluster which were stated and proved in [7] to develop a quantitative homogenization theory on the infinite percolation cluster. In subsections 2.5 and 2.6, we state the concentration inequality and the multiscale Poincaré inequality, which are the two key ideas in the proof of Theorem 1. In Section 3, we use the concentration inequality and the properties of the infinite cluster recorded in Section 2 to obtain an estimate on the spatial averages of the correctors. In Section 4, we use the result established in Section 3 combined with the multiscale Poincaré inequality to prove the optimal bound on the gradient of the correctors, stated in Theorem 1. In Section 5 we use the L^q bounds obtained in Section 4 to upgrade the bounds into an L^∞ bound, i.e, we prove Theorem 2. In Appendix A, we give a proof of the multiscale Poincaré inequality stated in subsection 2.6. In Appendix B, we give the proof of a technical lemma used in Section 3.

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2. PRELIMINARIES

In this section we record some properties about the infinite percolation cluster in the supercritical regime. Most of these properties were established in [7].

2.1. Triadic partitions of good cubes.

2.1.1. *A general scheme for partition of good cubes.* The construction of the partition is accomplished by a stopping time argument reminiscent of a Calderón-Zygmund-type decomposition. We are given a notion of “good cube” represented by an \mathcal{F} -measurable function which maps Ω into the set of all subsets of \mathcal{T} . In other words, for each $\mathbf{a} \in \Omega$, we are given a subcollection $\mathcal{G}(\mathbf{a}) \subseteq \mathcal{T}$ of triadic cubes. We think of $\square \in \mathcal{T}$ as being a good cube if $\square \in \mathcal{G}(\mathbf{a})$. As usual, we typically drop the dependence on \mathbf{a} and just write \mathcal{G} .

Proposition 2.1 (Proposition 2.1 of [7]). *Let $\mathcal{G} \subseteq \mathcal{T}$ be a random collection of triadic cubes, as above. Suppose that \mathcal{G} satisfies, for every $\square = z + \square_n \in \mathcal{T}$,*

$$(2.1) \quad \text{the event } \{\square \notin \mathcal{G}\} \text{ is } \mathcal{F}(z + \square_{n+1})\text{-measurable,}$$

and, for some constants $K, s > 0$,

$$\sup_{z \in 3^n \mathbb{Z}^d} \mathbb{P}[z + \square_n \notin \mathcal{G}] \leq K \exp(-K^{-1} 3^{ns}).$$

Then, \mathbb{P} -almost surely, there exists a partition $\mathcal{S} \subseteq \mathcal{T}$ of \mathbb{Z}^d into triadic cubes with the following properties:

(i) *All predecessors of elements of \mathcal{S} are good: for every $\square, \square' \in \mathcal{T}$,*

$$\square' \subseteq \square \text{ and } \square' \in \mathcal{S} \implies \square \in \mathcal{G}.$$

(ii) *Neighboring elements of \mathcal{S} have comparable sizes: for every $\square, \square' \in \mathcal{S}$ such that $\text{dist}(\square, \square') \leq 1$, we have*

$$\frac{1}{3} \leq \frac{\text{size}(\square')}{\text{size}(\square)} \leq 3.$$

(iii) *Estimate for the coarseness of \mathcal{S} : if we denote by $\square_{\mathcal{S}}(x)$ the unique element of \mathcal{S} containing a point $x \in \mathbb{Z}^d$, then there exists $C(s, K, d) < \infty$ such that, for every $x \in \mathbb{Z}^d$,*

$$\text{size}(\square_{\mathcal{S}}(x)) \leq \mathcal{O}_s(C).$$

(iv) *Minimal scale for \mathcal{S} . For each $t \in [1, \infty)$, there exists $C := C(t, s, K, d) < \infty$, an \mathbb{N} -valued random variable $\mathcal{M}_t(\mathcal{S})$ and exponent $r := r(t, s, K, d) > 0$ such that*

$$\mathcal{M}_t(\mathcal{S}) \leq \mathcal{O}_r(C)$$

and for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_t(\mathcal{S})$,

$$\frac{1}{|\square_m|} \sum_{x \in \square_m} \text{size}(\square_{\mathcal{S}}(x))^t \leq C \quad \text{and} \quad \sup_{x \in \square_m} \text{size}(\square_{\mathcal{S}}(x)) \leq 3^{\frac{dm}{d+t}}.$$

Moreover, a careful investigation of the proof of the proof of [7, Proposition 2.1] shows that the assumption (2.1) is only useful to prove (iv). In particular the following proposition can be extracted from the proof of [7, Proposition 2.1]. It will be useful to define the partition \mathcal{U} in Definition 3.7.

Proposition 2.2 (Proposition 2.1 of [7]). *Let $\mathcal{G} \subseteq \mathcal{T}$ be a random collection of triadic cubes, as above. Suppose that \mathcal{G} satisfies, for every $\square = z + \square_n \in \mathcal{T}$, and, for some constants $K, s > 0$,*

$$\sup_{z \in 3^n \mathbb{Z}^d} \mathbb{P}[z + \square_n \notin \mathcal{G}] \leq K \exp(-K^{-1} 3^{ns}).$$

Then, \mathbb{P} -almost surely, there exists a partition $\mathcal{S} \subseteq \mathcal{T}$ of \mathbb{Z}^d into triadic cubes with the following properties:

(i) *All predecessors of elements of \mathcal{S} are good: for every $\square, \square' \in \mathcal{T}$,*

$$\square' \subseteq \square \text{ and } \square' \in \mathcal{S} \implies \square \in \mathcal{G}.$$

- (ii) *Neighboring elements of \mathcal{S} have comparable sizes: for every $\square, \square' \in \mathcal{S}$ such that $\text{dist}(\square, \square') \leq 1$, we have*

$$\frac{1}{3} \leq \frac{\text{size}(\square')}{\text{size}(\square)} \leq 3.$$

- (iii) *Estimate for the coarseness of \mathcal{S} : if we denote by $\square_{\mathcal{S}}(x)$ the unique element of \mathcal{S} containing a point $x \in \mathbb{Z}^d$, then there exists $C(s, K, d) < \infty$ such that, for every $x \in \mathbb{Z}^d$,*

$$\text{size}(\square_{\mathcal{S}}(x)) \leq \mathcal{O}_s(C).$$

2.1.2. *The partition \mathcal{P} of well-connected cubes.* We apply the construction of the previous subsection to obtain a random partition \mathcal{P} of \mathbb{Z}^d which simplifies the geometry of the percolation cluster. This partition plays an important role in the rest of the paper. For bounds on the “good event” which allows us to construct the partition, we use the important results of Pisztorá [29], Penrose and Pisztorá [28] and Antal and Pisztorá [1]. We first recall some definitions introduced in those works.

Definition 2.3 (Crossability and crossing cluster). We say that a cube \square is *crossable* (with respect to $\mathbf{a} \in \Omega$) if each of the d pairs of opposite $(d-1)$ -dimensional faces of \square is joined by an open path in \square . We say that a cluster $\mathcal{C} \subseteq \square$ is a *crossing cluster* for \square if \mathcal{C} intersects each of the $(d-1)$ -dimensional faces of \square .

Definition 2.4 (Good cube). We say that a triadic cube $\square \in \mathcal{T}$ is *well-connected* if there exists a crossing cluster \mathcal{C} for the cube \square such that:

- (i) each cube \square' with $\text{size}(\square') \in \left[\frac{1}{10} \text{size}(\square), \frac{1}{2} \text{size}(\square)\right]$ and $\square' \cap \frac{3}{4}\square \neq \emptyset$ is crossable; and
- (ii) every path $\gamma \subseteq \square'$ with $\text{diam}(\gamma) \geq \frac{1}{10} \text{size}(\square)$ is connected to \mathcal{C} within \square' .

We say that $\square \in \mathcal{T}$ is a *good cube* if $\text{size}(\square) \geq 3$, \square is well-connected and each of the 3^d successors of \square are well-connected. We say that $\square \in \mathcal{T}$ is a *bad cube* if it is not a good cube.

The following estimate on the probability of the cube \square_n being good is a consequence [29, Theorem 3.2] and [28, Theorem 5], as recalled in [1, (2.24)].

Lemma 2.5 ([1, (2.24)]). *For each $\mathbf{p} \in (\mathbf{p}_c, 1]$, there exists $C(d, \mathbf{p}) < \infty$ such that, for every $m \in \mathbb{N}$,*

$$(2.2) \quad \mathbb{P}[\square_n \text{ is good}] \geq 1 - C \exp(-C^{-1}3^n).$$

It follows from Definition 2.4 that, for every good cube \square , there exists a unique maximal crossing cluster for \square which is contained in \square . We denote this cluster by $\mathcal{C}_*(\square)$. In the next lemma, we record the observation that adjacent triadic cubes which have similar sizes and are both good have connected clusters.

Lemma 2.6 (Lemma 2.8 of [7]). *Let $n, n' \in \mathbb{N}$ with $|n - n'| \leq 1$ and $z, z' \in 3^n \mathbb{Z}^d$ such that*

$$\text{dist}(\square_n(z), \square_{n'}(z')) \leq 1.$$

Suppose also that $\square_n(z)$ and $\square_{n'}(z')$ are good cubes. Then there exists a cluster \mathcal{C} such that

$$\mathcal{C}_*(\square_n(z)) \cup \mathcal{C}_*(\square_{n'}(z')) \subseteq \mathcal{C} \subseteq \square_n(z) \cup \square_{n'}(z').$$

We next define our partition \mathcal{P} .

Definition 2.7. We let $\mathcal{P} \subseteq \mathcal{T}$ be the partition \mathcal{S} of \mathbb{Z}^d obtained by applying Proposition 2.1 to the collection

$$\mathcal{G} := \{\square \in \mathcal{T} : \square \text{ is good}\}.$$

More generally, for each $y \in \mathbb{Z}^d$, we let be $\mathcal{P}_y \subseteq \mathcal{T}$ be the partition \mathcal{S} of \mathbb{Z}^d obtained by applying Proposition 2.1 to the collection

$$\mathcal{G} := \{y + \square : \square \in \mathcal{T} \text{ and } y + \square \text{ is good}\}.$$

From the construction of \mathcal{P} and \mathcal{P}_y , we also have

$$\mathcal{P}_y = y + \mathcal{P}(\tau_{-y}\mathbf{a}) = \{y + \square : \square \in \mathcal{P}(\tau_{-y}\mathbf{a})\}.$$

The (random) partition \mathcal{P} plays an important role throughout the rest of the paper. We also denote by \mathcal{P}_* the collection of triadic cubes which contains some element of \mathcal{P} , that is,

$$\mathcal{P}_* := \{\square : \square \text{ is a triadic cube and } \square \supseteq \square' \text{ for some } \square' \in \mathcal{P}\}.$$

Notice that every element of \mathcal{P}_* can be written in a unique way as a disjoint union of elements of \mathcal{P} . According to Proposition 2.1(i), every triadic cube containing an element of \mathcal{P} is good. By Proposition 2.1(iii) and Lemma 2.5, there exists $C(d, \mathbf{p}) < \infty$ such that, for every $x \in \mathbb{Z}^d$,

$$(2.3) \quad \text{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_1(C).$$

By the properties of \mathcal{P} given in Proposition 2.1(i) and (ii) and Lemma 2.6, the maximal crossing cluster $\mathcal{C}_*(\square)$ of an element $\square \in \mathcal{P}_*$ must satisfy $\mathcal{C}_*(\square) \subseteq \mathcal{C}_\infty$, since the union of all crossing clusters of elements of \mathcal{P} is unbounded and connected. Notice also that, although we may not have $\mathcal{C}_*(\square) = \mathcal{C}_\infty \cap \square$, by definition of the partition \mathcal{P} and (ii) of Definition 2.4, we have that, for every cube $\square \in \mathcal{P}$, there exists a cluster \mathcal{C} such that

$$(2.4) \quad \mathcal{C}_\infty \cap \square \subseteq \mathcal{C} \subseteq \bigcup_{\square' \in \mathcal{P}, \text{dist}(\square, \square') \leq 1} \square'.$$

In other words, for any cube $\square \in \mathcal{P}$ and every $x, y \in \mathcal{C}_\infty \cap \square$, there exists a path linking x to y staying in \square or in its neighbors.

It is also interesting to note that, for $m \in \mathbb{N}$ such that $3^m \geq \mathcal{M}_{2d}(\mathcal{P})$, $\mathcal{C}_*(\square_m)$, $\mathcal{C}_\infty \cap \square_m$ and \square_m are of comparable size, precisely, there exists a constant $C := C(d, \mathbf{p}) < \infty$ such that

$$(2.5) \quad C^{-1}|\square_m| \leq |\mathcal{C}_*(\square_m)| \leq |\mathcal{C}_\infty \cap \square_m| \leq |\square_m|.$$

This result is a consequence of the Cauchy-Schwarz inequality and the three relations, under the assumption $3^m \geq \mathcal{M}_{2d}(\mathcal{P})$, which implies in particular that \square_m is good,

$$\sum_{\square \in \mathcal{P}, \square \subseteq \square_m} 1 \leq |\mathcal{C}_*(\square_m)|, \quad \sum_{\square \in \mathcal{P}, \square \subseteq \square_m} \text{size}(\square_{\mathcal{P}})^d = |\square_m| \quad \text{and} \quad \sum_{\square \in \mathcal{P}, \square \subseteq \square_m} \text{size}(\square_{\mathcal{P}})^{2d} \leq C|\square_m|.$$

The first inequality comes from the fact that each cube of \mathcal{P} contained in \square_m must have non-empty intersection with $\mathcal{C}_*(\square_m)$, the second is the preservation of the volume and the third is where we use the assumption $3^m \geq \mathcal{M}_{2d}(\mathcal{P})$.

Given $\square \in \mathcal{P}$, we let $\bar{z}(\square)$ denote the element of $\mathcal{C}_*(\square) \cap \square_n(z)$ which is closest to z in the Manhattan distance; if this is not unique, then we break ties by the lexicographical order.

Definition 2.8. Given a function $u : \mathcal{C}_\infty \rightarrow \mathbb{R}$, we define *the coarsened function with respect to \mathcal{P}* to be

$$[u]_{\mathcal{P}} : \mathbb{Z}^d \rightarrow \mathbb{R} \\ x \mapsto u(\bar{z}(\square_{\mathcal{P}}(x))).$$

The reason we use the coarsened function is that it is defined on the entire lattice \mathbb{Z}^d and not on the infinite cluster. This allows to make use of the simpler and more favorable geometric structure of \mathbb{Z}^d . The price to pay is the difference between u and $[u]_{\mathcal{P}}$. This depends on the coarseness of the partition \mathcal{P} and the control one has on ∇u in a way that is made precise in the following proposition. The dependence on the coarseness of \mathcal{P} is present via the size of the cubes of the partition. Recall that the notation $|F|(x)$ for a vector field F is defined in (1.6).

Proposition 2.9 (Lemma 3.2 of [7]). *For every triadic cube $\square \in \mathcal{P}_*$, $1 \leq s < \infty$ and $w : \mathcal{C}_\infty \rightarrow \mathbb{R}$,*

$$(2.6) \quad \int_{\mathcal{C}_*(\square)} |w(x) - [w]_{\mathcal{P}}(x)|^s dx \leq C^s \int_{\mathcal{C}_*(\square)} \text{size}(\square_{\mathcal{P}}(x))^{sd} |\nabla w|^s(x) dx.$$

More generally, for any family of disjoint cubes $\{\square_i\}_{i \in I} \in (\mathcal{P}_)^I$, we have*

$$(2.7) \quad \int_{\mathcal{C}_*(\cup_{i \in I} \square_i)} |w(x) - [w]_{\mathcal{P}}(x)|^s dx \leq C^s \int_{\mathcal{C}_*(\cup_{i \in I} \square_i)} \text{size}(\square_{\mathcal{P}}(x))^{sd} |\nabla w|^s(x) dx,$$

where $\mathcal{C}_(\cup_{i \in I} \square_i)$ denotes the union of the maximal clusters of each connected component of $\cup_{i \in I} \square_i$.*

Remark 2.10. Unfortunately, we do not have $\mathcal{C}_*(\cup_{i \in I} \square_i) = \cup_{i \in I} \mathcal{C}_*(\square_i)$. The problem is the same than the one we had in (2.4) and thus (2.7) can not be directly obtained from (2.6). Nevertheless, thanks to this equation, we do have the inclusion

$$(2.8) \quad \mathcal{C}_\infty \cap \square \subseteq \mathcal{C} \left(\bigcup_{\substack{\square' \in \mathcal{P}, \\ \text{dist}(\square, \square') \leq 1}} \square' \right).$$

Moreover we can control the L^s norm of the vector field $\nabla[w]_{\mathcal{P}}$ depending on the L^s norm of ∇w and the coarseness of the partition \mathcal{P} thanks to the following Proposition.

Proposition 2.11 (Lemma 3.3 of [7]). *For every triadic cube $\square \in \mathcal{P}_*$, $1 \leq s < \infty$ and $w : \mathcal{C}_\infty \rightarrow \mathbb{R}$,*

$$(2.9) \quad \int_{\mathcal{C}_*(\square)} |\nabla[w]_{\mathcal{P}}|^s(x) dx \leq C^s \int_{\mathcal{C}_*(\square)} \text{size}(\square_{\mathcal{P}}(x))^{sd-1} |\nabla w|^s(x) dx.$$

More generally, for any family of disjoint cubes $\{\square_i\}_{i \in I} \in (\mathcal{P}_*)^I$, we have

$$(2.10) \quad \int_{\mathcal{C}_*(\cup_{i \in I} \square_i)} |\nabla[w]_{\mathcal{P}}|^s(x) dx \leq C^s \int_{\mathcal{C}_*(\cup_{i \in I} \square_i)} \text{size}(\square_{\mathcal{P}}(x))^{sd-1} |\nabla w|^s(x) dx.$$

2.2. Elliptic inequalities on the supercritical percolation cluster. In this section, we record some simple elliptic inequalities, the Caccioppoli inequality and the Meyers estimate. These inequalities were written in [7] for harmonic functions. In our context, we need to apply these results when the right-hand term is not 0 but the divergence of a vector field. The inequalities are consequently written in this more general setting.

Proposition 2.12 (Caccioppoli inequality). *Assume that we are given a function $u : \mathcal{C}_\infty \rightarrow \mathbb{R}$ and a vector field $\xi : E_d \rightarrow \mathbb{R}$ satisfying the following condition*

$$(2.11) \quad \xi(x, y) = 0 \text{ if } \mathbf{a}(x, y) = 0 \text{ or } x, y \notin \mathcal{C}_\infty.$$

In particular, gradients of functions defined on the infinite cluster satisfy this condition by (1.5). Assume additionally that u and ξ satisfy the following equation,

$$-\nabla \cdot (\mathbf{a} \nabla u) = -\nabla \cdot \xi \text{ in } \mathcal{C}_\infty.$$

Select two open bounded sets $U, V \subseteq \mathbb{R}^d$ such that $V \subseteq U$ and $\text{dist}(V, \partial U) \geq r \geq 1$. Then there exists $C(\lambda) < \infty$ such that

$$(2.12) \quad \int_{\mathcal{C}_\infty \cap V} |\nabla u|^2(x) dx \leq \frac{C}{r^2} \int_{\mathcal{C}_\infty \cap (U \setminus V)} |u(x)|^2 dx + C \int_{\mathcal{C}_\infty \cap U} |\xi|^2(x) dx.$$

Remark 2.13. This version of the Caccioppoli inequality is more general than the one proved in [7, Lemma 3.5], since a divergence form right-hand term was added. This extra term will be useful in the proof of the Meyers estimate, Proposition 2.14, which will be an important ingredient in the proof of Theorem 1.

Proof. Select a function $\eta : \mathbb{Z}^d \rightarrow \mathbb{R}$ satisfying

$$(2.13) \quad \mathbf{1}_V \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } \mathbb{R}^d \setminus U, \quad \text{and } \forall x, y \in \mathbb{Z}^d \text{ such that } x \sim y, \quad |\eta(x) - \eta(y)|^2 \leq \frac{C(\eta(x) + \eta(y))}{r^2}.$$

We also denote by $E_V^{\mathbf{a}} := \{(x, y) \in E_d : x, y \in \mathcal{C}_\infty \cap U \text{ and } \mathbf{a}(x, y) \neq 0\}$. Testing the equation satisfied by u against ηu yields

$$\begin{aligned} & \sum_{(x, y) \in E_V^{\mathbf{a}}} (\eta(x)u(x) - \eta(y)u(y)) \xi(x, y) \\ &= \sum_{(x, y) \in E_V^{\mathbf{a}}} (\eta(x)u(x) - \eta(y)u(y)) \mathbf{a}(\{x, y\}) (u(x) - u(y)) \end{aligned}$$

But we have

$$\begin{aligned} \sum_{(x,y) \in E_U^{\mathbf{a}}} (\eta(x)u(x) - \eta(y)u(y)) \xi(x,y) \\ = \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) (u(x) - u(y)) \xi(x,y) + \sum_{(x,y) \in E_U^{\mathbf{a}}} u(y) (\eta(x) - \eta(y)) \xi(x,y) \end{aligned}$$

and

$$\begin{aligned} \sum_{(x,y) \in E_U^{\mathbf{a}}} (\eta(x)u(x) - \eta(y)u(y)) \mathbf{a}(\{x,y\}) (u(x) - u(y)) \\ = \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) (u(x) - u(y)) \mathbf{a}(\{x,y\}) (u(x) - u(y)) \\ + \sum_{(x,y) \in E_U^{\mathbf{a}}} u(y) (\eta(x) - \eta(y)) \mathbf{a}(\{x,y\}) (u(x) - u(y)). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) \mathbf{a}(\{x,y\}) (u(x) - u(y))^2 \\ \leq \sum_{(x,y) \in E_U^{\mathbf{a}}} |u(y)| |\eta(x) - \eta(y)| \mathbf{a}(\{x,y\}) |u(x) - u(y)| \\ + \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) |u(x) - u(y)| |\xi(x,y)| \\ + \sum_{(x,y) \in E_U^{\mathbf{a}}} |u(y)| |\eta(x) - \eta(y)| |\xi(x,y)|. \end{aligned}$$

We first estimate the first term in the right-hand side according to the following computation

$$\begin{aligned} \sum_{(x,y) \in E_U^{\mathbf{a}}} |u(y)| |\eta(x) - \eta(y)| \mathbf{a}(\{x,y\}) (u(x) - u(y)) \\ \leq C \sum_{(x,y) \in E_U^{\mathbf{a}}} \frac{|\eta(x) - \eta(y)|^2}{\eta(x) + \eta(y)} |u(y)|^2 \\ + \frac{1}{4} \sum_{(x,y) \in E_U^{\mathbf{a}}} (\eta(x) + \eta(y)) \mathbf{a}(\{x,y\})^2 (u(x) - u(y))^2 \\ \leq \frac{C}{r^2} \sum_{(x,y) \in E_U^{\mathbf{a}}} \mathbb{1}_{\{\eta(x) \neq \eta(y)\}} (u(y))^2 + \frac{1}{2} \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) \mathbf{a}(\{x,y\})^2 (u(x) - u(y))^2. \end{aligned}$$

The second term in the right-hand side can be estimated similarly, using the assumption $\mathbf{a} \geq \lambda \mathbb{1}_{\{\mathbf{a} \neq 0\}}$,

$$\begin{aligned} \sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) (u(x) - u(y)) \xi(x,y) \\ \leq \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) (u(x) - u(y))^2 \right)^{\frac{1}{2}} \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) |\xi(x,y)|^2 \right)^{\frac{1}{2}} \\ \leq \frac{1}{\sqrt{\lambda}} \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) \mathbf{a}(\{x,y\}) (u(x) - u(y))^2 \right)^{\frac{1}{2}} \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) |\xi(x,y)|^2 \right)^{\frac{1}{2}} \\ \leq C \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} \eta(x) \mathbf{a}(\{x,y\}) (u(x) - u(y))^2 \right)^{\frac{1}{2}} \left(\sum_{(x,y) \in E_U^{\mathbf{a}}} |\xi(x,y)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We then estimate the third term in the right-hand side similarly

$$\begin{aligned}
\sum_{(x,y) \in E_V^{\mathbf{a}}} |u(y)| |\eta(x) - \eta(y)| |\xi(x,y)| &\leq \frac{1}{2} \sum_{(x,y) \in E_V^{\mathbf{a}}} \frac{|\eta(x) - \eta(y)|^2}{\eta(x) + \eta(y)} |u(y)|^2 \\
&\quad + \frac{1}{2} \sum_{(x,y) \in E_V^{\mathbf{a}}} (\eta(x) + \eta(y)) \xi(x,y)^2 \\
&\leq \frac{C}{r^2} \sum_{(x,y) \in E_V^{\mathbf{a}}} \mathbf{1}_{\{\eta(x) \neq \eta(y)\}} (u(y))^2 \\
&\quad + \sum_{(x,y) \in E_V^{\mathbf{a}}} \xi(x,y)^2.
\end{aligned}$$

Combining the previous displays and denoting $X := \left(\sum_{(x,y) \in E_V^{\mathbf{a}}} \eta(x) \mathbf{a}(\{x,y\}) (u(x) - u(y))^2 \right)^{\frac{1}{2}}$, we obtain the inequality

$$X^2 \leq C \left(\sum_{(x,y) \in E_V^{\mathbf{a}}} |\xi(x,y)|^2 \right)^{\frac{1}{2}} X + \frac{C}{r^2} \sum_{(x,y) \in E_V^{\mathbf{a}}} \mathbf{1}_{\{\eta(x) \neq \eta(y)\}} (u(y))^2 + \sum_{(x,y) \in E_V^{\mathbf{a}}} |\xi(x,y)|^2.$$

This implies

$$X^2 \leq \frac{C}{r^2} \sum_{(x,y) \in E_V^{\mathbf{a}}} \mathbf{1}_{\{\eta(x) \neq \eta(y)\}} (u(y))^2 + \sum_{(x,y) \in E_V^{\mathbf{a}}} |\xi(x,y)|^2.$$

We obtain (2.12) after rewriting the previous inequality, using $\mathbf{a} \geq \lambda \mathbf{1}_{\{\mathbf{a} \neq 0\}}$ and (2.13). \square

The second important elliptic estimate needed in this article is the Meyers estimate. This estimate was also proved in [7] in the case of \mathbf{a} -harmonic functions but is needed in the proof of Theorem 1 in the more general case when the right-hand term is the divergence of a vector field.

Proposition 2.14 (Meyers estimate). *There exist a constant $C := C(d, \lambda, \mathbf{p}) < \infty$, two exponents $s := s(d, \lambda, \mathbf{p}) > 0$ and $\varepsilon := \varepsilon(d, \lambda, \mathbf{p}) > 0$ and a random variable $\mathcal{M}_{\text{Meyers}} \leq \mathcal{O}_s(C)$ such that for each $m \in \mathbb{N}$ with $3^m \geq \mathcal{M}_{\text{Meyers}}$, and each function $v : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying*

$$-\nabla \cdot (\mathbf{a} \nabla v) = -\nabla \cdot \xi \text{ in } \mathcal{C}_\infty,$$

for some vector field $\xi : E_d \rightarrow \mathbb{R}$ satisfying (2.11), the following estimate holds,

$$\begin{aligned}
(2.14) \quad &\left(\frac{1}{|\square_m|} \int_{\square_m \cap \mathcal{C}_\infty} |\nabla v|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}} \\
&\leq C \left(\frac{1}{|\frac{4}{3}\square_m|} \int_{\frac{4}{3}\square_m \cap \mathcal{C}_\infty} |\nabla v|^2(x) dx \right)^{\frac{1}{2}} + C \left(\frac{1}{|\frac{4}{3}\square_m|} \int_{\frac{4}{3}\square_m \cap \mathcal{C}_\infty} |\xi|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}}.
\end{aligned}$$

Proof of Proposition 2.14. The results of Proposition 3.8 and Definition 3.9 of [7] can be adapted in our context to prove the following result: there exist a constant $C := C(d, \lambda, \mathbf{p}) < \infty$, two exponents $s := s(d, \lambda, \mathbf{p}) > 0$ and $\varepsilon := \varepsilon(d, \lambda, \mathbf{p}) > 0$ and a random variable $\mathcal{M} \leq \mathcal{O}_s(C)$ such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}$, each function $v : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying

$$-\nabla \cdot (\mathbf{a} \nabla v) = -\nabla \cdot \xi \text{ in } \mathcal{C}_\infty$$

for some vector field $\xi : E_d \rightarrow \mathbb{R}$ satisfying (2.11), the following estimate holds,

$$(2.15) \quad \left(\frac{1}{|\square_m|} \int_{\square_m \cap \mathcal{C}_* \left(\frac{4}{3} \square_m \right)} |\nabla v|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}} \\ \leq C \left(\frac{1}{\left| \frac{4}{3} \square_m \right|} \int_{\mathcal{C}_* \left(\frac{4}{3} \square_m \right)} |\nabla v|^2(x) dx \right)^{\frac{1}{2}} + C \left(\frac{1}{\left| \frac{4}{3} \square_m \right|} \int_{\mathcal{C}_* \left(\frac{4}{3} \square_m \right)} |\xi|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}}$$

where $\mathcal{C}_* \left(\frac{4}{3} \square_m \right)$ denotes the largest cluster included in $\frac{4}{3} \square_m$. The Meyers estimate is indeed a consequence of the three following ingredients: the Caccioppoli inequality, the Sobolev inequality and the Gehring lemma. But Proposition 2.12 provides a version of the Caccioppoli inequality well-suited to deal with a divergence form right-hand side. The Sobolev inequality is valid for any function (and not simply for \mathbf{a} -harmonic functions). The usual version of the Gehring Lemma, see for instance Theorem 6.6 & Corollary 6.1 of [15], is general enough to be applied in our context.

We now show how to pass from (2.15) to (2.14). By Proposition 2.1 and (2.4), there exists an exponent $t := t(d) < \infty$ such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_t(\mathcal{P})$, we have

$$\square \cap \mathcal{C}_* \left(\frac{4}{3} \square_m \right) = \mathcal{C}_\infty \cap \square_m.$$

This also gives

$$\mathcal{C}_* \left(\frac{4}{3} \square_m \right) \subseteq \mathcal{C}_\infty \cap \frac{4}{3} \square_m.$$

Thus, if we set $\mathcal{M}_{\text{Meyers}} = \max(\mathcal{M}, \mathcal{M}_t(\mathcal{P})) \leq \mathcal{O}_s(C)$, we have, for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_{\text{Meyers}}$,

$$\left(\frac{1}{|\square_m|} \int_{\square_m \cap \mathcal{C}_\infty} |\nabla v|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}} \\ \leq C \left(\frac{1}{\left| \frac{4}{3} \square_m \right|} \int_{\frac{4}{3} \square_m \cap \mathcal{C}_\infty} |\nabla v|^2(x) dx \right)^{\frac{1}{2}} + C \left(\frac{1}{\left| \frac{4}{3} \square_m \right|} \int_{\frac{4}{3} \square_m \cap \mathcal{C}_\infty} |\xi|^{2+\varepsilon}(x) dx \right)^{\frac{1}{2+\varepsilon}},$$

which is the desired estimate. The proof of Lemma 2.14 is complete. \square

2.3. Solving the Poisson equation with divergence form source term. In this section we study the existence and uniqueness of the equation $-\nabla \cdot \mathbf{a} \nabla u = \nabla \cdot \xi$ on the infinite cluster \mathcal{C}_∞ . Recall the notation $E_d^{\mathbf{a}}$ introduced in the proof of the Caccioppoli inequality, Proposition 2.12. We then denote by

$$E_d^{\mathbf{a}} := \{(x, y) \in E_d : x, y \in \mathcal{C}_\infty \text{ and } \mathbf{a}(x, y) \neq 0\} = E_{\mathbb{Z}^d}^{\mathbf{a}}.$$

The results of this section can be summarized in the two following propositions.

Proposition 2.15 (Gradient of Green's function). *Let $\mathbf{a} \in \Omega$ an environment in a set of probability 1 such that there exists a unique maximal infinite cluster. Let $e = (x, y)$ be an edge of $E_d^{\mathbf{a}}$, there exist a constant $C := C(d, \lambda) < \infty$ and a function $G^e : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying*

$$(2.16) \quad \sup_{e' \in E_d^{\mathbf{a}}} |\nabla G^e(e')| \leq C \quad \text{and} \quad \langle \nabla G^e, \nabla G^e \rangle_{\mathcal{C}_\infty} \leq C,$$

solution of the equation

$$-\nabla \cdot \mathbf{a} \nabla G^e = \delta_x - \delta_y \text{ in } \mathcal{C}_\infty,$$

in the sense that for each function $h : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying $\langle \nabla h, \nabla h \rangle_{\mathcal{C}_\infty} < \infty$, we have

$$\langle \nabla G^e, \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \nabla h(e).$$

Moreover, we have, for each $e, e' \in E_d^{\mathbf{a}}$,

$$(2.17) \quad \nabla G^e(e') = \nabla G^{e'}(e).$$

Proposition 2.16. *Let $\mathbf{a} \in \Omega$ an environment in a set of probability 1 such that there exists a unique maximal infinite cluster. Let $\xi : E_d \rightarrow \mathbb{R}$ be a vector field satisfying*

$$(2.18) \quad \xi(x, y) = 0 \text{ if } \mathbf{a}(x, y) \neq 0 \text{ or } x, y \notin \mathcal{C}_\infty.$$

If ξ satisfies $\langle \xi, \xi \rangle_{\mathcal{C}_\infty} < \infty$ then there exists a unique (up to a constant) solution w_ξ of

$$-\nabla \cdot \mathbf{a} \nabla w_\xi = -\nabla \cdot \xi \text{ in } \mathcal{C}_\infty,$$

in the sense that for each function $h : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying $\langle \nabla h, \nabla h \rangle_{\mathcal{C}_\infty} < \infty$, we have

$$\langle \nabla w_\xi, \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \langle \xi, \nabla h \rangle_{\mathcal{C}_\infty}.$$

Additionally, we have the equality

$$(2.19) \quad \nabla w_\xi(\cdot) = \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla G^e(\cdot).$$

Proof of Proposition 2.15. In this proof, we denote by \dot{H}^1 the space of functions defined on the infinite cluster whose gradient is L^2 , i.e

$$\dot{H}^1 := \{u : \mathcal{C}_\infty \rightarrow \mathbb{R} : \langle \nabla u, \nabla u \rangle_{\mathcal{C}_\infty} < \infty\},$$

and look at the minimization problem

$$\inf_{u \in \dot{H}^1} \frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathcal{C}_\infty} - \nabla u(e).$$

Let u_n be a minimizing sequence. For this sequence, we have,

$$\langle \nabla u_n, \nabla u_n \rangle_{\mathcal{C}_\infty} \leq C.$$

By a diagonal extraction argument, one can assume that for each edge e' in \mathcal{C}_∞ , $\nabla u_n(e')$ converges to a limit denoted by $F(e')$. By integrating on loops of the cluster, one can see that the vector field $F(e')$ is in fact a gradient vector-field thus there exists a function $G^e : \mathcal{C}_\infty \rightarrow \mathbb{R}$ such that, for each $e' \in E_d(\mathcal{C}_\infty)$,

$$\nabla u_n(e') \xrightarrow{n \rightarrow \infty} \nabla G^e(e').$$

By the previous result and Fatou's Lemma, we obtain

$$\frac{1}{2} \langle \nabla G^e, \mathbf{a} \nabla G^e \rangle_{\mathcal{C}_\infty} - \nabla G^e(e) = \inf_{u \in \dot{H}^1} \frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathcal{C}_\infty} - \nabla u(e)$$

and

$$\langle \nabla G^e, \nabla G^e \rangle_{\mathcal{C}_\infty} \leq C.$$

By the first variations, we have, for each $h \in \dot{H}^1$,

$$\langle \nabla G^e, \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \nabla h(e).$$

To prove (2.17), test the function G^e against the function $G^{e'}$, this yields

$$\nabla G^{e'}(e) = \langle \nabla G^e, \mathbf{a} \nabla G^{e'} \rangle_{\mathcal{C}_\infty} = \nabla G^e(e').$$

This also yields, by the Cauchy-Schwarz inequality

$$|\nabla G^{e'}(e)| \leq (\langle \nabla G^e, \nabla G^e \rangle_{\mathcal{C}_\infty})^{\frac{1}{2}} \left(\langle \nabla G^e, \nabla G^{e'} \rangle_{\mathcal{C}_\infty} \right)^{\frac{1}{2}} \leq C.$$

The proof of Proposition 2.15 is complete. \square

Proof of Proposition 2.16. The first part of the proof of Proposition 2.16 is similar to the proof of Proposition 2.15 and left to the reader. We only prove (2.19). First note that the right-hand side of (2.19) is well defined. Indeed, we have, for each $e' \in E_d^{\mathbf{a}}$,

$$\sum_{e \in E_d^{\mathbf{a}}} |\xi(e) \nabla G^e(e')| = \sum_{e \in E_d^{\mathbf{a}}} |\xi(e) \nabla G^{e'}(e)| \leq \left(\sum_{e \in E_d^{\mathbf{a}}} |\xi(e)|^2 \right)^{\frac{1}{2}} \left(\sum_{e \in E_d^{\mathbf{a}}} |\nabla G^{e'}(e)|^2 \right)^{\frac{1}{2}} < \infty.$$

Then note that, by integrating on loops of the cluster, the right hand side of (2.19) is a gradient vector-field, thus there exists a function $f : \mathcal{C}_\infty \rightarrow \mathbb{R}$ such that $\nabla f(\cdot) = \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla G^e(\cdot)$.

We first make the extra assumption (which will be removed later)

$$\sum_{e \in E_d^{\mathbf{a}}} |\xi(e)| < \infty.$$

With this extra assumption, we compute, for each vector field $\gamma : E_d \rightarrow \mathbb{R}$,

$$\begin{aligned} \sum_{e \in E_d^{\mathbf{a}}} |\nabla f(e) \mathbf{a}(e) \gamma(e)| &\leq \sum_{e, e' \in E_d^{\mathbf{a}}} |\xi(e') \nabla G^e(e') \gamma(e)| \\ &\leq \sum_{e' \in E_d^{\mathbf{a}}} |\xi(e')| \left(\sum_{e \in E_d^{\mathbf{a}}} |\nabla G^e(e')|^2 \right)^{\frac{1}{2}} \left(\sum_{e \in E_d^{\mathbf{a}}} |\gamma(e)|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{e' \in E_d^{\mathbf{a}}} |\xi(e')| \left(\sum_{e \in E_d^{\mathbf{a}}} |\nabla G^{e'}(e)|^2 \right)^{\frac{1}{2}} \left(\sum_{e \in E_d(\mathcal{C}_\infty)} |\gamma(e)|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{e' \in E_d^{\mathbf{a}}} |\xi(e')| \right) \left(\sum_{e \in E_d^{\mathbf{a}}} |\gamma(e)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

In particular, we deduce from this computation that $f \in \dot{H}^1$ and

$$\left(\sum_{e \in E_d^{\mathbf{a}}} |\nabla f(e)|^2 \right)^{\frac{1}{2}} \leq C \left(\sum_{e' \in E_d^{\mathbf{a}}} |\xi(e')| \right).$$

Then by Fubini's Theorem

$$\langle \nabla f, \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \left\langle \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla G^e, \mathbf{a} \nabla h \right\rangle_{\mathcal{C}_\infty} = \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \langle \nabla G^e, \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla h(e).$$

By uniqueness, we obtain

$$\nabla w_\xi = \nabla f = \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla G^e(e').$$

We then remove the assumption $\sum_{e \in E_d^{\mathbf{a}}} |\xi(e)| < \infty$. To do so, let $\xi : E_d \rightarrow \mathbb{R}$ be a vector field satisfying (2.18) such that $\sum_{e \in E_d^{\mathbf{a}}} |\xi(e)|^2 < \infty$ and let ξ_n be a sequence of vector fields satisfying (2.18) such that

$$\sum_{e \in E_d^{\mathbf{a}}} |\xi_n(e)| < \infty \quad \text{and} \quad \sum_{e \in E_d^{\mathbf{a}}} |\xi_n(e) - \xi(e)|^2 \xrightarrow{n \rightarrow \infty} 0.$$

But we have

$$\begin{aligned} \sum_{e \in E_d^{\mathbf{a}}} |(\xi(e) - \xi_n(e)) \nabla G^e(e')| &= \sum_{e \in E_d^{\mathbf{a}}} |(\xi(e) - \xi_n(e)) \nabla G^{e'}(e)| \\ &\leq \left(\sum_{e \in E_d^{\mathbf{a}}} |\xi(e) - \xi_n(e)|^2 \right)^{\frac{1}{2}} \left(\sum_{e \in E_d^{\mathbf{a}}} |\nabla G^{e'}(e)|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{e \in E_d^{\mathbf{a}}} |\xi(e) - \xi_n(e)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus for each $e' \in \langle \nabla G^e, \mathbf{a} \nabla G^{e'} \rangle_{\mathcal{C}_\infty}$,

$$\nabla w_{\xi_n}(e') = \sum_{e \in E_d^{\mathbf{a}}} \xi_n(e) \nabla G^e(e') \xrightarrow{n \rightarrow \infty} \sum_{e \in E_d^{\mathbf{a}}} \xi(e) \nabla G^e(e').$$

To complete the proof, we show that for each $e' \in E_d^{\mathbf{a}}$,

$$\nabla w_{\xi_n}(e') \xrightarrow{n \rightarrow \infty} \nabla w_\xi(e').$$

More precisely, we show that

$$(2.20) \quad \sum_{e' \in E_d^{\mathbf{a}}} (\nabla w_\xi(e') - \nabla w_{\xi_n}(e'))^2 \xrightarrow{n \rightarrow \infty} 0.$$

To prove this note that for each function $h \in \dot{H}^1$,

$$\langle \nabla(w_\xi - w_{\xi_n}), \mathbf{a} \nabla h \rangle_{\mathcal{C}_\infty} = \langle \xi - \xi_n, \nabla h \rangle_{\mathcal{C}_\infty}.$$

This allows the following computation

$$\begin{aligned} \sum_{e' \in E_d^{\mathbf{a}}} (\nabla w_\xi(e') - \nabla w_{\xi_n}(e'))^2 &\leq C \langle \nabla(w_\xi - w_{\xi_n}), \mathbf{a} \nabla(w_\xi - w_{\xi_n}) \rangle_{\mathcal{C}_\infty} \\ &\leq C \langle \xi - \xi_n, \nabla w_\xi - \nabla w_{\xi_n} \rangle_{\mathcal{C}_\infty} \\ &\leq C \left(\sum_{e' \in E_d^{\mathbf{a}}} (\nabla w_\xi(e') - \nabla w_{\xi_n}(e'))^2 \right)^{\frac{1}{2}} \left(\sum_{e' \in E_d^{\mathbf{a}}} (\xi(e') - \xi_n(e'))^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This implies (2.20) and completes the proof of Proposition 2.16. \square

2.4. Regularity theory. In this subsection, we record a result from the regularity theory established in [7] giving a Lipschitz bound for the gradient of \mathbf{a} -harmonic functions. This result is only a small part of the regularity theory established in [7, Theorem 2], but is the only result needed in the proof of Theorems 1 and 2.

Proposition 2.17 (Regularity theory). *There exist a constant $C := C(d, \mathbf{p}, \lambda) > 0$, an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a random variable \mathcal{X} satisfying*

$$(2.21) \quad \mathcal{X} \leq \mathcal{O}_s(C),$$

such that for each $u : \mathcal{C}_\infty \mapsto \mathbb{R}$ solution of the equation

$$(2.22) \quad -\nabla \cdot \mathbf{a} \nabla u = 0$$

and each $R \geq r \geq \mathcal{X}$, we have

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}.$$

We also introduce the notation, for each $x \in \mathbb{Z}^d$

$$\mathcal{X}(x) := \mathcal{X} \circ \tau_x.$$

This proposition is much weaker than Theorem 2 of [7], it is indeed a consequence of the Caccioppoli inequality and Theorem 2 (iii) of [7] for $k = 0$. For the sake of completeness, we write down how to deduce this proposition from the two tools we just mentioned.

Proof. First, Theorem 2 (iii) of [7] for $k = 0$ tells us that there exists a random variable \mathcal{X} satisfying (2.21) such that for every $u : \mathcal{C}_\infty \mapsto \mathbb{R}$ solution of (2.22) and each $R \geq r \geq \mathcal{X}$,

$$(2.23) \quad \inf_{c \in \mathbb{R}} \|u - c\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \frac{r}{R} \inf_{c \in \mathbb{R}} \|u - c\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}$$

We then estimate the term on the right-hand side by the Caccioppoli inequality (Proposition 2.12 with $\xi = 0$), this yields

$$(2.24) \quad \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq \frac{C}{r} \inf_{c \in \mathbb{R}} \|u - c\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{2r})}.$$

We then estimate the second term on the right-hand side. The idea is to apply the Poincaré inequality, unfortunately due to the non Euclidean structure of the infinite percolation cluster, this inequality cannot be directly applied. Instead, we apply the Sobolev inequality, Proposition 3.4 of [7], to obtain

$$\begin{aligned} \inf_{c \in \mathbb{R}} \|u - c\|_{L^2(\mathcal{C}_\infty \cap B_R)} &= \|u - (u)_{\mathcal{C}_\infty \cap B_R}\|_{L^2(\mathcal{C}_\infty \cap B_R)} \\ &\leq C \left(\sum_{\square' \in \mathcal{P}, \square' \cap B_R \neq \emptyset} \text{size}(\square')^{2d} \int_{\square' \cap \mathcal{C}_\infty} |\nabla u|^{\frac{2d}{d+2}}(x) dx \right)^{\frac{d+2}{2d}}. \end{aligned}$$

The inequality presented in Proposition 3.4 of [7] is written for triadic cubes, but can be easily adapted to our context by making \mathcal{X} larger so that B_R contains a cube of \mathcal{P} if necessary. In view of (2.3), this extra assumption does not impact the condition (2.21).

We then apply the Hölder inequality with exponents $p = \frac{d+2}{2}$ and its Hölder conjugate $q = \frac{d+2}{d}$ to the previous computation to obtain

$$\inf_{c \in \mathbb{R}} \|u - c\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)} \leq C \left(\sum_{\square' \in \mathcal{P}, \square' \cap B_R \neq \emptyset} \text{size}(\square')^{3pd+d} \right)^{\frac{1}{d}} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}.$$

Without loss of generality one can assume $\mathcal{X} \geq \mathcal{M}_{3pd}(\mathcal{P})$, with this assumption, we have

$$\left(\sum_{\square' \in \mathcal{P}, \square' \cap B_R \neq \emptyset} \text{size}(\square')^{3pd+d} \right)^{\frac{1}{d}} \leq CR.$$

Combining the previous displays yields

$$(2.25) \quad \inf_{c \in \mathbb{R}} \|u - c\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)} \leq CR \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}.$$

A combination of (2.23), (2.24) and (2.25) shows for each $R \geq 2r \geq \mathcal{X}$

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}.$$

But for each $2r \geq R \geq r$, we have

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq 2^{\frac{d}{2}} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}.$$

Combining the two previous displays shows for each $R \geq r \geq \mathcal{X}$,

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R)}$$

and the proof is complete. \square

We now present the two main tools to prove Theorem 1. The first one is a concentration inequality, thanks to which we can obtain some quantitative control on the spatial average of the gradient at scale R , cf Proposition 3.3. We then deduce Theorem 1 from Proposition 3.3 thanks to the Poincaré inequality.

2.5. Concentration inequality for stretched exponential moments. The following concentration inequality is the key point in the proof of Proposition 3.3 in the next section. A proof of this inequality can be found in [6, Proposition 2.2].

Proposition 2.18 (Proposition 2.2 of [6]). *Fix $\beta \in (0, 2)$. Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and set for each $e \in \mathcal{B}_d(\mathbb{Z}^d)$,*

$$X'_e = \mathbb{E}[X | \mathcal{F}(\mathcal{B}_d \setminus \{e\})] \text{ and } \mathbb{V}[X] = \sum_{e \in \mathcal{B}_d} (X - X'_e)^2,$$

then there exists $C := C(d, \beta) < \infty$ such that

$$\mathbb{E} \left[\exp \left(|X - \mathbb{E}[X]|^\beta \right) \right] \leq C \mathbb{E} \left[\exp \left((C \mathbb{V}[X])^{\frac{2\beta}{2-\beta}} \right) \right].$$

2.6. Multiscale Poincaré inequality. The next proposition is a version of the multiscale Poincaré. It controls the oscillation of a function in the L^q norm (left-hand term of (2.26)) by the spatial average of the gradient of the function (right-hand term of (2.26)).

Proposition 2.19 (Multiscale Poincaré inequality, heat kernel version). *For each $r > 0$, we define*

$$\begin{aligned} \Phi_r &: \mathbb{R}^d \rightarrow \mathbb{R} \\ x &\mapsto r^{-d} \exp \left(-\frac{|x|^2}{r^2} \right). \end{aligned}$$

For each $q \geq 1$, there exists a constant $C := C(d, q) < \infty$ such that for each tempered distribution $u \in W_{\text{loc}}^{1,q}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,

$$(2.26) \quad \|u - (u)_{B_R}\|_{L^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|^2}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Moreover the dependence on the variable q of the constant C can be estimated as follows, for each $q \geq 2$

$$C(d, q) \leq Aq^{\frac{3}{2}}$$

for some constant $A := A(d) < \infty$.

The proof of this proposition heavily relies on [5, Proposition D.1 and Remark D.6] and is presented in Appendix A.

3. ESTIMATES OF THE SPATIAL AVERAGES OF THE FIRST-ORDER CORRECTORS

We now have all the necessary tools to prove the optimal L^q bounds of the corrector, stated in Theorem 1. The idea is to first prove Proposition 3.3 thanks to the concentration inequality, Proposition 2.18. We then deduce the bound on the coarsened correctors thanks to the multiscale Poincaré inequality, Proposition 2.19 and remove the coarsening thanks to Proposition 2.9. This eventually yields (1.17).

Definition 3.1. Fix a function $\eta \in C_c^\infty(B_{\frac{1}{2}})$ satisfying

$$\forall x \in \mathbb{R}^d, \eta(x) \geq 0 \text{ and } \int_{\mathbb{R}^d} \eta(x) dx = 1.$$

Given a function $w : \mathcal{C}_\infty \mapsto \mathbb{R}$, we consider the function $[w]_{\mathcal{P}}$ defined on the entire lattice \mathbb{Z}^d . We then extend this function to a function constant by part on \mathbb{R}^d by setting, for each $x \in \mathbb{Z}^d$ and each $y \in x + B_{\frac{1}{2}}$, $[w]_{\mathcal{P}}(y) = [w]_{\mathcal{P}}(x)$. We then smoothen this function by convoluting against η and define

$$\begin{aligned} [w]_{\mathcal{P}}^\eta &: \mathbb{R}^d \rightarrow \mathbb{R} \\ x &\mapsto ([w]_{\mathcal{P}} * \eta)(x). \end{aligned}$$

This creates a smooth function defined on \mathbb{R}^d . This property will be convenient when we apply the multiscale Poincaré inequality, to obtain Theorem 1. This is the only reason we need to go from a discrete function defined on \mathbb{Z}^d to a continuous function defined on \mathbb{R}^d . Additionally, this function satisfies a number of convenient properties, recorded in the following proposition.

Proposition 3.2. *Given a function $w : \mathcal{C}_\infty \mapsto \mathbb{R}$, the function $[w]_{\mathcal{P}}^\eta$ defined in Definition 3.1 satisfies*

(i) *For each $x \in \mathbb{Z}^d$,*

$$[w]_{\mathcal{P}}^\eta(x) = [w]_{\mathcal{P}}(x).$$

(ii) *For each $x \in \mathbb{Z}^d$ and each $y \in \mathbb{Z}^d + B_{\frac{1}{2}}$, we have,*

$$|\nabla [w]_{\mathcal{P}}^\eta(y)| \leq C |\nabla [w]_{\mathcal{P}}(x)|,$$

for some constant C depending only on d and η .

We now state the main technical proposition of this article.

Proposition 3.3. *For each $R \geq 1$, and each $x \in \mathbb{R}^d$, the quantity $\nabla(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta)(x)$ is well-defined and there exist an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a constant $C := C(d, \mathbf{p}, \lambda) < \infty$ such that it satisfies*

$$(3.1) \quad |\nabla(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta)(x)| \leq \mathcal{O}_s(C|p|R^{-\frac{d}{2}}),$$

where we used the notation introduced earlier for the Gaussian function

$$\begin{aligned} \Phi_R &: \mathbb{R}^d \rightarrow \mathbb{R} \\ x &\mapsto R^{-d} \exp\left(-\frac{|x|^2}{R^2}\right). \end{aligned}$$

Before starting the proof, note that the application $p \rightarrow \chi_p$ is linear. Thus, we can assume without loss of generality that $|p| = 1$ to simplify the computations. To verify that the quantity $\nabla(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta)(x)$ is well-defined, it is sufficient to check that, for each $x \in \mathbb{R}^d$ and each $R > 0$,

$$\int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{R^2}\right) |\nabla [\chi_p]_{\mathcal{P}}^\eta(y)| dy < \infty \quad \mathbb{P}\text{-almost surely}$$

To prove this, we actually show the stronger estimate, for each $x \in \mathbb{R}^d$

$$(3.2) \quad |\nabla [\chi_p]_{\mathcal{P}}^\eta(x)| \leq \mathcal{O}_s(C).$$

By (ii) of Proposition 3.2, it is enough to prove, for each edges $e = (x, y) \in \mathbb{R}^d$,

$$(3.3) \quad |\nabla [\chi_p]_{\mathcal{P}}(e)| \leq \mathcal{O}_s(C).$$

By Proposition 2.11, we have

$$\begin{aligned} |\nabla [\chi_p]_{\mathcal{P}}(e)| &\leq \int_{\mathcal{C}_*(\square_{\mathcal{P}}(x) \cup \square_{\mathcal{P}}(y))} |\nabla [\chi_p]_{\mathcal{P}}(x')| dx' \\ &\leq C \int_{\mathcal{C}_*(\square_{\mathcal{P}}(x) \cup \square_{\mathcal{P}}(y))} \text{size}(\square_{\mathcal{P}}(x'))^{d-1} |\nabla \chi_p(x')| dx' \\ &\leq C \int_{\mathcal{C}_\infty \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \text{size}(\square_{\mathcal{P}}(x'))^{d-1} |\nabla \chi_p(x')| dx' \\ &\leq C \sum_{x' \in \mathbb{Z}^d} \mathbb{1}_{\{x' \in \mathcal{C}_\infty \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \text{size}(\square_{\mathcal{P}}(x'))^{d-1} |\nabla \chi_p(x')|. \end{aligned}$$

Moreover for each $x \in \mathbb{Z}^d$, $\text{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_s(C)$, we have

$$(3.4) \quad \mathbb{1}_{\{x' \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \leq C \frac{\text{size}(\square_{\mathcal{P}}(x))^{d+1}}{|x-x'|^{d+1} \vee 1} \leq \frac{\mathcal{O}_s(C)}{|x-x'|^{d+1} \vee 1},$$

where we used the notation $a \vee b = \max(a, b)$. Applying Proposition (2.17) to the \mathbf{a} -harmonic function $u(x) = p \cdot x + \chi_p(x)$, the Caccioppoli inequality and using the assumption $|p| = 1$, we obtain, for each $x' \in \mathbb{Z}^d$,

$$\begin{aligned} |\nabla \chi_p|(x') \mathbb{1}_{\{x' \in \mathcal{C}_\infty\}} &\leq C + C \mathcal{X}(x')^{\frac{d}{2}} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{\mathcal{X}(x')}(x'))} \\ &\leq C + C \mathcal{X}(x')^{\frac{d}{2}} \liminf_{R \rightarrow \infty} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R(x'))} \\ &\leq C + \frac{C}{R} \mathcal{X}(x')^{\frac{d}{2}} \liminf_{R \rightarrow \infty} \|u - (u)_{\mathcal{C}_\infty \cap B_R(x')}\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R(x'))}. \end{aligned}$$

By (1.14), we have

$$\limsup_{R \rightarrow \infty} \frac{1}{R} \|u - (u)_{\mathcal{C}_\infty \cap B_R(x')}\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_R(x'))} \leq 1 \quad \text{a.s.}$$

This shows, for each $x' \in \mathbb{Z}^d$

$$(3.5) \quad |\nabla \chi_p|(x') \mathbb{1}_{\{x' \in \mathcal{C}_\infty\}} \leq \mathcal{O}_s(C).$$

Combining the previous estimate with (3.4), we obtain, for each $x' \in \mathbb{Z}^d$

$$\mathbb{1}_{\{x' \in \mathcal{C}_\infty \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \text{size}(\square_{\mathcal{P}}(x'))^{d-1} |\nabla \chi_p|(x') \leq \frac{\mathcal{O}_s(C)}{|x-x'|^{d+1} \vee 1}.$$

Since $\sum_{x \in \mathbb{Z}^d \setminus \{y\}} \frac{1}{|x-y|^{d+1}} < \infty$, we can use (1.10) to get

$$\sum_{x' \in \mathbb{Z}^d} \mathbb{1}_{\{x' \in \mathcal{C}_\infty \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \text{size}(\square_{\mathcal{P}}(x'))^{d-1} |\nabla \chi_p|(x') \leq \mathcal{O}_s(C).$$

Combining the previous displays shows that for each $e \in E_d$,

$$|\nabla [\chi_p]_{\mathcal{P}}(e)| \leq \mathcal{O}_s(C).$$

By (1.10) and (3.2), we have, for each $x \in \mathbb{R}^d$ and each $R > 0$,

$$\int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{R^2}\right) |\nabla [\chi_p]_{\mathcal{P}}^\eta(y)| dy \leq \mathcal{O}_s(CR^d).$$

This implies in particular the desired estimate.

We now turn to the proof of (3.1). Fix some $x' \in \mathbb{R}^d$ and denote by $X = \nabla(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta)(x')$. We are going to prove

$$|X| \leq \mathcal{O}_s\left(CR^{-\frac{d}{2}}\right),$$

The main idea of the proof is to apply Proposition 2.18 to X . To do so, we need to prove the two following results.

Result 1. $|\mathbb{E}[X]| \leq CR^{-\frac{d}{2}}$.

Result 2. $\mathbb{V}[X] \leq \mathcal{O}_s(CR^{-d})$.

Proof of Result 1. For each $x, y, z \in \mathbb{Z}^d$ with $x \sim y$, denote by $\tau_z \mathbf{a}$ the translated environment defined by

$$\tau_z \mathbf{a}(\{x, y\}) = \mathbf{a}(\{x-z, y-z\}).$$

By the uniqueness of the gradient of the corrector, we have, for almost every environment and each $x, y, z \in \mathbb{Z}^d$ with $x, y \in \mathcal{C}_\infty$, $x \sim y$, $\mathbf{a}(\{x, y\}) \neq 0$,

$$(3.6) \quad \nabla \chi_p((x+z, y+z))(\tau_z \mathbf{a}) = \nabla \chi_p((x, y))(\mathbf{a}).$$

For $k \in \mathbb{N}$, we construct the $3^k \mathbb{Z}^d$ -stationnary partition $\mathcal{P}_{\text{stat}}^k$ by applying Proposition 2.1 to the collection of triadic cubes

$$\mathcal{G}_{\text{stat}}^k := \mathcal{G} \cup \left(\bigcup_{n=k}^{\infty} \mathcal{T}_n \right).$$

This collection of cube is $3^k \mathbb{Z}^d$ -stationnary: for every environment \mathbf{a} , every $x \in \mathbb{Z}^d$, $z \in 3^k \mathbb{Z}^d$,

$$(3.7) \quad \text{size} \left(\square_{\mathcal{P}_{\text{stat}}^k} (x+z) \right) (\tau_z \mathbf{a}) = \text{size} \left(\square_{\mathcal{P}_{\text{stat}}^k} (x) \right) (\mathbf{a}).$$

With a proof similar to the proof of [7, Proposition 2.1 (iv)], we derive

$$(3.8) \quad \mathbb{P} \left[\exists x \in \square_k, \square_{\mathcal{P}}(x) \neq \square_{\mathcal{P}_{\text{stat}}^k}(x) \right] \leq C \exp(-C^{-1} 3^k).$$

We then define, for $u : \mathcal{C}_{\infty} \rightarrow \mathbb{R}$, the coarsened function with respect to the partition $\mathcal{P}_{\text{stat}}^k$ by the formula

$$[u]_{\mathcal{P}_{\text{stat}}^k} := u \left(\bar{z}_{\text{stat}} \left(\square_{\mathcal{P}_{\text{stat}}^k} (x) \right) \right)$$

with the notation, for $\square \in \mathcal{T}$,

$$(3.9) \quad \bar{z}_{\text{stat}}(\square) := \begin{cases} \bar{z}(\square) & \text{if } \bar{z}(\square) \in \mathcal{C}_{\infty} \text{ and } \square \text{ is a good cube,} \\ \underset{z \in \mathcal{C}_{\infty}}{\text{argmin dist}}(z, \square) & \text{otherwise.} \end{cases}$$

If there is more than one choice in the argument of the minima, we select the one which is minimal for the lexicographical order. In particular, combining (3.6) and (3.7) yields

$$(3.10) \quad \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^{\eta} \text{ is } 3^k \mathbb{Z}^d \text{-stationnary,}$$

where $[\chi_p]_{\mathcal{P}_{\text{stat}}^k}^{\eta}$ is defined from $[\chi_p]_{\mathcal{P}_{\text{stat}}^k}$ by a convolution with η , as in Definition 3.1. We fix $k \in \mathbb{Z}^d$ such that $3^k \leq R^{\frac{1}{2}} \leq 3^{k+1}$ and split the proof of Result 1 into three steps.

(i) In Step 1, we prove

$$\mathbb{E} \left[\left| \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^{\eta} \right) (x') - \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^{\eta} \right) (x') \right| \right] \leq CR^{-\frac{d}{2}}.$$

(ii) In Step 2, we prove

$$\mathbb{E} \left[\int_{x' + \left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^{\eta} (x) dx \right] = 0.$$

Note that we wrote $\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d$ and not \square_k because we are referring to the continuous cube and not the discrete one as it was defined in (1.8).

(iii) In Step 3, we use the result obtained in Step 2 to show

$$\left| \mathbb{E} \left[\nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^{\eta} \right) (x') \right] \right| \leq CR^{-\frac{d}{2}}.$$

Result 1 is then a consequence of the main results of Steps 1 and 3.

Step 1. The main result of this step is a consequence of the following computation, by (1.5), Proposition 3.2 and Proposition 2.11,

(3.11)

$$\begin{aligned} & \mathbb{E} \left[\left| \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta \right) (x') - \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right) (x') \right| \right] \\ & \leq \mathbb{E} \left[\left| \int_{B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta (x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta (x) \right) \Phi_R(x-x') dx \right| \mathbb{1}_{\left\{ \exists x \in B_{R^2}(x') : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right\}} \right] \\ & \quad + \mathbb{E} \left[\left| \int_{\mathbb{R}^d \setminus B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta (x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta (x) \right) \Phi_R(x-x') dx \right| \right]. \end{aligned}$$

The first term on the right-hand side can be estimated crudely the following way. We denote U_0 the set

$$U_0 := \bigcup_{x \in B_{R^2}(x')} \square_{\mathcal{P}}(x),$$

we then enlarge this set by adding two additional layers of cubes and define

$$U_1 := \bigcup_{\square \in \mathcal{P}, \text{dist}(\square, U) \leq 1} \square \quad \text{and} \quad U := \bigcup_{\square \in \mathcal{P}, \text{dist}(\square, U_1) \leq 1} \square.$$

With these definitions, the definition of $\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta$, which essentially amounts to (3.9), and (2.4), we have, for each $x \in B_{R^2}(x')$,

$$\left| \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta (x) \right| \leq \int_{\mathcal{L}_\infty \cap U} |\nabla \chi_p|(y) dy.$$

Similarly, by definition of $\nabla [\chi_p]_{\mathcal{P}}^\eta$ and the properties of the partition \mathcal{P} , we have, for each $x \in B_{R^2}(x')$,

$$\left| \nabla [\chi_p]_{\mathcal{P}}^\eta (x) \right| \leq \int_{\mathcal{L}_\infty \cap U} |\nabla \chi_p|(y) dy.$$

This leads to the estimate

$$\begin{aligned} (3.12) \quad & \left| \int_{B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta (x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta (x) \right) \Phi_R(x-x') dx \right| \\ & \leq C \left(\int_{\mathcal{L}_\infty \cap U} |\nabla \chi_p|(y) dy \right) \int_{B_{R^2}(x')} \Phi_R(x-x') dx \\ & \leq C \int_{\mathcal{L}_\infty \cap U} |\nabla \chi_p|(y) dy. \end{aligned}$$

This term can be estimated thanks to the regularity theory, Proposition 2.17 applied to the \mathbf{a} -harmonic function $x \mapsto p \cdot x + \chi_p$. Denote

$$R' := \inf \{ R > 0 : U \subseteq B_R(x') \},$$

fix some $z \in \mathbb{Z}^d$ such that $U \subseteq B_{R'}(z)$. With this notation, we can compute

$$\begin{aligned} (3.13) \quad & \int_{\mathcal{L}_\infty \cap U} |\nabla \chi_p|(y) dy \leq C|U| + C \int_{\mathcal{L}_\infty \cap U} |p + \nabla \chi_p|(y) dy \\ & \leq C|U| + C \int_{\mathcal{L}_\infty \cap B_{R'}(x')} |p + \nabla \chi_p|(y) dy \\ & \leq C|U| + CR'^d \left(\int_{\mathcal{L}_\infty \cap B_{R'}(x')} |p + \nabla \chi_p|^2(y) dy \right)^{\frac{1}{2}} \\ & \leq C|U| + CR'^{\frac{d}{2}} \max(\mathcal{X}(x'), R')^{\frac{d}{2}} \liminf_{R \rightarrow \infty} \left(\int_{\mathcal{L}_\infty \cap B_R(x')} |p + \nabla \chi_p|^2(y) dy \right)^{\frac{1}{2}} \end{aligned}$$

But, by (1.15) and the Caccioppoli inequality, we have

$$\liminf_{R \rightarrow \infty} \left(\int_{\mathcal{C}_\infty \cap B_R(x')} |p + \nabla \chi_p|^2(y) dy \right)^{\frac{1}{2}} \leq 1,$$

thus

$$\int_{\mathcal{C}_\infty \cap U} |\nabla \chi_p|(y) dy \leq C|U| + CR'^{\frac{d}{2}} \max(\mathcal{X}(z), R')^{\frac{d}{2}}.$$

By the properties of the partition \mathcal{P} , and (1.10),

$$|U| = C|U_1| \leq C|U_0| \leq C \sum_{x \in B_{R^2}(x')} \text{size}(\square_{\mathcal{P}}(x))^d \leq \mathcal{O}_s(CR^{2d}).$$

Similarly, we have the crude bound

$$R' \leq C \sum_{x \in B_{R^2}(x')} \text{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_s(CR^{2d}).$$

Combining the previous displays shows

$$\left| \int_{B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right) \Phi_R(x - x') dx \right| \leq \mathcal{O}_s(CR^{2d^2}).$$

By (3.8), we also have

$$\begin{aligned} \mathbb{P} \left[\exists x \in B_{R^2}(x') : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right] &\leq \sum_{z \in 3^k \mathbb{Z}^d \cap B_{R^2}(x')} \mathbb{P} \left[\exists x \in z + \square_k : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right] \\ &\leq \frac{R^{2d}}{3^{dk}} \mathbb{P} \left[\exists x \in \square_k : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right] \\ &\leq \frac{CR^{2d}}{3^{dk}} \exp(-C^{-1}3^k). \end{aligned}$$

In particular, since k has been chosen such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$, for each $q > 0$, there exists a constant $C := C(d, \mathbf{p}, \lambda, q) < \infty$ and an exponent $s := s(d, \mathbf{p}, \lambda, q) > 0$ such that

$$\mathbb{1}_{\left\{ \exists x \in B_{R^2}(x') : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right\}} \leq \mathcal{O}_s(CR^{-q}).$$

Combining the three previous displays with q chosen large enough (here we need $q \geq 2d^2 + \frac{d}{2}$), the Cauchy-Schwarz inequality and (1.12), we obtain

$$\left| \int_{B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right) \Phi_R(x - x') dx \right| \mathbb{1}_{\left\{ \exists x \in B_{R^2}(x') : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right\}} \leq \mathcal{O}_s \left(CR^{-\frac{d}{2}} \right),$$

which yields in particular

$$\mathbb{E} \left[\left| \int_{B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right) \Phi_R(x - x') dx \right| \mathbb{1}_{\left\{ \exists x \in B_{R^2}(x') : \square_{\mathcal{P}_{\text{stat}}^k}(x) \neq \square_{\mathcal{P}}(x) \right\}} \right] \leq CR^{-\frac{d}{2}}.$$

We now focus on estimating the second term on the right-hand side of (3.11). With the same computation as the one we just wrote, we obtain,

$$\int_{B_{R^2}(x')} \left| \nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right| dx \leq \mathcal{O}_s \left(CR^{2d^2+2d} \right),$$

indeed the proof is the same, we only need to replace $\int_{B_{R^2}(x')} \Phi_R(x-x') dx$ by CR^{2d} in (3.12). Since this result is true for any $R \geq 1$, we obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathcal{E}_\infty \cap (\square_{n+1} \setminus \square_n)} \left| \nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right| dx &\leq \int_{\mathcal{E}_\infty \cap B_{3^n}} \left| \nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right| dx \\ &\leq \mathcal{O}_s \left(C 3^{n(d^2+d)} \right). \end{aligned}$$

This allows the computation

$$\begin{aligned} &\mathbb{E} \left[\left| \int_{\mathbb{R}^d \setminus B_{R^2}(x')} \left(\nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right) \Phi_R(x-x') dx \right| \right] \\ &\leq \sum_{n=2\log_3(R)}^{+\infty} \mathbb{E} \left[\exp\left(-\frac{3^{2n}}{R^2}\right) R^{-d} \int_{\mathcal{E}_\infty \cap (\square_{n+1} \setminus \square_n)} \left| \nabla [\chi_p]_{\mathcal{P}}^\eta(x) - \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right| dx \right] \\ &\leq \sum_{n=2\log_3(R)}^{+\infty} \exp\left(-\frac{3^{2n}}{R^2}\right) R^{-d} |p| 3^{n(d^2+d)} \\ &\leq C \exp\left(-C^{-1}R^2\right). \end{aligned}$$

Combining the estimates of the first and the second terms of the right-hand side completes the proof of Step 1.

Remark 3.4. The same proof shows the stronger result : for each $q > 0$, there exists $C := C(d, \mathbf{p}, \lambda, q) < \infty$ such that for each $R \geq 1$ and $k \in \mathbb{N}$ such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$,

$$\mathbb{E} \left[\left| \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^\eta \right) (x') - \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right) (x') \right| \right] \leq CR^{-q}$$

but the proof of Theorem 1 only requires the result with $q = \frac{d}{2}$.

Step 2. We prove the main result of this step by combining the stationarity property (3.10) with the sublinear growth of the corrector. First notice that by (1.14), we have, for each $R > 1$,

$$\text{osc}_{\mathcal{E}_\infty \cap B_R} \chi_p \leq \mathcal{O}_s \left(CR^{1-\delta} \right).$$

By the Stokes formula we have, for each $n \in \mathbb{N}$, large enough (depending on the environment)

$$\begin{aligned} \left| \int_{x'+(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2})^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right| &= \left| \int_{\partial(x'+(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2})^d)} [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \mathbf{n}(x) dx \right| \\ &\leq 3^{kn(d-1)} \text{osc}_{\mathcal{E}_\infty \cap B_{3^{kn}}(x')} \chi_p \\ &\leq 3^{kn(d-1)} \mathcal{O}_s \left(C 3^{kn(1-\delta)} \right) \\ &\leq \mathcal{O}_s \left(C 3^{kn(d-\delta)} \right). \end{aligned}$$

This yields in particular

$$\left| \mathbb{E} \left[\int_{x'+(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2})^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right] \right| \leq C 3^{kn(d-\delta)}$$

Or we also have, by (3.10).

$$\begin{aligned} \mathbb{E} \left[\int_{x'+(-\frac{3^k}{2}, \frac{3^k}{2})^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right] &= \sum_{z \in x'+(3^k \mathbb{Z}^d \cap \square_{kn})} \mathbb{E} \left[\int_{z+(-\frac{3^k}{2}, \frac{3^k}{2})^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right] \\ &= \frac{3^{dkn}}{3^{dk}} \mathbb{E} \left[\int_{x'+(-\frac{3^k}{2}, \frac{3^k}{2})^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right]. \end{aligned}$$

Combining the two previous results shows

$$\left| \mathbb{E} \left[\int_{x' + \left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right] \right| \leq C 3^{dk} 3^{-kn\delta}.$$

Sending n to ∞ shows

$$\left| \mathbb{E} \left[\int_{x' + \left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) dx \right] \right| = 0$$

and the proof is complete.

Step 3. First notice that

$$\mathbb{E} \left[\left(\Phi_R * \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right) (x') \right] = \left(\Phi_R * \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right] \right) (x').$$

By (3.10), the function

$$\begin{aligned} f &: \mathbb{R}^d \rightarrow \mathbb{R}^d \\ x &\mapsto \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right] \end{aligned}$$

is $3^k \mathbb{Z}^d$ -periodic, consequently there exists $(a_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d} \in (\mathbb{C}^d)^{\mathbb{Z}^d}$ such that

$$f(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \exp\left(\frac{2i\pi \mathbf{n} \cdot x}{3^k}\right).$$

With this formula, the previous display can be rewritten

$$\left(\Phi_R * \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right] \right) (x') = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \int_{\mathbb{R}^d} \Phi_R(x - x') \exp\left(\frac{2i\pi \mathbf{n} \cdot x}{3^k}\right) dx.$$

Or, we know that

$$\int_{\mathbb{R}^d} \Phi_R(x - x') \exp\left(\frac{2i\pi \mathbf{n} \cdot x}{3^k}\right) dx = \pi^{\frac{d}{2}} \exp\left(\frac{2i\pi R \mathbf{n} \cdot x'}{3^k}\right) \exp\left(-\frac{|\pi R \mathbf{n}|^2}{3^{2k}}\right).$$

Combining the previous displays proves the inequality

$$\left(\Phi_R * \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right] \right) (x') = \sum_{\mathbf{n} \in \mathbb{Z}^d} a_{\mathbf{n}} \pi^{\frac{d}{2}} \exp\left(\frac{2i\pi R \mathbf{n} \cdot x'}{3^k}\right) \exp\left(-\frac{|\pi R \mathbf{n}|^2}{3^{2k}}\right).$$

Notice that the main result of Step 2 is equivalent to the following equality

$$a_{(0, \dots, 0)} = 0.$$

Using this relation and the Cauchy-Schwarz inequality, we obtain

$$(3.14) \quad \left| \left(\Phi_R * \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta \right] \right) (x') \right|^2 \leq C \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus (0, \dots, 0)} |a_{\mathbf{n}}|^2 \right) \left(\sum_{\mathbf{n} \in \mathbb{Z}^d \setminus (0, \dots, 0)} \exp\left(-2\frac{|\pi R \mathbf{n}|^2}{3^{2k}}\right) \right).$$

In particular, since k was chosen such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$, we have

$$(3.15) \quad \sum_{\mathbf{n} \in \mathbb{Z}^d \setminus (0, \dots, 0)} \exp\left(-2\frac{|\pi R \mathbf{n}|^2}{3^{2k}}\right) \leq C \exp(-C^{-1}R).$$

Moreover, we have

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{Z}^d} |a_{\mathbf{n}}|^2 &= \int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \left| \mathbb{E} \left[\nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right] \right|^2 dx \\ &\leq \mathbb{E} \left[\int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \left| \nabla [\chi_p]_{\mathcal{P}_{\text{stat}}^k}^\eta(x) \right|^2 dx \right]. \end{aligned}$$

As in Step 1, we define

$$U_0 := \bigcup_{x \in \square_k} \square_{\mathcal{P}}(x), \quad U_1 := \bigcup_{\square \in \mathcal{P}, \text{dist}(\square, U) \leq 1} \square \quad \text{and} \quad U := \bigcup_{\square \in \mathcal{P}, \text{dist}(\square, U_1) \leq 1} \square$$

to obtain the estimate, for each $x \in \left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d$,

$$\left| \nabla [\chi_{\mathcal{P}}]_{\mathcal{P}_{\text{stat}}^k}^{\eta}(x) \right| \leq \int_{\mathcal{C}_{\infty} \cap U} |\nabla \chi_{\mathcal{P}}(y)| dy.$$

As in Step 1, we define

$$R' := \inf \{R > 0 : U \subseteq B_R\},$$

and we fix some $z \in \mathbb{Z}^d$ such that $U \subseteq B_{R'}(z)$. The same computation as (3.13) yields

$$\int_{\mathcal{C}_{\infty} \cap U} |\nabla \chi_{\mathcal{P}}|^2(y) dy \leq C|U||p|^2 + C|p|^2 \max(\mathcal{X}, R')^d$$

and

$$|U| \leq C \sum_{x \in \square_k} \text{size}(\square_{\mathcal{P}}(x))^d \leq \mathcal{O}_s(C3^{kd}).$$

Similarly, we have the crude bound

$$R' \leq C(3^k + \text{size}(\square_{\mathcal{P}}(0))) \leq \mathcal{O}_s(C3^k).$$

Combining the previous displays shows

$$\begin{aligned} \int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \left| \nabla [\chi_{\mathcal{P}}]_{\mathcal{P}_{\text{stat}}^k}^{\eta}(x) \right|^2 dx &\leq 3^{kd} \left(\int_{\mathcal{C}_{\infty} \cap U} |\nabla \chi_{\mathcal{P}}(y)| dy \right)^2 \\ &\leq 3^{kd} |U| \int_{\mathcal{C}_{\infty} \cap U} |\nabla \chi_{\mathcal{P}}|^2(y) dy \\ &\leq C3^{kd} |U| (|U| + \max(\mathcal{X}, R')^d) \end{aligned}$$

By (1.12), this gives

$$\int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \left| \nabla [\chi_{\mathcal{P}}]_{\mathcal{P}_{\text{stat}}^k}^{\eta}(x) \right|^2 dx \leq \mathcal{O}_s(C|p|^2 3^{kd}).$$

Taking the expectation yields

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} |a_{\mathbf{n}}|^2 \leq \mathbb{E} \left[\int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \left| \nabla [\chi_{\mathcal{P}}]_{\mathcal{P}_{\text{stat}}^k}^{\eta}(x) \right|^2 dx \right] \leq C3^{3kd}.$$

Combining this with (3.14) and (3.15), we obtain

$$\left| \left(\Phi_R * \mathbb{E} \left[\nabla [\chi_{\mathcal{P}}]_{\mathcal{P}_{\text{stat}}^k}^{\eta} \right] \right) (x') \right|^2 \leq CR^{\frac{3d}{2}} \exp(-C^{-1}R) \leq C \exp(-C^{-1}R),$$

where we increased the value of the constant C in the second inequality to absorb the term $R^{\frac{3d}{2}}$. This implies in particular the main result of Step 3 and completes the proof of *Result 1*.

Proof of Result 2. We recall Proposition 2.18 and the notation $X = \nabla(\Phi_R * [\chi_{\mathcal{P}}]_{\mathcal{P}}^{\eta})(x')$. Given an environment $\mathbf{a} \in \Omega$ and an edge $e = (x, y) \in E_d$, we want to estimate $(X - X'_e)^2$. To do so, we need to understand how changing the value of the edge e can impact the infinite cluster \mathcal{C}_{∞} and the partition \mathcal{P} . This is studied in the following lemma.

Lemma 3.5. *There exist two constants $C_0 := C(d) < \infty$ and $C := C_0(d) < \infty$ such that for each edge $e = (x, y) \in E_d$, environments $\mathbf{a}, \tilde{\mathbf{a}} \in \Omega$ satisfying $\mathbf{a}(e') = \tilde{\mathbf{a}}(e')$ for each edge $e' \in E_d \setminus \{e\}$ and for every $z \in \mathbb{Z}^d \setminus B(x, C_0 \text{size}(\square_{\mathcal{P}}(x)))$,*

$$\text{size}(\square_{\mathcal{P}(\tilde{\mathbf{a}})}(z)) \leq C \text{size}(\square_{\mathcal{P}(\mathbf{a})}(x)).$$

Moreover, if $z \in \mathbb{Z}^d \setminus B(x, C_0 \text{size}(\square_{\mathcal{P}}(x)))$ then

$$\text{size}(\square_{\mathcal{P}(\tilde{\mathbf{a}})}(z)) = \text{size}(\square_{\mathcal{P}(\mathbf{a})}(z)).$$

Proof of Lemma 3.5. The main ingredients of the proof are the following:

- (1) If a good cube $\square \in \mathcal{P}_*$ is such that $3\square \cap \{x, y\} = \emptyset$ then \square is a good cube under the environment $\tilde{\mathbf{a}}$.
- (2) By the properties of the partition \mathcal{P} , every cube $\square \in \mathcal{P}$ which does not contain x nor y is crossable under the environment $\tilde{\mathbf{a}}$. The predecessors of $\square_{\mathcal{P}}(x)$ and $\square_{\mathcal{P}}(y)$ are also crossable under the environment $\tilde{\mathbf{a}}$.
- (3) Notice that by resampling one edge we cannot create an isolated cluster which is not connected to \mathcal{C}_∞ of size larger than $C \text{size}(\square_{\mathcal{P}}(x))$, for some $C_0 := C_0(d) < \infty$. In particular, there exists a constant $C := C(d) < \infty$ such that every good cube of size larger than $C \text{size}(\square_{\mathcal{P}}(x))$ under the environment \mathbf{a} satisfies (ii) of Definition 2.6 under the environment $\tilde{\mathbf{a}}$.
- (4) There exists a constant $C := C(d) < \infty$ such that every cube of size larger than $C \text{size}(\square_{\mathcal{P}}(x))$ intersecting $\square_{\mathcal{P}}(x)$ is crossable by a cluster which does not intersect $\square_{\mathcal{P}}(x)$.
- (5) If, for $y \in B(x, C_0 \text{size}(\square_{\mathcal{P}}(x)))$, $\square_{\mathcal{P}}(y)$ is larger than $C \text{size}(\square_{\mathcal{P}}(x))$, then x belongs to $\square_{\mathcal{P}}(y)$ or one of its neighbors and thus $\text{size}(\square_{\mathcal{P}}(y)) \leq C \text{size}(\square_{\mathcal{P}}(x))$.

Combining these properties shows that every good cube \square under the environment \mathbf{a} satisfying $\text{size}(\square) \geq C \text{size}(\square_{\mathcal{P}}(x))$ is a good cube under the environment $\tilde{\mathbf{a}}$. It is then straightforward to see from the previous remarks and the construction of the partition \mathcal{P} in the proof of Proposition 2.1 that the conclusion of the lemma is valid. \square

To estimate $(X - X'_e)^2$, we introduce an extended probability space by doubling the variables $(\Omega', \mathcal{F}', \mathbb{P}') = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$. For a given environment $(\mathbf{a}(e), \tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d} \in \Omega'$, we denote by pr_1 (resp. pr_2) the first (resp. second) projection, i.e. $\text{pr}_1((\mathbf{a}(e), \tilde{\mathbf{a}}(e))) = (\mathbf{a}(e))_{e \in \mathcal{B}_d}$ (resp. $\text{pr}_2((\mathbf{a}(e), \tilde{\mathbf{a}}(e))) = (\tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}$). Any random variable Z defined on $(\Omega, \mathcal{F}, \mathbb{P})$ can be seen as a random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ by the formula $Z = Z \circ \text{pr}_1$, i.e. $Z((\mathbf{a}(e), \tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}) = Z((\mathbf{a}(e))_{e \in \mathcal{B}_d})$. All the random variables in this proof must be considered as random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$, unless explicitly stated.

We will denote \mathbb{E}' the expectation of a random variable $Z : \Omega' \rightarrow \mathbb{R}$ with respect to the measure \mathbb{P}' . For a constant $C \in (0, \infty)$ and an exponent $s > 0$, we denote

$$Z \leq \mathcal{O}'_s(C) \text{ if and only if } \mathbb{E}' \left[\exp \left(\left(\frac{Z}{C} \right)^s \right) \right] \leq 2.$$

In particular any random variable Z defined on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $Z \leq \mathcal{O}_s(C)$ satisfies, as a random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$, $Z \leq \mathcal{O}'_s(C)$.

Given an enlarged environment given the enlarged environment $(\mathbf{a}(e), \tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}$, we denote by \mathbf{a} the environment $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ and by $\mathbf{a}^{e'}$ the environment $((\mathbf{a}(e))_{e \in \mathcal{B}_d \setminus \{e'\}}, \tilde{\mathbf{a}}(e'))$. Similarly, given Z a random variable defined on Ω and $e' \in \mathcal{B}_d$ an edge, we denote by $Z^{e'}$ the random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ by the formula, for each $(\mathbf{a}(e), \tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d} \in \Omega'$

$$Z^{e'}((\mathbf{a}(e), \tilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}) := Z(\mathbf{a}^{e'}).$$

We give a similar definition for partitions, $\mathcal{P}^{e'}$ will denote the random partition of \mathbb{Z}^d under the environment $\mathbf{a}^{e'}$, and for the infinite cluster $(\mathbb{P}'$ almost surely there exists a unique infinite cluster under the environment $\mathbf{a}^{e'}$ which will be denoted $\mathcal{C}_\infty^{e'}$).

To prove Result 2, we will prove the following estimate

$$(3.16) \quad \sum_{e \in \mathcal{B}_d} |\nabla(\Phi_R * ([\chi_{\mathcal{P}}]_{\mathcal{P}}^{\eta} - [\chi_{\mathcal{P}}]_{\mathcal{P}}^{\eta, e'}))(x')|^2 \leq \mathcal{O}'_s \left(\frac{C}{R^d} \right).$$

This is enough to prove Result 2, indeed with the same argument as in [5, Lemma 2.3],

$$\begin{aligned}
& \mathbb{E} \left[\exp \left(\left(\frac{\sum_{e \in \mathcal{B}_d} (X - X'_e)^2}{CR^{-d}} \right)^s \right) \right] \\
&= \int_{\Omega} \exp \left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left| \int_{\Omega} \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') d\mathbb{P}(\tilde{\mathbf{a}}) \right|^2}{CR^{-d}} \right)^s \right) d\mathbb{P}(\mathbf{a}) \\
&\leq \int_{\Omega} \exp \left(\left(\int_{\Omega} \frac{\sum_{e \in \mathcal{B}_d} \left| \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') \right|^2}{CR^{-d}} d\mathbb{P}(\tilde{\mathbf{a}}) \right)^s \right) d\mathbb{P}(\mathbf{a}) \\
&\leq C \int_{\Omega} \int_{\Omega} \exp \left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left| \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') \right|^2}{CR^{-d}} \right)^s \right) d\mathbb{P}(\mathbf{a}) d\mathbb{P}(\tilde{\mathbf{a}}) \\
&\leq C \mathbb{E}' \left[\exp \left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left| \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') \right|^2}{CR^{-d}} \right)^s \right) \right] \\
&\leq 2C.
\end{aligned}$$

This yields, after redefinition of the constant C ,

$$\sum_{e \in \mathcal{B}_d} (X - X'_e)^2 \leq \mathcal{O}_s \left(\frac{C}{R^d} \right).$$

Before starting the proof of (3.16), we need to give a meaning to the quantity, for $e = \{x, y\} \in \mathcal{B}_d$,

$$[\chi_p]_{\mathcal{P}}^\eta$$

as a random variable defined on Ω' . Since we do not necessarily have $\mathcal{C}_\infty = \mathcal{C}_\infty^e$, we cannot simply write $[\chi_p]_{\mathcal{P}}^\eta(x) = \chi_p^e(z(\square_{\mathcal{P}}(x)))$. Nevertheless, since the two environments $((\mathbf{a}(e'))_{e' \in \mathcal{B}_d \setminus \{e\}}, \tilde{\mathbf{a}}(e))$ and $(\mathbf{a}(e'))_{e' \in \mathcal{B}_d}$ are only different by one edge, we have either $\mathcal{C}_\infty \subseteq \mathcal{C}_\infty^e$ or $\mathcal{C}_\infty^e \subseteq \mathcal{C}_\infty$. In the former case, we can define $[\chi_p]_{\mathcal{P}}^\eta(x) = \chi_p^e(z(\square_{\mathcal{P}}(x)))$. In the latter case, $\mathcal{C}_\infty \setminus \mathcal{C}_\infty^e$ is connected to \mathcal{C}_∞ by the edge e . Without loss of generality, we denote by $e = (x, y)$ and assume that $x \in \mathcal{C}_\infty^e$. One can then check that the function

$$(3.17) \quad \begin{array}{ccc} w & : & \mathcal{C}_\infty \rightarrow \mathbb{R} \\ & & z \mapsto \chi_p^e(z) \mathbb{1}_{\{z \in \mathcal{C}_\infty^e\}} + (p \cdot (x - z) + \chi_p^e(x)) \mathbb{1}_{\{z \notin \mathcal{C}_\infty^e\}} \end{array}$$

is a solution of

$$-\nabla \cdot (\mathbf{a} \nabla (p \cdot x + w)) = 0$$

and more precisely that $x \rightarrow p \cdot x + w(x) \in \mathcal{A}_1(\mathcal{C}_\infty)$. In particular, this gives

$$w = \chi_p + c$$

Thus we can define,

$$[\chi_p^e]_{\mathcal{P}} = [w]_{\mathcal{P}}.$$

We then extend $[\chi_p^e]_{\mathcal{P}}$ to a piecewise constant function on \mathbb{R}^d and convolve it with η , as in Definition 3.1, to obtain a smooth function $[\chi_p^e]_{\mathcal{P}}^\eta$ and it satisfies in particular

$$(3.18) \quad \nabla [\chi_p^e]_{\mathcal{P}}^\eta = \nabla [\chi_p]_{\mathcal{P}}^\eta.$$

To prove the estimate (3.16), we split the sum into two terms

$$(3.19) \quad \begin{aligned} & \left| \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') \right|^2 \\ & \leq 2 \left| \nabla (\Phi_R * ([\chi_p^e]_{\mathcal{P}}^\eta - [\chi_p]_{\mathcal{P}^e}^\eta)) (x') \right|^2 + 2 \left| \nabla (\Phi_R * ([\chi_p]_{\mathcal{P}}^\eta - [\chi_p^e]_{\mathcal{P}}^\eta)) (x') \right|^2. \end{aligned}$$

Step 1. We estimate the first term on the right-hand side the following way, using Proposition 3.2, Lemma 3.5 and Proposition 2.9 with $s = 1$,

$$\begin{aligned} & |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \\ & \leq \left(\int_{\mathbb{Z}^d \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} |\nabla [\chi_p^e]_{\mathcal{P}}(y)| + |\nabla [\chi_p^e]_{\mathcal{P}^e}(y)| dy \right)^2 \sup_{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(y - x') \\ & \leq C \left(\int_{\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \text{size}(\square_{\mathcal{P}}(x))^{d-1} (|\nabla \chi_p^e|(y) + 1) dy \right)^2 \sup_{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(y - x'). \end{aligned}$$

The "+1" term on the right-hand side comes from the fact that we assumed $|p| = 1$ combined with the definition of w in (3.17), in the case when $\mathcal{C}_\infty^e \subseteq \mathcal{C}_\infty$. This gives

$$\begin{aligned} & |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \\ & \leq C \text{size}(\square_{\mathcal{P}}(x))^{3d-2} \int_{\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} (|\nabla \chi_p^e|^2(y) + |p|^2) dy \sup_{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(y - x'). \end{aligned}$$

Moreover, there exists a constant $C(d) < \infty$ such that, for each $x \in \mathbb{Z}^d$,

$$(3.20) \quad \exp(-|x|^2) \leq C \frac{1}{|x|^{\frac{d+1}{2}}} \wedge 1.$$

We denote by ζ the function on the right-hand side, i.e, $\zeta(x) := C \frac{1}{|x|^{\frac{d+1}{2}}} \wedge 1$. We similarly denote $\zeta_R(x) = \frac{1}{R^d} \zeta\left(\frac{x}{R}\right)$. We will use this function instead of Φ_R to complete the estimate of the term on the right-hand side because, since it is decreasing slower than Φ_R , it satisfies the following properties

$$\sup_{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \zeta_R^2(y - x') \leq C \text{size}(\square_{\mathcal{P}}(x))^{d+1} \inf_{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \zeta_R^2(y - x')$$

and

$$\sum_{x \in \mathbb{Z}^d} \zeta(x - x')^2 < \infty.$$

In particular, the previous estimate can be rewritten

$$\begin{aligned} & |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \\ & \leq C \text{size}(\square_{\mathcal{P}}(x))^{4d-1} \int_{\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \zeta_R(y - x')^2 (|\nabla \chi_p^e|^2(y) + 1) dy. \end{aligned}$$

Summing over all the edges $e \in \mathcal{B}_d$ gives

$$\begin{aligned} (3.21) \quad & \sum_{e \in \mathcal{B}_d} |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \\ & \leq C \sum_{x \in \mathbb{Z}^d} \text{size}(\square_{\mathcal{P}}(x))^{4d-1} \int_{\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} \zeta_R(y - x')^2 (|\nabla \chi_p^e|^2(y) + 1) dy \\ & \leq C \sum_{y \in \mathcal{C}_\infty^e} \zeta_R(y)^2 (|\nabla \chi_p^e|^2(y) + 1) \left(\sum_{x \in \mathbb{Z}^d} \text{size}(\square_{\mathcal{P}}(x))^{4d-1} \mathbb{1}_{\{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \right). \end{aligned}$$

But, since for each $x \in \mathbb{Z}^d$, $\text{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}'_s(C)$, we have

$$\mathbb{1}_{\{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \leq C \frac{\text{size}(\square_{\mathcal{P}}(x))^{d+1}}{|x - y|^{d+1}} \leq \frac{\mathcal{O}'_s(C)}{|x - y|^{d+1}}$$

and thus, by (1.12)

$$\text{size}(\square_{\mathcal{P}}(x))^{4d-1} \mathbb{1}_{\{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \leq \frac{\mathcal{O}'_s(C)}{|x - y|^{d+1}}.$$

Since $\sum_{x \in \mathbb{Z}^d \setminus \{y\}} \frac{1}{|x-y|^{d+1}} < \infty$, we can use (1.10) to obtain

$$(3.22) \quad \sum_{x \in \mathbb{Z}^d} \text{size}(\square_{\mathcal{P}}(x))^{4d-1} \mathbf{1}_{\{y \in B(x, C \text{size}(\square_{\mathcal{P}}(x)))\}} \leq \mathcal{O}'_s(C).$$

Moreover, by (1.10) applied to the function $u(x) = p \cdot x + \chi_p(x)$ and the Caccioppoli inequality, we have, for each $y \in \mathbb{Z}^d$, and each $R \geq \mathcal{X}^e(y) := \mathcal{X}^e((\tau_y \mathbf{a}, \tau_y \tilde{\mathbf{a}}))$,

$$\begin{aligned} |\nabla \chi_p^e(y)| \mathbf{1}_{\{y \in \mathcal{C}_\infty^e\}} &\leq C + C \mathcal{X}^e(y)^{\frac{d}{2}} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B_{\mathcal{X}^e(y)})} \\ &\leq C + C \mathcal{X}^e(y)^{\frac{d}{2}} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B_R)} \\ &\leq C + \frac{C}{R} \mathcal{X}^e(y)^{\frac{d}{2}} \|u - (u)_{\mathcal{C}_\infty^e \cap B_R}\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B_R)}. \end{aligned}$$

By (1.14), we have

$$\limsup_{R \rightarrow \infty} \frac{1}{R} \|u - (u)_{\mathcal{C}_\infty^e \cap B_R}\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B_R)} \leq 1 \quad \text{a.s.}$$

Combining the previous displays yields

$$|\nabla \chi_p^e(y)| \mathbf{1}_{\{y \in \mathcal{C}_\infty^e\}} \leq C + C \mathcal{X}^e(y)^{\frac{d}{2}},$$

which can be rewritten

$$(3.23) \quad |\nabla \chi_p^e(y)| \mathbf{1}_{\{y \in \mathcal{C}_\infty^e\}} \leq \mathcal{O}'_s(C).$$

Injecting (3.22) and (3.23) into the previous computation (3.21) gives

$$\sum_{e \in \mathcal{B}_d} |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \leq C \sum_{y \in \mathbb{Z}^d} \zeta_R(y)^2 \mathcal{O}'_s(C).$$

Since $\sum_{y \in \mathbb{Z}^d} \zeta_R(y)^2 \leq \frac{C}{R^d}$, we can use (1.10) to obtain

$$\sum_{e \in \mathcal{B}_d} |(\Phi_R * (\nabla [\chi_p^e]_{\mathcal{P}}^\eta - \nabla [\chi_p^e]_{\mathcal{P}^e}^\eta))(x')|^2 \leq \mathcal{O}'_s\left(\frac{C}{R^d}\right).$$

This completes the proof of the estimate of the first term on the right-hand side of (3.19).

Step 2. We now estimate the second term on the right-hand side of (3.19)

$$|(\Phi_R * (\nabla [\chi_p - \chi_p^e]_{\mathcal{P}}^\eta))(x')|^2 \leq \mathcal{O}'_s\left(\frac{C}{R^d}\right).$$

To prove this estimate, we need to distinguish three cases, we recall that we denoted $e = (x, y)$.

Case 1. ($x \notin \mathcal{C}_\infty$ and $y \notin \mathcal{C}_\infty$) or $\mathbf{a} = \mathbf{a}^e$. In that case, $\mathcal{C}_\infty = \mathcal{C}_\infty^e$, and for each $z \in \mathcal{C}_\infty$, $\chi_p(z) = \chi_p^e(z)$. In particular, this yields

$$|(\Phi_R * (\nabla [\chi_p - \chi_p^e]_{\mathcal{P}}^\eta))(x')|^2 = 0.$$

Case 2. $\mathcal{C}_\infty \neq \mathcal{C}_\infty^e$. In that case, (3.18) is true. It implies

$$|(\Phi_R * (\nabla [\chi_p - \chi_p^e]_{\mathcal{P}}^\eta))(x')|^2 = 0.$$

Case 3. $x, y \in \mathcal{C}_\infty$ and $\mathcal{C}_\infty = \mathcal{C}_\infty^e$ and $\mathbf{a} \neq \mathbf{a}^e$. We compute

$$-\nabla \cdot (\mathbf{a} \nabla (\chi_p - \chi_p^e)) = \nabla \cdot ((\mathbf{a} - \mathbf{a}^e) \nabla (p \cdot z + \chi_p^e)),$$

which can be rewritten

$$(3.24) \quad -\nabla \cdot (\mathbf{a} \nabla (\chi_p - \chi_p^e)) = (\mathbf{a} - \mathbf{a}^e)(x, y) (p \cdot (x - y) + \chi_p^e(x) - \chi_p^e(y)) (\delta_x - \delta_y).$$

Recall the notation G^e introduced in Proposition 2.15. In the rest of the proof, we use the following notation. If the edge $e = (x, y)$ does not belong to the infinite cluster, i.e if $\mathbf{a}(e) = 0$, then denote by e_1, \dots, e_n a path of edges of the infinite cluster linking x to y and denote by

$$(3.25) \quad G^e := \sum_{i=1}^n G^{e_i}.$$

This function is the unique solution (up to a constant) of the equation

$$-\nabla \cdot \mathbf{a} \nabla G^e = \delta_x - \delta_y.$$

We can solve (3.24) by using Proposition 2.15. Indeed the function $\chi_p - \chi_p^e - (\mathbf{a} - \mathbf{a}^e)(x, y)(p \cdot (x - y) + \chi_p^e(x) - \chi_p^e(y))G^e$ is \mathbf{a} -harmonic. Moreover, by the sublinear growth of the corrector (1.15), the L^2 bound on the gradient of the function G^e (2.16) and a version of the Poincaré inequality on the percolation cluster (see for instance the proof of Proposition 2.17), one can show that the function $\chi_p - \chi_p^e - (\mathbf{a} - \mathbf{a}^e)(x, y)(p \cdot (x - y) + \chi_p^e(x) - \chi_p^e(y))G^e$ has a sublinear growth. This implies, by [7, Theorem 2] that this function is constant. In particular, this shows

$$\nabla \chi_p - \nabla \chi_p^e = (\mathbf{a} - \mathbf{a}^e)(x, y)(p \cdot (x - y) + \chi_p^e(x) - \chi_p^e(y)) \nabla G^e.$$

But, if $\mathbf{a}^e(e) = \tilde{\mathbf{a}}(e) \neq 0$, we have the estimate, by (3.23),

$$|\chi_p^e(x) - \chi_p^e(y)| \leq |\nabla \chi_p^e|(x) \leq C(1 + \mathcal{X}^e(x))^{\frac{d}{2}}$$

If $\mathbf{a}^e(e) = \tilde{\mathbf{a}}(e) = 0$, then there exists a path going from x to y which lays in $\square_{\mathcal{P}^e}(x)$ and its neighbors (its neighbors because we may not have $\square_{\mathcal{P}^e}(x) = \square_{\mathcal{P}^e}(y)$ or we may have $x, y \in \mathcal{C}_\infty \setminus \mathcal{C}_*(\square_{\mathcal{P}^e}(x))$). Combining this remark with Lemma 3.5, we obtain

$$\begin{aligned} |\chi_p^e(x) - \chi_p^e(y)| &\leq C \int_{\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x)))} |\nabla \chi_p^e|(z) dz \\ &\leq C \text{size}(\square_{\mathcal{P}}(x))^d \|\nabla \chi_p^e\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x))))}. \end{aligned}$$

With (1.14), the Caccioppoli inequality applied to the function $u(z) = p \cdot z + \chi_p^e(z)$ and a similar computation to the one we ran to get (3.23), we obtain

$$\|\nabla \chi_p^e\|_{\underline{L}^2(\mathcal{C}_\infty^e \cap B(x, C \text{size}(\square_{\mathcal{P}}(x))))} \leq C(1 + \mathcal{X}^e(x))^{\frac{d}{2}}.$$

Combining the two previous displays yields

$$|\chi_p^e(x) - \chi_p^e(y)| \leq C \text{size}(\square_{\mathcal{P}}(x))^d (1 + \mathcal{X}^e(x))^{\frac{d}{2}}.$$

Thus

$$(3.26) \quad \begin{aligned} &|(\Phi_R * (\nabla [\chi_p - \chi_p^e]_{\mathcal{P}}^\eta))(x')|^2 \\ &\leq C|p|^2 |(\Phi_R * (\nabla [G^e]_{\mathcal{P}}^\eta))(x')|^2 \text{size}(\square_{\mathcal{P}}(x))^{2d} (1 + \mathcal{X}^e(x))^d. \end{aligned}$$

The next step in the proof consists in getting rid of the coarsening in the right-hand side. To do so, we prove that there exist a constant $C := C(d) < \infty$ and a (random) vector field $\gamma_R : E_d \rightarrow \mathbb{R}$ satisfying, for each $e' = (x, y) \in E_d$,

$$|\gamma_R(e')| \leq C \text{size}(\square_{\mathcal{P}}(x))^{2d} \zeta_R(x - x')$$

such that for each function $u : \mathcal{C}_\infty \rightarrow \mathbb{R}$ satisfying $\langle \nabla u, \nabla u \rangle_{\mathcal{C}_\infty}$,

$$(3.27) \quad (\Phi_R * \nabla [u]_{\mathcal{P}}^\eta)(x') = \langle \gamma_R, \nabla u \rangle_{\mathcal{C}_\infty}.$$

We first compute

$$\begin{aligned} (\Phi_R * \nabla [u]_{\mathcal{P}}^{\eta})(x') &= \int_{\mathbb{R}^d} \Phi_R(t-x') \nabla [u]_{\mathcal{P}}^{\eta}(t) dx \\ &= \int_{\mathbb{R}^d} \Phi_R(t-x') \int_{B_{\frac{1}{2}}} [u]_{\mathcal{P}}(t-s) \nabla \eta(s) ds dt \\ &= \int_{\mathbb{R}^d} \Phi_R(t-x') \int_{B_{\frac{1}{2}}} ([u]_{\mathcal{P}}(t-s) - [u]_{\mathcal{P}}(t)) \nabla \eta(s) ds dt. \end{aligned}$$

But $[u]_{\mathcal{P}}(t)$ and $[u]_{\mathcal{P}}(t-s)$ are only different if t and $t-s$ belong to two different cubes of the partition \mathcal{P} , in that case, we have

$$[u]_{\mathcal{P}}(t-s) - [u]_{\mathcal{P}}(t) = u(\bar{z}(\square_{\mathcal{P}}(t-s))) - u(\bar{z}(\square_{\mathcal{P}}(t))).$$

Recall that there exists a path between $\bar{z}(\square_{\mathcal{P}}(t))$ and $\bar{z}(\square_{\mathcal{P}}(t-s))$ which lies entirely in $\square_{\mathcal{P}}(t) \cup \square_{\mathcal{P}}(t-s)$, which will be denoted $p_{t,t-s} \subseteq E_d$. Summing over the edges along this path, we find that

$$u(\bar{z}(\square_{\mathcal{P}}(t))) - u(\bar{z}(\square_{\mathcal{P}}(t-s))) = \sum_{e' \in p_{t,t-s}} \nabla u(e') = \sum_{e' \in E_d} \nabla u(e') \mathbb{1}_{\{e' \in p_{t,t-s}\}}.$$

If t and $t-s$ belongs to the same cube of the partition \mathcal{P} , we keep the same notation with the convention $p_{t,t-s} = \emptyset$. Consequently, we have for each $(t, s) \in \mathbb{R}^d \times B_{\frac{1}{2}}$,

$$[u]_{\mathcal{P}}(t-s) - [u]_{\mathcal{P}}(t) = \sum_{e' \in E_d} \nabla u(e') \mathbb{1}_{\{e' \in p_{t,t-s}\}}.$$

With this formula, we can rewrite

$$\begin{aligned} (\Phi_R * \nabla [u]_{\mathcal{P}}^{\eta})(x') &= \int_{\mathbb{R}^d} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') ([u]_{\mathcal{P}}(t-s) - [u]_{\mathcal{P}}(t)) \nabla \eta(s) ds dt \\ &= \int_{\mathbb{R}^d} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') \sum_{e' \in E_d} \nabla u(e') \mathbb{1}_{\{e' \in p_{t,t-s}\}} \nabla \eta(s) ds dt \\ &= \sum_{e' \in E_d} \nabla u(e') \int_{\mathbb{R}^d} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') \mathbb{1}_{\{e' \in p_{t,t-s}\}} \nabla \eta(s) ds dt \\ &= \langle \gamma_R, \nabla u \rangle_{\mathcal{E}_{\infty}} \end{aligned}$$

with for each $e' \in E_d$

$$\gamma_R(e') = \int_{\mathbb{R}^d} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') \mathbb{1}_{\{e' \in p_{t,t-s}\}} \nabla \eta(s) ds dt.$$

But, for each $(t, s) \in \mathbb{R}^d \times B_{\frac{1}{2}}$ such that $\square_{\mathcal{P}}(t-s) \neq \square_{\mathcal{P}}(t)$, the path between $\bar{z}(\square_{\mathcal{P}}(t-s))$ and $\bar{z}(\square_{\mathcal{P}}(t))$ lies entirely in $\square_{\mathcal{P}}(t-s) \cup \square_{\mathcal{P}}(t)$. In particular $e' = (x, y) \in p_{t,t-s}$ only if $t \in \partial \square_{\mathcal{P}}(x) + B_{\frac{1}{2}}$, this shows

$$\gamma_R(e') = \int_{\partial \square_{\mathcal{P}}(x) + B_{\frac{1}{2}}} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') \mathbb{1}_{\{e' \in p_{t,t-s}\}} \nabla \eta(s) ds dt$$

and thus, we have the estimate,

$$\begin{aligned} |\gamma_R(e')| &\leq C \int_{\partial \square_{\mathcal{P}}(x) + B_{\frac{1}{2}}} \int_{B_{\frac{1}{2}}} \Phi_R(t-x') ds dt \\ &\leq \int_{\partial \square_{\mathcal{P}}(x) + B_{\frac{1}{2}}} \Phi_R(t-x') dt. \end{aligned}$$

As in (3.20), one has for each $x \in \mathbb{R}^d$,

$$\exp(-|r|^2) \leq \zeta(x).$$

The function ζ satisfies the inequality, for each triadic cube $\square \in \mathcal{T}$,

$$\sup_{\partial\square+B_{\frac{1}{2}}} \zeta_R \leq C \text{size}(\square)^{\frac{d+1}{2}} \inf_{\square} \zeta_R.$$

As a consequence of the two previous displays, we can rewrite the previous estimate

$$\begin{aligned} (3.28) \quad |\gamma_R(e')| &\leq \int_{\partial\square_{\mathcal{P}}(x)+B_{\frac{1}{2}}} \Phi_R(t-x') dt \\ &\leq C \text{size}(\square_{\mathcal{P}}(x))^{d-1} \sup_{\partial\square+B_{\frac{1}{2}}} \zeta_R(\cdot-x') \\ &\leq C \text{size}(\square_{\mathcal{P}}(x))^{2d} \text{size}(\square)^{d+1} \inf_{\square} \zeta_R(\cdot-x') \\ &\leq C \text{size}(\square_{\mathcal{P}}(x))^{2d} \zeta_R(x-x'), \end{aligned}$$

which is the desired estimate. This completes the proof of (3.27).

Applying this property with $u = \nabla G^e$, the inequality (3.26) becomes

$$|(\Phi_R * (\nabla[\chi_p - \chi_p^e]_{\mathcal{P}})) (x')|^2 \leq C |p|^2 |\langle \gamma_R, \nabla G^e \rangle_{\mathcal{C}_\infty}|^2 \text{size}(\square_{\mathcal{P}}(x))^{2d} (1 + \mathcal{X}^e(x))^d.$$

Applying Proposition 2.16, we denote by $w_{\gamma_R} : \mathcal{C}_\infty \rightarrow \mathbb{R}$ the solution of

$$-\nabla \cdot (\mathbf{a} \nabla w_{\gamma_R}) = -\nabla \cdot \gamma_R \text{ in } \mathcal{C}_\infty$$

so that, for each $e' \in E_d^{\mathbf{a}}$

$$\nabla w_{\gamma_R}(e') = \sum_{e \in E_d^{\mathbf{a}}} \gamma_R(e) \nabla G^e(e') = \sum_{e \in E_d^{\mathbf{a}}} \gamma_R(e) \nabla G^{e'}(e) = \langle \gamma_R, \nabla G^{e'} \rangle_{\mathcal{C}_\infty}.$$

This implies in particular, in both cases $\mathbf{a}(e) = 0$ and $\mathbf{a}(e) \neq 0$,

$$w_{\gamma_R}(x) - w_{\gamma_R}(y) = \langle \gamma_R, \nabla G^e \rangle_{\mathcal{C}_\infty}.$$

This gives consequently

$$|(\Phi_R * (\nabla[\chi_p - \chi_p^e]_{\mathcal{P}})) (x')|^2 \leq C |w_{\gamma_R}(x) - w_{\gamma_R}(y)|^2 \text{size}(\square_{\mathcal{P}}(x))^{2d} (1 + \mathcal{X}^e(x))^d.$$

We now combine cases 1, 2 and 3 to obtain the following estimate, using the new notation, for each $x \in \mathbb{Z}^d$, $\mathcal{B}_d^x := \{\{x, y\} : y \in \mathbb{Z}^d, y \sim x\}$ the set of bonds linking x to another vertex of \mathbb{Z}^d .

$$\begin{aligned} \sum_{e \in \mathcal{B}_d} |(\Phi_R * (\nabla[\chi_p - \chi_p^e]_{\mathcal{P}})) (x')|^2 \\ \leq C \sum_{x, y \in \mathcal{C}_\infty, |x-y|_1=1} |w_{\gamma_R}(x) - w_{\gamma_R}(y)|^2 \text{size}(\square_{\mathcal{P}}(x))^{2d} \sum_{e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d. \end{aligned}$$

Using that for each $x, y \in \mathcal{C}_\infty$ with $|x-y|_1 = 1$, there exists a path which is contained in \mathcal{C}_∞ , $\square_{\mathcal{P}}(x)$ and its neighbors (the path is simply (x, y) if $\mathbf{a}(\{x, y\}) \neq 0$), we obtain

$$\begin{aligned} (3.29) \quad \sum_{e \in \mathcal{B}_d} |(\Phi_R * (\nabla[\chi_p - \chi_p^e]_{\mathcal{P}})) (x')|^2 \\ \leq C \int_{\mathcal{C}_\infty} |\nabla w_{\gamma_R}|^2(z) \text{size}(\square_{\mathcal{P}}(z))^{3d} \\ \times \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right) dz. \end{aligned}$$

To estimate the term on the right-hand side, we first notice

$$\begin{aligned} \int_{\mathcal{C}_\infty} |\nabla w_{\gamma_R}|^2(z) dz &\leq \langle \nabla w_{\gamma_R}, \mathbf{a} \nabla w_{\gamma_R} \rangle_{\mathcal{C}_\infty} \\ &= \langle \gamma_R, \nabla w_{\gamma_R} \rangle_{\mathcal{C}_\infty} \\ &\leq \left(\int_{\mathcal{C}_\infty} \gamma_R^2(z) dz \right)^{\frac{1}{2}} \left(\int_{\mathcal{C}_\infty} |\nabla w_{\gamma_R}|^2(z) dz \right)^{\frac{1}{2}}. \end{aligned}$$

This yields by (3.28) and (1.10)

$$\begin{aligned} \int_{\mathcal{C}_\infty} |\nabla w_{\gamma_R}|^2(z) dz &\leq C \int_{\mathcal{C}_\infty} \gamma_R^2(z) dz \\ &\leq C \int_{\mathbb{Z}^d} \gamma_R^2(z) dz \\ &\leq C \int_{\mathbb{Z}^d} C \text{size}(\square_{\mathcal{P}}(x))^{4d} \zeta_R(x-x') dx \\ &\leq \mathcal{O}'_s\left(\frac{C}{R^d}\right). \end{aligned}$$

There remains to estimate the term $\text{size}(\square_{\mathcal{P}}(z))^{3d} \sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d$ on the right-hand side of (3.29). To do so, we need to prove a minimal scale statement and a Meyers estimate as stated below.

Lemma 3.6 (Minimal scale). *There exists a constant $C := C(d, \mathbf{p}, \lambda) < \infty$, an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a random variable $\mathcal{M}_1 \leq \mathcal{O}'_s(C)$ such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_1$,*

$$(3.30) \quad 3^{-dm} \sum_{z \in \square_m} \text{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \leq C$$

where $\varepsilon := \varepsilon(d, \mathbf{p}, \lambda)$ is the exponent which appears in Proposition 2.14.

Definition 3.7 (The partition \mathcal{U}). We define the following family of “good cubes”

$$\mathcal{G} := \{\square \in \mathcal{T} : (2.14) \text{ and } (3.30) \text{ hold}\}$$

in which a deterministic Meyers estimate and a minimal scale inequality hold. By Lemma 3.6 and Proposition 2.14, this family satisfies the assumption of Proposition 2.2 (but not the assumption (2.1) of Proposition 2.1 due to the random variable \mathcal{X} which appears (3.30), consequently Proposition 2.1 cannot be applied). We denote by \mathcal{U} the partition thus obtained. Moreover, by (iii) of Proposition 2.2, one has the inequality

$$\text{size}(\square_{\mathcal{U}}(x)) \leq \mathcal{O}_s(C),$$

for some exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and some constant $C := C(d, \mathbf{p}, \lambda) < \infty$.

We postpone the proof of Lemma 3.6 and complete the proof of Result 2. Using the partition \mathcal{U} , we have

$$\begin{aligned}
& \int_{\mathcal{L}_\infty} |\nabla w|^2(z) \text{size}(\square_{\mathcal{P}}(z))^{3d} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right) dz \\
&= \sum_{\square \in \mathcal{U}} \int_{\square \cap \mathcal{L}_\infty} |\nabla w|^2(z) \\
&\quad \times \text{size}(\square_{\mathcal{P}}(z))^{3d} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right) dz \\
&\leq \sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{|\square|} \int_{\square \cap \mathcal{L}_\infty} |\nabla w|^{2+\varepsilon}(z) dz \right)^{\frac{2}{2+\varepsilon}} \\
&\quad \times \left(\frac{1}{|\square|} \sum_{z \in \square} (\text{size}(\square_{\mathcal{P}}(z)))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
&\leq C \sum_{\square \in \mathcal{U}} \left(\int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} |\nabla w|^2(x) dx + |\square| \left(\frac{1}{|\frac{4}{3}\square|} \int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} \gamma_R^{2+\varepsilon}(x) dx \right)^{\frac{2}{2+\varepsilon}} \right).
\end{aligned}$$

To estimate the term on the right-hand side notice that since $\frac{4}{3}\square$ is included in $\bigcup_{\square' \in \mathcal{U}, \text{dist}(\square', \square) \leq 1} \square'$ and the cardinality of the set $\{\square' \in \mathcal{U} : \text{dist}(\square', \square) \leq 1\}$ is bounded by a constant depending only on the dimension d . This leads to

$$\begin{aligned}
\sum_{\square \in \mathcal{U}} \int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} |\nabla w|^2(x) dx &\leq C \int_{\mathcal{L}_\infty} |\nabla w|^2(x) dx \\
&\leq \mathcal{O}'_s \left(\frac{C}{R^d} \right).
\end{aligned}$$

To estimate the second term on the right-hand side, we use the following inequality: for any finite sequence of positive numbers $(b_i)_{0 \leq i \leq n} \in \mathbb{R}_+^{n+1}$ and any $t \geq 1$,

$$\sum_{i=0}^n b_i^t \leq \left(\sum_{i=0}^n b_i \right)^t$$

to obtain

$$\begin{aligned}
\sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{|\frac{4}{3}\square|} \int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} \gamma_R^{2+\varepsilon}(x) dx \right)^{\frac{2}{2+\varepsilon}} &\leq C \sum_{\square \in \mathcal{U}} |\square|^{1-\frac{2}{2+\varepsilon}} \int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} \gamma_R^2(x) dx \\
&\leq C \sum_{x \in \mathcal{L}_\infty} \gamma_R(x)^2 \text{size}(\square_{\mathcal{U}}(x))^{d(1-\frac{2}{2+\varepsilon})} \\
&\leq C \sum_{x \in \mathbb{Z}^d} \zeta_R(x)^2 \text{size}(\square_{\mathcal{P}}(x))^{4d} \text{size}(\square_{\mathcal{U}}(x))^{d(1-\frac{2}{2+\varepsilon})}.
\end{aligned}$$

Using that $\text{size}(\square_{\mathcal{U}}(x)), \text{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}'_s(C)$ and (1.10) we obtain

$$\sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{|\frac{4}{3}\square|} \int_{\frac{4}{3}\square \cap \mathcal{L}_\infty} \gamma_R^{2+\varepsilon}(x) dx \right)^{\frac{2}{2+\varepsilon}} \leq \mathcal{O}'_s \left(\frac{C}{R^d} \right)$$

and the proof of Result 2, and thus of Proposition 3.3, is complete.

4. OPTIMAL L^q ESTIMATES FOR FIRST ORDER CORRECTORS

We now show how to obtain the L^q optimal scaling bounds on the correctors, Theorem 1, from Proposition 3.3. Theorem 1, is restated below and proved in this section.

Theorem 1 (Optimal L^q estimates for first order correctors). *There exist two exponents $s := s(d, \mathbf{p}, \lambda) > 0$, $k := k(d, \mathbf{p}, \lambda) < \infty$ and a constant $C(d, \mathbf{p}, \lambda) < \infty$ such that for each $R \geq 1$, each $q \geq 1$ and each $p \in \mathbb{R}^d$,*

$$(4.1) \quad \left(R^{-d} \int_{\mathcal{C}_\infty \cap B_R} \left| \chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C |p| q^k \log^{\frac{1}{2}} R \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C |p| q^k \right) & \text{if } d \geq 3. \end{cases}$$

Before starting the proof, we mention an important caveat. In this section we need to keep track of the dependence on the parameter q of the constants. We will thus be careful to track every dependence in the q variable. This will be useful in the next section to obtain the L^∞ bounds on the corrector. In particular in this section the exponent k may vary from line to line but will always remain finite and will depend solely on the variables d, \mathbf{p}, λ .

Proof of Theorem 1. As in the proof of Proposition 3.3, we assume that $|p| = 1$ to simplify the computations. Additionally, note that by the Jensen inequality, it is enough to prove Theorem 1 in the case $q \geq 2$. We consequently make this assumption for the rest of the proof. The proof of this theorem is split into two steps.

- In Step 1, we use Proposition 3.3 and the multiscale Poincaré inequality, Proposition 2.19, to show, for each $R \geq 1$,

$$\left(\int_{B_R} \left| [\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{B_R} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \log^{\frac{1}{2}} R \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3, \end{cases}$$

with C, k and s depending only on s, \mathbf{p}, λ .

- In Step 2, we remove the coarsening, thanks to Proposition 2.9, to eventually obtain

$$\left(R^{-d} \int_{\mathcal{C}_\infty \cap B_R} \left| \chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \log^{\frac{1}{2}} R \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3. \end{cases}$$

This is precisely (4.1).

Step 1. Fix some $R \geq 1$. The main idea of this step is to apply Proposition 2.19 to the function $u = [\chi_p]_{\mathcal{P}}^\eta$. The assumption of Proposition 2.19 is clearly satisfied (it is a consequence of the construction of $[\chi_p]_{\mathcal{P}}^\eta$ and of the sublinearity property (1.15)). Consequently, we have, for each $R \geq 1$,

$$\left\| [\chi_p]_{\mathcal{P}}^\eta - ([\chi_p]_{\mathcal{P}}^\eta)_{B_R} \right\|_{L^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}.$$

To study the term on the right-hand side, we split the interior integral into two terms

$$(4.2) \quad \int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr = \int_0^1 r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr + \int_1^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr.$$

But, by Proposition 3.3, we know that for each $r \geq 1$ and each $x \in \mathbb{R}^d$,

$$|\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)| \leq \mathcal{O}_s \left(C r^{-\frac{d}{2}} \right).$$

This implies,

$$|\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 \leq \mathcal{O}_s \left(C r^{-d} \right).$$

The second term on the right-hand side can be estimated by using Proposition 3.3 and the inequality (1.10), this yields

$$\int_1^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \leq \begin{cases} \mathcal{O}_s(C \log R) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \geq 3. \end{cases}$$

To estimate the first term on the right-hand side of (4.2), we use (3.2) which reads, for each $x \in \mathbb{R}^d$

$$|\nabla [\chi_p]_{\mathcal{P}}^\eta(x)| \leq \mathcal{O}_s(C).$$

By this and (1.10), we obtain

$$\int_0^1 r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \leq \mathcal{O}_s(C).$$

Combining the previous displays shows

$$\int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \leq \begin{cases} \mathcal{O}_s(C \log R) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \geq 3. \end{cases}$$

for some exponent $s = s(d, \mathbf{p}, \lambda) > 0$ and some constant $C := C(d, \mathbf{p}, \lambda) < \infty$ depending only on the parameters d, \mathbf{p} and λ . We then obtain

$$\left(\int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \right)^{\frac{q}{2}} \leq \begin{cases} \mathcal{O}_{\frac{2s}{q}} \left(C^{\frac{q}{2}} (\log R)^{\frac{q}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_{\frac{2s}{q}} \left(C^{\frac{q}{2}} \right) & \text{if } d \geq 3. \end{cases}$$

We then apply (1.10) and keep track of the constant thanks to (1.11), we obtain

$$\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \right)^{\frac{q}{2}} dx \leq \begin{cases} \mathcal{O}_{\frac{2s}{q}} \left(\left(\frac{q}{s \ln(2)} \right)^{\frac{q}{s}} C^{\frac{q}{2}} (\log R)^{\frac{q}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_{\frac{2s}{q}} \left(\left(\frac{q}{s \ln(2)} \right)^{\frac{q}{s}} C^{\frac{q}{2}} \right) & \text{if } d \geq 3. \end{cases}$$

This eventually yields

$$\left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla [\chi_p]_{\mathcal{P}}^\eta(x)|^2 dr \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(q^{\frac{1}{s}} C (\log R)^{\frac{1}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(q^{\frac{1}{s}} C \right) & \text{if } d \geq 3. \end{cases}$$

For some constant $C := C(d, \mathbf{p}, \lambda) < \infty$ and some exponent $s := s(d, \mathbf{p}, \lambda) > 0$ depending only on d, \mathbf{p}, λ and not on q . We now set $k := \frac{1}{s} + \frac{3}{4}$. This exponent depends only on the parameters d, \mathbf{p}, λ . By applying Proposition 2.19, we obtain

$$(4.3) \quad \left(\int_{B_R} |[\chi_p]_{\mathcal{P}}^\eta - ([\chi_p]_{\mathcal{P}}^\eta)_{B_R}|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \log^{\frac{1}{2}} R \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3, \end{cases}$$

for some exponents $s := s(d, \mathbf{p}, \lambda) > 0$, $k := k(d, \mathbf{p}, \lambda) > 0$ and a constant $C := C(d, \mathbf{p}, \lambda) < \infty$. The next goal is to remove the regularization by convolution par η . We first apply (3.2) to obtain

$$\left(\int_{B_R} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{B_R}|^q \right)^{\frac{1}{q}} \leq C \left(\int_{B_R} |[\chi_p]_{\mathcal{P}}^\eta - ([\chi_p]_{\mathcal{P}}^\eta)_{B_R}|^q \right)^{\frac{1}{q}}.$$

Note that by the triangle inequality and Jensen inequality, we have

$$\begin{aligned} \left(\int_{B_R} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{B_R}|^q \right)^{\frac{1}{q}} &\leq 2 \inf_{a \in \mathbb{R}} \left(\int_{B_R} |[\chi_p]_{\mathcal{P}} - a|^q dx \right)^{\frac{1}{q}} \\ &\leq 2 \left(\int_{B_R} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{B_R}|^q \right)^{\frac{1}{q}} \\ &\leq C \left(\int_{B_R} |[\chi_p]_{\mathcal{P}}^\eta - ([\chi_p]_{\mathcal{P}}^\eta)_{B_R}|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Combining the previous estimates completes the proof of Step 1.

Step 2. We remove the coarsening, thanks to Proposition 2.9. We split the L^q norm of the corrector into two terms,

$$\begin{aligned} \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} &\leq \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\quad + \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

The reason we use the indicator $\mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}}$ is to be able to apply (2.5) in the computation below. But first, we estimate the first term on the right-hand side, to do so we can use the L^∞ bound (1.14) (applied with $\delta = 0$ to simplify the computation), this gives:

$$\begin{aligned} \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} &\leq \|\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}\|_{L^\infty(\mathcal{C}_\infty \cap \square_m)} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\leq \mathcal{O}_s(C3^m) \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

Since $\mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \leq \mathcal{O}_s(3^{-m})$, we obtain, for some possibly smaller exponent s (depending only on d, \mathfrak{p}),

$$(4.4) \quad \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \leq \mathcal{O}_s(C).$$

To estimate the second term in the right-hand side, we compute

$$\begin{aligned} (4.5) \quad &\left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\leq C \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - [\chi_p]_{\mathcal{P}}|^q(x) dx \right)^{\frac{1}{q}} \\ &\quad + C \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \left(\int_{\mathcal{C}_\infty \cap \square_m} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}}. \end{aligned}$$

To estimate the first term on the right-hand side, we first use (2.4) and Proposition 2.9, to obtain for each $m \in \mathbb{N}$ such that $\square_m \in \mathcal{P}_*$

$$\begin{aligned} \int_{\mathcal{C}_\infty \cap \square_m} |\chi_p - [\chi_p]_{\mathcal{P}}|^q(x) dx &\leq \int_{\mathcal{C}_*(\square_{m+1})} |\chi_p - [\chi_p]_{\mathcal{P}}|^q(x) dx \\ &\leq C \int_{\mathcal{C}_*(\square_{m+1})} \text{size}(\square_{\mathcal{P}}(x))^{qd} |\nabla \chi_p|^q(x) dx \\ &\leq C \int_{\mathcal{C}_\infty \cap \square_{m+1}} \text{size}(\square_{\mathcal{P}}(x))^{qd} |\nabla \chi_p|^q(x) dx. \end{aligned}$$

By the bound on the gradient of the corrector (3.5) and the property of the partition \mathcal{P} (2.3), we have, for each $x \in \mathbb{Z}^d$

$$\text{size}(\square_{\mathcal{P}}(x))^d |\nabla \chi_p|(x) \mathbb{1}_{\{x \in \mathcal{C}_\infty\}} \leq \mathcal{O}_s(C),$$

and thus

$$\text{size}(\square_{\mathcal{P}}(x))^{qd} |\nabla \chi_p|^q(x) \mathbb{1}_{\{x \in \mathcal{C}_\infty\}} \leq \mathcal{O}_{\frac{s}{q}}(C^q).$$

Consequently, by (1.10) and using (1.11) to keep track of the dependence of the constants in the q variable

$$\begin{aligned} \int_{\mathcal{C}_\infty \cap \square_{m+1}} \text{size}(\square_{\mathcal{P}}(x))^{qd} |\nabla \chi_{\mathcal{P}}|^q(x) dx &= \sum_{x \in \square_{m+1}} \text{size}(\square_{\mathcal{P}}(x))^{qd} |\nabla \chi_{\mathcal{P}}|^q(x) \mathbb{1}_{\{x \in \mathcal{C}_\infty\}} \\ &\leq \mathcal{O}_{\frac{s}{q}} \left(3^{d(m+1)} \left(\frac{q}{s \ln(2)} \right)^{\frac{q}{s}} C^q \right) \\ &\leq \mathcal{O}_{\frac{s}{q}} \left(3^{d(m+1)} q^{\frac{q}{s}} C^q \right). \end{aligned}$$

In particular, if 3^m is larger than $\mathcal{M}_{2d}(\mathcal{P})$, then the cube \square_m belongs to \mathcal{P}_* , the previous computations consequently show

$$\mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \int_{\mathcal{C}_\infty \cap \square_m} |\chi_{\mathcal{P}} - [\chi_{\mathcal{P}}]_{\mathcal{P}}|^q(x) dx \leq \mathcal{O}_{\frac{s}{q}} \left(3^{d(m+1)} q^{\frac{q}{s}} C^q \right).$$

Then by (2.5), we obtain

$$\mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_{\mathcal{P}} - [\chi_{\mathcal{P}}]_{\mathcal{P}}|^q(x) dx \right)^{\frac{1}{q}} \leq \mathcal{O}_s \left(q^{\frac{1}{s}} C \right).$$

To estimate the second term on the right-hand side of (4.5), we compute, by (2.5)

$$\begin{aligned} &\left(\int_{\mathcal{C}_\infty \cap \square_m} \left| [\chi_{\mathcal{P}}]_{\mathcal{P}} - ([\chi_{\mathcal{P}}]_{\mathcal{P}})_{\mathcal{C}_\infty \cap \square_m} \right|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\leq 2 \inf_{a \in \mathbb{R}} \left(\int_{\mathcal{C}_\infty \cap \square_m} |[\chi_{\mathcal{P}}]_{\mathcal{P}} - a|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\leq 2 \left(\int_{\mathcal{C}_\infty \cap \square_m} \left| [\chi_{\mathcal{P}}]_{\mathcal{P}} - ([\chi_{\mathcal{P}}]_{\mathcal{P}})_{\square_m} \right|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\leq C \left(\int_{\square_m} \left| [\chi_{\mathcal{P}}]_{\mathcal{P}} - ([\chi_{\mathcal{P}}]_{\mathcal{P}})_{\square_m} \right|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

We then apply (4.3) and obtain

$$\left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \leq \begin{cases} \mathcal{O}_s \left(q^k C m^{\frac{1}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(q^k C \right) & \text{if } d \geq 3, \end{cases}$$

for some exponents $k := k(d, \mathbf{p}, \lambda)$, $s := s(d, \mathbf{p}, \lambda) > 0$ and some constant $C := C(d, \mathbf{p}, \lambda) < \infty$. Combining this with (4.4), we obtain

$$\left(\int_{\mathcal{C}_\infty \cap \square_m} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap \square_m}|^q(x) dx \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(q^k C m^{\frac{1}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(q^k C \right) & \text{if } d \geq 3, \end{cases}$$

For a general $R \geq 1$, let m be the integer such that $3^m < R \leq 3^{m+1}$, and compute

$$\begin{aligned} \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} &\leq \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\quad + \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

The first term is estimated as in (4.4) and we obtain

$$\left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_{\mathcal{P}} - (\chi_{\mathcal{P}})_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \leq \mathcal{O}_s(C).$$

recall that

$$\left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \leq 2 \inf_{a \in \mathbb{R}} \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - a|^q(x) dx \right)^{\frac{1}{q}}.$$

To estimate the second term, we use (2.5) and compute

$$\begin{aligned} & \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \\ & \leq 2 \inf_{a \in \mathbb{R}} \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - a|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \\ & \leq 2 \left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_{m+1}}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}} \\ & \leq C \left(\int_{\mathcal{C}_\infty \cap \square_{m+1}} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap \square_{m+1}}|^q(x) dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \leq \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

Combining the few previous displays shows

$$\left(\int_{\mathcal{C}_\infty \cap B_R} |\chi_p - (\chi_p)_{\mathcal{C}_\infty \cap B_R}|^q(x) dx \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(q^k C (\log R)^{\frac{1}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_s (q^k C) & \text{if } d \geq 3, \end{cases}$$

and completes the proof of Theorem 1. \square

5. OPTIMAL L^∞ ESTIMATES FOR THE FIRST ORDER CORRECTORS

In this section, we prove the L^∞ bound on the corrector, Theorem 2.

Theorem 2 (Optimal L^∞ estimates for first order correctors). *There exists an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a constant $C := C(d, \mathbf{p}, \lambda) < \infty$ such that for each $x, y \in \mathbb{Z}^d$ and each $p \in \mathbb{R}^d$,*

$$|\chi_p(x) - \chi_p(y)| \mathbb{1}_{\{x, y \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s \left(C \log^{\frac{1}{2}} |p| |x - y| \right) & \text{if } d = 2, \\ \mathcal{O}_s (C |p|) & \text{if } d \geq 3. \end{cases}$$

Proof of Theorem 2. First by the stationnarity of the gradient of the corrector, we can assume without loss of generality that $y = 0$. Without loss of generality, we can also assume $|p| = 1$, as it was done in the proofs of Proposition 3.3 and of Theorem 1. We thus want to prove, for each $x \in \mathbb{Z}^d$,

$$|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0, x \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s \left(C \log^{\frac{1}{2}} |x| \right) & \text{if } d = 2, \\ \mathcal{O}_s (C) & \text{if } d \geq 3. \end{cases}$$

Before starting the proof, note that, for every $q > 0$ and every $x \in \mathbb{R}_+$

$$\exp(x) \geq \frac{x^q}{q^q \exp(-q)}.$$

This implies, for each $s, q, \theta > 0$,

$$(5.1) \quad X \leq \mathcal{O}_s(\theta) \implies \mathbb{E}[X^q] \leq 2\theta^q \left(\frac{q}{s} \right)^{\frac{q}{s}} \exp\left(\frac{q}{s} \right).$$

We split the proof into six steps.

- In Step 1, we prove that for each $q \geq 1$ and each $m \in \mathbb{N}$,

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in y + \square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \leq \begin{cases} C^q q^q m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^q & \text{if } d \geq 3. \end{cases}$$

- In Step 2, we use the result of Step 1 to prove that for each $q \geq 1$ and each $m \in \mathbb{N}$,

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

Note that this statement is not just a consequence of Step 1 and the stationarity of the corrector since the partition \mathcal{P} is not stationary. One additional argument is needed to conclude.

- In Step 3, we prove that for each $q \geq 1$ and $m \in \mathbb{N}$, chosen such that $3^m \leq |x| < 3^{m+1}$,

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

- In Step 4, we combine Steps 2 and 3 to obtain, for each $q \geq 1$

$$(5.2) \quad \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

- In Step 5, we prove that there exist an exponent $s := s(d, \mathfrak{p}, \lambda) > 0$ and a constant $C := C(d, \mathfrak{p}, \lambda) < \infty$ such that

$$(5.3) \quad \left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right| \leq \begin{cases} \mathcal{O}_s \left(C \log^{\frac{1}{2}} |x| \right) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \geq 3. \end{cases}$$

- In Step 6, we remove the coarsening and eventually show that

$$\left| \chi_p(x) - \chi_p(0) \right| \mathbb{1}_{\{0, x \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s \left(C \log^{\frac{1}{2}} |x| \right) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \geq 3. \end{cases}$$

Step 1. The main tool of this step is the following inequality which was proved in Step 1 of the proof of Theorem 1, for each $m \in \mathbb{N}$, and each $q \geq 1$,

$$(5.4) \quad \left(\int_{\square_m} \left| [\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{\square_m} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \sqrt{m} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3. \end{cases}$$

Note that this implies, by increasing the values of C and k ,

$$(5.5) \quad \mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{\square_m} \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

For some fixed $y \in \mathbb{Z}^d$, note that by stationarity of the corrector (1.16), for almost every $\mathbf{a} \in \Omega$, one has

$$\left([\chi_p]_{\mathcal{P}}(-y) - ([\chi_p]_{\mathcal{P}})_{\square_m} \right) (\mathbf{a}) = \left([\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} \right) (\tau_y \mathbf{a}),$$

where we recall the notation $\mathcal{P}_y = y + \mathcal{P}(\tau_{-y} \mathbf{a})$. Using the stationarity property (1.4), we obtain, for each $q \geq 1$,

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} \right|^q \right] = \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(-y) - ([\chi_p]_{\mathcal{P}})_{\square_m} \right|^q \right].$$

Since this is true for each $y \in \mathbb{Z}^d$, we can integrate over y to obtain

$$\int_{\square_m} \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} \right|^q \right] dy = \int_{\square_m} \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(-y) - ([\chi_p]_{\mathcal{P}})_{\square_m} \right|^q \right] dy.$$

Thus, by (5.5),

$$(5.6) \quad \mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} \right|^q dy \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

We now remove the translation of the partition and prove, for each $z \in \mathbb{Z}^d$

$$(5.7) \quad \left| [\chi_p]_{\mathcal{P}_y}(z) - [\chi_p]_{\mathcal{P}}(z) \right| \leq \mathcal{O}_s(C).$$

To prove this, note that, by definition of the coarsening (2.8), we have

$$[\chi_p]_{\mathcal{P}_y}(z) - [\chi_p]_{\mathcal{P}}(z) = \chi_p(\bar{z}(\square_{\mathcal{P}_y}(z))) - \chi_p(\bar{z}(\square_{\mathcal{P}}(z))),$$

and by definition of the two partitions \mathcal{P} and \mathcal{P}_y , there exists a path linking $\square_{\mathcal{P}_y}(z)$ to $\square_{\mathcal{P}}(z)$ which lies in $B(z, C \max(\text{size}(\square_{\mathcal{P}_y}(z)), \text{size}(\square_{\mathcal{P}}(z))))$. To simplify the notation in the following computation, we denote $R' = C \max(\text{size}(\square_{\mathcal{P}_y}(z)), \text{size}(\square_{\mathcal{P}}(z)))$. As a consequence, we have the estimate

$$\left| [\chi_p]_{\mathcal{P}_y}(z) - [\chi_p]_{\mathcal{P}}(z) \right| \leq \int_{\mathcal{E}_\infty \cap B_{R'}(z)} |\nabla \chi_p|(x) dx.$$

We then use the regularity theory to estimate (this computation is very similar to the one we run in (3.13))

$$(5.8) \quad \begin{aligned} \int_{\mathcal{E}_\infty \cap B_{R'}(z)} |\nabla \chi_p|(y) dy &\leq CR'^d + C \int_{\mathcal{E}_\infty \cap U} |p + \nabla \chi_p|(y) dy \\ &\leq CR'^d + C \int_{\mathcal{E}_\infty \cap B_{R'}(z)} |p + \nabla \chi_p|(y) dy \\ &\leq CR'^d + CR'^d \left(\int_{\mathcal{E}_\infty \cap B_{R'}(z)} |p + \nabla \chi_p|^2(y) dy \right)^{\frac{1}{2}} \\ &\leq CR'^d + CR'^{\frac{d}{2}} \max(\mathcal{X}(z), R')^{\frac{d}{2}} \liminf_{R \rightarrow \infty} \left(\int_{\mathcal{E}_\infty \cap B_R(z)} |p + \nabla \chi_p|^2(y) dy \right)^{\frac{1}{2}} \\ &\leq CR'^d + CR'^{\frac{d}{2}} \max(\mathcal{X}(z), R')^{\frac{d}{2}}. \end{aligned}$$

Now since $R' \leq \mathcal{O}_s(C)$ and $\mathcal{X}(z) \leq \mathcal{O}_s(C)$, we have

$$\int_{\mathcal{E}_\infty \cap B_{R'}(z)} |\nabla \chi_p|(y) dy \leq \mathcal{O}_s(C).$$

Combining the previous displays completes the proof of (5.7). To remove the parameter y in (5.6), we compute

$$(5.9) \quad \begin{aligned} \mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right|^q dy \right] \\ \leq 2^q \mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} \right|^q dy \right] \\ + 2^q \mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right|^q dy \right]. \end{aligned}$$

By (5.7) and (1.10), we have, for each $y \in \square_m$,

$$\left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right| \leq \mathcal{O}_s(C),$$

and thus

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right|^q \right] \leq C^q q^{qk}.$$

Integrating over $y \in \square_m$ yields

$$\mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}_y}(0) - ([\chi_p]_{\mathcal{P}_y})_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right|^q dy \right] \leq C^q q^{qk}.$$

By the previous display and (5.6), we have

$$\mathbb{E} \left[\int_{\square_m} \left| [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} \right|^q dy \right] \leq \begin{cases} C^q q^{qk} m^{\frac{d}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

By the Jensen inequality, we obtain

$$\mathbb{E} \left[\left| \int_{\square_m} [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} dy \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

but notice that

$$\int_{\square_m} [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+\square_m} dy = [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in y+\square_m} [\chi_p]_{\mathcal{P}}(z).$$

Combining the two previous displays completes the proof of Step 1.

Step 2. By the stationarity of the corrector (1.16), for almost every $\mathbf{a} \in \Omega$, every $y, z \in \mathbb{Z}^d$,

$$[\chi_p]_{\mathcal{P}}(z)(\mathbf{a}) = [\chi_p]_{\mathcal{P}_y}(z+y)(\tau_y \mathbf{a}).$$

Using this property, we have

$$\begin{aligned} \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_x}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}_x}(z) \right|^q \right] &= \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \\ &\leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases} \end{aligned}$$

Doing the same computation as in (5.9), we can replace \mathcal{P}_x by \mathcal{P} in the previous display, this yields

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

This completes the proof of the main estimate of Step 2.

Step 3. This step is similar to Step 1, but the main tool of this step is slightly different and presented below. For $m \in \mathbb{N}$ such that $3^m \leq |x| < 3^{m+1}$, and for each $q \geq 1$,

$$\left(\int_{\square_m} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m}|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s(Cq^k \sqrt{m}) & \text{if } d = 2, \\ \mathcal{O}_s(Cq^k) & \text{if } d \geq 3. \end{cases}$$

To prove this result, we note that $x + \square_m \subseteq \square_{m+2}$. With this in mind, we can compute

$$\begin{aligned} \left(\int_{\square_m} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m}|^q \right)^{\frac{1}{q}} &\leq C \left(\int_{\square_{m+2}} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m}|^q \right)^{\frac{1}{q}} \\ &\leq C \left(\int_{\square_{m+2}} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{\square_{m+2}}|^q \right)^{\frac{1}{q}} \\ &\quad + C \left| ([\chi_p]_{\mathcal{P}})_{\square_{m+2}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m} \right|. \end{aligned}$$

We estimate the second term on the right-hand side as follows

$$\begin{aligned} \left| ([\chi_p]_{\mathcal{P}})_{\square_{m+2}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m} \right| &= \left| \int_{x+\square_m} [\chi_p]_{\mathcal{P}}(x) - ([\chi_p]_{\mathcal{P}})_{\square_{m+2}} dx \right| \\ &\leq \int_{x+\square_m} |[\chi_p]_{\mathcal{P}}(x) - ([\chi_p]_{\mathcal{P}})_{\square_{m+2}}| dx \\ &\leq C \int_{\square_{m+2}} |[\chi_p]_{\mathcal{P}}(x) - ([\chi_p]_{\mathcal{P}})_{\square_{m+2}}| dx \\ &\leq C \left(\int_{\square_{m+2}} |[\chi_p]_{\mathcal{P}}(x) - ([\chi_p]_{\mathcal{P}})_{\square_{m+2}}|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Combining the two previous displays with (5.4) shows

$$\left(\int_{\square_m} |[\chi_p]_{\mathcal{P}} - ([\chi_p]_{\mathcal{P}})_{x+\square_m}|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s(Cq^k \sqrt{m}) & \text{if } d = 2, \\ \mathcal{O}_s(Cq^k) & \text{if } d \geq 3. \end{cases}$$

With the same proof as in Step 1, we obtain, for each $q \geq 1$

$$\mathbb{E} \left[\left| \int_{\square_m} [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+x+\square_m} dy \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

But note that

$$[\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) = \int_{\square_m} [\chi_p]_{\mathcal{P}}(0) - ([\chi_p]_{\mathcal{P}})_{y+x+\square_m} dy.$$

Combining the two previous displays completes the proof of Step 3.

Step 4. In this step, we first split the integral,

$$\begin{aligned} \mathbb{E} \left[|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|^q \right] &\leq 2^q \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] \\ &\quad + 2^q \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right]. \end{aligned}$$

Combining the results of Step 2 and Step 3, we have, for $m \in \mathbb{N}$ chosen such that $3^m \leq |x| \leq 3^{m+1}$ and for each $q \geq 1$,

$$\mathbb{E} \left[|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

Since $m \leq \frac{\log|x|}{\log 3}$, the proof of Step 3 is complete.

Step 5. First we extend the result of Step 4 to the case $0 < q < 1$. By the Jensen inequality, we have, for each $0 < q < 1$

$$\mathbb{E} \left[|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|^q \right] \leq \mathbb{E} \left[|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|^2 \right]^{\frac{q}{2}} \leq \begin{cases} C^q m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q & \text{if } d \geq 3. \end{cases}$$

We now prove the main result of this step. We first deal with the case $d = 2$. Select an exponent $s > 0$ depending only on d, \mathfrak{p}, λ such that $s < \frac{1}{k}$, where k is the exponent (depending only on d, \mathfrak{p}, λ) which appears in (5.2).

We then compute

$$\begin{aligned} \mathbb{E} \left[\exp \left(\left(\frac{|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|}{\log^{\frac{1}{2}} |x|} \right)^s \right) \right] &= \sum_{l=0}^{\infty} \frac{1}{l!} \mathbb{E} \left[\frac{|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|^{sl}}{\log^{\frac{sl}{2}} |x|} \right] \\ &\leq \sum_{l=0}^{\lfloor \frac{1}{s} \rfloor} \frac{C^{sl}}{l!} + \sum_{l=\lfloor \frac{1}{s} \rfloor}^{\infty} \frac{C^{sl} (sl)^{skl}}{l!} \\ &< \infty, \end{aligned}$$

by the Stirling formula. We now set $\sigma := \max \left(\frac{\log 2}{\log \left(\sum_{l=0}^{\lfloor \frac{1}{s} \rfloor} \frac{C^{sl}}{l!} + \sum_{l=\lfloor \frac{1}{s} \rfloor}^{\infty} \frac{C^{sl} (sl)^{skl}}{l!} \right)}, 1 \right) > 0$. Note that σ depends only on d, \mathfrak{p}, λ . We have

$$\begin{aligned} \mathbb{E} \left[\exp \left(\sigma \left(\frac{|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|}{\log^{\frac{1}{2}} |x|} \right)^s \right) \right] &\leq \mathbb{E} \left[\exp \left(\left(\frac{|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|}{\log^{\frac{1}{2}} |x|} \right)^s \right) \right]^{\sigma} \\ &\leq \left(\sum_{l=0}^{\lfloor \frac{1}{s} \rfloor} \frac{C^{sl}}{l!} + \sum_{l=\lfloor \frac{1}{s} \rfloor}^{\infty} \frac{C^{sl} (sl)^{skl}}{l!} \right)^{\sigma} \\ &\leq 2. \end{aligned}$$

From this computation, we obtain

$$|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)| \leq \mathcal{O}_s \left(\sigma^{-\frac{1}{s}} \log^{\frac{1}{2}} |x| \right).$$

Setting $C := \sigma^{-\frac{1}{s}}$, we obtain (5.3). The proof of Step 5 is complete.

Step 6. In this step, we remove the coarsening. To do so, we prove, for each $y \in \mathbb{Z}^d$

$$|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)| \mathbb{1}_{\{y \in \mathcal{C}_\infty\}} \leq \mathcal{O}_s(C).$$

To prove this note that if $y \in \mathcal{C}_\infty$ then there exists a path linking y to $\bar{z}(\square_{\mathcal{P}}(y))$ which lies in $\square_{\mathcal{P}}(y)$ and its neighbors. Consequently we have the estimate

$$|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)| \mathbb{1}_{\{y \in \mathcal{C}_\infty\}} \leq \int_{\mathcal{C}_\infty \cap B_{C \text{size}(\square_{\mathcal{P}}(y))}(y)} |\nabla \chi_p|(x) dx.$$

Then the same computation as in (5.8) shows

$$|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)| \mathbb{1}_{\{y \in \mathcal{C}_\infty\}} \leq \mathcal{O}_s(C).$$

From this we deduce

$$\begin{aligned} & |\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0, x \in \mathcal{C}_\infty\}} \\ & \leq |\chi_p(0) - [\chi_p]_{\mathcal{P}}(0)| \mathbb{1}_{\{0 \in \mathcal{C}_\infty\}} + |\chi_p(x) - [\chi_p]_{\mathcal{P}}(x)| \mathbb{1}_{\{x \in \mathcal{C}_\infty\}} + |[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)|. \end{aligned}$$

Combining the result of Step 5 with the previous displays shows

$$|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0, x \in \mathcal{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s \left(C \log^{\frac{1}{2}} |x| \right) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \geq 3. \end{cases}$$

The proof of Step 6 is complete. \square

APPENDIX A. PROOF OF THE L^q MULTISCALE POINCARÉ INEQUALITY

In this appendix, we prove the L^q multiscale Poincaré inequality.

Proposition 2.19 (Multiscale Poincaré inequality, heat kernel version). *For each $r > 0$, we define*

$$\begin{aligned} \Phi_r & : \mathbb{R}^d \rightarrow \mathbb{R} \\ x & \mapsto r^{-d} \exp\left(-\frac{|x|^2}{r^2}\right). \end{aligned}$$

for each $q \geq 2$, there exists a constant $C := C(d, q) < \infty$ such that for each tempered distribution $u \in W_{\text{loc}}^{1,q}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,

$$(A.1) \quad \|u - (u)_{B_R}\|_{\underline{L}^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|^2}{R^2}} \left(\int_0^R r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}.$$

Moreover the dependence on the variable p of the constant C can be estimated as follows, for each $q \geq 2$,

$$C(d, q) \leq Aq^{\frac{1}{2}}$$

for some constant $A := A(d) < \infty$.

Before starting the proof, we need to state the following proposition from [5, Proposition D.1 and Remark D.6] and to record a result from the elliptic regularity theory.

Proposition A.1 (Proposition D.1 and Remark D.6 of [5]). *For each $q \geq 2$, there exists a constant $C := C(d, q) < \infty$ such that for every tempered distribution $w \in \mathcal{S}'(\mathbb{R}^d)$,*

$$\|w\|_{W^{-1,q}(B_1)} \leq C \left(\int_{\mathbb{R}^d} e^{-|x|} \left(\int_0^1 r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Moreover the constant C satisfies, for each $q \geq 2$

$$C(d, q) \leq A\sqrt{q},$$

for some constant $A := A(d) < \infty$.

The dependence on the q variable of the constant C is not explicit in [5]. It can nevertheless be recovered by a careful investigation of the proof.

We then record a result from the theory of elliptic regularity, it can be found in [14, Lemma 7.12 and Proposition 9.9].

Proposition A.2 (Lemma 7.12 and Proposition 9.9 of [14]). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain of \mathbb{R}^d . Let $f \in L^p(\Omega)$, $1 < p < \infty$, and let w be the Newtonian potential of f , i.e.,*

$$w(x) := \int_{\Omega} \Gamma(x-y)f(y) dy,$$

where Γ is the fundamental solution of the Laplace equation, i.e.,

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \log|x| & \text{if } d = 2, \\ \frac{1}{d(2-d)\omega_d} |x|^{2-d} & \text{if } d \geq 3, \end{cases}$$

where ω_d is the volume of the unit sphere in \mathbb{R}^d . Then $w \in W^{2,p}(\Omega)$, $\Delta w = f$ a.e.,

$$\|\nabla^2 w\|_{L^p(\Omega)} \leq C_0 \|f\|_{L^p(\Omega)}$$

and

$$\|w\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)} \leq C_1 \|f\|_{L^p(\Omega)},$$

for some constants $C_1 := C_1(d, \Omega) < \infty$ and $C_0 := C_0(d, p, \Omega) < \infty$. Moreover, the dependence on p of the constant C_0 can be explicited:

$$C_0(d, p, \Omega) \leq Ap, \text{ if } p \geq 2 \quad \text{and} \quad C_0(d, p, \Omega) \leq A \frac{1}{p-1} \text{ if } 1 < p \leq 2,$$

for some $A := A(d, \Omega) < \infty$.

Before starting the proof, we mention that the dependence on the p variable is not explicit in [14, Proposition 9.9], but can be recovered by keeping track of the constant p in the application of the Marcinkiewicz interpolation theorem. Also the case of the logarithmic potential is not considered in [14, Lemma 7.12] (it is useful to obtain the estimate of the L^p norm of w in dimension 2). Nevertheless their proof is general enough to be applied in this setting.

Proof of Proposition 2.19. Let $\psi \in C_c^\infty(B_{\frac{1}{4}}, \mathbb{R})$ and $2 \leq q < \infty$. We denote by p the conjugate exponent of q , i.e, $p := \frac{q}{q-1} \in (1, 2]$. We split the proof into 5 steps.

- In Step 1, we show that there exists a constant $C := C(d, \psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,

$$\|u - \psi * u\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

- In Step 2, we prove that there exists a constant $C := C(d, \psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,

$$(A.2) \quad \|u - \psi * u(0)\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

- In Step 3, we prove that there exists a constant $C := C(d, q, \psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,

$$\|u\|_{L^q(B_{\frac{1}{2}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)} + C \|u\|_{W^{-1,q}(B_{\frac{3}{4}})}$$

and that the constant C satisfies $C(d, \psi, q) \leq Aq$ for some $A := A(d, \psi) < \infty$.

- In Step 4, we show that there exists a constant $C := C(d, q, \psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,

$$\left\| u - (u)_{B_{\frac{1}{2}}} \right\|_{L^q(B_{\frac{1}{2}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}$$

and that the constant C satisfies $C(d, \psi, q) \leq Aq$ for some $A := A(d, \psi) < \infty$.

- In Step 5, we show that for each tempered distribution $u \in W_{\text{loc}}^{1,q}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,

$$\|u - (u)_{B_R}\|_{L^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Step 1. We prove that there exists a constant $C := C(d) < \infty$ such that

$$\|u - u * \psi\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

Let $\psi \in C_c^\infty(B_{\frac{1}{4}}, \mathbb{R})$ and define, for $n \in \mathbb{N}$,

$$\psi_n := 2^{-dn} \psi \left(\frac{\cdot}{2^n} \right).$$

Since $\psi_n * u \rightarrow u$ in $L^q(B_{\frac{3}{4}})$, we can use the triangle inequality to bound

$$(A.3) \quad \|u - \psi * u\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq \sum_{n=0}^{\infty} \|\psi_{n+1} * u - \psi_n * u\|_{W^{-1,q}(B_{\frac{3}{4}})}.$$

Since the function $\psi_1 - \psi_0$ is compactly supported in $B_{\frac{1}{4}}$ and of mean 0, we can apply [5, Lemma 5.7], to show that there exists a function $\Psi \in C_c^\infty(B_{\frac{1}{4}}, \mathbb{R})$ satisfying

$$\nabla \cdot \Psi = \psi_1 - \psi_0.$$

For each $n \in \mathbb{N}$, we denote

$$\Psi_n := 2^{-dn} \Psi \left(\frac{\cdot}{2^n} \right),$$

by scaling invariance we also have

$$2^{-n} \nabla \cdot \Psi_n = \psi_{n+1} - \psi_n.$$

For each function $g \in W_0^{1,p}(B_{\frac{3}{4}})$, we have

$$\begin{aligned} \int_{(B_{\frac{3}{4}})} (\psi_{n+1} - \psi_n) * u(x) g(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\psi_{n+1} - \psi_n)(x-y) u(y) g(x) dx dy \\ &= 2^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \cdot \Psi_n(x-y) u(y) g(x) dx dy \\ &= 2^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi_n(x-y) \cdot \nabla u(y) g(x) dx dy \\ &= 2^{-n} \int_{\mathbb{R}^d} \nabla u(y) \cdot \left(\int_{\mathbb{R}^d} \Psi_n(x-y) g(x) dx \right) dy. \end{aligned}$$

By construction, the function $y \rightarrow \left(\int_{\mathbb{R}^d} \Psi_n(x-y)g(x) dx \right)$ is supported in B_1 , we can thus estimate

$$\left| \int_{B_{\frac{3}{4}}} (\psi_{n+1} - \psi_n) * u(x)g(x) dx \right| \leq 2^{-n} \left\| \left(\int_{\mathbb{R}^d} \Psi_n(x-\cdot)g(x) dx \right) \right\|_{W_0^{1,p}(B_1)} \|\nabla u\|_{W^{-1,q}(B_1)}.$$

Moreover, one can check that there exists a constant $C := C(d, \psi) < \infty$ such that

$$\left\| \left(\int_{\mathbb{R}^d} \Psi_n(x-\cdot)g(x) dx \right) \right\|_{W_0^{1,p}(B_1)} \leq C \|g\|_{W_0^{1,p}(B_1)} = C \|g\|_{W_0^{1,p}(B_{\frac{3}{4}})}.$$

Taking the supremum over $g \in W_0^{1,p}(B_{\frac{3}{4}})$ of norm 1 and combining this with (A.3), we obtain

$$\|u - \psi * u\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)},$$

for some constant $C := C(d) < \infty$. The proof of Step 1 is complete.

Step 2. We split the norm

$$(A.4) \quad \|u - \psi * u(0)\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq \|u - \psi * u\|_{W^{-1,q}(B_{\frac{3}{4}})} + \|\psi * u - \psi * u(0)\|_{W^{-1,q}(B_{\frac{3}{4}})}.$$

But note that, for each $x \in B_{\frac{3}{4}}$,

$$(A.5) \quad |\psi * u(x) - \psi * u(0)| \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

The proof of this estimate is very similar to the previous step, only simpler: by [5, Lemma 5.7], we represent $\psi(\cdot - x) - \psi$ in the form

$$\nabla \Psi_x = \psi(\cdot - x) - \psi$$

with $\Psi_x \in C_c^\infty(B_1, \mathbb{R})$ and then prove (A.5) thanks to an integration by parts. From this we deduce

$$\|\psi * u - \psi * u(0)\|_{W^{-1,q}(B_{\frac{3}{4}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

Combining this estimate with (A.4) and the estimate proved in the previous step completes the proof of Step 2.

Step 3. Let $\eta \in C_c^\infty(B_1)$ be a cutoff function satisfying

$$\mathbf{1}_{B_{\frac{1}{2}}} \leq \eta \leq \mathbf{1}_{B_{\frac{3}{4}}}, \quad \text{and} \quad |\nabla^2 \eta| + |\nabla \eta| \leq C.$$

For any function $f \in L^p(B_1)$, we denote by w_f the Newtonian potential of f introduced in Proposition A.2 with $\Omega = B_1$. We then compute

$$\begin{aligned} \int_{B_1} \eta(x)u(x)f(x) dx &= \int_{B_1} \eta(x)u(x)\Delta w_f(x) dx \\ &= \int_{B_1} \nabla(\eta u)(x)\nabla w_f(x) dx \\ &= \int_{B_1} \nabla \eta(x)u(x)\nabla w_f(x) + \eta(x)\nabla u(x)\nabla w_f(x) dx \\ &\leq \|u\|_{W^{-1,q}(B_{\frac{3}{4}})} \|\nabla \eta \nabla w_f\|_{W_0^{1,p}(B_{\frac{3}{4}})} + \|\nabla u\|_{W^{-1,q}(B_{\frac{3}{4}})} \|\eta \nabla w_f\|_{W_0^{1,p}(B_{\frac{3}{4}})}. \end{aligned}$$

By the properties of η and by Proposition A.2, we have

$$\|\nabla \eta \nabla w_f\|_{W_0^{1,p}(B_{\frac{3}{4}})} + \|\eta \nabla w_f\|_{W_0^{1,p}(B_{\frac{3}{4}})} = \|\nabla \eta \nabla w_f\|_{W_0^{1,p}(B_1)} + \|\eta \nabla w_f\|_{W_0^{1,p}(B_1)} \leq C \|f\|_{L^p(B_1)},$$

for some constant $C := C(d, p, \eta) < \infty$ satisfying

$$C(d, p, \eta) \leq A \frac{1}{p-1},$$

with $A := A(d, \eta) < \infty$. Consequently

$$\begin{aligned} \|u\|_{L^q(B_{\frac{1}{2}})} &\leq \|\eta u\|_{L^q(B_1)} = \sup_{f \in L^p(B_1), \|f\|_{L^p(B_1)}=1} \int_{B_1} \eta(x) u(x) f(x) dx \\ &\leq C \left(\|u\|_{W^{-1,q}(B_{\frac{3}{4}})} + \|\nabla u\|_{W^{-1,q}(B_{\frac{3}{4}})} \right) \\ &\leq C \left(\|u\|_{W^{-1,q}(B_{\frac{3}{4}})} + \|\nabla u\|_{W^{-1,q}(B_1)} \right). \end{aligned}$$

The proof of Step 3 is complete.

Step 4. Applying the main result of the previous step to the function $u - \psi * u(0)$, we have

$$\|u - \psi * u(0)\|_{L^q(B_{\frac{1}{2}})} \leq C \left(\|u - \psi * u(0)\|_{W^{-1,q}(B_{\frac{3}{4}})} + \|\nabla u\|_{W^{-1,q}(B_1)} \right).$$

Then by Step 2, we obtain

$$\|u - \psi * u(0)\|_{L^q(B_{\frac{1}{2}})} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

But we have by the Jensen inequality, for each $a \in \mathbb{R}$

$$\begin{aligned} \left\| u - (u)_{B_{\frac{1}{2}}} \right\|_{L^q(B_{\frac{1}{2}})} &\leq \|u - a\|_{L^q(B_{\frac{1}{2}})} + \left| a - (u)_{B_{\frac{1}{2}}} \right| \\ &\leq 2 \|u - a\|_{L^q(B_{\frac{1}{2}})}. \end{aligned}$$

Thus

$$\left\| u - (u)_{B_{\frac{1}{2}}} \right\|_{L^q(B_{\frac{1}{2}})} \leq 2 \inf_{a \in \mathbb{R}} \|u - a\|_{L^q(B_{\frac{1}{2}})} \leq 2 \|u - \psi * u(0)\|_{L^q(B_{\frac{1}{2}})}.$$

Combining the previous displays completes the proof of Step 4.

Step 5. Applying the result of Step 4 and Proposition A.1, we obtain, for each $q \geq 2$ and each $u \in \mathcal{S}'(\mathbb{R}^d) \cap W_{\text{loc}}^{1,q}(\mathbb{R}^d)$,

$$\left\| u - (u)_{B_{\frac{1}{2}}} \right\|_{L^q(B_{\frac{1}{2}})} \leq C \left(\int_{\mathbb{R}^d} e^{-|x|} \left(\int_0^1 r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}$$

For some constant $C := C(d, q)$ satisfying $C(d, q) \leq Aq^{\frac{3}{2}}$. Rescaling the previous estimates eventually shows

$$\|u - (u)_{B_R}\|_{L^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r |\Phi_r * \nabla u(x)|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

and the proof of Proposition 2.19 is complete. \square

APPENDIX B. PROOF OF LEMMA 3.6

In this appendix, we prove Lemma 3.6. We first restate the lemma.

Lemma 3.6 (Minimal scale). *There exists a constant $C := C(d, \mathbf{p}, \lambda) < \infty$, an exponent $s := s(d, \mathbf{p}, \lambda) > 0$ and a random variable $\mathcal{M}_1 \leq \mathcal{O}'_s(C)$ such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_1$,*

$$3^{-dm} \sum_{z \in \square_m} \text{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \leq C$$

where $\varepsilon := \varepsilon(d, \mathbf{p}, \lambda)$ is the exponent which appears in Proposition 2.14.

Proof of Lemma 3.6. First, notice that one can rewrite

$$\begin{aligned} & 3^{-dm} \sum_{z \in \square_m} \text{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \\ & \leq C 3^{-dm} \sum_{z \in \square_m} \text{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)+2}{\varepsilon}} \sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^{d \frac{2+\varepsilon}{\varepsilon}} \\ & \leq C 3^{-dm} \sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1, e \in \mathcal{B}_d^x} \text{size}(\square_{\mathcal{P}}(x))^{\frac{3d(2+\varepsilon)+2}{\varepsilon}} (1 + \mathcal{X}^e(x))^{d \frac{2+\varepsilon}{\varepsilon}} \end{aligned}$$

By (iv) of Proposition 2.1 applied with $t = \frac{6d(2+\varepsilon)+4}{\varepsilon}$, it is clear that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_t(\mathcal{P})$, we have

$$(1) \sup_{x \in \square_{m+1}} \text{size}(\square_{\mathcal{P}}(x)) \leq 3^{\frac{dm}{d+t}}, \text{ this implies in particular}$$

$$\{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1\} \subseteq \square_{m+1}.$$

(2) the following estimate

$$\begin{aligned} & \left(3^{-dm} \sum_{x \in \mathbb{Z}^d, \text{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1} \text{size}(\square_{\mathcal{P}}(x))^{\frac{6d(2+\varepsilon)+4}{\varepsilon}} \right)^{\frac{1}{2}} \leq C \left(3^{-d(m+1)} \sum_{x \in \square_{m+1}} \text{size}(\square_{\mathcal{P}}(x))^{\frac{6d(2+\varepsilon)+4}{\varepsilon}} \right)^{\frac{1}{2}} \\ & \leq C. \end{aligned}$$

Thus by the Cauchy-Schwarz inequality, it is enough to prove that there exists a random variable \mathcal{M} satisfying $\mathcal{M} \leq \mathcal{O}'_s(C)$, such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}$,

$$(B.1) \quad 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} (\mathcal{X}^e(x))^{\frac{d(4+2\varepsilon)}{\varepsilon}} \leq C.$$

Unfortunately, we cannot prove this exact statement but we will prove a slightly weaker estimate, Lemma B.1, which is still strong enough to prove Proposition 3.3. Define, for each $C > 0$, the random variable

$$\mathcal{X}_C := \inf \left\{ r \in [1, \infty) : \forall r', R' \in [r, \infty), \text{ with } r' \leq R', \forall u \in \mathcal{A}(\mathcal{C}_\infty \cap B_{R'}) \right. \\ \left. \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{r'})} \leq C \frac{r'}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{R'})} \right\},$$

and we similarly define, for each $x \in \mathbb{Z}^d$,

$$\mathcal{X}_C(x) := \mathcal{X}_C \circ \tau_x.$$

Denote by $C_0 := C_0(d, \mathbf{p}, \lambda) < \infty$ the constant appearing in Proposition 2.17. By definition we have

$$\mathcal{X}_{C_0} = \mathcal{X}.$$

Note also that \mathcal{X}_C is decreasing in C . With this new notation in mind, we have the following lemma.

Lemma B.1. *For every $t > 0$, there exist a constant $C(d, \mathbf{p}, \lambda, t) < \infty$, an exponent $s(d, \mathbf{p}, \lambda, t) > 0$ and a random variable \mathcal{M}_t satisfying*

$$\mathcal{M}_t^{\mathcal{X}} \leq \mathcal{O}'_s(C)$$

such that for every $m \in \mathbb{N}$ satisfying

$$3^m \geq \mathcal{M}_t^{\mathcal{X}}$$

the following inequality holds

$$3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{C_0}^e(x) \right|^t \leq C.$$

Remark B.2. (1) This statement is weaker than (B.1) since, for each $x \in \mathbb{Z}^d$ and $e \in \mathcal{B}_d^x$,

$$\mathcal{X}_{C_0^e}^e(x) \leq \mathcal{X}_{C_0}^e(x) = \mathcal{X}^e(x).$$

Nevertheless it is enough to prove Result 2, since we only need to replace C_0 by C_0^2 in every computation involving the estimates on the random variables $\mathcal{X}_p^e(x)$ and the result remain the same, only the value of the constants will be increased.

(2) Applying this result with $t = \frac{d(4+2\varepsilon)}{\varepsilon}$ completes the proof of Lemma 3.6. □

There remains to prove Lemma B.1 but before starting the proof, we need to introduce a few ingredients and preliminary results. First define, for $R, C \in [1, \infty)$, the random variable $\mathcal{X}_{R,C}$ by the formula,

$$(B.2) \quad \mathcal{X}_{R,C} := \inf \left\{ r \in [1, R] : \forall r', R' \in [r, R], \text{ with } r' \leq R', \forall u \in \mathcal{A}(\mathcal{C}_{\max}(B_R) \cap B_{R'}) \right. \\ \left. \|\nabla u\|_{\underline{L}^2(\mathcal{C}_{\max}(B_R) \cap B_{R'})} \leq C \frac{r'}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_{\max}(B_R) \cap B_{R'})} \right\},$$

Where $\mathcal{C}_{\max}(B_R)$ denotes the largest cluster contained in B_R . Similarly we define, for each $x \in \mathbb{Z}^d$,

$$\mathcal{X}_C(x) := \mathcal{X}_C \circ \tau_x.$$

Note that this random variable is defined on the enlarged probability space $\Omega \times \Omega$ and is measurable with respect to $\mathcal{F}(x + B_R) \otimes \{\emptyset, \Omega\}$ (it depends on the edges in the ball $x + B_R$ of the first variable and does not depend on the edges of the second variable).

The reason why we were careful to write $\mathcal{C}_{\max}(B_R)$ in (B.2) and not $\mathcal{C}_*(B_R)$ or $\mathcal{C}_\infty \cap B_R$ (these three clusters are morally equal for large R), is to constrain the random variable $\mathcal{X}_{R,C}$ be measurable with respect to $\mathcal{F}(B_R) \otimes \{\emptyset, \Omega\}$.

The only incentive of this quantity is that the random variable $\mathcal{X}_{R,C}$ is local (or is measurable with respect to $\mathcal{F}(B_R) \otimes \{\emptyset, \Omega\}$) and in particular the random variables $\mathcal{X}_{R,C}(x)$ and $\mathcal{X}_{R,C}(y)$ are independent as soon as $|x - y| > 2R$.

Note also that $\mathcal{X}_{R,C}$ is decreasing in the C variable and, for $R \geq \mathcal{M}_t(\mathcal{P})$, it is increasing in the R variable. We thus denote by, for each $C \geq 1$

$$\mathcal{X}_C := \lim_{R \rightarrow \infty} \mathcal{X}_{R,C} = \limsup_{R \geq 1} \mathcal{X}_{R,C} \in [1, \infty].$$

By Proposition 2.17, we know that there exists a constant $C_0 := C_0(d, \mathfrak{p}, \lambda) < \infty$ such that

$$(B.3) \quad \mathcal{X}_{C_0} = \mathcal{X} \leq \mathcal{O}'_s(C).$$

thus, for each $R > \mathcal{M}_t(\mathcal{P})$,

$$\mathcal{X}_{R,C_0} \leq \mathcal{X}_{C_0} \leq \mathcal{O}'_s(C).$$

Moreover, for each $R \in [1, \mathcal{M}_t(\mathcal{P})]$, we have

$$\mathcal{X}_{R,C_0} \leq \mathcal{M}_t(\mathcal{P}) \leq \mathcal{O}'_s(C).$$

Combining the two previous displays yields, for each $R \geq 1$,

$$\mathcal{X}_{R,C_0} \leq \mathcal{O}'_s(C).$$

We now prove the following inequality, for each $R, C > 1$,

$$(B.4) \quad \mathcal{X}_{C^2} \leq \mathcal{X}_{R,C} + R \mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbb{1}_{\{\mathcal{X}_C > R\}}.$$

We split the proof of this inequality into two cases.

Case 1. If $\mathcal{X}_C > R$, then since $C \geq 1$ and \mathcal{X}_C is decreasing in C , the inequality (B.4) follows from the computation

$$\mathcal{X}_{C^2} \leq \mathcal{X}_C \leq \mathcal{X}_{R,C} + R \mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbb{1}_{\{\mathcal{X}_C > R\}}.$$

Case 2. If $\mathcal{X}_C \leq R$ and $R \leq \mathcal{M}_t(\mathcal{P})$, then

$$\mathcal{X}_{C^2} \leq R \mathbf{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} \leq \mathcal{X}_{R,C} + R \mathbf{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbf{1}_{\{\mathcal{X}_C > R\}}.$$

Case 3. If $\mathcal{X}_C \leq R$ and $R \geq \mathcal{M}_t(\mathcal{P})$ then $\mathcal{C}_{\max}(B_R)$ is equal to the maximal connected component of $\mathcal{C}_\infty \cap B_R$ and we have, for each $r, R' > R$ with $R' \geq r$

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{R'})}.$$

Moreover, for each $r, R' \in [\mathcal{X}_{R,C}, R]$ with $R' \geq r$, we have

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{R'})}.$$

Recall that we picked C under the assumption $C \geq 1$ so that $C^2 \geq C$. Combining the two previous displays yields for each $r, R' \geq \mathcal{X}_{R,C}$ with $R' \geq r$,

$$\|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_r)} \leq C^2 \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_\infty \cap B_{R'})}$$

and thus by definition of \mathcal{X}_{C^2} ,

$$\mathcal{X}_{C^2} \leq \mathcal{X}_{R,C}$$

and the proof of the inequality (B.4) is complete.

For $x \in \mathbb{Z}^d$, $e = \{x, y\} \in \mathcal{B}_d$, $C, R \in [1, \infty)$, denote by $\mathcal{X}_{R,C}^e(x)$ the translated and resampled random variable

$$\mathcal{X}_{R,C}^e(x) := \inf \left\{ r \in [1, R] : \text{such that } \forall 1 \leq r' \leq R' \leq R, u \in \mathcal{A}^e(\mathcal{C}_{\max}^e(B_R(x)) \cap B_{R'}(x)) \right. \\ \left. \|\nabla u\|_{\underline{L}^2(\mathcal{C}_{\max}^e(B_R(x)) \cap B_{R'}(x))} \leq C \frac{r'}{R'} \|\nabla u\|_{\underline{L}^2(\mathcal{C}_{\max}^e(B_R(x)) \cap B_{R'}(x))} \right\}.$$

We also define, for each $x \in \mathbb{Z}^d$

$$\mathcal{X}_C^e(x) := \lim_{R \rightarrow \infty} \mathcal{X}_{R,C}^e(x) = \limsup_{R \geq 1} \mathcal{X}_{R,C}^e(x) \in [1, \infty].$$

The second ingredient in the proof of Lemma B.1 is the following minimal scale lemma. It is a slight modification of [7, Lemma 2.3] and will be used at the very end of the proof of Lemma B.1.

Lemma B.3. *Fix $K \geq 1$, $s > 0$ and $\beta > 0$ and suppose that $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of random variables satisfying, for every $n \in \mathbb{N}$,*

$$X_n \leq K + \mathcal{O}_s(K 3^{-n\beta}).$$

Then there exists $C(s, \beta, K) < \infty$ such that the random scale

$$M := \sup \{3^n \in \mathbb{N} : X_n \geq K + 1\}$$

satisfies the estimate

$$M \leq \mathcal{O}_{s\beta}(C).$$

Proof. This result can be deduced by applying [7, Lemma 2.3] to the sequence of random variables $X'_n = \max(X_n - C_0, 0)$. \square

We now turn to the proof of Lemma B.1.

Proof of Lemma B.1. Fix $t \in (0, \infty)$ and $m, n \in \mathbb{N}$ with $m > n$. Using (B.4), we have

$$(B.5) \quad 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{C_0}^e(x) \right|^t \\ \leq C 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0}^e(x) \right|^t + C 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{C_0}^e(x) \right|^t \mathbf{1}_{\{\mathcal{X}_{C_0}^e(x) > 3^n\}} \\ + C 3^{-dm} \sum_{x \in \square_m} 3^{tn} \mathbf{1}_{\{3^n \leq \mathcal{M}_t(\mathcal{P})\}} \circ \tau_x.$$

Since $\mathcal{X}_{C_0}^e(x) \leq \mathcal{O}'_s(C)$, for every $t, t' > 0$, there exists an exponent $s'(d, \mathbf{p}, \lambda, t, t') > 0$ and a constant $C'(d, \mathbf{p}, \lambda, t, t') < \infty$ such that

$$3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{C_0}^e(x)|^t \mathbb{1}_{\{\mathcal{X}_{C_0}^e(x) > 3^n\}} \leq \mathcal{O}'_{s'}(C' 3^{-nt'})$$

and

$$3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} 3^{nt} \mathbb{1}_{\{3^n \geq \mathcal{M}_t(\mathcal{P})\}} \circ \tau_x \leq \mathcal{O}'_{s'}(C' 3^{-nt'}).$$

Combining the previous displays yields

$$3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{C_0}^e(x)|^t \leq C 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t + \mathcal{O}'_{s'}(C' 3^{-nt'}).$$

Moreover, notice that by definition of the localized random variable $\mathcal{X}_{3^n, C_0}^e(x)$, we have for each $x \in \mathbb{Z}^d$

$$\sum_{e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t \leq 2d \times 3^{nt}.$$

The proof of the lemma is then the same as the proof of Steps 1 and 2 of [7, Proposition 2.1] with $3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{C_0}^e(x)|^t dx$ instead of $\Lambda_t(z + \square_m, \mathcal{S})$ and $3^{-dm} \sum_{x \in z + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t dx$ instead of $\Lambda_t(z' + \square_n, \mathcal{S}_{\text{loc}}(z'))$. We rewrite it for completeness.

We denote

$$Z := 3^{-dm} \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t = \frac{|\square_n|}{|\square_m|} \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} 3^{-dm} \sum_{x \in z + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t.$$

We first prove that there exists a constant $C := C(d, \mathbf{p}, \lambda, t) < \infty$ such that

$$(B.6) \quad Z \leq C + \mathcal{O}'_1(3^{nt-d(m-n)}).$$

To do so, choose an enumeration $\{z^j : 1 \leq j \leq 3^{d(m-n-2)}\}$ of the elements of the set $3^{n+2} \mathbb{Z}^d \cap \square_m$. Next, for each $1 \leq j \leq 3^{d(m-n-2)}$, we let $\{z^{i,j} : 1 \leq i \leq 3^{2d}\}$ be an enumeration of the elements of the set $3^n \mathbb{Z}^d \cap (z^j + \square_{n+2})$, such that, for every $1 \leq j, j' \leq 3^{d(m-n-2)}$ and $1 \leq i \leq 3^{2d}$, $z^j - z^{j'} = z^{i,j} - z^{i,j'}$. The point of this is that, for every $1 \leq i \leq 3^{2d}$ and $1 \leq j < j' \leq 3^{d(m-n-2)}$, we have $\text{dist}(z^{i,j} + \square_n, z^{i,j'} + \square_n) \geq 3^{n+1}$ and therefore, $3^{-dm} \sum_{x \in z^{i,j} + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t$ and $3^{-dm} \sum_{x \in z^{i,j'} + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t$ are independent. Now fix $h > 0$ and compute, using the Hölder inequality and the independence

$$\begin{aligned} & \log \mathbb{E}[\exp(h 3^{-nt} Z)] \\ &= \log \mathbb{E} \left[\prod_{i=1}^{3^{2d}} \prod_{j=1}^{3^{d(m-n-2)}} \exp \left(h 3^{-nt-d(m-n)} 3^{-dm} \sum_{x \in z^{i,j} + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t \right) \right] \\ &\leq 3^{-2d} \sum_{i=1}^{3^{2d}} \log \mathbb{E} \left[\prod_{j=1}^{3^{d(m-n-2)}} \exp \left(h 3^{-nt-d(m-n-2)} 3^{-dm} \sum_{x \in z^{i,j} + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t \right) \right] \\ &\leq 3^{-2d} \sum_{i=1}^{3^{2d}} \sum_{j=1}^{3^{d(m-n-2)}} \log \mathbb{E} \left[\exp \left(h 3^{-nt-d(m-n-2)} 3^{-dm} \sum_{x \in z^{i,j} + \square_n, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t \right) \right]. \end{aligned}$$

This inequality can be rewritten

$$\log \mathbb{E} \left[\exp(h3^{-nt} Z) \right] \leq 3^{-2d} \sum_{z' \in 3^n \mathbb{Z}^d \cap (z + \square_m)} \log \mathbb{E} \left[\exp \left(h3^{-nt-d(m-n-2)} 3^{-dm} \sum_{x \in z' + \square_n, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0^2}^e(x) \right|^t \right) \right].$$

Next we use the elementary inequality

$$\forall y \in [0, 1], \quad \exp(y) \leq 1 + 2y$$

to get, for every $h \in [0, (2d)^{-t} 3^{d(m-n-2)}]$,

$$\exp \left(h3^{-nt-d(m-n-2)} \sum_{x \in z' + \square_n, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0^2}^e(x) \right|^t \right) \leq 1 + 2h3^{-nt-d(m-n-2)} \sum_{x \in z' + \square_n, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0^2}^e(x) \right|^t.$$

Taking the expectation of this, applying the previous display and using the elementary inequality

$$\forall y \geq 0, \quad \log(1 + y) \leq y,$$

we get

$$\begin{aligned} \log \mathbb{E} \left[\exp(h3^{-nt} Z) \right] &\leq 3^{d(m-n)} \log \left(1 + 2h3^{-nt-d(m-n-1)} \mathbb{E} \left[\sum_{x \in z' + \square_n, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0^2}^e(x) \right|^t \right] \right) \\ &\leq 2h3^{-nt+d} \mathbb{E} \left[\sum_{x \in z' + \square_n, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0^2}^e(x) \right|^t \right] \\ &\leq Ch3^{-nt}. \end{aligned}$$

Taking $h := (2d)^{-t} 3^{d(m-n-2)}$ yields

$$\mathbb{E} \left[\exp \left((2d)^{-t} 3^{d(m-n-2)-nt} Z \right) \right] \leq \exp \left(C3^{d(m-n)-nt} \right).$$

From this and Chebyshev's inequality, we obtain a constant C such that

$$\mathbb{P} [Z \geq C + h] \leq \exp \left(-hC^{-1} 3^{d(m-n)-nt} \right)$$

This implies (B.6).

Step 2. We complete the proof by applying union bounds. Combining (B.5) and (B.6) yields

$$\sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0}^e(x) \right|^t \leq C + \mathcal{O}_1 \left(C3^{nt-d(m-n)} \right) + \mathcal{O}_{s'} \left(C3^{-nt'} \right).$$

Choosing

$$n := \left\lceil \frac{dm}{d+t+1} \right\rceil \text{ and } t' = 1$$

so that the previous line becomes

$$\sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0}^e(x) \right|^t dx \leq C + \mathcal{O}'_1 \left(C3^{-\frac{d}{d+t+1}m} \right) + \mathcal{O}'_{s'} \left(C3^{-\frac{d}{d+t+1}m} \right).$$

Thus, by (1.13) and (1.9), we obtain that there exist two exponents $s := s(d, \mathbf{p}, \lambda, t) > 0$, $\beta := \beta(d, \mathbf{p}, \lambda, t) > 0$ and a constant $C_0 := C_0(d, \mathbf{p}, \lambda, t) < \infty$ such that

$$\sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0}^e(x) \right|^t dx \leq C_0 + \mathcal{O}'_s \left(C_0 3^{-\beta m} \right).$$

Define

$$\mathcal{M}_t^{\mathcal{X}} := \sup \left\{ 3^m : \sum_{x \in \square_m, e \in \mathcal{B}_d^x} \left| \mathcal{X}_{3^n, C_0}^e(x) \right|^t dx \geq C_0 + 1 \right\}$$

We want to prove

$$\mathcal{M}_t^{\mathcal{X}} \leq \mathcal{O}_{s\beta} (C)$$

This is exactly Lemma B.3 with $X_n = \sum_{x \in \square_m, e \in \mathcal{B}_d^x} |\mathcal{X}_{3^n, C_0}^e(x)|^t dx$ and $K = C_0$.

□

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